CERN-TH/97-37
US-FT-9/97, UB-ECM-PF 97/03
hep-th/9703072
March 1997

SOFTLY BROKEN $N = 2$ QCD WITH MASSIVE QUARK HYPERMULTIPLETS, I

Luis Álvarez-Gaumé[a], Marcos Mariño[b] and Frederic Zamora[a,c].

[a] *Theory Division, CERN, 1211 Geneva 23, Switzerland.*

[b] *Departamento de Física de Partículas, Universidade de Santiago de Compostela, E-15706 Santiago de Compostela, Spain.*

[c] *Departament d'Estructura i Constituents de la Materia, Facultat de Física, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain.*

Abstract

We present a general analysis of all possible soft breakings of $N = 2$ supersymmetric QCD preserving the analytic properties of the Seiberg-Witten solutions for the $SU(2)$ group with $N_f = 1, 2, 3$ hypermultiplets. We obtain all the couplings of the spurion fields in terms of properties of the Seiberg-Witten periods, which we express in terms of elementary elliptic functions by uniformizing the elliptic curves associated to each number of flavors. We analyze in detail the monodromy properties of the softly broken theory, and obtain them by a particular embedding into a pure gauge theory with higher rank group. This allows to write explicit expressions of the effective potential which are close to the exact answer for moderate values of the supersymmetry breaking parameters. The vacuum structures and phases of the broken theories will be analyzed in the forthcoming second part of this paper.

1 Introduction

This is the first of two papers where we analyze the general possible soft breakings of $N = 2$-rigid supersymmetry compatible with the special properties of the Seiberg-Witten solution for the $N = 2$ generalization of QCD with up to four massive flavours of quark hypermultiplets [3, 4]. We will also discuss the straightforward generalization to other groups, in particular $SU(N_c)$ [8, 9, 10, 11]. In [3, 4] Seiberg and Witten use the holomorphy properties of the prepotential describing the low energy action of $N = 2$ supersymmetry with vector multiplets, some beautiful physical arguments, and a large colection of self-consistency conditions to obtain the exact expression for the effective action up to two derivatives with and without multiplets. The solution is described by associating to each theory a Riemann surface of genus one which is determined in the Coulomb branch by the physical scale of the theory Λ_{N_f} and the hypermultiplet masses. In terms of a unique meromorphic differential on this Riemann surface they are able to write down the prepotential and all the couplings of the theory, and to determine the exact mass spectrum of BPS states (as the original solution will be crucial in what follows, we will present a brief summary of the relevant results of [4] including the analyticity and monodromy properties of their solutions at the beginning of section two). Since so much information follows from the solution of the pure $N = 2$ one would like to be able to break $N = 2$ supersymmetry down to $N = 0$ in terms of a collection of soft breaking terms preserving all the holomorphy and monodromy properties which were crucial to determine the solutions in [3, 4]. This was begun in [14, 15] (for $N = 1$ similar soft breaking analysis was done in [12, 13] to understand the implications of the non-perturbative superpotentials obtained by Seiberg and collaborators [1, 2] in the absence of supersymmetry). In this paper we study the most general soft breakings in $N = 2$ QCD-like theories including massive hypermultiplets. We find that for a theory with N_f hypermultiplets there are $3(1 + N_f)$ soft-breaking parameters satisfying all requirements and they are associated with making the dilaton a spurion (as in [14]) plus converting the quark masses into spurion vector multiplets. This is analogous to gauging the quark number symmetries and then freezing the vector multiplets to become spurions. Since each $N = 2$-vector multiplet contains three real auxiliary fields, this gives a $3(1 + N_f)$-parameter space of softly broken theories all preserving the holomorphy properties of [3, 4]. There are a number of interesting features in the analysis of these theories and their corresponding monodromies, and for this reason we have preferred to collect all the relevant information in this paper. In the second part [34] we will analyze the vacuum structure, patterns of symmetry breaking, low energy Goldstone boson Lagrangians and other interesting physical issues which one would like to understand in ordinary QCD but which here can be given explicit answers for moderate sypersymmetry breaking parameters compared to the physical scale of the theory. We also postpone to [34] the analysis of possible modifications of our results coming from some recent proposals for the supersymmetric effective action up to four derivatives [32]. For the numerical analysis of the effective potential for these theories we need to have simple expressions for the Seiberg-Witten solutions in the massive cases in terms of controllable functions. We obtain explicit expressions for these solutions in terms of elementary elliptic integrals using the uniformization of elliptic curves by Weierstrass functions.

The structure of this paper is as follows: In section two we begin by a quick summary of some of the salient features of the Seiberg-Witten analysis for gauge group $SU(2)$ with up to three hypermultiplets. In particular to each theory one can associate a given elliptic curve and a unique abelian differential of the third kind whose periods parametrize the Seiberg-Witten solution as a function of the modulus in the Coulomb phase of the theory. We then present a general framework to obtain explicit expressions for these periods by transforming the curves to Weierstrass canonical form, and then using uniformization. This gives very explicit results in terms of simple elliptic functions. We test our expressions in several ways. First the residues at their poles are given by the expected linear combinations of the bare masses of the quark hypermultiplets, second we verify that the weak coupling behavior follows when the modulus goes to infinity, and third, we verify that the BPS formula in terms of our expressions leads to a vanishing monopole mass where it is expected for $N_f = 1, 2, 3$. The explicit expressions derived are crucial for the numerical analysis in [34]. In section three we analyze the general form of soft breaking compatible with monodromy invariance and holomorphy. We present three different ways of looking at the problem giving the same answer. We can begin with the theory with masses, assume that the masses are now given vector multiplets associated with the gauging of quark numbers plus the dilaton vector multiplets, and derive the monodromy transformations associated to the theory and all relevant dynamical variables.

We can also consider this theory as obtained from a Seiberg-Witten treatment of a $SU(2) \times U(1)^{N_f}$ theory in the infrared. Since in this theory there are no "baryonic" monopoles, together with some simple physical constraints we again obtain the properties derived using the first method. Finally, we consider the theory with $SU(2) \times U(1)^{N_f}$ group as obtained by looking at the Seiberg-Witten curve for $SU(2+N_p)$ [8], with N_p the number of poles of the Seiberg-Witten abelian differential with N_f massive hypermultiplets ($N_p = 1, 2, 4$, for $N_f = 1, 2, 3$, respectively), and study it in a particular singular region of its moduli space. Namely, to this group we associate a hyperelliptic curve of genus $2N_p + 1$, given by a polynomial of degree $2N_p + 3$. If its roots are $e_1, e_2, e_3, r_1, s_1, r_2, s_2, ...r_{N_p}, s_{N_p}$ we can take the limit where r_i, s_i coalesce precisely at the positions which are given by the poles of the Seiberg-Witten differential in the massive case, and the other three roots serve to determine the embedding of $SU(2)$ in $SU(2+N_p)$. This reproduces precisely the same results as before including the monodromies. We also present in section three an explicit computation of the coupling matrices of the spurion fields to themselves and to physical fields because these are necessary to determine the low-energy effective action. We obtain these couplings by directly studying the integrals involved. For $SU(N)$ the authors in [33] have obtained similar expressions using the theory of integrable systems. Here, since we have a simple uniformization at hand, it is possible to derive the results directly. Finally in section four we write down the effective potential of the theory including all the spurion fields. Here we present the expressions that need to be analyzed numerically to determine all the possible vacuum structures of the different theories. The result of this analysis will appear in [34].

2 $N = 2$ QCD with Massive Quark Hypermultiplets

2.1 The Seiberg-Witten Solution

In this section we will focus on the features of the Seiberg-Witten solution which appear when bare masses for the hypermultiplets are included. For a general review of the work of Seiberg and Witten, see [5, 6, 7].

Matter hypermultiplets in the fundamental representation of $SU(N_c)$ can be given $N=2$ invariant bare masses through the $N=1$ superpotential

$$W_m = \sum_f m_f \widetilde{Q}_f Q_f, \tag{2.1}$$

where Q_f, $\widetilde{Q}_f$ denote as usual the $N=1$ chiral multiplets corresponding to an $N=2$ hypermultiplet. In the case of a gauge group $SU(2)$, the flavour symmetry group $SO(2N_f)$ is explicitly broken by (2.1) down to an $U(1)^{N_f}$ subgroup, where the $U(1)$ for each hypermultiplet is the baryon number symmetry. For $SU(2)$ with $N_f \leq 3$ hypermultiplets in the fundamental representation, the low-energy effective action for $N=2$ QCD was determined in [4] for arbitrary masses. The explicit solution of the theory is encoded, as in the massless case, in a genus one algebraic curve, in such a way that a and a_D (the fields entering into the BPS mass formula) are obtained as contour integrals of an abelian differential defined on the curve. However, there are three remarkable differences with the massless case that give these theories their characteristic richness and complexity:

i) When bare masses for the quarks are introduced, the flavor symmetry group is generically broken to an abelian subgroup that can contribute to the central charge of the $N=2$ algebra. The central charge reads now

$$Z = n_e a + n_m a_D + \sum_{f=1}^{N_f} S^f \frac{m_f}{\sqrt{2}}, \tag{2.2}$$

where n_e, n_m and S_f are respectively the electric, magnetic and baryonic quantum numbers. Notice that the physical electric charge Q_e is different from the electric quantum number n_e due to Witten's effect. In the same way, it has been recently shown that a similar phenomenon holds for the baryon number S^f [17]. In particular, the values of S^f in the massive case can be different from the ones in the massless case. In [17] this was argued from the point of view of the renormalization group flow from the massive to the massless theories, and in [18] this was explicitly shown in the $N_f=1$ case. In principle, the solutions we give in the next subsection for a and a_D determine the values of the baryon quantum numbers in the massive case. These will be needed in the numerical analysis we will perform in a forthcoming paper.

ii) The Seiberg-Witten abelian differential λ_{SW} is determined by the equation

$$\frac{\partial \lambda_{SW}}{\partial u} = \frac{\sqrt{2}}{8\pi}\frac{dx}{y}, \tag{2.3}$$

as this requirement guarantees the positivity of the metric. The relation between y and x determines an elliptic curve (see below for explicit formulae). In the massless case, λ_{SW} is an abelian differential of the second kind. However, in the massive case, λ_{SW} is of the third kind, *i.e.* it is a meromorphic one-form with single poles and non-zero residues. This implies that the quantities

$$a_D = \oint_{\alpha_1} \lambda_{SW}, \qquad a = \oint_{\alpha_2} \lambda_{SW}, \tag{2.4}$$

where $\alpha_{1,2}$ are the homology one-cycles of the genus one algebraic curve, are not invariant under deformations of the cycles across the poles of λ_{SW}. The jumps in a_D, a are of the form $2\pi i$ times the residue of λ_{SW}. In fact [4], the residues are linear combinations of the bare masses. In the next subsection we will compute these residues for the $N_f \leq 3$ and we will see that they have the expected structure.

iii) As a consequence of the points above, the monodromy transformations of a, a_D are no longer elements in $SL(2,\mathbf{Z})$. One must take into account the possibility of jumps in a, a_D depending on the residues of λ_{SW}. The description of the duality structure requires now the introduction of an N_f+2-dimensional column vector $(m_f/\sqrt{2}, a_D, a)$ which transforms as

$$\begin{pmatrix} m_f/\sqrt{2} \\ a_D \\ a \end{pmatrix} \to \mathcal{M} \begin{pmatrix} m_f/\sqrt{2} \\ a_D \\ a \end{pmatrix}, \tag{2.5}$$

where

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ p^f & \alpha & \beta \\ q^f & \gamma & \delta \end{pmatrix} \tag{2.6}$$

and α, β, γ, δ form an $Sl(2,\mathbf{Z})$-matrix. The numbers p^f, q^f are determined by the structure of the residues; they are in general integer multiples of the baryonic charges. The invariance of the central charge (2.2) gives in addition the transformation properties of the quantum numbers appearing in (2.2). If we consider the (N_f+2)-dimensional row vector $W = (S^f, n_m, n_e)$, under a monodromy transformation it must change as $W \to W\mathcal{M}^{-1}$.

2.2 A General Procedure to Compute a, a_D

In this subsection we will present a general procedure to compute a and a_D starting from the abelian differential λ_{SW} of Seiberg and Witten, and we will present the results for the asymptotically free $SU(2)$ theories with massive matter hypermultiplets in the fundamental representation[1]. For the theories with massless hypermultiplets, a and a_D can be computed using the Picard-Fuchs equations [9, 31, 27, 20], and the problem of integrating the Seiberg-Witten abelian differential along the two cycles of the torus is reduced to the problem of solving a differential equation. However, when we introduce masses for the hypermutiplets, the Picard-Fuchs equations are of third order and the solutions are not easy to obtain with this procedure (although they have been obtained numerically in the semiclassical regime in [21]). Explicit solutions have been obtained for the periods (the derivatives of a_D, a with respect to u) in [22, 23]. Our approach (which can be used in the massless case as well) is to use Abel's theorem and the Jacobi inversion to uniformize the Seiberg-Witten curve. In this way the contour integrals defining a and a_D can be obtained in terms of elementary elliptic funcions.

As it is well known, every algebraic curve of genus one can be written in the Weierstrass form [28],

$$Y^2 = 4X^3 - g_2 X - g_3 = 4(X-e_1)(X-e_2)(X-e_3), \tag{2.7}$$

where the coefficients g_2, g_3 are related to the roots e_i, $i = 1, 2, 3$ by the equations

$$g_2 = -4(e_2e_3 + e_3e_1 + e_1e_2), \quad g_3 = 4e_1e_2e_3. \tag{2.8}$$

To uniformize the curve, we use Abel's theorem, which states that an algebraic curve of genus one like (2.7) is of the form $\mathbf{C}/\Lambda$ for some lattice $\Lambda \subset \mathbf{C}$. The map from $\mathbf{C}/\Lambda$ to the curve (2.7) is given by

$$\psi(z) = (\wp(z), \wp'(z)) = (X, Y), \tag{2.9}$$

where we consider (X, Y) as inhomogeneous coordinates in $\mathbf{CP}^2$, and the Weierstrass function $\wp(z)$ verifies the differential equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{2.10}$$

[1]When this paper was being written a paper appeared [19] where a similar approach is used in the $N_f = 4$ theory, and a paper in preparation by A. Bilal and F. Ferrari is announced where they analyze the $N_f \leq 3$ theories with similar methods.

Under this correspondence, the half periods of the lattice Λ, $\omega_i/2$, $i = 1, 2, 3$, $\omega_3 = \omega_1 + \omega_2$, are mapped to the roots $e_i = \wp(\omega_i/2)$ of the cubic equation in (2.7), and the differential dz on $\mathbf{C}/\Lambda$ is mapped to the abelian differential of the first kind dX/Y. The map (2.9) has an inverse given by

$$z = \psi^{-1}(p) = \int_{\infty}^{p} \frac{dX}{Y}, \tag{2.11}$$

which is defined modulo Λ. We can obtain an explicit expression for the inverse map (2.11), doing the change of variable $t^2 = (e_2 - e_1)/(X - e_1)$, to obtain

$$z = -\frac{1}{\sqrt{e_2 - e_1}} F(\phi, k), \tag{2.12}$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind, with modulus $k^2 = (e_3 - e_1)/(e_2 - e_1)$, and $\sin^2\phi = (e_2 - e_1)/(\wp(z) - e_1)$.

In fact, all these functions can be computed in terms of the roots e_i and elliptic functions. First of all we have the periods of the abelian differential dX/Y. We take the branch cut on the X-plane from e_1 to e_3, and from e_2 to infinity (see fig. 1), so that the α_1 and α_2 periods are given by

$$\begin{aligned} \omega_1 &= \oint_{\alpha_1} \frac{dX}{Y} = \int_{e_2}^{e_3} \frac{dX}{\sqrt{(X - e_1)(X - e_2)(X - e_3)}}, \\ \omega_2 &= \oint_{\alpha_2} \frac{dX}{Y} = \int_{e_1}^{e_3} \frac{dX}{\sqrt{(X - e_1)(X - e_2)(X - e_3)}}. \end{aligned} \tag{2.13}$$

Introducing now the complementary modulus $k'^2 = 1 - k^2$, we obtain a representation of the periods in terms of the complete elliptic integral of the first kind,

$$\begin{aligned} \omega_1 &= \frac{2i}{\sqrt{e_2 - e_1}} K(k'), \\ \omega_2 &= \frac{2}{\sqrt{e_2 - e_1}} K(k). \end{aligned} \tag{2.14}$$

We will also need the Weierstrass ζ-function, which is defined by the equation $\zeta'(z) = -\wp(z)$. Because of this property, we have that

$$\zeta(\frac{\omega_i}{2}) = -\int_{\omega_j/2}^{\omega_3/2} dz \wp(z), \qquad i, j = 1, 2,\ j \neq i, \tag{2.15}$$

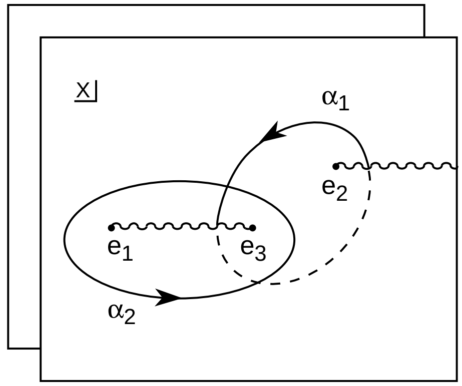

Figure 1: The choice of the nontrivial one-cycles α_1 and α_2.

hence their values at the half periods can be computed in terms of complete elliptic integrals,

$$\begin{aligned} \zeta(\frac{\omega_1}{2}) &= -i\frac{e_1}{\sqrt{e_2-e_1}}K(k') - i\sqrt{e_2-e_1}E(k'), \\ \zeta(\frac{\omega_2}{2}) &= -\frac{e_2}{\sqrt{e_2-e_1}}K(k) + \sqrt{e_2-e_1}E(k). \end{aligned} \tag{2.16}$$

We can also give the value of $\zeta(z)$, for general z, in terms of incomplete elliptic integrals of first and second kind:

$$\zeta(z) = \zeta(\frac{\omega_1}{2}) - \frac{e_2}{\sqrt{e_2-e_1}}F(\varphi,k) + \sqrt{e_2-e_1}E(\varphi,k), \tag{2.17}$$

where $\sin^2\varphi = (\wp(z)-e_1)/(e_3-e_1)$.

As in the massive theories the Seiberg-Witten differential is of the third kind (with residues depending on the bare masses for the quark hypermultiplets), we need an expression for integrals on $\mathbf{C}/\Lambda$ of this kind of differentials. First of all we have the relation [24, 25]

$$-\frac{\wp'(z_0)}{\wp(z)-\wp(z_0)} = \zeta(z+z_0) + \zeta(z-z_0) - 2\zeta(z_0). \tag{2.18}$$

We can now use the fact that the Weierstrass ζ-function is the logarithmic derivative of the σ-function to obtain the expression

$$\int \frac{dz}{\wp(z)-\wp(z_0)} = \frac{1}{\wp'(z_0)}\Big(2z\zeta(z_0) + \ln\frac{\sigma(z-z_0)}{\sigma(z+z_0)}\Big). \tag{2.19}$$

As the σ-function is odd and verifies

$$\sigma(z+\omega_i) = -\sigma(z)\mathrm{e}^{2\zeta(\frac{\omega_i}{2})(z+\frac{1}{2}\omega_i)}, \tag{2.20}$$

we can obtain from (2.19) a simple expression for contour integrals of quotients like (2.18) over the homology one-cycles:

$$\int_{\omega_i/2}^{\omega_3/2} \frac{dz}{\wp(z)-\wp(z_0)} = \frac{1}{\wp'(z_0)}(\omega_j\zeta(z_0) - 2\zeta(\frac{\omega_j}{2})z_0). \tag{2.21}$$

Where $\omega_3 = \omega_1 + \omega_2$ and $i, j = 1, 2$, with $j \neq i$. The Weierstrass function $\wp(z)$ is an even elliptic function of order two. This means that all the points $\pm z_0 + \Lambda$ give the same value to the Weierstrass function. Using Legendre's relation,

$$\omega_1\zeta(\frac{\omega_2}{2}) - \omega_2\zeta(\frac{\omega_1}{2}) = i\pi, \tag{2.22}$$

one can verify that if we substitute z_0 by $\pm z_0 + n_1\omega_1 + n_2\omega_2$, with n_1, n_2 integer numbers, in the integral (2.21), we just obtain the additional term $\pm(-1)^i 2\pi i n_i/\wp'(z_0)$. On the other hand, notice that $-1/\wp'(z_0)$ is precisely the residue of $1/(\wp(z)-\wp(z_0))$. As a contour integral, (2.21) is then defined up to $\pi i n$ times the residue of $1/(\wp(z)-\wp(z_0))$, where n is an integer. We will see that this is the behaviour we need in order to reproduce the jumps in a_D, a that one expects in $N=2$ QCD with massive hypermutliplets. Also notice that the ambiguity in the choice of z_0 has the same structure that the one coming from the residue of the pole.

When one of the periods of the lattice goes to infinity, say ω_2 (this is precisely the case in the strong coupling singularities), one can derive expressions for the roots e_i and $\zeta(\omega_1/2)$ in terms of ω_1:

$$e_3 = e_2 = -\frac{e_1}{2} = -\frac{\pi^2}{3\omega_1^2}, \qquad \zeta(\frac{\omega_1}{2}) = \frac{\pi^2}{6\omega_1}. \tag{2.23}$$

These are all the expressions we need to give explicit formulae for a and a_D.

2.3 a, a_D for $N=2$ $SU(2)$ with $N_f \leq 3$

We will now use the approach previously described to obtain explicit expressions for a and a_D for the $N=2$ $SU(2)$ gauge theory with $N_f \leq 3$ hypermultiplets. To do this, we will use the cubic curves presented in [4].

As it should be clear from the discussion above, the first thing to do is to write the Seiberg-Witten curves in the Weierstrass form. Consider then the general form of the curves for the massive theories as described in [4]:

$$y^2 = x^3 + Bx^2 + Cx + D, \tag{2.24}$$

where the coefficients B, C and D depend on the gauge-invariant parameter u, the dynamical scale of the theory Λ_{N_f}, and on the bare quark masses. To put this curve in the Weierstrass form, it suffices to redefine the variables as

$$y = 4Y, \qquad x = 4X - \frac{1}{3}B, \tag{2.25}$$

and the curve (2.24) has now the form given in (2.7) with

$$g_2 = -\frac{1}{4}(C - \frac{1}{3}B^2), \qquad g_3 = -\frac{1}{16}(D + \frac{2B^3}{27} - \frac{CB}{3}). \tag{2.26}$$

Notice that with the redefinition given in (2.25), the abelian differential of the first kind is $dX/Y = dx/y$.

2.3.1 $N_f \leq 3$ Massless

Before analyzing the three massive cases of the $SU(2)$ theory, we give a review of the massless case within our approach. For the $SU(2)$ gauge theory with N_f massless hypermultiplets, the Seiberg-Witten abelian differential is [26]

$$\lambda_{SW} = \frac{\sqrt{2}}{8\pi}\frac{dx}{y}(2u - (4 - N_f)x). \tag{2.27}$$

By the uniformization method, the integrals to do are (a_D and a will be denoted by a_1, a_2, respectively):

$$a_i = \frac{\sqrt{2}}{\pi}\int_{\omega_j/2}^{\omega_3/2} dz\Big((\frac{N_f+2}{12})u - \frac{\delta_{N_f,3}\Lambda_3^2}{768} - (4 - N_f)\wp(z)\Big). \tag{2.28}$$

And just using formula (2.15) and that $\zeta(\frac{\omega_3}{2}) = \zeta(\frac{\omega_1}{2}) + \zeta(\frac{\omega_2}{2})$, we obtain

$$a_i = \frac{\sqrt{2}}{\pi}\Big((4 - N_f)\zeta(\frac{\omega_i}{2}) + (\frac{N_f+2}{24})u\,\omega_i - (\frac{\delta_{N_f,3}\Lambda_3^2}{1536})\omega_i\Big) \tag{2.29}$$

In particular we immediately obtain the identity [29, 30, 31]

$$a\Big(\frac{da_D}{du}\Big) - a_D\Big(\frac{da}{du}\Big) = \frac{i(4-N_f)}{4\pi} \tag{2.30}$$

which holds for the massless case, just by direct application of the Legendre's relation (2.22). From the analysis of the periods (2.13) at weak coupling,

$$\omega_1 = \frac{i(4-N_f)}{\sqrt{u}}\Big(\log\frac{u}{\Lambda_{N_f}^2} + 2 + O\Big(\frac{\Lambda_{N_f}^2}{u}\Big)\Big), \tag{2.31}$$

$$\omega_2 = \frac{2\pi}{\sqrt{u}}\Big(1 + O\Big(\frac{\Lambda_{N_f}^2}{u}\Big)\Big), \tag{2.32}$$

we have

$$\zeta\Big(\frac{\omega_1}{2}\Big) \sim \frac{i(4-N_f)}{24}\sqrt{u}\log\frac{u}{\Lambda_{N_f}^2} - \frac{i(2+N_f)}{12}\sqrt{u}; \tag{2.33}$$

$$\zeta\Big(\frac{\omega_2}{2}\Big) \sim \frac{\pi}{12}\sqrt{u}. \tag{2.34}$$

Where we have used the homogeneity relation of the Weierstrass ζ-function,

$$\zeta(\lambda z; \lambda\omega_i) = \lambda^{-1}\zeta(z;\omega_i), \tag{2.35}$$

and the fact that $\zeta(\pi) = \pi/12$ for a lattice of periods $(2\pi, \infty)$, to compute the asymptotic expression of $\zeta(\omega_2/2)$ in the weak coupling region. By the Legendre's relation (2.22) we have obtained the behaviour of $\zeta(\omega_1/2)$ in this region. Substituting these expressions in (2.29), we obtain the expected weak coupling expressions

$$a = \frac{\sqrt{2u}}{2}\Big(1 + O\Big(\frac{\Lambda_{N_f}^2}{u}\Big)\Big), \tag{2.36}$$

$$a_D = \frac{i(4-N_f)}{4\pi}\sqrt{2u}\Big(\log\frac{u}{\Lambda_{N_f}^2} + O\Big(\frac{\Lambda_{N_f}^2}{u}\Big)\Big). \tag{2.37}$$

Finally, we show a strong coupling test. On the strong coupling singularities, ω_2 diverges and we have, using (2.23), that

$$a_D = \frac{\sqrt{2}}{\pi}\Big((4-N_f)\frac{\pi^2}{6\omega_1} + \big(\frac{N_f+2}{24}\big)u\,\omega_1 - \big(\frac{\delta_{N_f,3}}{64}\big)\omega_1\Big), \tag{2.38}$$

where we have gone to adimensional units, choosing $\Lambda_{N_f}^{(4-N_f)} = 8$. For instance, on the monopole singularity, we have that

$$(u, \omega_1) = (e^{-\frac{i\pi}{3}}, 2\pi e^{\frac{2\pi i}{3}}),\ (1, \frac{2\pi i}{\sqrt{2}}),\ (0, -2\pi)$$

for $N_f = 1, 2, 3$, respectively. With these values (2.38) implies that

$$a_D = 0, \tag{2.39}$$

as expected.

2.3.2 $N_f = 1$ Massive

The Seiberg-Witten curve is in this case

$$y^2 = x^2(x-u) + \frac{1}{4} m \Lambda_1^3 x - \frac{1}{64} \Lambda_1^6, \tag{2.40}$$

and with the redefinitions in (2.25), $x = 4X + u/3$. This corresponds to a curve in the Weierstrass form (2.7) with

$$\begin{aligned} g_2(u,m) &= \frac{1}{4}\Big(\frac{u^2}{3} - \frac{1}{4} m \Lambda_1^3\Big), \\ g_3(u,m) &= \frac{1}{16}\Big(-\frac{1}{12} m u \Lambda_1^3 + \frac{1}{64}\Lambda_1^6 + \frac{2}{27} u^3\Big). \end{aligned} \tag{2.41}$$

An explicit representative of the Seiberg-Witten differential can be easily obtained in this case:

$$\lambda_{SW} = -\frac{\sqrt{2}}{8\pi} \frac{dx}{y} (3x - 2u + \frac{m\Lambda_1^4}{4x}), \tag{2.42}$$

up to an exact differential. This differential has a pole at $x = 0$, with residue

$$\mathrm{Res}_{x=0} \lambda_{SW} = -\frac{1}{2\pi i} \frac{m}{\sqrt{2}}, \tag{2.43}$$

in the positive Riemann sheet (*i.e.* for the positive sign of the square root). To obtain a_D and a, we will simply use the correspondence between X and $\wp(z)$ given by the Abel map in (2.9), and the integrals (2.15) and (2.21).

Notice that the pole at $x = 0$ corresponds to a pole at z_0, verifying $4\wp(z_0) + u/3 = 0$. In the case at hand, we can easily compute the residue of this pole, $1/\wp'(z_0)$, taking into account that Y corresponds to $\wp'(z)$ under the Abel map (2.9). In this way we obtain

$$16(\wp'(z_0))^2 = y^2(0) = -\frac{1}{64}\Lambda_1^6, \tag{2.44}$$

hence $\wp'(z_0) = i\Lambda_1^3/32$, where the point z_0 corresponds to the pole $x = 0$ which lives in the positive Riemann sheet. The final expression for the a_i is

$$a_i = \frac{\sqrt{2}}{4\pi}\left(12\zeta(\frac{\omega_i}{2}) + u\frac{\omega_i}{2} + 2im[\omega_i\zeta(z_0) - 2z_0\zeta(\frac{\omega_i}{2})]\right). \tag{2.45}$$

Notice that the coefficient of the term in square brackets in (2.45) is precisely twice the residue of λ_{SW} (with a positive sign in the square root). This is a general fact for all the expressions we will find below, and is a direct consequence of the structure of the abelian differential of third kind. We have seen that the integral (2.21) is defined up to πin times $1/\wp'(z_0)$. This implies that a_D, a will be defined up to $2\pi in$ times the residue of λ_{SW}. This is the expected structure of the constant shifts in the monodromy transformations, as we recalled in the previous subsection following [4]. One should be careful with the global structure of the solutions on the u-plane. In general one must analytically continue the expressions of a_D, a through the cuts. This simply amounts, in this formalism, to a permutation of the roots of (2.22) for each case and they are given by the Weierstrass function at the half-periods of the torus. These issues will be discussed in the forthcoming [34] second part of this paper.

2.3.3 $N_f = 2$ Massive

In this case the explicit expressions are slightly more involved. The Seiberg-Witten curve is given by

$$y^2 = (x^2 - \frac{1}{64}\Lambda_2^4)(x-u) + \frac{1}{4}m_1m_2\Lambda_2^2 x - \frac{1}{64}(m_1^2 + m_2^2)\Lambda_2^4, \tag{2.46}$$

which can be written in the Weierstrass form with $x = 4X + \frac{u}{3}$ and coefficients

$$g_2(u, m_1, m_2) \quad = \quad \frac{1}{4}\Big(\frac{u^2}{3} + \frac{1}{64}\Lambda_2^4 - \frac{1}{4}m_1m_2\Lambda_2^2\Big),$$

$$g_3(u,m_1,m_2) = \frac{1}{16}\Big(\frac{2u^3}{27} + \frac{1}{64}(m_1^2+m_2^2-u)\Lambda_2^4 - \frac{u\Lambda_2^2}{12}(m_1m_2 - \frac{1}{16}\Lambda_2^2)\Big). \qquad (2.47)$$

The abelian Seiberg-Witten differential can be easily computed in this case [4]:

$$\lambda_{SW} = -\frac{\sqrt{2}}{4\pi}\frac{dx}{y}\left(x - u + \frac{1}{16}\Lambda_2^2\Big(\frac{m_+^2}{x+\frac{1}{8}\Lambda_2^2} - \frac{m_-^2}{x-\frac{1}{8}\Lambda_2^2}\Big)\right), \qquad (2.48)$$

where we have defined $m_\pm = m_1 \pm m_2$. We see that there are poles at $x_\pm = \mp\Lambda_2^2/8$, with residues

$$\mathrm{Res}_{x=x_\pm}\lambda_{SW} = \mp\frac{1}{4\pi i}\frac{m_\pm}{\sqrt{2}}, \qquad (2.49)$$

for the positive Riemann sheet. These poles are mapped to the points $z_\pm$ on $\mathbf{C}$ given by $4\wp(z_\pm) + u/3 - x_\pm = 0$. Taking into account the structure of the curve in the Weierstrass form one can compute

$$\wp'(z_\pm) = \frac{i}{32}\Lambda_2^2 m_\pm, \qquad (2.50)$$

where, again, the $z_\pm$ are chosen in order to map on the positive Riemann sheet. One then obtains,

$$a_i = \frac{\sqrt{2}}{4\pi}\Big(8\zeta(\frac{\omega_i}{2}) + \frac{2u}{3}\omega_i + i\Big[m_+(\omega_i\zeta(z_+) - 2z_+\zeta(\frac{\omega_i}{2})) - m_-(\omega_i\zeta(z_-) - 2z_-\zeta(\frac{\omega_i}{2}))\Big]\Big), \qquad (2.51)$$

where we can see the same pattern as in the $N_f = 1$ case.

2.3.4 $N_f = 3$ Massive

This is the more involved case. The Seiberg-Witten curve is given by

$$y^2 = x^2(x-u) - \frac{1}{64}\Lambda_3^2(x-u)^2 - \frac{1}{64}\Lambda_3^2 t_2(x-u) + \frac{1}{4}\Lambda_3 t_3 x - \frac{1}{64}\Lambda_3^2 t_4, \qquad (2.52)$$

where t_2, t_3, t_4 denote the polynomials

$$t_2 = m_1^2 + m_2^2 + m_3^2, \quad t_3 = m_1 m_2 m_3,$$
$$t_4 = m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2. \tag{2.53}$$

The curve (2.52) can be written in the Weierstrass form with

$$x = 4X + \frac{1}{3}\Big(u + \frac{1}{64}\Lambda_3^2\Big). \tag{2.54}$$

The computation of the appropriate Seiberg-Witten differential is somewhat involved, and we will explain it in some detail. First of all we write the curve in terms of the u-variable (since, because of (2.3), one must integrate with respect to it). Letting $c = \Lambda_3/8$, we have:

$$y^2 = -c^2 u^2 + p(x)u + q(x), \tag{2.55}$$

where

$$p(x) = -x^2 + 2c^2 x + t_2 c^2, \quad q(x) = x^3 - c^2(x^2 + t_2 x + t_4) + 2ct_3 x. \tag{2.56}$$

Up to an exact differential, and a term which is u-independent (and then does not spoil the requirement in (2.3)), we obtain the explicit expression

$$\lambda_{SW} = -\frac{\sqrt{2}}{16\pi}\frac{xdx}{y}\frac{\frac{1}{2}pq' - p'q - u\Big(c^2 q + \frac{1}{4}p^2\Big)'}{c^2 q + \frac{1}{4}p^2}, \tag{2.57}$$

where the prime denotes derivative with respect to x. The polynomial in the denominator has the roots

$$\begin{aligned} x_1 &= \frac{1}{8}\Lambda_3(-m_1 + m_2 + m_3), \qquad (2.58)\\ x_2 &= \frac{1}{8}\Lambda_3(m_1 - m_2 + m_3),\\ x_3 &= \frac{1}{8}\Lambda_3(m_1 + m_2 - m_3),\\ x_4 &= \frac{1}{8}\Lambda_3(-m_1 - m_2 - m_3), \end{aligned}$$

which are in one-to-one correspondence with the weights of the negative-chirality spinor representation of $SO(6)$. The abelian differential simplifies and becomes:

$$\lambda_{SW} = -\frac{\sqrt{2}}{16\pi}\frac{dx}{y}\Big(2x - 4u - \sum_{n=1}^{4}\frac{y_n x_n}{x - x_n}\Big), \tag{2.59}$$

where the coefficients y_n verify $y^2(x_n) = -c^2 y_n^2$ and are given explicitly by

$$\begin{aligned} y_1 &= u - m_1 m_2 - m_1 m_3 + m_2 m_3 - x_1, \\ y_2 &= u - m_1 m_2 + m_1 m_3 - m_2 m_3 - x_2, \\ y_3 &= u + m_1 m_2 - m_1 m_3 - m_2 m_3 - x_3, \\ y_4 &= u + m_1 m_2 + m_1 m_3 + m_2 m_3 - x_4. \end{aligned} \tag{2.60}$$

As a partial check of our result, notice that when all the three masses are zero, we recover precisely the abelian differential of the massless case [26]. With this explicit expression, we can easily compute the residues of the abelian differential at the poles x_n:

$$\mathrm{Res}_{x=x_n}\lambda_{SW} = \frac{1}{8\pi i}\frac{\hat{x}_n}{\sqrt{2}}, \tag{2.61}$$

where $\hat{x}_n = x_n/c$. The computation of a, a_D is now straightforward. The poles x_n correspond to the poles z_n in $\mathbf{C}$ through the equation (2.54), and one easily obtains $\wp'(z_n) = icy_n/4$. The final result is

$$a_i = \frac{\sqrt{2}}{8\pi}\left(8\zeta(\frac{\omega_i}{2}) + \frac{\omega_i}{3}\Big(5u - \frac{1}{64}\Lambda_3^2\Big) - i\sum_{n=1}^{4}\hat{x}_n[\omega_i\zeta(z_n) - 2z_n\zeta(\frac{\omega_i}{2})]\right). \tag{2.62}$$

3 Soft Breaking with Massive Quark Hypermultiplets

In this section we study the general soft breaking of $N = 2$ $SU(2)$ gauge theory with massive quark hypermultiplets, keeping the holomorphicity of the low energy Lagrangian. In the first subsection we analyze the possibility of promoting the hypermultiplets bare masses to the status of $N = 2$ spurion vector superfields. In the second subsection we give computable expressions for the dual spurion masses and dilaton, and all the couplings.

3.1 The Hypermultiplet Masses as $N = 2$ Vector Superfields

The mass dependence of the Lagrangian (2.1) suggests the interpretation of the bare hypermultiplet masses as frozen $U(1)$ $N = 2$ vector multiplets of the baryonic flavour symmetries. Regarding them from this point of view, we could turn on their auxiliary fields obtaining additional supersymmetry breaking parameters. When the hypermultiplet masses are promoted to the status of abelian $N = 2$ vector multiplets $\mathcal{M}_f$, we are really dealing with an effective Seiberg-Witten model with $SU(2) \times U(1)^{N_f}$ gauge group where the N_f $U(1)$ factors come from the gauging of the baryon numbers[2]. At low energies, the $N = 2$ effective Lagrangian of the vector multiplets includes the superfield $\mathcal{A}$ of the unbroken abelian subgroup of $SU(2)$, the N_f mass spurions $\mathcal{M}_f$, with its scalar component giving the bare hypermultiplet masses m_f[3], and the dilaton $\mathcal{S}$, whose scalar component gives the dynamically generated scale $\Lambda = e^{is}$. The fact that the spurions are $N = 2$ vector multiplets keeps the holomorphy of the leading part of the low energy Lagrangian, explicitly manifest in $N = 2$ superspace,

$$\mathcal{L}_{\rm VM} = \frac{1}{4\pi}{\rm Im}\Big(\int d^4\theta \mathcal{F}(\mathcal{A}, \mathcal{M}_f, \mathcal{S})\Big). \tag{3.1}$$

It is not possible to generate a "magnetic" baryonic charge, S_D^f, from the abelian baryonic factors, as the only states in the spectrum are "electric" baryonic ones, with baryonic charges S_f. Then, the central charge of the $N = 2$ supersymmetry algebra of the $SU(2) \times U(1)^{N_f}$ gauge group coincides with that of the $SU(2)$ gauge group with N_f massive hypermultiplets (2.2). For a light $N = 2$ BPS hypermultiplet $\{H, \widetilde{H}\}$, with (n_m, n_e, S^f) charges, its effective Lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm HM} &= \int d^4\theta (H^* {\rm e}^{2V' + 2S^f V_f} H + \widetilde{H}^* {\rm e}^{-2V' - 2S^f V_f} \widetilde{H}) \\ &+ \int d^2\theta (\sqrt{2} A' H \widetilde{H} + \sqrt{2} S^f M_f H \widetilde{H}) + {\rm h.c.},\end{aligned} \tag{3.2}$$

[2]Effective model because the $SU(2) \times U(1)^{N_f}$ gauge group does not have a good ultraviolet behaviour. At the end of the section we will interpret this gauge group as the low energy effective gauge group of some asymptotically free gauge theory at some subdomain of its moduli space.

[3]In this section we rescale the quark masses by a $\sqrt{2}$ factor, and all repeated indices are summed.

where $\mathcal{A}' = n_m \mathcal{A}_D + n_e \mathcal{A}$.

The $N=1$ Kähler potential of (3.1),

$$K(a, m_f, s; \bar{a}, \bar{m}_f, \bar{s}) = \frac{1}{4\pi} \mathrm{Im}\Big(\frac{\partial \mathcal{F}}{\partial a} \bar{a} + \frac{\partial \mathcal{F}}{\partial m_f} \bar{m}_f + \frac{\partial \mathcal{F}}{\partial s} \bar{s} \Big), \tag{3.3}$$

because of its symplectic structure, is formally invariant under general $Sp(4+2N_f, \mathbf{R})$ transformations Γ,

$$\begin{pmatrix} s_D \\ s \\ m_D^f \\ m_f \\ a_D \\ a \end{pmatrix} \to \begin{pmatrix} \widetilde{s}_D \\ \widetilde{s} \\ \widetilde{m}_D^f \\ \widetilde{m}_f \\ \widetilde{a}_D \\ \widetilde{a} \end{pmatrix} = \Gamma \begin{pmatrix} s_D \\ s \\ m_D^g \\ m_g \\ a_D \\ a \end{pmatrix}; \tag{3.4}$$

where we have defined

$$m_D^f = \left(\frac{\partial \mathcal{F}}{\partial m_f} \right)_{a,s}, \quad s_D = \left(\frac{\partial \mathcal{F}}{\partial s} \right)_{a,m_f}. \tag{3.5}$$

These transformations are the isometries of the corresponding Kähler metric, given by the imaginary part of the $(2+N_f) \times (2+N_f)$ matrix of couplings

$$\begin{gathered} \tau^{aa} = \frac{\partial^2 \mathcal{F}}{\partial a^2}, \quad \tau^{fa} = \frac{\partial^2 \mathcal{F}}{\partial a \partial m_f}, \quad \tau^{0a} = \frac{\partial^2 \mathcal{F}}{\partial a \partial s}, \\ \tau^{fg} = \frac{\partial^2 \mathcal{F}}{\partial m_f \partial m_g}, \quad \tau^{0f} = \frac{\partial^2 \mathcal{F}}{\partial m_f \partial s}, \quad \tau^{00} = \frac{\partial^2 \mathcal{F}}{\partial s^2}. \end{gathered} \tag{3.6}$$

The monodromy group of the $SU(2) \times U(1)^{N_f}$ model will be a subgroup of $Sp(4+2N_f, \mathbf{R})$. But there are some physical conditions that the monodromy transformations must fulfill, in order to preserve the structure of the BPS mass formula: 1) they should preserve the monodromy invariance of the hypermultiplet masses and the dilaton, 2) they should not generate a "magnetic" baryon number different from zero, 3) they should not produce either "electric" or "magnetic" charges for the dilaton vector multiplet, and 4) they should preserve the integrality of the electric and magnetic charges. If we restrict the action of $Sp(4+2N_f, \mathbf{R})$ to the subgroup that satisfies our physical inputs, we get that the possible monodromy transformations of the $SU(2)\times U(1)^{N_f}$ are:

$$\begin{pmatrix} \widetilde{s}_D \\ \widetilde{s} \\ \widetilde{m}_D^f \\ \widetilde{m}_f \\ \widetilde{a}_D \\ \widetilde{a} \end{pmatrix} = \begin{pmatrix} s_D \\ s \\ m_D^f + p^f(\gamma a_D + \delta a) - q^f(\alpha a_D + \beta a) + r^{fg} m_g \\ m_f \\ \alpha a_D + \beta a + p^f m_f \\ \gamma a_D + \delta a + q^f m_f \end{pmatrix}, \tag{3.7}$$

with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbf{Z})$, and $p^f, q^f, r^{fg} \in \mathbf{Q}$. The reason why p^f, q^f, r^{fg} may be rational instead of integers has to do with the possible fractionalization of the quark numbers. For this reason it is more convenient to state that the monodromy is a subgroup of the group of symplectic rational matrices, and in each case determine the subgroup by looking at the explicit fractionalization (if any) of the quark numbers.

Observe that (a_D, a) transform as in the $SU(2)$ Seiberg-Witten theory with N_f massive hypermultiplets (2.5). This follows from conditions 2), 3) and 4) above. The fact that the transformation is an element of $Sp(4+2N_f, \mathbf{Q})$ fixes the monodromy transformations of m_D^f and s_D. Notice that the dual dilaton spurion, s_D, is still a monodromy invariant, as in the case of the theory with massless hypermultiplets [14]. But the dual mass spurions, m_D^f, have a quite involved monodromy transformation. If the $SU(2)\times U(1)^{N_f}$ model with dilaton spurion is equivalent to the $SU(2)$ theory with N_f massive hypermultiplets, we must be able to reproduce the same monodromy behaviour of s_D and m_D^f just from the $SU(2)$ Seiberg-Witten solution with N_f massive hypermultiplets. To check this, we should know the monodromy behaviour of the effective prepotential of massive $N=2$ QCD. It can be derived by integrating the identity

$$\begin{aligned} \Big(\frac{\partial \widetilde{\mathcal{F}}(\widetilde{a}, m_f, s)}{\partial a}\Big)_{m_f,s} &= \Big(\frac{\partial \widetilde{\mathcal{F}}}{\partial \widetilde{a}}\Big)_{m_f,s}\Big(\frac{\partial \widetilde{a}}{\partial a}\Big)_{m_f,s} \\ &= \frac{\partial}{\partial a}\Big(\mathcal{F}(a, m_f, s) + \frac{1}{2}\beta\delta a^2 + \frac{1}{2}\alpha\gamma a_D^2 + \beta\gamma a a_D \\ &\quad + p^f m_f(\gamma a + \delta a_D)\Big)|_{m_f,s}, \end{aligned} \tag{3.8}$$

where we choose the a-independent constant to be zero. On the other hand, from the Jacobian of the transformation $\{a, m_f, s\} \to \{\gamma a + \delta + q^f m_f, m_f, s\}$

we obtain the relations

$$\Big(\frac{\partial}{\partial\widetilde{a}}\Big)_{\Gamma-\mathrm{basis}} = \frac{1}{\gamma\tau^{aa}+\delta}\frac{\partial}{\partial a}, \tag{3.9}$$

$$\Big(\frac{\partial}{\partial s}\Big)_{\Gamma-\mathrm{basis}} = \frac{\partial}{\partial s} - \frac{\gamma\tau^{a0}}{\gamma\tau^{aa}+\delta}\frac{\partial}{\partial a}, \tag{3.10}$$

$$\Big(\frac{\partial}{\partial m_f}\Big)_{\Gamma-\mathrm{basis}} = \frac{\partial}{\partial m_f} - \frac{q^f+\gamma\tau^{af}}{\gamma\tau^{aa}+\delta}\frac{\partial}{\partial a}. \tag{3.11}$$

Acting with (3.10) and (3.11) on $\widetilde{\mathcal{F}}$ one obtains respectively the monodromy transformations of m_D^f and s_D, defined by equations (3.5):

$$\begin{aligned}\widetilde{m}_D^f &= \Big(\frac{\partial\widetilde{\mathcal{F}}}{\partial m_f}\Big)_{\Gamma-\mathrm{basis}} \\ &= m_D^f + p^f(\gamma a_D+\delta a) - q^f(\alpha a_D+\beta a) - q^f p^g m_g; \end{aligned}\tag{3.12}$$

$$\widetilde{s}_D = \Big(\frac{\partial\widetilde{\mathcal{F}}}{\partial s}\Big)_{\Gamma-\mathrm{basis}} = s_D. \tag{3.13}$$

Hence, we rederive the monodromy behaviour of s_D in the $SU(2)$ massive theory[4]. The monodromy transformation for the dual mass of the $SU(2)\times U(1)^{N_f}$ model, given in (3.7), coincides with that of the $SU(2)$ massive theory in (3.12) if we put $r^{fg}=-q^f p^g$. With this choice, the prepotentials of the $SU(2)\times U(1)^{N_f}$ model and the $SU(2)$ massive theory are exactly the same[5].

Finally, we can obtain the monodromy transformations of the couplings (3.6) acting with the derivatives (3.9), (3.10) and (3.11) on $\{\widetilde{a}_D, \widetilde{m}_D^f, \widetilde{s}_D\}$:

$$\begin{aligned}\widetilde{\tau}^{aa} &= \frac{\alpha\tau^{aa}+\beta}{\gamma\tau^{aa}+\delta}, \\ \widetilde{\tau}^{0a} &= \frac{\tau^{0a}}{\gamma\tau^{aa}+\delta}, \\ \widetilde{\tau}^{00} &= \tau^{00} - \frac{\gamma(\tau^{0a})^2}{\gamma\tau^{aa}+\delta},\end{aligned}$$

[4] We will explicitly see that the expresion of s_D turns out to be essentially the same as in the massles case.

[5] Remember that there is an ambiguity in the definition of the prepotential by terms quadratic in the vector multiplets, which allows to redefine a_D, m_D^f and s_D by linear terms in a, m_f and s.

$$
\begin{aligned}
\widetilde{\tau}^{af} &= \frac{\tau^{af}}{\gamma\tau^{aa}+\delta} - q^f(\frac{\alpha\tau^{aa}+\beta}{\gamma\tau^{aa}+\delta}) + p^f, \\
\widetilde{\tau}^{f0} &= \tau^{f0} - (\frac{q^f+\gamma\tau^{fa}}{\gamma\tau^{aa}+\delta})\tau^{0a}, \\
\widetilde{\tau}^{fg} &= \tau^{fg} - \frac{\gamma\tau^{fa}\tau^{ga}}{\gamma\tau^{aa}+\delta} - \frac{q^f\tau^{ga}}{\gamma\tau^{aa}+\delta} - \frac{q^g\tau^{fa}}{\gamma\tau^{aa}+\delta} \\
&\quad -p^f q^g - p^g q^f + q^f q^g(\frac{\alpha\tau^{aa}+\beta}{\gamma\tau^{aa}+\delta}).
\end{aligned} \tag{3.14}
$$

In this subsection we have shown the equivalence between the $N=2$ $SU(2)$ theory with N_f massive hypermultiplets and a pure $N=2$ $SU(2)\times U(1)^{N_f}$ gauge model, supporting the interpretation of the bare quark masses as the scalar part of some frozen $N=2$ vector multipets $\mathcal{M}_f$. This opens the possibility of promoting them to the status of $N=2$ supersymmetry breaking spurion superfields, following the strategy of [14], where the breaking was induced only by a dilaton spurion. At the same time we still keep the holomorphic structure of the low energy Lagrangian, as the $N=2$ prepotential has a holomorphic dependence on the hypermultiplet masses. Notice that, in contrast to s_D, the dual spurion mass is not a monodromy invariant. But this is not relevant as far as the m_D^f dual spurion fields do not appear in the BPS mass formula. In other words, there are no "magnetic" baryonic charges associated to the monodromy invariant hypermultiplet masses. The important thing in this softly broken $N=2$ QCD is that the extended monodromy transformations (3.7) belong to a subgroup of $Sp(4+2N_f,\mathbf{Q})$. This in turn guarantees the monodromy invariance of the vacuum energy induced by the supersymmetry breaking, since this energy is determined by the monodromy invariant $N=1$ Kähler potential (3.3).

We have focused, just for simplicity, on the analysis of the soft breaking of $N=2$ $SU(2)$ with spurion masses and dilaton. But the logic is straightforwardly extended for $SU(N_c)$ gauge groups with N_f massive hypermultiplets ($N_f\leq 2N_c$). The monodromy transformations of the $r=1,\cdots,N_c-1$ abelian vector multiplets of the $SU(N_c)$ gauge group are

$$
\left(\begin{array}{c} a_D^r \\ a^r \end{array}\right) \rightarrow \left(\begin{array}{c} \widetilde{a}_D^r \\ \widetilde{a}^r \end{array}\right) = \left(\begin{array}{c} A_s^r a_D^s + B_s^r a^s + p^{rf} m_f \\ C_s^r a_D^s + D_s^r a^s + q^{rf} m_f \end{array}\right), \tag{3.15}
$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2r,\mathbf{Z})$ and p^{rf}, q^{rf} are integer multiples of the baryonic charges. The effective prepotential has the monodromy behaviour

$$
\begin{aligned}
\widetilde{\mathcal{F}}(\widetilde{a}^r, m_f) = \quad & \mathcal{F}(a^r, m_f) + \frac{1}{2} a^T (D^T B) a + \frac{1}{2} a_D^T (C^T A) a_D + a_D^T (C^T B) a \\
& + (p \cdot m)^T (C a_D + D a),
\end{aligned} \tag{3.16}
$$

where $(p \cdot m)^r = p^{rf} m_f$. Computing the Jacobian of the monodromy transformations on the variables $\{a^r, m_f, s\}$, one obtains that

$$
\Big(\frac{\partial}{\partial \widetilde{a}^r} \Big)_{\Gamma-\text{basis}} = \Big[(C\tau + D)^{-1} \Big]_r^s \frac{\partial}{\partial a^s}, \tag{3.17}
$$

$$
\Big(\frac{\partial}{\partial s} \Big)_{\Gamma-\text{basis}} = \frac{\partial}{\partial s} - \Big[(C\tau + D)^{-1} \Big]_s^r C_t^s \tau^{0t} \frac{\partial}{\partial a^r} \tag{3.18}
$$

$$
\Big(\frac{\partial}{\partial m_f} \Big)_{\Gamma-\text{basis}} = \frac{\partial}{\partial m_f} - \Big[(C\tau + D)^{-1} \Big]_s^r (q^{sf} + C_t^s \tau^{tf}) \frac{\partial}{\partial a^r}. \tag{3.19}
$$

where we define, as usual, the couplings

$$
\begin{aligned}
\tau^{rs} &= \frac{\partial^2 \mathcal{F}}{\partial a^r \partial a^s}, \quad \tau^{rf} = \frac{\partial^2 \mathcal{F}}{\partial a^r \partial m_f}, \quad \tau^{r0} = \frac{\partial^2 \mathcal{F}}{\partial a^r \partial s}, \\
\tau^{fg} &= \frac{\partial^2 \mathcal{F}}{\partial m_f \partial m_g}, \quad \tau^{0f} = \frac{\partial^2 \mathcal{F}}{\partial m_f \partial s}, \quad \tau^{00} = \frac{\partial^2 \mathcal{F}}{\partial s^2}.
\end{aligned} \tag{3.20}
$$

Acting with (3.17, 3.18, 3.19) on (3.16), one obtains the following monodromy transformations of the $N_f + 1$ dual spurion fields:

$$
\begin{aligned}
\widetilde{m}_D^f = \Big(\frac{\partial \widetilde{\mathcal{F}}}{\partial m_f} \Big)_{\Gamma-\text{basis}} &= m_D^f + p^{rf} (C_s^r a_D^s + D_s^r a^s) \\
&- q^{rf} (A_s^r a_D^s + B_s^r a^s) - q^{rf} p^{rg} m_g,
\end{aligned} \tag{3.21}
$$

$$
\widetilde{s}_D = \Big(\frac{\partial \widetilde{\mathcal{F}}}{\partial s} \Big)_{\Gamma-\text{basis}} = s_D. \tag{3.22}
$$

Notice that s_D continues to be a monodromy invariant. One can easily verify that the monodromy transformations of the $(2N_c + 2N_f)$ dimensional vector $(s_D, s, m_D^f, m_f, a_D^r, a^r)$ given by (3.15, 3.21, 3.22), with s and m_f invariants, are in a subgroup of the group $Sp(2N_c + 2N_f, \mathbf{Q})$ that leaves invariant the corresponding low energy Kähler potential. This guarantees the monodromy invariance of the vacuum energy.

3.2 Expressions for the Couplings

In this subsection, we rederive some results of [33], where an approach via integrable sistems was used.

To find the expression of m_D^f, we regard a and m_f as independent variables and consider the second derivatives of the prepotential with respect to them:

$$\Big(\frac{\partial m_D^f}{\partial a}\Big)_{m_g} = \Big(\frac{\partial a_D}{\partial m_f}\Big)_a = \oint_{\alpha_1} \Big(\frac{\partial \lambda_{SW}}{\partial m_f}\Big)_a. \tag{3.23}$$

The Riemann bilinear relation applied to the vanishing $(2,0)$-form

$$\Big(\frac{\partial \lambda_{SW}}{\partial a}\Big)_{m_f} \wedge \Big(\frac{\partial \lambda_{SW}}{\partial m_f}\Big)_a$$

gives

$$\oint_{\alpha_1} \Big(\frac{\partial \lambda_{SW}}{\partial m_f}\Big)_a = 2\pi i \sum_{n=1}^{N_p} \mathrm{Res}_{x_n^+}\Big[\Big(\frac{\partial \lambda_{SW}}{\partial m_f}\Big)_a\Big] \int_{x_n^-}^{x_n^+} \Big(\frac{\partial \lambda_{SW}}{\partial a}\Big)_{m_f}. \tag{3.24}$$

The points x_n^+ and x_n^- $(n = 1, \cdots, N_p)$ are the simple poles of λ_{SW} at each of the two Riemann sheets. Remember that the number of simple poles is $N_p = 1, 2, 4$ for $N_f = 1, 2, 3$, respectively. In (3.24) we have taken into account that

$$\oint_{\alpha_2} \Big(\frac{\partial \lambda_{SW}}{\partial m_f}\Big)_a = \Big(\frac{\partial}{\partial m_f}\Big)_a \oint_{\alpha_2} \lambda_{SW} = 0; \tag{3.25}$$

$$\oint_{\alpha_2} \Big(\frac{\partial \lambda_{SW}}{\partial a}\Big)_{m_f} = \Big(\frac{\partial}{\partial a}\Big)_{m_f} \oint_{\alpha_2} \lambda_{SW} = 1. \tag{3.26}$$

Since the poles x_n and its corresponding residues are a-independent, the expression (3.24) can easily be integrated with respect to a, to obtain the formulae for the dual mass spurions (see fig. 2):

$$m_D^f = \sum_{n=1}^{N_p} S_n^f \int_{x_n^-}^{x_n^+} \lambda_{SW} + (a - \text{indep. const.}), \tag{3.27}$$

where S_n^f are $2\pi i$ times the residues in (3.24), *i.e.* the coefficients of the masses in the residues of λ_{SW} corresponding to the poles x_n. The resulting

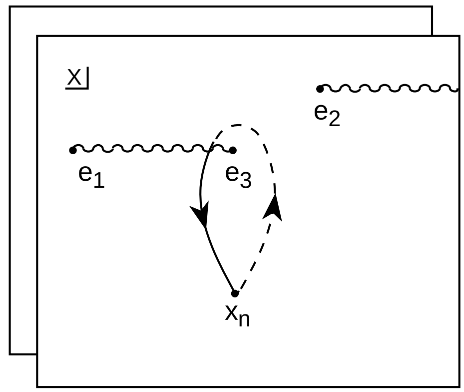

Figure 2: The integral associated to the dual spurion masses.

expression can be interpreted as an integral around some dual nontrivial one-cycle associated to the pole x_n. To see this, consider the pure $N=2$ $SU(2+N_p)$ gauge theory. The low energy description is encoded in a genus $1+N_p$ hyperelliptic curve described by a $(2N_p+3)$-order polynomial [8]. The roots of this polynomial will be denoted by $\hat{e}_1^n$, $\hat{e}_2^n$, e_1, e_2 and e_3, where $n=1,\cdots,N_p$. We define

$$\hat{m}_n = \oint_{\hat{\alpha}_2^n} \hat{\lambda}_{SW}, \tag{3.28}$$

$$\hat{m}_D^n = \oint_{\hat{\alpha}_1^n} \hat{\lambda}_{SW}, \tag{3.29}$$

where $\hat{\lambda}_{SW}$ is the Seiberg-Witten abelian differential of the $SU(2+N_p)$ theory, $\hat{\alpha}_2^n$ is a one-cycle going from $\hat{e}_1^n$ to $\hat{e}_2^n$, and $\hat{\alpha}_1^n$ is the corresponding dual one-cycle, going from $\hat{e}_2^n$ to e_3. The remaining roots, e_1 and e_2, together with e_3, define the a and a_D variables of an embedded $SU(2) \subset SU(2+N_p)$ theory, with the conventions for the branch cuts as in section 2 (see fig. 3). Now imagine going to a singular submanifold of the moduli space of the $SU(2+N_p)$ theory, where the roots $\hat{e}_1^n$ and $\hat{e}_2^n$ coincide with the values x_n of the poles of the massive $SU(2)$ theory, and $\hat{\lambda}_{SW}$ goes to λ_{SW}. Then $\hat{m}_n$ and $\hat{m}_D^n$ will become $S_n^f m_f$ and $\int_{x_n^-}^{x_n^+} \lambda_{SW}$, respectively. In this region of the moduli space, we have $SU(2+N_p) \to SU(2) \times U(1)^{N_f}$ and we recover the

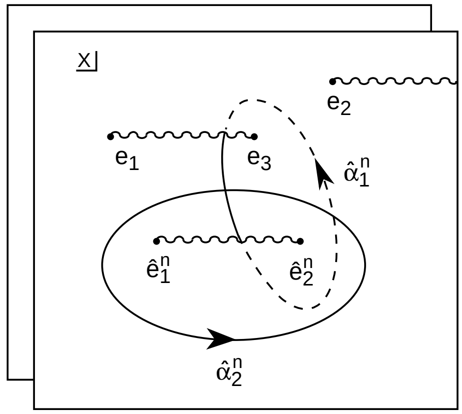

Figure 3: The $\hat{\alpha}_1^n$ and $\hat{\alpha}_2^n$ one-cycles associated to the x_n pole.

$SU(2)$ Seiberg-Witten model with N_f massive hypermultiplets[6].

The integral (3.27) can be computed by the uniformization method given in section 2, but we must be careful with the integration limits. They are on the poles of λ_{SW} and the integral diverges. This is why there are no "magnetic" baryonic charges appearing in the central charge of this $SU(2)$ massive theory embedded in a larger $SU(2+N_p)$ pure gauge theory. The corresponding "baryonic monopoles" of the $SU(2+N_p)$ theory have become infinitely massive in that singular region of the $SU(2+N_p)$ moduli space, and they have been decoupled. Effectively we should be able to substract this divergence using the ambiguity in the prepotential with respect to the addition of quadratic terms in a and m_f. This is possible because the residues of the poles are only linear in the masses. We regularize the integral and focus on the part which gives the divergence:

$$\begin{aligned} \mathrm{div}(m_D^f) &= \wp'(z_n^+)\frac{c_n}{4}\int_{z_n^- -\epsilon}^{z_n^+ +\epsilon} dz \frac{1}{\wp(z)-\wp(z_n^+)} \\ &= c_n\Big(z_n^+\zeta(z_n^+) + z_n^+\zeta(\frac{\omega_3}{2}) - \Big(\frac{\omega_3}{2}\Big)\zeta(z_n^+) \\ &\qquad -\frac{1}{2}\log\sigma(2z_n^+) + \frac{1}{2}\log\sigma(\epsilon)\Big). \end{aligned} \tag{3.30}$$

where we have chosen z_n^+ and $z_n^- = -z_n^+ + \omega_3$, both in the fundamental

[6]Notice that the S_n^f matrix has rank N_f.

domain of the lattice. As we already know, the constant c_n appearing in the previous integral is just proportional to $S_n^f m_f$, the corresponding residue of the x_n-pole. This means that we can cancel the divergence just by a quadratic redefinition of the prepotential. This is equivalent to choosing the a-independent constant in (3.27) to include $-\frac{1}{2}c_n \log\sigma(\epsilon)$.

Now we derive the expression of s_D for massive $N=2$ QCD. The Euler relation for the prepotential gives

$$2\mathcal{F} = a a_D + m_f m_D^f + \Lambda\frac{\partial\mathcal{F}}{\partial\Lambda}. \tag{3.31}$$

If we differentiate with respect to the moduli parameter u, we obtain

$$\frac{\partial}{\partial u}\Big(m_f m_D^f + \Lambda\frac{\partial\mathcal{F}}{\partial\Lambda}\Big) = \oint_{\alpha_2}\frac{dx}{y}\oint_{\alpha_1}\lambda_{SW} - \oint_{\alpha_1}\frac{dx}{y}\oint_{\alpha_2}\lambda_{SW}. \tag{3.32}$$

Applying the Riemann bilinear relation to the vanishing $(2,0)$-form

$$\frac{dx}{y}\wedge\lambda_{SW},$$

we have contributions from the residues of the x_n simple poles and from the constant factor

$$\pm\frac{\sqrt{2}}{8\pi}(4-N_f)$$

associated to the infinity points of the two Riemann sheets. The result is

$$\oint_{\alpha_1}\frac{dx}{y}\oint_{\alpha_2}\lambda_{SW} - \oint_{\alpha_2}\frac{dx}{y}\oint_{\alpha_1}\lambda_{SW} \tag{3.33}$$

$$= \frac{(4-N_f)}{4\pi i} + \sum_{n=1}^{N_p} S_n^f m_f \int_{x_n^-}^{x_n^+}\Big(\frac{\partial\lambda_{SW}}{\partial u}\Big). \tag{3.34}$$

Integrating with respect to u and using (3.27), we identify

$$s_D = i\Lambda\frac{\partial\mathcal{F}}{\partial\Lambda} = \frac{(4-N_f)}{4\pi}u \ + (a - \text{indep.term}), \tag{3.35}$$

showing its monodromy invariance. The last expression is defined up to an integration constant. This integration constant will depend on the regularization of the effective prepotential to remove the divergence of (3.27). This relation was obtained in [29, 30, 31] for the massless case; and in [33] for the massive case, where concrete quadratic terms in the hypermultiplet masses appeared in the a-independent term of formula (3.35) because they chose a concrete regularization of the prepotential. The choice of this integration constant will be done by convenience in the numerical analysis of the vacuum properties once supersymmetry is broken with soft terms [34].

Finally, we also give the explicit expressions for the couplings:

$$\tau^{fa} = \oint_{\alpha_1} \Big(\frac{\partial \lambda_{SW}}{\partial m_f}\Big)_a = \Big(\frac{\partial a_D}{\partial m_f}\Big)_u - \tau^{aa}\Big(\frac{\partial a}{\partial m_f}\Big)_u, \tag{3.36}$$

$$\tau^{0a} = \Big(\frac{\partial s_D}{\partial a}\Big)_{m_f} = \frac{(4-N_f)}{4\pi}\Big(\frac{\partial u}{\partial a}\Big)_{m_f}, \tag{3.37}$$

$$\tau^{0f} = \frac{(4-N_f)}{4\pi}\Big(\frac{\partial u}{\partial m_f}\Big)_a = -\frac{(4-N_f)}{4\pi}\Big(\frac{\partial u}{\partial a}\Big)_{m_f}\Big(\frac{\partial a}{\partial m_f}\Big)_u, \tag{3.38}$$

$$\tau^{00} = i(2s_D - a\tau^{a0} - m_f\tau^{0f}), \tag{3.39}$$

$$m_g\tau^{fg} = m_D^f - a\tau^{af} + i\tau^{0f}, \tag{3.40}$$

where τ^{00} and τ^{fg} are obtained from the Euler relation of the prepotential (3.31), acting with the appropriate derivatives. Remember posible shifts linear in the masses for τ^{0f}, coming from a concrete regularization of the prepotential.

4 The Effective Potential

In this section we will present the computation of the effective potential when mass and dilaton spurions break both supersymmetries. We will denote the $N=1$ chiral multiplets corresponding to both the $U(1)$ gauge field and the spurions by (A, S, M_f), $f=1,\cdots,N_f$, and the corresponding $N=1$ vector multiplets by (V_a, V_s, V_f). Roman capital letters refer to the mass and dilaton spurions only, $I, J = 0, 1, \cdots N_f$. $N=2$ supersymmetry will be broken down to $N=0$ by turning on VEVs for the auxiliaries of the dilaton and mass spurions. In terms of $N=1$ superfields, we have:

$$S = s + \theta^2 F^0, \qquad V_s = \frac{1}{2}D^0\theta^2\bar\theta^2,$$
$$M_f = \frac{m_f}{\sqrt{2}} + \theta^2 F^f, \qquad V_f = \frac{1}{2}D^f\theta^2\bar\theta^2, \tag{4.1}$$

where m_f are the bare quark masses, and s is related to the dynamical scale of the theory in the effective Lagrangian or to the classical gauge coupling in the bare Lagrangian. At the classical level, the breaking has the following effects. The dilaton spurion gives mass to the gauginos of the $N=2$ vector mutiplet and to the imaginary part of the Higgs field ϕ, the scalar component of the non-abelian $N=2$ vector multiplet [15]; it also gives couplings between the squarks and the Higgs field of the form

$$\overline{F}^0 \tilde{q}_f(\mathrm{Im}\phi)q_f + \mathrm{h.c.}, \qquad D^0\Big(q_f^\dagger(\mathrm{Im}\phi)q_f - \tilde{q}_f(\mathrm{Im}\phi)\tilde{q}_f^\dagger\Big), \tag{4.2}$$

while the mass spurions give terms for the squarks with the structure

$$F^f \tilde{q}_f q_f + \mathrm{h.c.}, \qquad D_f(|q_f|^2 - |\tilde{q}_f|^2). \tag{4.3}$$

In both cases, no new terms appear for the fermionic quarks. We see that the disadvantage of working with an $N=2$ spurion is that we do not generate a diagonal mass term for the squarks as in the soft breaking of $N=1$ supersymmetry. However, we have a better analytic control of the theory (at least for small supersymmetry breaking parameters). We will denote the couplings in the effective Lagrangian by

$$b_{IJ} = \frac{1}{4\pi}\mathrm{Im}\tau_{IJ}, \tag{4.4}$$

which are (up to a normalization) the components of the Kähler metric. As in section 2, we suppose that we are in the variables (electric, magnetic or dyonic) adequate to light BPS states. As all the states in the Seiberg-Witten singularities have the same charge (and normalized to one) with respect to the corresponding "photon", we have to introduce only the baryon numbers appearing in the central charge (2.2). For k light BPS states with S_i^f baryonic charges, $i = 1, \cdots, k$, the terms in (3.1) and (3.2) contributing to the effective potential are

$$\begin{aligned} V &= b_{AB}F^A\overline{F}^B + b_{aA}(F^A\overline{F}^a + \overline{F}^A F^a) + b_{aa}|F_a|^2 \\ &+ \frac{1}{2}b_{AB}D^A D^B + +b_{aA}D^a D^A + \frac{1}{2}b_{aa}D_a^2 \\ &+ (D_a + D_f S_i^f)(|h_i|^2 - |\tilde{h}_i|^2) + \sum_{i=1}^k \Big(|F_{h_i}|^2 + |F_{\tilde{h}_i}|^2\Big) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \sqrt{2}\Big(F^a h_i \widetilde{h}_i + a h_i F_{\widetilde{h}_i} + a \widetilde{h}_i F_{h_i} + \text{h.c.}\Big) \\
&+ \quad \sqrt{2}\Big(F^f S_i^f h_i \widetilde{h}_i + \frac{m_f}{\sqrt{2}} S_i^f h_i F_{\widetilde{h}_i} + \frac{m_f}{\sqrt{2}} S_i^f \widetilde{h}_i F_{h_i} + \text{h.c.}\Big), \qquad (4.5)
\end{aligned}
$$

where all repeated indices are summed. We eliminate the auxiliary fields and obtain:

$$
\begin{aligned}
D_a &= -\frac{1}{b_{aa}}\Big(b_{aA} D^A + \sum_{i=1}^{k}(|h_i|^2 - |\widetilde{h}_i|^2)\Big), \\
F_a &= -\frac{1}{b_{aa}}\Big(b_{aA} F^A + \sqrt{2}\overline{h}_i \overline{\widetilde{h}}_i\Big), \\
F_{h_i} &= -\sqrt{2}(\overline{a} + S_i^f \frac{\overline{m}_f}{\sqrt{2}})\overline{\widetilde{h}}_i, \qquad F_{\widetilde{h}_i} = -\sqrt{2}(\overline{a} + S_i^f \frac{\overline{m}_f}{\sqrt{2}})\overline{h}_i. \qquad (4.6)
\end{aligned}
$$

Which substituted in (4.5) yields:

$$
\begin{aligned}
V \quad = & \quad \Big(\frac{b_{aA} b_{aB}}{b_{aa}} - b_{AB}\Big)\Big(\frac{1}{2} D^A D^B + F^A \overline{F}^B\Big) + \frac{b_{aA}}{b_{aa}} D^A \sum_{i=1}^{k}(|h_i|^2 - |\widetilde{h}_i|^2) \\
+ & \quad \frac{\sqrt{2} b_{aA}}{b_{aa}}\Big(F^A h_i \widetilde{h}_i + \overline{F}^A \overline{h}_i \overline{\widetilde{h}}_i\Big) + \frac{2}{b_{aa}} \overline{h}_i \overline{\widetilde{h}}_i h_j \widetilde{h}_j \\
+ & \quad \frac{1}{2 b_{aa}} \sum_{i,j=1}^{k} (|h_i|^2 - |\widetilde{h}_i|^2)(|h_j|^2 - |\widetilde{h}_j|^2) - D_f S_i^f (|h_i|^2 - |\widetilde{h}_i|^2) \\
+ & \quad 2|a + S_i^f \frac{m_f}{\sqrt{2}}|^2 (|h_i|^2 + |\widetilde{h}_i|^2) \\
- & \quad \sqrt{2}\Big(S_i^f F^f h_i \widetilde{h}_i + S_i^f \overline{F}^f \overline{h}_i \overline{\widetilde{h}}_i\Big), \qquad (4.7)
\end{aligned}
$$

where $(b_{aA} b_{aB}/b_{aa}) - b_{AB}$ is the cosmological term. This term in the potential must be a monodromy invariant, as we expect from the analysis in section 3. One can indeed check it explicitly using the monodromy transformations of the couplings given in (3.14).

The potential (4.7) has a very rich structure of vacua for the different flavors, and there are a number of interesting questions in ordinary QCD that can be translated to this context. We will present the detailed analysis in [34].

5 Conclusions

In this paper we have shown that it is possible to softly break $N = 2$-supersymmetric extensions of QCD in a way that preserves the analytic properties of the solutions in [3, 4]. We obtain a $3(N_f + 1)$ space of parameters associated to the auxiliary fields of $N_f + 1$ spurion vector superfields. One of them is the dilaton multiplet of $N = 2$ supergravity, and the others are associated to the gauging of the quark number symmetries for each hypermultiplet. Although the analysis has been carried out for the gauge group $SU(2)$, the generalization is straightforward for other groups. We have obtained explicit expressions for the Seiberg-Witten periods using the standard uniformization of elliptic curves, and verified our expressions at strong, and weak coupling and by computing the residues of their poles which are linear combinations of the bare quark masses. An important ingredient in the determination of the effective action in the presence of spurions is the question of monodromy invariance. We have analyzed this issue by three different procedures giving the same answer. In particular we have obtained our results by embedding the $SU(2)$ theory in the $SU(2 + N_p)$ theory looking at a singular subset of its moduli space where some of its monopoles and dyon excitations have infinite mass. We have obtained explicit expressions for the couplings of spurions to other multiplets, and as a consequence we can write down the exact form of the effective potential for moderate values of the supersymmetry breaking parameters with respect to the dynamical scale of the theory. It remains now to look at the possible vacuum structures, patterns of symmetry breaking and low energy phenomena encoded in this effective action. This is currently under investigation.

It is clear that although the theories we have treated are still far from real QCD, one can nevertheless pose many analogous questions in these softly broken theories and obtain explicit answers. The issue is how one could eventually compare with the real theory. There are a number of obstacles that remain to be dealt with, most notably the fact that we have a scalar field in the adjoint representation. Only in the limit when this field decouples it seems possible to obtain the structure of global symmetries in QCD. It is difficult to see how to do this in the case at hand, but it might be possible in some large N limit version of our results. This as well as the possible effects of higher derivative terms in the effective action, and how they could modify the vacuum structures, are being investigated. We plan to report on our results elsewhere.

Acknowledgements

We acknowledge J.L.F. Barbón for valuable discussions and a critical reading of the manuscript. We also thank Isabelle Canon for helping with the figures. L.A.G. would like to thank the Physics Department of Tokyo University where part of this work was done. M.M. and F.Z. would like to thank the Theory Division at CERN for its hospitality. The work of M.M. is supported in part by DGICYT under grant PB93-0344 and by CICYT under grant AEN94-0928. The work of F.Z. is supported by a fellowship from Ministerio de Educación.

References

[1] N. Seiberg, Phys. Lett. **B318** (1993) 469, hep-ph/9309335; Phys. Rev. **D49** (1994) 6857, hep-th/9402044;
K. Intriligator, R. Leigh and N. Seiberg, Phys. Rev. **D50** (1994) 1052, hep-th/9403198;
K. Intriligator, Phys. Lett. **B336** (1994) 409, hep-th/9407106;
K. Intriligator and N. Seiberg, Nucl. Phys. **B431** (1994) 551, hep-th/9408155.

[2] N. Seiberg, Nucl. Phys. **B435** (1995) 129, hep-th/9411149.

[3] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19, hep-th/9407087.

[4] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484, hep-th/9408099.

[5] L. Álvarez-Gaumé and S.F. Hassan, hep-th/9701169.

[6] W. Lerche, hep-th/9611190.

[7] A. Bilal, hep-th/9601007.

[8] A. Klemm, W. Lerche. S. Theisen and S. Yankielowicz, Phys. Lett. **B344** (1995) 169, hep-th/9411048;
P.C. Argyres and A.E. Faraggi, Phys. Rev. Lett. **74** (1995) 3931, hep-th/9411057.

[9] A. Klemm, W. Lerche. and S. Theisen, Int. J. Mod. Phys. **A11** (1996) 1929, hep-th/9505150.

[10] M. Douglas and S.H. Shenker, Nucl. Phys. **B447** (1995) 271, hep-th/9503163.

[11] A. Brandhuber and K. Landsteiner, Phys. Lett. **B358** (1995) 73, hep-th/9507008;
U.H. Danielsson and B. Sundborg, Phys. Lett. **B358** (1995) 273, hep-th/9504102;
A. Hanany and Y. Oz, Nucl. Phys. **B452** (1995) 283, hep-th/9505075;
P.C. Argyres, M.R. Plesser and A.D. Shapere, Phys. Rev. Lett. **75** (1995) 1699, hep-th/9505100.

[12] O. Aharony, J. Sonnenschein, M.E. Peskin and S. Yankielowicz, Phys. Rev. **D52** (1995) 6157, hep-th/9507013.

[13] N. Evans, S.D.H. Hsu and M. Schwetz, Phys. Lett. **B355** (1995) 475, hep-th/9503186;
N. Evans, S.D.H. Hsu, M. Schwetz, S.B. Selipsky, Nucl. Phys. **B456** (1995) 205, hep-th/9508002.

[14] L. Álvarez-Gaumé, J. Distler, C. Kounnas and M. Mariño, Int. J. Mod. Phys. **A11** (1996) 4745, hep-th/9604004; L. Álvarez-Gaumé and M. Mariño, hep-th/9606168; L. Álvarez-Gaumé and M. Mariño, Int. J. Mod. Phys. **A12** (1997) 975, hep-th/9606191.

[15] N. Evans, S.D.H. Hsu and M. Schwetz, hep-th/9608135.

[16] E. Witten, Phys. Lett. **B86** (1979) 283.

[17] F. Ferrari, Phys. Rev. Lett. **78** (1997) 795, hep-th/9609191.

[18] A. Brandhuber and S. Stieberger, hep-th/9610053.

[19] F. Ferrari, hep-th/9702166.

[20] J.M. Isidro, A. Mukherjee, J.P. Nunes and H.J. Schnitzer, hep-th/9609116.

[21] Y. Ohta, J. Math. Phys. **37** (1996) 6074, hep-th/9604051; Y. Ohta, hep-th/9604059.

[22] T. Masuda and H. Suzuki, hep-th/9609066.

[23] A. Brandhuber and S. Stieberger, hep-th/9609130.

[24] K. Chandrasekharan, Elliptic functions (Springer-Verlag, 1985).

[25] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series and products (Academic Press, 1994).

[26] K. Ito and S.-K. Yang, Phys. Lett. **B366** (1996) 165, hep-th/9507144.

[27] A. Bilal and F. Ferrari, Nucl. Phys. **B480** (1996) 589, hep-th/9605101.

[28] See for instance F. Kirwan, Complex algebraic curves (Cambridge University Press, 1992).

[29] M. Matone, Phys. Lett. **B357** (1995) 342, hep-th/9506102.

[30] J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. **B367** (1996) 145, hep-th/9510129.

[31] T. Eguchi and S.-K. Yang, Mod. Phys. Lett. **A11** (1996) 131, hep-th/9510183.

[32] M. Henningson, Nucl. Phys. **B458** (1996) 445;

B. de Wit, M.T. Grisaru and M. Rocek, Phys. Lett **B374** (1996) 297;

M. Matone, hep-th/9610204.

[33] E. D'Hoker, I.M. Krichever and D.H. Phong, hep-th/9610156.

[34] L. Álvarez-Gaumé, M. Mariño and F. Zamora, in preparation.