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AN EVALUATION OF THE ANALYTIC CONTINUATION BY DUALITY TECHNIQUE

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Abstract

In Ref. 1, Sundrum and Hsu estimated the value of the oblique correction parameter S for walking technicolor theories using a technique called *Analytic Continuation by Duality* (ACD). We apply the ACD technique to the perturbative vacuum polarization function and find that it fails to reproduce the well known result $S = 1/6\pi$. This brings into question the reliability of the ACD technique and the Sundrum and Hsu estimate of S.

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In Ref. 1, Sundrum and Hsu estimated the value of the oblique correction parameter S for walking technicolor theories using a technique called *Analytic Continuation by Duality* (ACD). We apply the ACD technique to the perturbative vacuum polarization function and find that it fails to reproduce the well known result $S = 1/6\pi$. This brings into question the reliability of the ACD technique and the Sundrum and Hsu estimate of S.

1 Introduction

The *analytic continuation by duality* (ACD) technique was proposed by Sundrum and Hsu in Ref. 1 as a reliable method to compute the oblique correction parameter S for technicolor theories. The advertised advantage of the ACD technique was that it could be applied to both QCD–like and walking technicolor [2] theories whereas the dispersion relation technique used by Peskin and one of us in Ref. 3 could only be applied to the former. Furthermore, the ACD estimate of S for walking technicolor implied that walking dynamics could render S negative, making it compatible with the current experimental limit. [4] This was in contrast to the result of Harada and Yoshida [5] who used the Bethe–Salpeter equation approach to conclude that S was positive even for walking theories.

In this talk, we investigate the reliability of the ACD technique to see whether the Sundrum–Hsu estimate should be taken seriously or not. In section 2, we review the definition of the S parameter and explain the ACD technique. In section 3, we apply the ACD technique to the perturbative spectral function to see if the famous result $1/6\pi$ could be reproduced. Discussions and conclusions are stated in Section 4.

2 The ACD technique

The S parameter, as defined in Ref. 3, is equal to a certain linear combination of electroweak vacuum polarization functions evaluated at zero momentum

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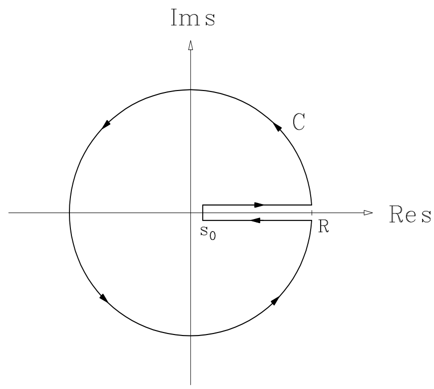

Figure 1: The contour C which avoids the branch cut along the real s–axis.

transfer. We represent this schematically as

$$S = \Pi(0).$$

(The precise definition of $\Pi(s)$ is irrelevant to our ensuing discussion.) The vacuum polarization function $\Pi(s)$ is analytic in the entire complex s plane except for a branch cut along the positive real s axis starting from the lowest particle threshold contributing to $\Pi(s)$. Applying Cauchy's theorem to the contour C shown in Fig. 1, we find

$$S = \frac{1}{\pi}\int_{s_0}^{R} ds \frac{\mathrm{Im}\Pi(s)}{s} + \frac{1}{2\pi i}\oint_{|s|=R} ds \frac{\Pi(s)}{s}. \qquad (1)$$

If the radius of the contour R is taken to infinity, the integral around the circle at $|s| = R$ can be shown to vanish and we obtain the dispersion relation

$$S = \frac{1}{\pi}\int_{s_0}^{R} ds \frac{\mathrm{Im}\Pi(s)}{s},$$

which was used in Ref. 3 to calculate S. However, the dispersion relation approach requires the knowledge of $\mathrm{Im}\Pi(s)$ along the real s axis which is only available for QCD–like technicolor theories.

The basic idea of the ACD technique, on the other hand, is to approximate the kernel $1/s$ by a polynomial

$$\frac{1}{s} \approx p_N(s) = \sum_{n=0}^{N} a_n(N) s^n, \qquad s \in [s_0, R]$$

and use it to make the integral along the real s axis vanish instead. Applying Cauchy's theorem to the product $p_N(s)\Pi(s)$ over the same contour C yields

$$0 = \frac{1}{\pi}\int_{s_0}^{R} ds\, p_N(s)\mathrm{Im}\Pi(s) + \frac{1}{2\pi i}\oint_{|s|=R} ds\, p_N(s)\Pi(s). \tag{2}$$

Subtracting Eq. (2) from Eq. (1), we obtain

$$S = S_N + \Delta_{\mathrm{fit}},$$

where

$$\begin{aligned} S_N &\equiv \frac{1}{2\pi i}\oint_{|s|=R} ds \left[\frac{1}{s} - p_N(s)\right]\Pi(s), \\ \Delta_{\mathrm{fit}} &\equiv \frac{1}{\pi}\int_{s_0}^{R} ds \left[\frac{1}{s} - p_N(s)\right]\mathrm{Im}\Pi(s). \end{aligned}$$

For sufficiently large N, we can expect Δ_{fit} to be negligibly small. In fact, it converges to zero in the limit $N \to \infty$ (though how quickly the convergence occurs depends on the interval $[s_0, R]$). We can therefore neglect it and approximate S with S_N which is an integral around the circle $|s| = R$ only. We call Δ_{fit} the *fit error*.

If the radius of the contour R is taken to be sufficiently large, the function $\Pi(s)$ can be approximated on $|s| = R$ by a large momentum expansion:

$$\Pi(s) \approx \sum_{m=1}^{M} \frac{b_m(s)}{s^m}. \tag{3}$$

This expression is obtained by analytically continuing the operator product expansion (OPE) of $\Pi(s)$ from the deep Euclidean region where it can be calculated for both QCD–like and walking technicolor theories. Therefore, we can write

$$S_N = S_{N,M} + \Delta_{\mathrm{tr}},$$

where

$$\begin{aligned} S_{N,M} &\equiv \frac{1}{2\pi i}\oint_{|s|=R} ds \left[\frac{1}{s} - p_N(s)\right]\sum_{m=1}^{M}\frac{b_m(s)}{s^m}, \\ \Delta_{\mathrm{tr}} &\equiv \frac{1}{2\pi i}\oint_{|s|=R} ds \left[\frac{1}{s} - p_N(s)\right]\left[\Pi(s) - \sum_{m=1}^{M}\frac{b_m(s)}{s^m}\right], \end{aligned}$$

and approximate S_N with $S_{N,M}$. The neglected term $\Delta_{\rm tr}$ is called the *truncation error.*

Sundrum and Hsu take their approximation one step further and neglect the s–dependence of expansion coefficients in Eq. (3), *i.e.*

$$b_m(s) \approx b_m(-R) \equiv \hat{b}_m.$$

This is obviously a dangerous approximation to make since the analytic structure of the integrand will be completely altered. Define

$$S_{N,M} = S_{\rm ACD} + \Delta_{\rm AC}$$

where

$$\begin{aligned} S_{\rm ACD} &\equiv \frac{1}{2\pi i}\oint_{|s|=R} ds \left[\frac{1}{s} - p_N(s)\right] \sum_{m=1}^{M} \frac{\hat{b}_m}{s^m}, \\ \Delta_{\rm AC} &\equiv \frac{1}{2\pi i}\oint_{|s|=R} ds \left[\frac{1}{s} - p_N(s)\right] \sum_{m=1}^{M} \frac{b_m(s) - \hat{b}_m}{s^M}. \end{aligned}$$

Sundrum and Hsu argue that $\Delta_{\rm AC}$ can be expected to be highly suppressed and thus negligible since the difference $1/s - p_N(s)$ is approximately zero in the vicinity of the positive real s axis where the difference $b_m(s) - \hat{b}_m$ can be expected to be most pronounced. Thus:

$$S \approx S_{\rm ACD}.$$

In this approximation, the integral for $S_{\rm ACD}$ will only pick up the residues of the single poles inside the integration contour and we find,

$$S_{\rm ACD} = -\sum_{n=0}^{\min\{N,M-1\}} a_n(N)\hat{b}_{n+1}.$$

We will call $\Delta_{\rm AC}$ the *analytical continuation error.*

To summarize, the ACD technique uses the relation

$$S = S_{\rm ACD} + \Delta_{\rm AC} + \Delta_{\rm tr} + \Delta_{\rm fit},$$

and assumes that all three types of error can be neglected and approximates S with $S_{\rm ACD}$.

Table 1: $S_{\rm ACD}$ and the fit, truncation, and analytical continuation errors for the perturbative vacuum polarization function. The cutoffs are $[s_0, R] = [4m^2, 25m^2]$, and the fit routine was the least square fit. The exact value of S is $1/6\pi = 0.0531$.

N	M	$S_{\rm ACD}$	$S_{N,M} = S_{\rm ACD} + \Delta_{\rm AC}$	$\Delta_{\rm fit}$	$\Delta_{\rm tr}$
3	2	0.2930	0.0580	−0.0002	−0.0048
	3	0.2883	0.0530		0.0002
	4	0.2884	0.0532		−0.0000
4	2	0.4330	0.0632	−0.0001	−0.0101
	3	0.4203	0.0521		−0.0010
	4	0.4211	0.0532		−0.0000
	5	0.4211	0.0531		0.0000
5	3	0.5731	0.0506	−0.0000	0.0025
	4	0.5759	0.0533		−0.0002
	5	0.5757	0.0531		0.0000
	6	0.5757	0.0531		−0.0000

3 The Perturbative Spectral Function

To check validity of the approximation $S \approx S_{\rm ACD}$, we calculate $S_{\rm ACD}$ for the one–loop contribution of a massive fermion doublet to S. The vacuum polarization function $\Pi(s)$ in this case is given by:

$$\Pi_{\rm pert}(s) = -\frac{1}{\pi}\frac{m^2}{s}\int_0^1 dx \log\left[1 - x(1-x)\frac{s}{m^2}\right]. \tag{4}$$

Evaluating this expression at $s = 0$, we find the well known result $S = 1/6\pi$.

The function $\Pi_{\rm pert}(s)$ is analytic in the entire complex s plane except for a branch cut along the positive real s axis starting from $s = 4m^2$. The imaginary part of this function along the cut is given by

$${\rm Im}\Pi_{\rm pert}(s) = \frac{m^2}{s}\beta\theta(s - 4m^2), \qquad \beta = \sqrt{1 - \frac{4m^2}{s}}. \tag{5}$$

The first few terms of the large s-expansion of $\Pi_{\rm pert}(s)$ are given by

$$\begin{aligned}&\pi\Pi_{\rm pert}(s)\\ &\quad = \quad x\left\{-\ln\left(-\frac{1}{x}\right) + 2\right\} + x^2\left\{2\ln\left(-\frac{1}{x}\right) + 2\right\} + x^3\left\{2\ln\left(-\frac{1}{x}\right) - 1\right\} + \ldots,\end{aligned}$$

where $x \equiv 4m^2/s$. Using these expressions, we calculated $S_{\rm ACD}$, $\Delta_{\rm AC}$, $\Delta_{\rm tr}$, and $\Delta_{\rm fit}$. The results of our calculations for several values of N and M are shown in Table 3. The fit interval was $[s_0, R] = [4m^2, 25m^2]$, and the fit routine was the least square fit.

As is evident from Table. 3, the fit and truncation errors are under excellent control and $S_{N,M}$ reproduces the exact value of S accurately already at $N = M = 3$. However, the analytic continuation error is not. For the $N = 5$ case, for instance, S_{ACD} is larger than the exact value by more than an order of magnitude. In fact, we find that S_{ACD} and Δ_{AC} diverge as $N \to \infty$.

We conclude that neglecting the s–dependence of the $b_m(s)$'s fails miserably as an approximation. The reason for this can be traced to the fact that even though the difference $1/s - p_N(s)$ converges to zero within its radius of convergence, outside it diverges. Therefore, the handwaving argument of Sundrum and Hsu was wrong: the error induced by the neglect of the s–dependence of the $b_m(s)$'s may be suppressed near the real s axis, but it is actually *enhanced* away from it.

4 Discussion and Conclusions

The application of the ACD technique to the perturbative vacuum polarization function has shown that the analytic continuation error Δ_{AC} is not under control and that the approximation $S \approx S_{\text{ACD}}$ cannot be trusted. This brings into doubt the reliability of the ACD estimate of S obtained by Sundrum and Hsu in Ref. 1.

A natural question to ask next is whether the ACD technique can be improved by including the s–dependence of the large momentum expansion coefficients $b_m(s)$ and using $S_{N,M}$ as the estimate of S instead of S_{ACD}. In the perturbative case, we have seen that this is an excellent approximation. However, whether $S_{N,M}$ will reproduce the correct value of S for all cases is far from clear. If the large momentum expansion is an asymptotic series, the truncation error Δ_{tr} may not converge to zero in the limit $M \to \infty$. Even if it is a convergent series, the convergence may be too slow for the method to be practical. These questions, and all related problems will be addressed in subsequent papers. [6]

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