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The Effect of Gaps - An Exact Model for Short-Range Wakes

by

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This is really an Appendix to the report "Influence of gaps in the beam on single stream instabilities of the SPS bunches" by D. Möhl, this proceedings.

One expects that instabilities due to long-range wakes such as resistive-wall wakes would not be much affected by small gaps in the beam (bunched or continuous). On the other hand, we expect that instabilities due to short-range wakes would be affected, especially if the gap is long compared with the decay time of the wake. Surprisingly, this is not the case. Möhl shows that the growth rates remain substantially the same, even when the wake field decays several orders of magnitude during the gap. Because this result is so surprising, and because Möhl makes some approximations in his calculation, we present here an exact calculation for a simple model.

Consider M bunches and a wake force that couples a given bunch to the two preceding bunches. Then the amplitude of oscillation of bunch n satisfies

$$\ddot{\chi}_n + \omega_0^2 \chi_n = W \chi_{n+1} + \alpha W \chi_{n+2} \quad (1)$$

where W is the wake force due to bunch n + 1, and this is reduced by a factor  $\alpha$  for bunch n + 2. The eigenfrequencies for the coupled system are determined by

$$\begin{vmatrix} \lambda & 1 & \alpha & 0 & \dots & \dots & 0 \\ & \lambda & 1 & \alpha & 0 & \dots & \\ & & \lambda & 1 & \alpha & 0 & \\ & & & & \dots & \dots & \\ \alpha & 0 & & & & & \lambda & 1 \\ 1 & \alpha & 0 & \dots & \dots & \dots & \dots & \lambda \end{vmatrix} = 0 \quad (2)$$

where  $\lambda = \frac{\omega^2 - \omega_0^2}{W}$ . The exact solution of the cyclic determinant (2) is

$$\lambda_m = e^{2\pi i \frac{m}{M}} + a e^{4\pi i \frac{m}{M}} \quad (3)$$

Now make a gap by removing one bunch, in this case the  $M^{\text{th}}$  bunch. The eigenfrequencies are determined by the  $(M - 1)$  order determinant

$$D = \begin{vmatrix} \lambda & 1 & a & \dots & \dots & \dots & 0 \\ & \lambda & 1 & a & & & \\ & & \dots & \dots & \dots & & \\ & & & & & \dots & \lambda & 1 \\ a & 0 & \dots & \dots & \dots & \dots & \dots & \lambda \end{vmatrix} = 0 \quad (4)$$

This can be expanded as

$$D = \lambda \underbrace{\begin{vmatrix} \lambda & 1 & a & \dots & \dots & \dots & 0 \\ & \lambda & 1 & a & & & \\ & & \dots & \dots & \dots & & \\ & & & & & \dots & \lambda & 1 \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & \lambda \end{vmatrix}}_{\lambda^{M-2}} + (-1)^{M-2} a \underbrace{\begin{vmatrix} 1 & a & 0 & \dots & \dots & \dots & 0 \\ \lambda & 1 & a & & & & \\ & \lambda & 1 & a & & & \\ 0 & & & & & \dots & \lambda & 1 & a \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & \lambda & 1 \end{vmatrix}}_{P_{M-2}} \quad (5)$$

where the second  $(M - 2)$  order determinant is designated  $P_{M - 2}$ . In general

$$P_n = \frac{U^{n+1} - V^{n+1}}{U - V} \quad (6)$$

where  $U$  and  $V$  are the roots of  $x^2 - x + \alpha\lambda = 0$ . (See Thomas Muir, A Treatise on the Theory of Determinants, (Dover Publications, N.Y. 1960), p. 565, but his extra factor  $\frac{1}{2^{n+1}}$  is incorrect.) We find

$$x = \frac{1 \pm \sqrt{1 - 4\alpha\lambda}}{2} \quad (7)$$

We are interested in the case where  $\alpha \ll 1$ , so (7) becomes

$$x = \frac{1}{2} [1 \pm (1 - 2\alpha\lambda)]$$

and

$$\begin{aligned} U &= 1 - \alpha\lambda \\ V &= \alpha\lambda \end{aligned} \quad (8)$$

with

$$\begin{aligned} P_n &= \frac{(1 - \alpha\lambda)^{n+1} - (\alpha\lambda)^{n+1}}{1 - 2\alpha\lambda} \\ &\approx (1 - \alpha\lambda)^{n+1} \end{aligned}$$

The determinate D is therefore

$$D = \lambda^{M-1} - (-1)^{M-1} \alpha (1 - \alpha \lambda)^{M-1} = 0 \quad (9)$$

with the solution

$$\frac{\lambda_m}{1 - \alpha \lambda_m} = -\alpha^{\frac{1}{M-1}} e^{2\pi i \frac{m}{M-1}} \quad (10)$$

or

$$\lambda_m \approx -\alpha^{\frac{1}{M-1}} e^{2\pi i \frac{m}{M-1}} .$$

Because M is about 4000 for the SPS, the quantity  $\alpha^{\frac{1}{M-1}}$  is very near to unity unless  $\alpha$  is of order  $10^{-4000}$  or smaller. Thus, the gap will have no effect on the instability. Möhl arrives at a similar conclusion for the more general case where many bunches are missing and where the bunches are allowed to have different frequencies.

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