SI/Note BR/72-1
25.5.1972

TRANSVERSE RESISTIVE-WALL INSTABILITIES OF THE BUNCHED BEAM IN THE SPS

(prepared for the CERN Laboratory II 1972 Spring study)

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1. CONCLUSIONS

i) Single-bunch modes are harmless.

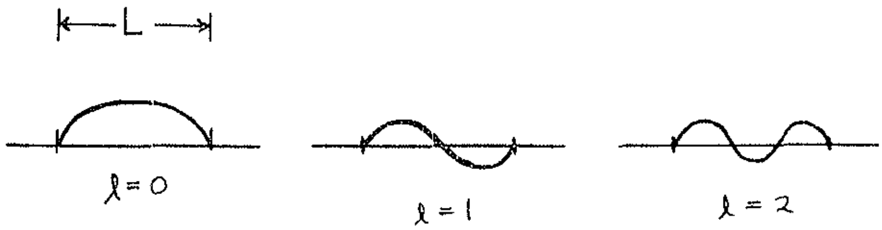
ii) Coupled-bunch modes are serious, with growth times of the order of 1.6 ms for 10^{13} particles at injection and increasing linearly with γ to 32 ms at 200 GeV.

iii) Sextupoles or the natural machine chromaticity have little or no effect on bunched beam instabilities.

iv) Octupole Q_z-spreads of order of $\pm$ 5 x 10^{-3} across the beam are sufficient to cure the instability.

2. REVIEW

The original calculations for bunched beam instabilities were made by Courant and Sessler[1)], who assumed that the bunch would move as a rigid unit. This was extended by Lee, Mills and Morton[2)] to include breathing motion and higher "throbbing beam" modes. Both calculations neglect synchrotron motion and require that the transverse motion be the same all along the bunch length. Further progress was made by Pellegrini[3)] and Sands[4)] who included synchrotron motion, the effect of machine chromaticity, and also the higher head-tail modes in which different parts of the bunch oscillate with different phases. Examples of these modes are the standing-wave patterns



(rigid-dipole mode)

that show the variation of transverse dipole moment along the bunch. The same is true for quadrupole and higher transverse multipole modes. If machine chromaticity or sextupole terms are included, the patterns acquire an additional traveling-wave component. As far as transverse oscillations are concerned, these modes form a complete set, which may be driven unstable by beam-equipment interactions such as resistive-wall, cavities, pick-up electrodes, etc.

If many bunches are present, all with the same frequency, the bunch phases will be locked together in patterns with an integral number of wave-lengths around the machine circumference. On the other hand, a sufficient spread in bunch frequencies prevents this phase-lock, and single-bunch modes result with a consequent reduction in growth rate.

3. FORMULA FOR SINGLE-BUNCH MODES

We consider only the dipole modes since they generally have the fastest growth rates. The resistance in a smooth, round vacuum chamber causes a growth rate

$$\frac{1}{\tau} = - \operatorname{Im} \Delta\omega$$

with

$$\Delta\omega = - \Delta\omega_W \, C_\ell \left[G(2\pi, Q) \, F_\ell'(\chi) + \sqrt{\frac{2\pi R}{L}} \, F_\ell^*(\chi) \right], \qquad (1)$$

where F_ℓ' and F_ℓ are form factors that depend on the type of mode - shown in Figs. 1, 2 and 3

$G(2\pi, Q)$ is the bunch function of Courant and Sessler

$C_\ell = 1$ for $\ell = 0$ and falls approximately as $\frac{1}{\ell + 1}$ for the higher modes

$$\chi = Q \frac{L}{R} \frac{\xi}{\eta} \qquad \begin{cases} \xi = \dfrac{P}{Q} \dfrac{\partial Q}{\partial P} = \text{machine chromaticity} \\ \eta = \dfrac{1}{\gamma_T^2} - \dfrac{1}{\gamma^2} \end{cases}$$

$$\Delta\omega_w = \frac{N_b \, r_o \, \beta \, c \, \delta}{\sqrt{2}\, \pi \, \gamma \, b^3 \, Q} \qquad \begin{cases} N_b = \text{particles/bunch} \\ r_o = e^2/m_o c^2 = 1.53 \times 10^{-18} \text{ m} \\ \delta = \text{skin depth at rev. freq.} \simeq 2.4 \text{ mm} \\ b = \text{vacuum chamber half-height} \simeq 2.6 \text{ cm} \end{cases}$$

If $\chi \to 0$, the lowest head-tail mode ($\ell = 0$) approaches the rigid-bunch mode of Courant and Sessler, and (1) becomes

$$\Delta\omega = -\Delta\omega_w \left[G(2\pi, Q) + (1.89 - i\, 0.377\chi) \sqrt{\frac{2\pi R}{L}} \right]. \tag{2}$$

This agrees with the C + S result except for

i) an error of $\sqrt{2\pi}$ in their eq. (4.4) that Morton[5] has pointed out previously

ii) they neglect synchrotron motion so that for them $\chi = Q \frac{L}{R}$.

Equation (1) also includes the head-tail results of Pellegrini and Sands except that more realistic modes are used here that result in somewhat larger growth rates. A report covering the derivation of (1) should be available soon.

4. FORMULA FOR COUPLED-BUNCH MODES

For M identical bunches Eq. (1) still applies if $G(2\pi,Q)$ is replaced by

$$\sqrt{M}\; G\left(2\pi, \frac{m + Q}{M}\right) \quad .$$

For large M (say $M > 5$), Hübner and Zotter[6] show that this approaches

$$\frac{1 + i \,\text{Sign}\,(n - Q)}{\sqrt{2}} \cdot \frac{M}{\sqrt{|n - Q|}}$$

where n is an integer. In the limit of large M, the near-field term in (1) approaches zero because $\chi \to 0$, and

$$\Delta\omega \to -\frac{1 + i\,\text{Sign}\,(n - Q)}{\sqrt{2}} \cdot \frac{M\Delta\omega_w}{\sqrt{|n - Q|}}\ , \tag{3}$$

which is identical to the coasting-beam result of LNS[7]. These frequencies are sketched in Fig. 4 for $Q = 28\frac{3}{4}$.

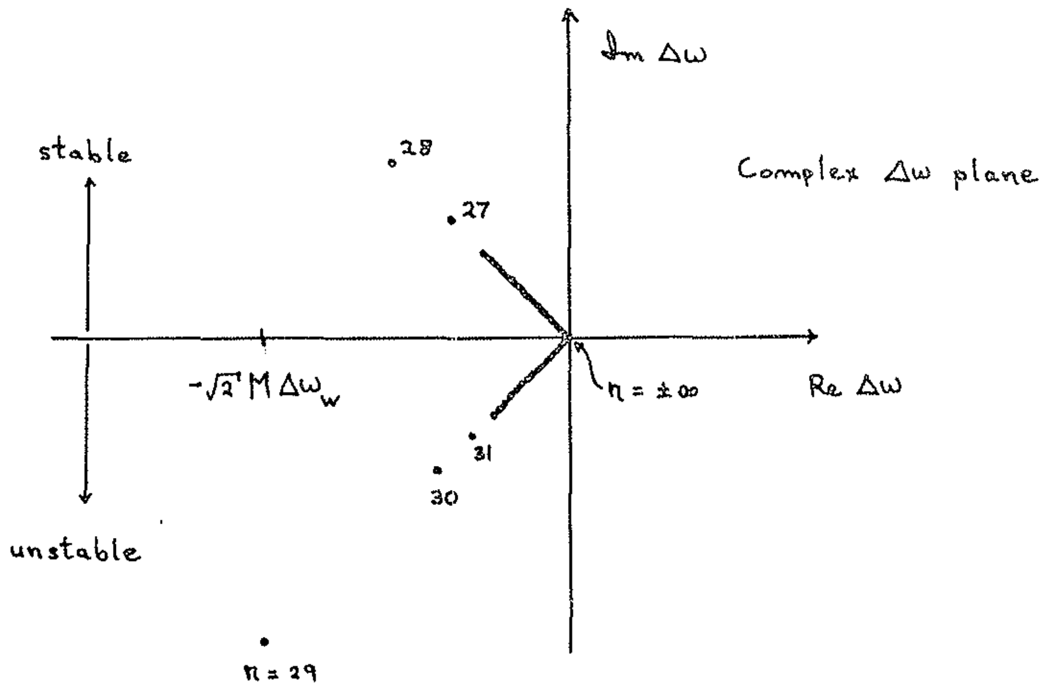


Fig. 4

The frequency diagram for 4,600 bunches differs from Fig. 4 only near the origin, and the differences are too small to be seen.

5. CRITERION FOR COUPLED MOTION

The most serious mode, n = 29, is also the most resistant to perturbation, while the modes with n much different from 29 are closely spaced and thus more easily modified. A criterion for the strength of perturbation required to destroy a given mode can be derived from perturbation theory. It is convenient to derive this criterion in the continuum limit $M \to \infty$, but the result is also valid for the discrete case. In the continuum limit, the bunch-coupling matrix

Eq. (3.20) of Courant and Sessler becomes

$$\lambda\Psi(\theta) = \int_0^{2\pi} K(\theta' - \theta)\ \Psi(\theta')\ d\theta', \tag{4}$$

where $\psi(\theta)$ is the amplitude of oscillation of the bunch at θ, and θ is measured from a reference bunch. The eigenfunctions and eigenvalues of (4) are

$$\Psi_n(\theta) = e^{in\theta}\ , \quad \lambda_n = \int_0^{2\pi} e^{in\theta}\ K(\theta)\ d\theta \tag{5}$$

$$= U + iV \text{ evaluated at } (n - Q)\omega_o .$$

For the resistive-wall interaction, $\lambda_n = \Delta\omega_n$ of Eq. (3).

We now perturb Eq. (4) by allowing the bunches to oscillate with different frequencies,

$$\lambda\Psi(\theta) = \omega(\theta)\Psi(\theta) + \int_0^{2\pi} K(\theta' - \theta)\ \Psi(\theta')\ d\theta'\ , \tag{6}$$

where $\omega(\theta)$ is the frequency of the bunch at θ, but measured with respect to the average frequency so that

$$\int_0^{2\pi} \omega(\theta)\ d\theta = 0\ . \tag{7}$$

The new eigenvalues are found from perturbation theory[8]. Define the matrix elements

$$\omega_{mn} = \frac{1}{2\pi}\int_0^{2\pi} \Psi_m^*\ (\theta)\ \omega(\theta)\ \Psi_n(\theta)\ d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n - m)\theta}\ \omega(\theta)\ d\theta\ , \tag{8}$$

where because of (7) $\omega_{mm} = 0$ for all m. Then to first order, there is no change in λ_n,

$$\lambda_n = \lambda_n^{\,0} + \omega_{nn}\ .$$

To second order,

$$\lambda_n = \lambda_n^o + \sum_m \frac{\omega_{nm}\omega_{mn}}{\lambda_n^o - \lambda_m^o} \quad . \tag{9}$$

This is valid provided the shift $\lambda_n - \lambda_n{}^0$ is much less than the spacing between level n and the rest. For the resistive-wall interaction, mode n = 29 (assuming $Q \simeq 28\frac{3}{4}$) has the largest spacing and is therefore the most difficult to destroy. The condition that this mode remain intact is

$$\sum_m \frac{\omega_{nm}\omega_{mn}}{\lambda_n^o - \lambda_m^o} \ll |\lambda_{29} - \lambda_{30}| \qquad \text{with } n = 29 \quad . \tag{10}$$

Because

$$\sum_m \frac{\omega_{nm}\omega_{mn}}{\lambda_n^o - \lambda_n^o} > \frac{1}{|\lambda_{29} - \lambda_{30}|} \sum_m \omega_{nm}\omega_{mn} \quad ,$$

and because $\omega_{mn} = \omega_{nm}^*$, condition (10) becomes

$$\sum_m |\omega_{nm}|^2 \ll |\lambda_{29} - \lambda_{30}| \qquad \text{with } n = 29 \quad . \tag{11}$$

From (8) we see that $\Sigma|\omega_{nm}|^2$ is the sum of Fourier components of $\omega(\theta)$. That is, if

$$\omega(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \qquad \text{with } a_k = \frac{1}{2\pi}\int_0^{2\pi} \omega(\theta)\, e^{-ik\theta} d\theta \ ,$$

then

$$\overline{\omega^2} = \frac{1}{2\pi}\int_0^{2\pi} \omega^2(\theta)\, d\theta = \sum_k |a_k|^2 = \sum_m |\omega_{nm}|^2 \quad . \tag{12}$$

Thus (10) becomes

$$\overline{\omega^2} << \Delta\lambda^2$$

or

$$\omega_{rms} < |\Delta\lambda| \qquad \text{for coupled motion,} \tag{13}$$

where ω_{rms} is the rms spread of bunch frequencies and $\Delta\lambda$ is the spacing between the mode in question and the next nearest mode. This criterion is completely general and applies to longitudinal as well as transverse motion, to coupling caused by cavities, pick-up electrodes, as well as resistive-wall interactions. It is analogous to the rule-of-thumb criterion for Landau damping in coasting beams, namely the spread in particle frequencies should exceed the frequency shift caused by the coherent motion.

The frequency spread may result from rf quadrupoles, or naturally from population differences between bunches via the coherent Laslett Q-shift $\Delta\omega_c$. In the latter case (13) becomes

$$\left(\frac{\Delta N}{N}\right)_{rms} \Delta\omega_c < \Delta\lambda \qquad \text{for coupled motion.} \tag{14}$$

As pointed out by D. Möhl[9], $\Delta\omega_c$ should not include the usual DC magnetic field terms since these shift the frequency of each bunch the same amount, independent of population differences. In this case[10]

$$\Delta\omega_c = -\frac{\xi_1}{\pi} \cdot \frac{N r_o \beta c}{Q B \beta^2 \gamma^3 b^2} \qquad \left\{ \begin{array}{l} N = \text{total number of particles} \\ B = \text{bunching factor} < 1 \\ b = \text{vacuum chamber half-height} \\ \xi_1 = \text{image coefficient} \simeq \frac{\pi^2}{16} \end{array} \right. \tag{15}$$

For large γ, other population dependent frequency shifts become important, including the resistive-wall term (1), the effect of cavities, dielectric or oxide coatings on the vacuum chamber, plus neutralizing electrons and ions.

6. NUMBERS FOR THE SPS

6.1 Normal acceleration in 4620 buckets

Unless stated otherwise we take

$$N = 10^{13} \text{ particles}$$
$$\gamma = 10$$
$$b = 2.6 \text{ cm}$$
$$L = 15 \text{ cm}$$
$$B = 0.1$$
$$Q = 28.75$$
$$\gamma_T = 28$$
$$\xi = -1.33$$
$$\omega_o = 2.74 \times 10^5 \text{ rad/s.}$$

Then $$\Delta\omega_W = 0.096$$
and the betatron phase shift χ is sketched in Fig. 5.

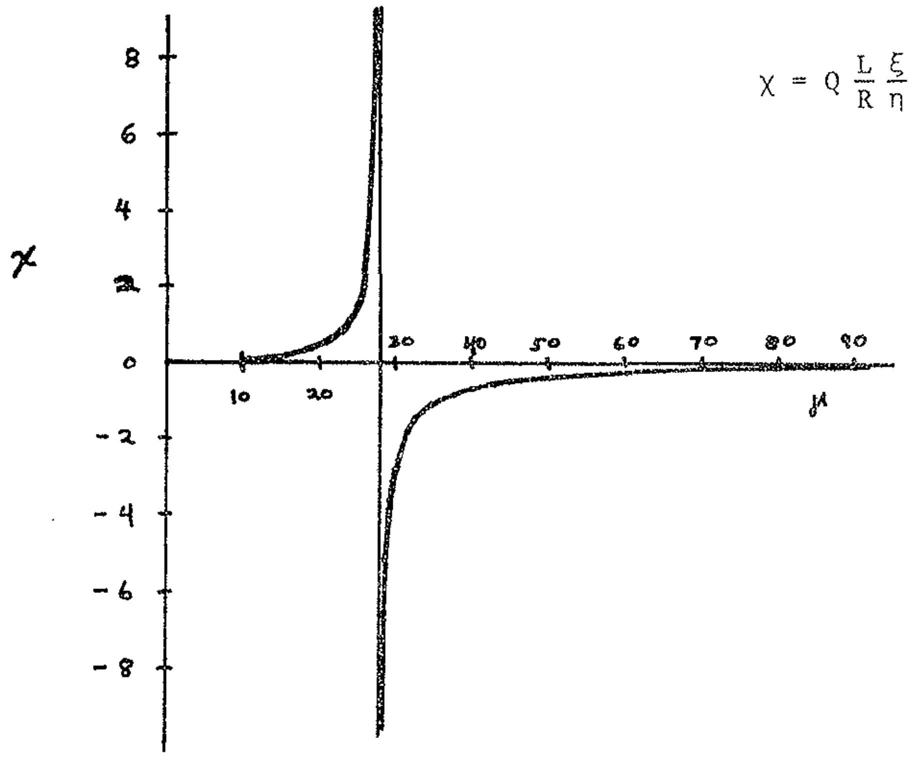


Fig. 5

If we ignore the non-adiabatic region near transition, then $\chi \leqslant 2$ rad and the $\ell = 0$ mode dominates,

$$\Delta\omega = - \Delta\omega_w [G + 215\ F_o(\chi)] \ ,$$

where the maximum value of Im $F_o(\chi)$ is

0.1 before transition (χ positive)

0.6 after transition (χ negative).

If the bunches were decoupled, $G = G(28.75) \simeq - 1 + i$, so near-fields dominate and

$$\frac{1}{\tau} = 0.096(1 + 215 \times 0.6) = 12.5 \quad ,$$

with $\tau = 80$ ms.

This is the maximum possible growth rate and occurs just after transition. We conclude that single bunch modes are not serious.

The more likely case is coupled motion with

$$G = \frac{1 + i}{\sqrt{2}} \cdot \frac{4620}{\sqrt{\frac{1}{4}}} = (1 + i)\ 6500$$

for the mode with 29 wavelengths around the machine circumference. Now near-fields are negligible and

$$\Delta\omega = - 0.096(1 + i)6500 = - 624(1 + i)$$

with $\tau = 1.6$ ms.

The criterion for coupled motion is

$$|\Delta\omega_c + \ldots| \left(\frac{\Delta N}{N}\right)_{rms} < |\Delta\lambda|$$

where we take

$$|\Delta\lambda| = |\Delta\omega| = 883$$

$$\Delta\omega_c = 475.$$

Thus to decouple the bunches requires a relative population spread of

$$\left(\frac{\Delta N}{N}\right)_{rms} > 1.85$$

or a full spread at half-height exceeding 370%. One expects $\left(\frac{\Delta N}{N}\right)_{rms}$ to be at most 0.1, corresponding to a 20% spread at half-height. Therefore, even large contributions to $\Delta\omega_c$ from electrons, ions, etc. are unlikely to decouple the bunches. This remains true at higher energies.

Sextupole terms or changes in the machine chromaticity change the betatron phase shift χ. As χ increases, the instability is shifted to the higher head-tail modes which have slower growth rates. However, to achieve a significant reduction in growth rate would require an order of magnitude increase in chromaticity. Because of the short bunch length, it is very difficult to have a large betatron phase shift between head and tail. We conclude that changes in chromaticity have a negligible effect.

Octupoles cure the instability if they produce enough frequency spread within a bunch to prevent its coherent motion, that is provided

$$\Delta Q_{oct} > \text{total frequency shift due to coherent motion}$$
$$= |U + iV|/\omega_o .$$

This has been computed by D. Möhl[11] who finds

$$\Delta Q_{oct} > 0.01$$

across the beam vertically. At $N = 10^{12}$ we need 0.001.

6.2 Injection and debunching of 20 PS bunches

Since there are no synchrotron forces, $\chi = Q \frac{L}{R}$. Equation (1) becomes

$$\Delta\omega = - 22.0\ C_\ell \left[G\ F_\ell'(\chi) + \sqrt{\frac{2\pi R}{L}}\ F^*(\chi)\right]$$

For decoupled motion, $G(28.75) \simeq - 1 + i$, and the frequency shifts $\Delta\omega_{SB}$ for single-bunch modes are given in Table I.

Table I - Single-bunch modes

Condition	L (m)	χ (rad)	ℓ	C_ℓ	$\sqrt{\frac{2\pi R}{L}}$	$F_\ell'(\chi)$	$F_\ell^*(\chi)$	$-\Delta\omega_{SB}$	τ (msec)
just injected	3	0.0765	0	1	48	1.0	0.80 + 0.013i	823 + 35.7i	28
1/10 debunched	35	0.88	0	1	14	0.95	0.85 + 0.11i	242 + 55.5i	18
½ debunched	173	4.4	1	0.5	6.3	0.56	0.90 + 0.16i	51.7 + 22i	45
debunched	345	8.8	4	0.2	4.5	0.20	0.55 + 0.17i	5.5 + 7.7i	130

The mode number ℓ is choosen for the fastest growing mode.

For coupled motion,

$$G = \frac{1 + i}{\sqrt{2}} \frac{20}{\sqrt{\frac{1}{4}}} = 28.3\ (1 + i),$$

and the frequency shifts $\Delta\omega_{CB}$ for the fastest growing coupled-bunch mode are given in Table II.

Table II - Coupled-bunch modes

Condition	$-\Delta\omega_{CB}$	τ (msec)
just injected	1470 + 631i	1.6
1/10 debunched	854 + 625i	1.6
½ debunched	231 + 184i	5.4
debunched	31.4 + 23.9i	42

The criterion for coupled motion is

$$|\Delta\omega_c + \Delta\omega_{SB} + \ldots| \left(\frac{\Delta N}{N}\right)_{rms} < |\Delta\omega_{CB}| ,$$

and it appears from Table III that the motion is coupled.

Table III - Criterion for coupled motion

Condition	B	$-\Delta\omega_C$	$\|\Delta\omega_c + \Delta\omega_{SB}\|$	$\|\Delta\omega_{CB}\|$	$\left(\frac{\Delta N}{N}\right)_{rms}$
just injected	0.01	4750	5673	1600	0.28
1/10 debunched	0.1	475	717	1060	1.48
½ debunched	0.5	95	147	294	2.0
debunched	1.0	47.5	53	39.4	0.75

The instability is cured by either sextupole or octupole terms. The spread in frequency due to the natural machine chromaticity is

$$\frac{\Delta\omega}{\omega} = 1.33 \times 1.3 \times 10^{-3} = 1.7 \times 10^{-3}$$

or

$$\Delta\omega = 1.7 \times 10^{-3} \times 28 \times 2.74 \times 10^{5} = 1.3 \times 10^{4}.$$

This spread is larger than the frequency shift $|\Delta\omega_c + \Delta\omega_{CB}|$ due to the coherent motion. We conclude that there should be no instability during injection and debunching, unless the natural chromaticity is reduced. The same conclusion is reached in Ref. 11 where a more conservative Landau damping criterion is employed.

Acknowledgement

I thank Dieter Möhl for reading the manuscript and suggesting several improvements.

Distribution

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REFERENCES

1. E.D. Courant and A.M. Sessler, Transverse coherent resistive instabilities of azimuthally bunched beams in particle accelerators, Rev. Sci. Instr. 37, 1579 (1966).

2. M.J. Lee, F.E. Mills and P.L. Morton, Throbbing beam transverse resistive instabilities in circular accelerators and storage rings, SLAC report 76 (1967).

3. C. Pellegrini, On a new instability in electron-positron storage rings (the head-tail effect), Nuovo Cimento 64A, 477 (1969).

4. M. Sands, The head-tail effect, an instability mechanism in storage rings, SLAC-TN-69/8, and head-tail effect II : from a resistive-wall wake, SLAC-TN-69/10 (1969).

5. P.L. Morton, An investigation of space-charge effects for the booster injector to the CPS, CERN Internal Report SI/Int. DL/68-3 (1968).

6. K. Hübner and B. Zotter, Transverse instabilities of the bunched beam in the ISR, CERN Internal Report CERN/ISR-TH/71-18 (1971).

7. L.J. Laslett, V.K. Neil and A.M. Sessler, Rev.Sci. Instr. 36, 426 (1965).

8. See, for example, L. Schiff, Quantum Mechanics, 2nd Ed., (McGraw-Hill Book Company, Inc., New York, 1955), p. 151.

9. D. Möhl, Transverse coherent instabilities in multi-bunch beams : a remark concerning the criterion for the absence of coherent bunch modes, Lawrence Berkeley Laboratory Report LBL-570 (1971).

10. L.J. Laslett and L. Resegotti, The space-charge limit imposed by coherent oscillation of a bunched beam, in Proceedings of the VI th International Conference on High Energy Accelerators, CEAL-2000, p. 150 (1967).

11. D. Möhl, Octupole and chromaticity required for transverse beam stability in the CERN - 200 GeV synchrotron. This proceedings.

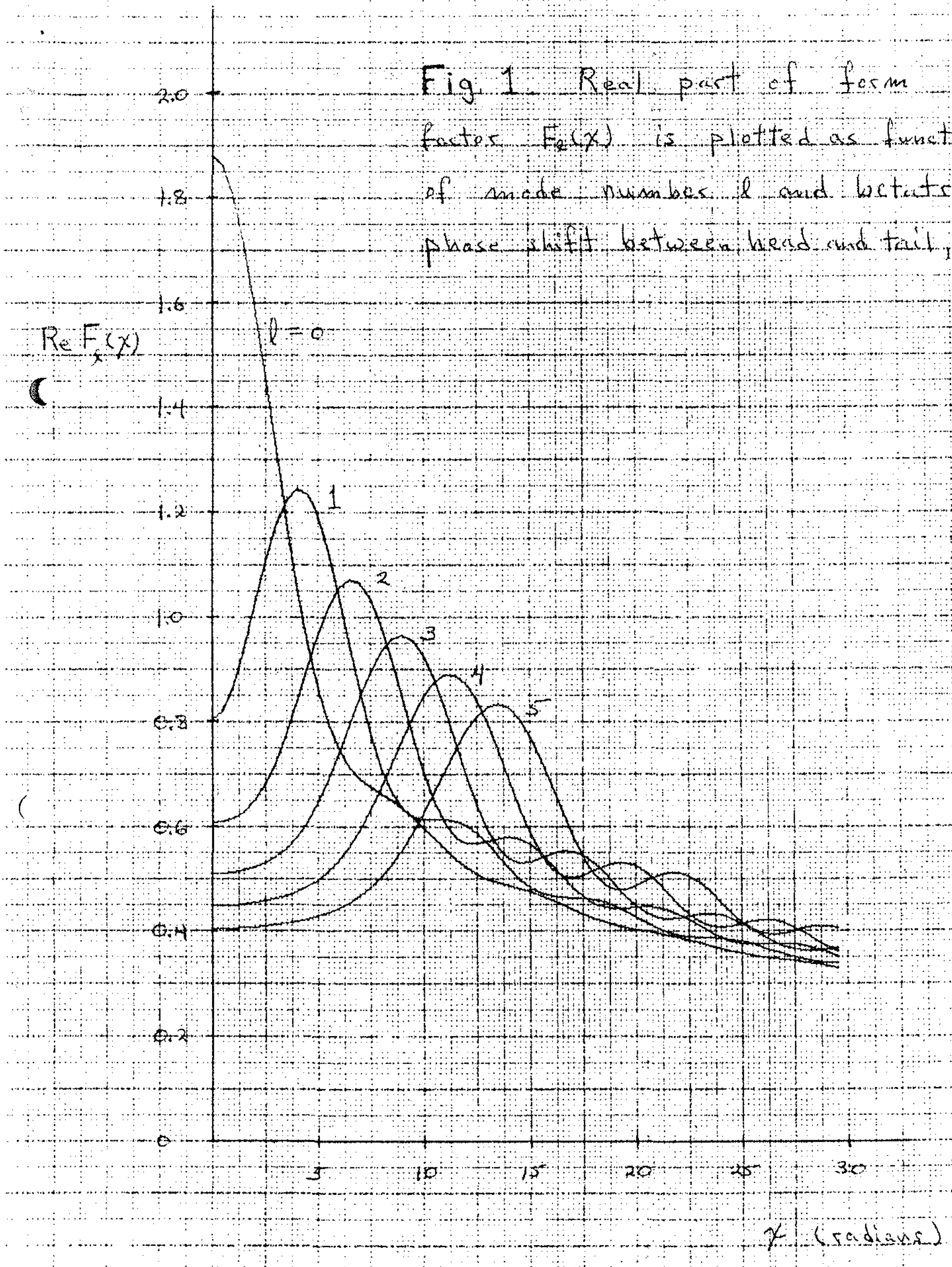

Fig. 1. Real part of form factor $F_\ell(\chi)$ is plotted as function of mode number ℓ and betatron phase shift between head and tail, χ.

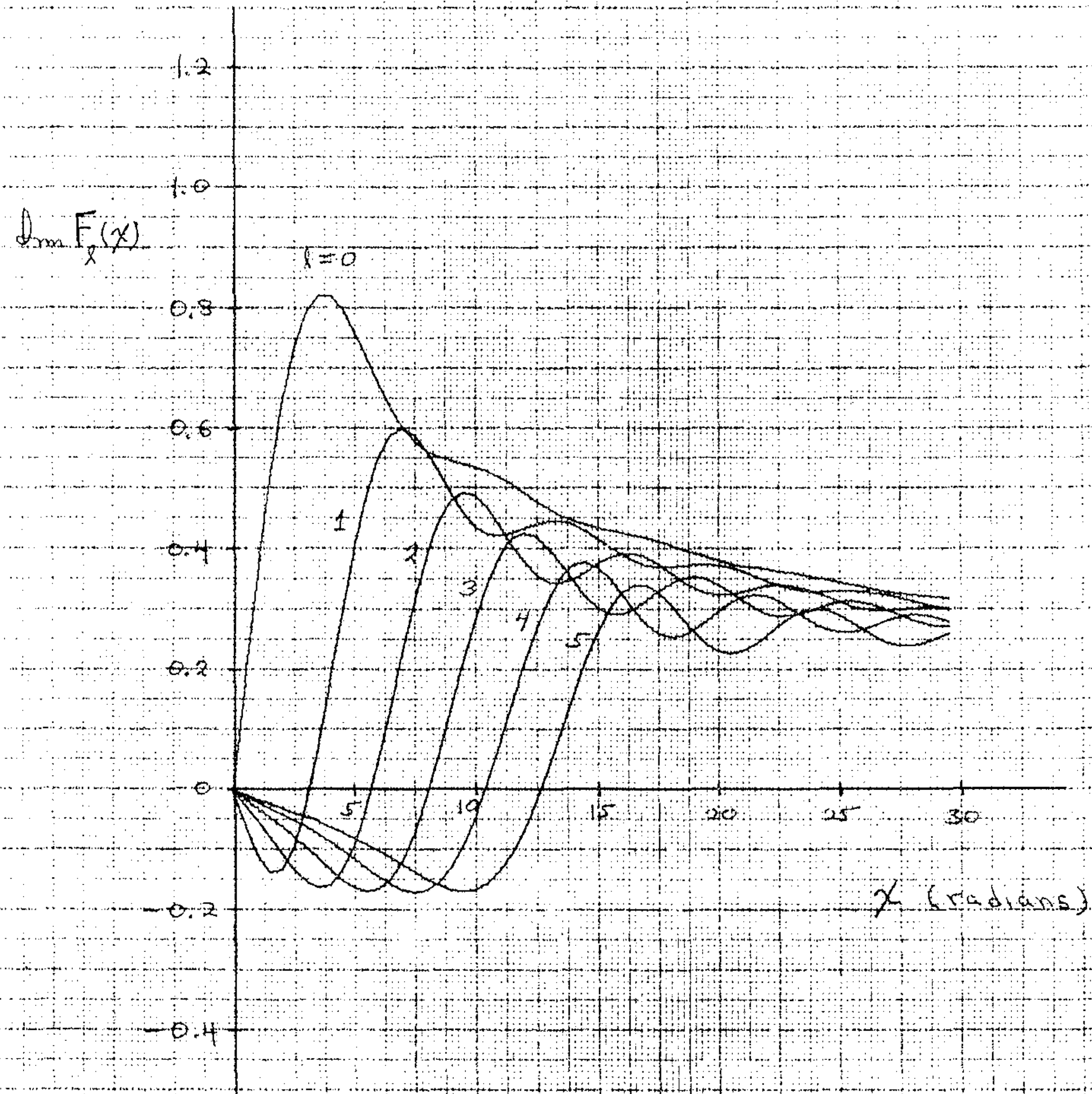

Fig. 2 Imaginary part of form factor $F_\ell(\chi)$ is plotted as function of mode number ℓ and betatron phase shift between head and tail, χ.

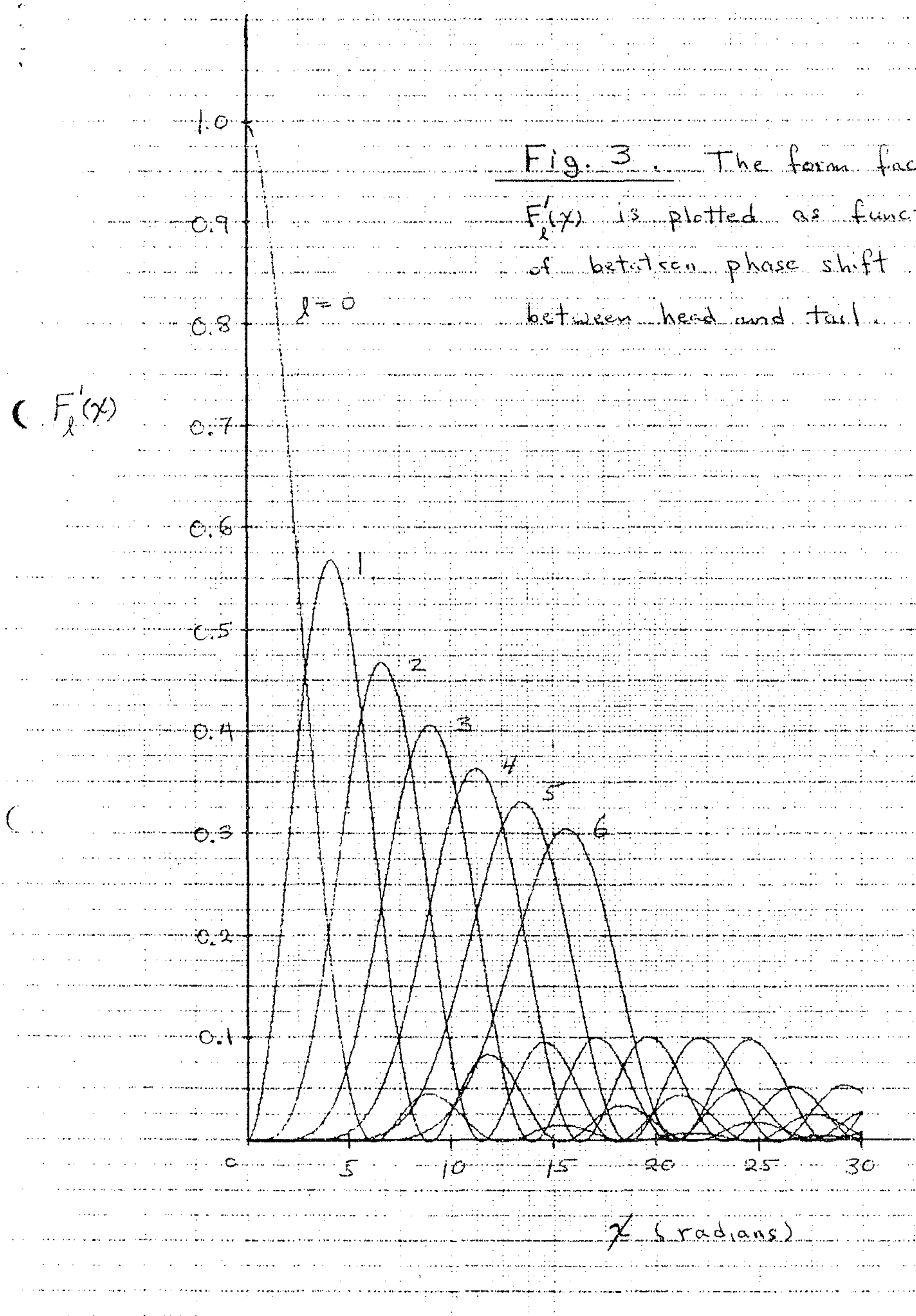

Fig. 3. The form factor $F_\ell'(\chi)$ is plotted as function of betatron phase shift χ between head and tail.