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The 4-loop quark mass anomalous dimension and the invariant quark mass.

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Abstract

We present the analytical calculation of the four-loop quark mass anomalous dimension in Quantum Chromodynamics within the minimal subtraction scheme. On the basis of this result we find that the so-called invariant quark mass is a very good reference mass for the accurate evolution of the running $\overline{\mathrm{MS}}$ quark mass in phenomenological applications. We also obtain for the first time a complete 4-th order perturbative QCD expression for a physical quantity, the total Higgs boson decay rate into hadrons, and analyze the infrared fixed point for this case.

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The renormalization group equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects that appear in these renormalization group equations.

The quark mass anomalous dimension in QCD was calculated at the 3-loop level in [1]. In a previous paper [2] we calculated analytically the 4-loop QCD β-function. Here we complete this 4-loop project by obtaining the quark mass anomalous dimension in the 4-loop order.

We will apply the obtained anomalous dimension to show the relevance of the invariant quark mass for an accurate evolution of the running quark masses in phenomenological applications. We also apply the 4-loop result to analyze a possible QCD infrared fixed point in the case of Higgs boson decay into hadrons.

Throughout the calculations in this article we use the technique of Dimensional Regularization [3] for the regularization of ultraviolet divergences and the Minimal Subtraction (MS) scheme [4] or its standard modification, the $\overline{\rm MS}$ scheme [5], for renormalization. The dimension of space-time is defined as $D = 4 - 2\varepsilon$, where ε is the regularzation parameter fixing the deviation of the space-time dimension from its physical value 4.

Let us write the full Lagrangian of Quantum Chromodynamics with massive quarks in the covariant gauge

$$
\begin{aligned}
L &= -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \sum_{q=1}^{n_f}\overline{\psi_q}\,(i\not{D} - m_q)\,\psi_q + L_{gf} + L_{gc} \\
G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\
[D_\mu]_{ij} &= \delta_{ij}\partial_\mu - igA^a_\mu [T^a]_{ij} \qquad (1)
\end{aligned}
$$

The gauge-fixing and gauge-compensating parts of the Lagrangian in the covariant gauge are

$$
\begin{aligned}
L_{gf} &= -\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \\
L_{gc} &= \partial^\mu \overline{\omega}^a(\partial_\mu \omega^a - g f^{abc}\omega^b A^c_\mu) \qquad (2)
\end{aligned}
$$

The fermion fields (the quark fields in QCD) ψ_q have a mass m_q and transform as the fundamental representation of a compact semi-simple Lie group, $q = 1, ..., n_f$ is the flavor index. The Yang-Mills fields (gluons in QCD) A^a_μ transform as the adjoint representation of this group. ω^a are the ghost fields, and ξ is the gauge parameter of the covariant gauge.

T^a are the generators of the fundamental representation and f^{abc} are the structure constants of the Lie algebra,

$$
T^aT^b - T^bT^a = if^{abc}T^c \qquad (3)
$$

In the case of QCD we have the gauge group SU(3) but we will perform the calculation for an arbitrary compact semi-simple Lie group G.

The beta function and the quark mass anomalous dimension are defined as

$$
\begin{aligned}
\frac{da}{d\ln\mu^2} &= \beta(a) \\
&= -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - \beta_3 a^5 + O(a^6) \qquad (4)
\end{aligned}
$$

$$\begin{aligned} \frac{d \ln m_q}{d \ln \mu^2} &= \gamma_m(a) \\ &= -\gamma_0 a - \gamma_1 a^2 - \gamma_2 a^3 - \gamma_3 a^4 + O(a^5) \end{aligned} \tag{5}$$

where $m_q = m_q(\mu^2)$ is the renormalized (running) quark mass and μ is the renormalization point in the $\overline{\rm MS}$ scheme. $a = \alpha_s/\pi = g^2/4\pi^2$ where $g = g(\mu^2)$ is the renormalized strong coupling constant of the standard QCD Lagrangian of Eq. (1).

This normalization for the basic perturbative expansion parameter a differs from the normalization $g^2/16\pi^2$ which naturally appears in multiloop calculations and was therefore used in [2]. Here we adopt the different normalization $a = \alpha_s/\pi$ because it is the commonly accepted normalization in applications.

The 4-loop QCD β-function in the $\overline{\rm MS}$ scheme reads

$$\begin{aligned} \beta_0 &= \frac{1}{4}\left[11 - \frac{2}{3}n_f\right] \\ \beta_1 &= \frac{1}{16}\left[102 - \frac{38}{3}n_f\right] \\ \beta_2 &= \frac{1}{64}\left[\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right] \\ \beta_3 &= \frac{1}{256}\left[\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f \right. \\ &\quad \left. + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right] \end{aligned} \tag{6}$$

Or in a numerical form

$$\begin{aligned} \beta_0 &\approx 2.75 - 0.166667 n_f \\ \beta_1 &\approx 6.375 - 0.791667 n_f \\ \beta_2 &\approx 22.3203 - 4.36892 n_f + 0.0940394 n_f^2 \\ \beta_3 &\approx 114.23 - 27.1339 n_f + 1.58238 n_f^2 + 0.0058567 n_f^3 \end{aligned} \tag{7}$$

To calculate the quark mass anomalous dimension γ_m we need to calculate the renormalization constant Z_m of the quark mass

$$m_B = Z_m m$$

where m_B is the bare (unrenormalized) quark mass. Since the anomalous dimension γ_m does not depend on masses within the MS scheme it is formally the same for all quarks. We will therefore omit the flavor index q in m_q when it is irrelevant. The expression of the γ_m-function via Z_m is given by the following chain of equations

$$\frac{d \ln m_B}{d \ln \mu^2} = 0 = \frac{d \ln Z_m}{d \ln \mu^2} + \frac{d \ln m}{d \ln \mu^2}$$

$$\Rightarrow \gamma_m = -\frac{d\ln Z_m}{d\ln\mu^2} = -\frac{\partial \ln Z_m}{\partial a}\frac{da}{d\ln\mu^2} = -\frac{\partial \ln Z_m}{\partial a}\left[-\varepsilon a + \beta(a)\right] = a\frac{\partial Z_m^{(1)}}{\partial a} \tag{8}$$

where one uses the fact that the bare mass m_B is invariant under the renormalization group transformations. Here $da/(d\ln\mu^2) = -\varepsilon a + \beta(a)$ is the β-function in D-dimensions, $Z_m^{(1)}$ is the coefficient of the first ε-pole in Z_m defined below.

Renormalization constants within the MS-scheme do not depend on dimensional parameters (masses, momenta) [6] and have the following structure:

$$Z_m(a, \frac{1}{\varepsilon}) = 1 + \sum_{n=1}^{\infty}\frac{Z_m^{(n)}(a)}{\varepsilon^n}, \tag{9}$$

Since Z_m does not depend explicitly on μ and m, the γ_m-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ. That is why the γ_m-function is the same in the MS-scheme [4] and in the $\overline{\text{MS}}$-scheme [5].

We obtained the mass renormalization constant Z_m by calculating the following two renormalization constants of the Lagrangian: $Z_{\overline{\psi}\psi}$ of the bilocal operator $\overline{\psi}\psi$ and Z_ψ of the quark field (i.e. of the inverted quark propagator). The connection between renormalized and bare quantities is defined as follows

$$\begin{aligned} [\overline{\psi}\psi]_R &= \frac{Z_{\overline{\psi}\psi}}{Z_\psi}\overline{\psi}_B\psi_B \\ \psi_B &= Z_\psi^{1/2}\psi \end{aligned} \tag{10}$$

Then $Z_m = Z_{\overline{\psi}\psi}/Z_\psi$. In order to calculate the 4-loop approximation of Z_m we needed to calculate the 4-loop ultraviolet counterterms of the quark propagator and of the Green function

$$G_{\overline{\psi}[\overline{\psi}\psi]\psi} = \int dx dy e^{iqx+ipy}\langle 0|T\{\overline{\psi}(x)[\overline{\psi}(y)\psi(y)]\psi(0)\}|0\rangle \tag{11}$$

These 4-loop ultraviolet counterterms were calculated using a technique which is based on the method of infrared rearrangement [7] and is described in Ref. [2]. This general technique reduces the calculation of the counterterms to the direct calculation of 4-loop massive vacuum (bubble) integrals and provides a procedure that is well suited for the automatic evaluation of large numbers of Feynman diagrams. The calculations were done with the symbolic manipulation program FORM [8] and a specially designed database program that stores the results of the individual diagrams and adds them in the end. For the present calculation we needed to evaluate of the order of 10000 4-loop diagrams. These diagrams were generated with the program QGRAF [9].

We obtained in this way the following result for the 4-loop γ_m-function in the $\overline{\text{MS}}$-scheme

$$\begin{aligned} \gamma_0 &= \frac{1}{4}\left[3C_F\right] \\ \gamma_1 &= \frac{1}{16}\left[\frac{3}{2}C_F^2 + \frac{97}{6}C_F C_A - \frac{10}{3}C_F T_F n_f\right] \end{aligned}$$

$$
\begin{aligned}
\gamma_2 &= \frac{1}{64}\left[\frac{129}{2}C_F^3 - \frac{129}{4}C_F^2C_A + \frac{11413}{108}C_FC_A^2\right.\\
&\left. +C_F^2T_Fn_f(-46+48\zeta_3) + C_FC_AT_Fn_f\left(-\frac{556}{27}-48\zeta_3\right) - \frac{140}{27}C_FT_F^2n_f^2\right]\\
\gamma_3 &= \frac{1}{256}\left[C_F^4\left(-\frac{1261}{8}-336\zeta_3\right) + C_F^3C_A\left(\frac{15349}{12}+316\zeta_3\right)\right.\\
&+C_F^2C_A^2\left(-\frac{34045}{36}-152\zeta_3+440\zeta_5\right) + C_FC_A^3\left(\frac{70055}{72}+\frac{1418}{9}\zeta_3-440\zeta_5\right)\\
&+C_F^3T_Fn_f\left(-\frac{280}{3}+552\zeta_3-480\zeta_5\right) + C_F^2C_AT_Fn_f\left(-\frac{8819}{27}+368\zeta_3-264\zeta_4+80\zeta_5\right)\\
&+C_FC_A^2T_Fn_f\left(-\frac{65459}{162}-\frac{2684}{3}\zeta_3+264\zeta_4+400\zeta_5\right)\\
&+C_F^2T_F^2n_f^2\left(\frac{304}{27}-160\zeta_3+96\zeta_4\right)\\
&+C_FC_AT_F^2n_f^2\left(\frac{1342}{81}+160\zeta_3-96\zeta_4\right) + C_FT_F^3n_f^3\left(-\frac{664}{81}+\frac{128}{9}\zeta_3\right)\\
&\left.+\frac{d_F^{abcd}d_A^{abcd}}{N_F}(-32+240\zeta_3) + n_f\frac{d_F^{abcd}d_F^{abcd}}{N_F}(64-480\zeta_3)\right] \qquad (12)
\end{aligned}
$$

Here ζ is the Riemann zeta function ($\zeta_3 = 1.2020569\cdots$, $\zeta_4 = 1.0823232\cdots$ and $\zeta_5 = 1.0369277\cdots$). $[T^aT^a]_{ij} = C_F\delta_{ij}$ and $f^{acd}f^{bcd} = C_A\delta^{ab}$ are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra and $\mathrm{tr}(T^aT^b) = T_F\delta^{ab}$ is the trace normalization of the fundamental representation. N_F is the dimension of the fermion representation (i.e. the number of quark colours) and n_f is the number of quark flavors. We expressed the higher order group invariants in terms of contractions between the following fully symmetrical tensors:

$$
\begin{aligned}
d_F^{abcd} &= \frac{1}{6}\mathrm{Tr}\left[T^aT^bT^cT^d + T^aT^bT^dT^c + T^aT^cT^bT^d\right.\\
&\left.+T^aT^cT^dT^b + T^aT^dT^bT^c + T^aT^dT^cT^b\right] \qquad (13)\\
d_A^{abcd} &= \frac{1}{6}\mathrm{Tr}\left[C^aC^bC^cC^d + C^aC^bC^dC^c + C^aC^cC^bC^d\right.\\
&\left.+C^aC^cC^dC^b + C^aC^dC^bC^c + C^aC^dC^cC^b\right] \qquad (14)
\end{aligned}
$$

where the matrices $[C^a]_{bc} \equiv -if^{abc}$ are the generators in the adjoint representation.

The result of Eq. (12) is valid for an arbitrary semi-simple compact Lie group. The result for QED (i.e. the group U(1)) is included in Eq. (12) by substituting $C_A = 0$, $d_A^{abcd} = 0$, $C_F = 1$, $T_F = 1$, $(d_F^{abcd})^2 = 1$, $N_F = 1$. The n_f^3 and n_f^2 terms in this 4-loop result for QED agree with the literature [10] where the leading and next-to-leading large-n_f terms for the QED gamma-function were calculated in all orders of the coupling constant. Furthermore it is interesting to note that for the choice $C_F = C_A = T_F, n_f = 1/2$ and $d_F = d_A$, which corresponds to the case of N=1 supersymmetry, all ζ terms and terms with the tensors d cancel. This is analogous to the case of the 4-loop β-function [2].

The result of Eq.(12) was obtained in an arbitrary covariant gauge for the gluon field. This means that we have kept the gauge parameter ξ that appears in the gluon propagator $i[-g^{\mu\nu} + (1-\xi)q^\mu q^\nu/(q^2+i\epsilon)]/(q^2+i\epsilon)$ as a free parameter in the calculations. The explicit cancellation of the gauge dependence in γ_m gives an important check of the results. The results for individual diagrams that contribute to γ_m also contain several constants specific for massive vacuum integrals. The cancellation of these constants at various stages in the calculation provides additional checks of the result. We note that at the 4-loop level ζ_4 and ζ_5 (Riemann zeta function of arguments 4 and 5) appear as new constants.

For the standard normalization of the SU(N) generators we find the following expressions for the color factors (more details about the color factors for various other groups can be found in [2])

$$T_F = \frac{1}{2}, \quad N_F = N, \quad C_A = N, \quad C_F = \frac{N^2-1}{2N},$$
$$\frac{d_F^{abcd}d_A^{abcd}}{N_F} = \frac{(N^2-1)(N^2+6)}{48}, \quad \frac{d_F^{abcd}d_F^{abcd}}{N_F} = \frac{(N^2-1)(N^4-6N^2+18)}{96N^3}$$

Substitution of these color factors for $N=3$ into Eq. (12) yields the following result for QCD

$$\begin{aligned}
\gamma_0 &= 1 \\
\gamma_1 &= \frac{1}{16}\left[\frac{202}{3} - \frac{20}{9}n_f\right] \\
\gamma_2 &= \frac{1}{64}\left[1249 + \left(-\frac{2216}{27} - \frac{160}{3}\zeta_3\right)n_f - \frac{140}{81}n_f^2\right] \\
\gamma_3 &= \frac{1}{256}\left[\frac{4603055}{162} + \frac{135680}{27}\zeta_3 - 8800\zeta_5 + \left(-\frac{91723}{27} - \frac{34192}{9}\zeta_3 + 880\zeta_4 + \frac{18400}{9}\zeta_5\right)n_f\right. \\
&\left. + \left(\frac{5242}{243} + \frac{800}{9}\zeta_3 - \frac{160}{3}\zeta_4\right)n_f^2 + \left(-\frac{332}{243} + \frac{64}{27}\zeta_3\right)n_f^3\right]
\end{aligned} \tag{15}$$

Or in a numerical form

$$\begin{aligned}
\gamma_0 &= 1 \\
\gamma_1 &\approx 4.20833 - 0.138889n_f \\
\gamma_2 &\approx 19.5156 - 2.28412n_f - 0.0270062n_f^2 \\
\gamma_3 &\approx 98.9434 - 19.1075n_f + 0.276163n_f^2 + 0.00579322n_f^3
\end{aligned} \tag{16}$$

Let us now consider the solution of the evolution (i.e. renormalization group) equation Eq.(5) for the quark mass

$$m_q(\mu^2) = m_q(\mu_0^2)\exp\left(\int_{a(\mu_0^2)}^{a(\mu^2)} da' \frac{\gamma_m(a')}{\beta(a')}\right) = \hat{m}_q \exp\left(\int^{a(\mu^2)} da' \frac{\gamma_m(a')}{\beta(a')}\right) \tag{17}$$

where we formally define the renormalization group invariant (i.e. independent of μ^2) quark mass $\hat{m}_q$ as

$$\hat{m}_q = m_q(\mu_0^2)\exp\left(-\int^{a(\mu_0^2)} da' \frac{\gamma_m(a')}{\beta(a')}\right) \tag{18}$$

One can expand Eq.(17) in a to obtain the following perturbative solution to the evolution equation

$$m_q(\mu^2) = \hat{m}_q a^{\gamma_0/\beta_0} \left[1 + A_1\ a + \left(A_1^2 + A_2\right)\frac{a^2}{2} + \left(\frac{1}{2}A_1^3 + \frac{3}{2}A_1 A_2 + A_3\right)\frac{a^3}{3} + O(a^4)\right] \quad (19)$$

where

$$\begin{aligned}
A_1 &= -\frac{\beta_1\gamma_0}{\beta_0^2} + \frac{\gamma_1}{\beta_0} \\
A_2 &= \frac{\gamma_0}{\beta_0^2}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right) - \frac{\beta_1\gamma_1}{\beta_0^2} + \frac{\gamma_2}{\beta_0} \\
A_3 &= \left[\frac{\beta_1\beta_2}{\beta_0} - \frac{\beta_1}{\beta_0}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right) - \beta_3\right]\frac{\gamma_0}{\beta_0^2} + \frac{\gamma_1}{\beta_0^2}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right) - \frac{\beta_1\gamma_2}{\beta_0^2} + \frac{\gamma_3}{\beta_0}
\end{aligned}$$

This gives us the following 4-loop expansions for the running $\overline{\mathrm{MS}}$ quark masses $m_q(\mu^2)$ for quark flavors q=s,c,b,t

$$\begin{aligned}
m_s(\mu^2) &= \hat{m}_s\left(\frac{\alpha_s}{\pi}\right)^{4/9}\left[1 + 0.895062\left(\frac{\alpha_s}{\pi}\right) + 1.37143\left(\frac{\alpha_s}{\pi}\right)^2 + 1.95168\left(\frac{\alpha_s}{\pi}\right)^3\right] \\
m_c(\mu^2) &= \hat{m}_c\left(\frac{\alpha_s}{\pi}\right)^{12/25}\left[1 + 1.01413\left(\frac{\alpha_s}{\pi}\right) + 1.38921\left(\frac{\alpha_s}{\pi}\right)^2 + 1.09054\left(\frac{\alpha_s}{\pi}\right)^3\right] \\
m_b(\mu^2) &= \hat{m}_b\left(\frac{\alpha_s}{\pi}\right)^{12/23}\left[1 + 1.17549\left(\frac{\alpha_s}{\pi}\right) + 1.50071\left(\frac{\alpha_s}{\pi}\right)^2 + 0.172478\left(\frac{\alpha_s}{\pi}\right)^3\right] \\
m_t(\mu^2) &= \hat{m}_t\left(\frac{\alpha_s}{\pi}\right)^{4/7}\left[1 + 1.39796\left(\frac{\alpha_s}{\pi}\right) + 1.79348\left(\frac{\alpha_s}{\pi}\right)^2 - 0.683433\left(\frac{\alpha_s}{\pi}\right)^3\right] \quad (20)
\end{aligned}$$

where $\alpha_s \equiv \alpha_s(\mu^2/\Lambda^2_{\overline{\mathrm{MS}}})$.

In phenomenological applications one uses the $\overline{\mathrm{MS}}$ quark masses at a characteristic scale μ of a considered process to sum large logarithms. From the expansions (20) we conclude that the invariant mass $\hat{m}_q$ is a good reference mass for the accurate evolution of the $\overline{\mathrm{MS}}$ quark masses to the necessary scale μ in phenomenological applications. The perturbative coefficients of the solutions (20) of evolution equations for the $\overline{\mathrm{MS}}$ quark masses are small up to and including the 4-loop level. On the contrary, it is known that the equations expressing the $\overline{\mathrm{MS}}$ quark masses m_q via the pole quark masses M_q have worse convergence of the perturbative expansions [11]; for a review see Ref. [12]. We should emphasize in this respect that the invariant mass is a fundamental gauge-invariant object that naturally appears in the solution of the renormalization group equation.

The analysis of the QCD infrared fixed point

Let us apply the obtained 4-loop quark mass anomalous dimension to the analysis of the QCD infrared fixed point in the case of the Higgs boson decay into hadrons. The infrared fixed point was analyzed at the third-order of perturbative QCD in Ref. [13] for the total hadronic Higgs decay and in [14, 15] for electron-positron annihilation into hadrons. Recently the total hadronic decay width of the Higgs boson was calculated at the 4-loop level of perturbative QCD [16] in the limit of massless quarks.

$$\begin{aligned}
\Gamma_H &= \frac{3G_F}{4\sqrt{2}\pi} M_H \sum_q m_q^2 \Bigg\{ 1 + \frac{17}{3}a + a^2 \Bigg[\frac{10801}{144} - \frac{19}{12}\pi^2 - \frac{39}{2}\zeta_3 \\
&\quad + n_f \left(-\frac{65}{24} + \frac{1}{18}\pi^2 + \frac{2}{3}\zeta_3 \right) \Bigg] + a^3 \Bigg[\frac{6163613}{5184} - \frac{3535}{72}\pi^2 - \frac{109735}{216}\zeta_3 + \frac{815}{12}\zeta_5 \\
&\quad + n_f \left(-\frac{46147}{486} + \frac{277}{72}\pi^2 + \frac{262}{9}\zeta_3 - \frac{5}{6}\zeta_4 - \frac{25}{9}\zeta_5 \right) + n_f^2 \left(\frac{15511}{11664} - \frac{11}{162}\pi^2 - \frac{1}{3}\zeta_3 \right) \Bigg] \Bigg\} \\
&\approx \frac{3G_F}{4\sqrt{2}\pi} M_H \sum_q m_q^2 \, [\, 1 + 5.66667a + (35.93996 - 1.35865 n_f)\, a^2 \\
&\quad + \left(164.1392 - 25.77119 n_f + 0.258974 n_f^2 \right) a^3 \,] \qquad (21)
\end{aligned}$$

where $a = \frac{\alpha_s(M_H)}{\pi}$, $m_q = m_q(M_H)$ and M_H is the Higgs mass. We should note that if the Higgs boson is lighter than the top quark, then additional contributions appear from the so-called singlet diagrams due to non-decoupling of the heavy top quark in this channel [17], see also Ref. [18]. We will neglect these contributions.

Together with the 4-loop quark mass anomalous dimension it allows for the first time to perform the analysis of the physical quantity at the fourth-order of perturbative QCD, i.e. with *four* known perturbative QCD α_s-terms.

For the analysis of the infrared fixed point it is convenient instead of Γ_H to introduce the function

$$\begin{aligned}
R(a) &= -\frac{1}{2}\frac{d\ln(\Gamma_H/M_H)}{d\ln M_H^2} = -\gamma_m(a) - \frac{1}{2}\beta(a)\frac{\partial \ln\Gamma(a)}{\partial a} \\
&= r_0 a(1 + r_1 a + r_2 a^2 + r_3 a^3) + O(a^5) \qquad (22)
\end{aligned}$$

where $\Gamma_H \equiv \frac{3G_F}{4\sqrt{2}\pi} M_H \sum_q m_q^2 \Gamma(a)$ with $\Gamma(a) \equiv 1 + \Gamma_1 a + \Gamma_2 a^2 + \Gamma_3 a^3$. This gives

$$\begin{aligned}
r_0 &= \gamma_0 = 1 \\
r_1 &= \gamma_1 + \frac{1}{2}\beta_0\Gamma_1 \\
r_2 &= \gamma_2 + \frac{1}{2}(\beta_1\Gamma_1 - \beta_0\Gamma_1^2 + 2\beta_0\Gamma_2) \\
r_3 &= \gamma_3 + \frac{1}{2}(\beta_0\Gamma_1^3 - \beta_1\Gamma_1^2 + \beta_2\Gamma_1 + 2\beta_1\Gamma_2 - 3\beta_0\Gamma_1\Gamma_2 + 3\beta_0\Gamma_3) \qquad (23)
\end{aligned}$$

And the full result for R becomes

$$\begin{aligned}
r_1 &= 12 - \frac{11}{18}n_f \\
&\approx 12 - 0.61111n_f
\end{aligned} \tag{24}$$

$$\begin{aligned}
r_2 &= \frac{7189}{36} - \frac{209}{48}\pi^2 - \frac{429}{8}\zeta_3 + n_f\left(-\frac{2995}{144} + \frac{5}{12}\pi^2 + \frac{17}{4}\zeta_3\right) \\
&\quad + n_f^2\left(\frac{275}{648} - \frac{1}{108}\pi^2 - \frac{1}{9}\zeta_3\right) \\
&\approx 92.26 - 11.578n_f + 0.19944n_f^2
\end{aligned} \tag{25}$$

$$\begin{aligned}
r_3 &= \frac{81937369}{20736} - \frac{11239}{64}\pi^2 - \frac{3014503}{1728}\zeta_3 + \frac{7865}{32}\zeta_5 \\
&\quad + n_f\left(-\frac{4313797}{6912} + \frac{15097}{576}\pi^2 + \frac{180343}{864}\zeta_3 - \frac{2945}{144}\zeta_5\right) \\
&\quad + n_f^2\left(\frac{1734463}{62208} - \frac{1043}{864}\pi^2 - \frac{71}{9}\zeta_3 + \frac{25}{36}\zeta_5\right) + n_f^3\left(-\frac{985}{2916} + \frac{11}{648}\pi^2 + \frac{5}{54}\zeta_3\right) \\
&\approx 376.12 - 135.72n_f + 7.2045n_f^2 - 0.05895n_f^3
\end{aligned} \tag{26}$$

One may notice that the ζ_4-terms coming separately from the 4-loop mass anomalous dimension and the 4-loop decay width cancel in the final expression for $R(a)$. This cancellation of ζ_4-terms provides an extra cross-check for both the calculations of γ_m and Γ_H. The remaining π^2-terms come from the imaginary parts of logarithms in the process of the analytical continuation from the Euclidean to the Minkowski region.

For the analysis of the infrared fixed point we will apply the approach of effective charges [19]. This approach to the scheme dependence of perturbative series is known, see e.g. Ref. [14], to give values for physical quantities close to another distinguished renormalization scheme – optimized perturbation theory [20]. Within the approach of effective charges one defines the whole calculated perturbative expansion as a new effective charge, i.e.

$$R(a) = a + r_1 a^2 + r_2 a^3 + r_3 a^4 \equiv a_{\rm eff} \tag{27}$$

The evolution equation for the effective charge (or equivalently for the physical quantity itself in the given order of perturbation theory) is governed by an effective β-function

$$\begin{aligned}
\frac{dR}{d\ln M_H^2} &= \beta_{\rm eff}(R) \\
&= -\beta_0 R^2 - \beta_1 R^3 - \beta_2^{\rm eff} R^4 - \beta_3^{\rm eff} R^5 + O(R^6)
\end{aligned} \tag{28}$$

Thus in the effective charge approach one deals with only one asymptotic perturbative series for $\beta_{\rm eff}(R)$ instead of two series for $R(a)$ and for $\beta(a)$ in the $\overline{\rm MS}$-scheme.

The third and the fourth coefficients of the effective β-function (which are scheme-invariant since they govern the evolution of the physical quantity $R(a)$) are expressed via the $\overline{\rm MS}$ coefficients as follows

$$\begin{aligned}\beta_2^{\rm eff} &= \beta_2 - \beta_1 r_1 + \beta_0(r_2 - r_1^2)\\ \beta_3^{\rm eff} &= \beta_3 - 2\beta_2 r_1 + \beta_1 r_1^2 + 2\beta_0(r_3 - 3r_1 r_2 + 2r_1^3)\end{aligned} \tag{29}$$

The numerical results are

$$\beta_2^{\rm eff} \approx -196.464 + 26.1453 n_f - 1.38317 n_f^2 + 0.0290035 n_f^3 \tag{30}$$

$$\beta_3^{\rm eff} \approx 3305.698 - 700.571 n_f + 65.1914 n_f^2 - 2.89465 n_f^3 + 0.0499219 n_f^4 \tag{31}$$

In the third order of QCD this $\beta_{\rm eff}$-function has a positive zero $R^{(0)} \approx 0.15$ [13] which practically does not depend on the number of quark flavors for $n_f = 3, 4, 5, 6$. In Ref. [13] it was considered to be a spurious fixed point, but it could have been interpreted as a real QCD infrared fixed point which would indicate the possibility of applying perturbation theory till zero energy. This is why it is important to check the stability of this zero $R^{(0)}$ under the inclusion of the fourth order of perturbative QCD. One can see from Eq. (31) that at the fourth order level the positive zero of the effective beta function disappears. This supports the idea that the infrared fixed point for the Higgs boson decay into hadrons at the third order of perturbative QCD is indeed a spurious fixed point.

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Just before completing this paper a similar paper appeared [21] with the mass anomalous dimension to four loops. It agrees completely with our formulae 15 and 20.

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