String–Unification, Universal One–Loop Corrections and Strongly Coupled Heterotic String Theory*

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Abstract

We derive the universal threshold corrections in heterotic string theory including a continuous Wilson line. Unification of gauge and gravitational couplings is shown to be possible even within perturbative string theory. The relative importance of gauge group dependent and independent thresholds on unification is clarified. Equipped with these results we can then attempt an extrapolation to the strongly coupled heterotic string — M–theory. We argue that such an extrapolation might be meaningful because of the holomorphic structure of the gauge coupling function and the close connection of the threshold corrections to the anomaly cancelation mechanism.

1 Introduction

The framework of string theory might ultimately give an explanation of the unification of all fundamental coupling constants: gauge and gravitational. At a very naive level such an unification is obtained, although one could still be dissatisfied with the numerical precision of this statement, in the heterotic theory the unification scale seems to be a factor 20 smaller than the string scale. Related to this question is the fact that we do not really understand the reason why the string coupling is so small to allow perturbative unification. Some aspects of unification might even look more "natural" when viewed from a strong coupling point of view [1], although our ability to do explicit calculations is very limited (if not nonexistent) in this region.

In that sense there is at the moment no alternative to threshold calculations in *perturbative* string theory. One might hope that such results could be extended to the region of stronger coupling. A study of these questions will be presented in this paper. By generalizing previous work, we shall first give the results of a full gauge coupling threshold calculation in the presence of a continous Wilson line background. Within this framework we show that unification can be achieved in the perturbative theory with rather natural values of the moduli. This can be seen as a consequence of the so–called gauge group dependent threshold corrections [2]. The new results of this paper concern also the gauge group independent thresholds which are found less relevant for the mechanism of unification. They are, however, very important for the understanding of the structure of the holomorphic gauge kinetic function $f(\phi)$ and its extrapolation to the region of stronger coupling.

There are reasons to believe that these threshold functions computed explicitly in perturbative heterotic string theory contain crucial information about the full underlying string or M–theory. The first reason is the intimate connection to the mechanism of anomaly cancelation in heterotic string theory [3]. The existence of the thresholds can be deduced from the presence of the Green–Schwarz terms required by the anomalies [4, 5]. The second reason resides in the holomorphic structure of the f–function that, due to nonrenormalization theorems [6, 7], is easily controllable in perturbation theory. This could then imply that the results obtained here might be of more general validity beyond the weak coupling regime.

In this paper we shall present the threshold calculation in the presence of a continuous Wilson line in detail, discuss its relevance for the question of unification and argue that some aspects of it will carry over to the M–theory [8, 9] domain. The paper will be organized as follows. In section 2 we give for completeness a discussion of the definition of the string coupling constant g_{string} at the 1-loop level while in section 3 we summarize the constraints for models that are consistent with the requirements of gauge coupling unification. Section 4 contains the new results of the threshold calculation with some technical details relegated to the appendix. We comment especially on the universal one–loop corrections and its implication for unification. In section 5 we consider the possible relevance of this calculation in the framework of M–theory. We shall clarify the connection to the cancelation of anomalies and the possible uses of the holomorphicity of the gauge

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kinetic function. We also study the range of validity of the linearized approximation concerning the T, U as well as the C (Wilson line) moduli. Various limits of strong coupling – large radii are examined in detail. Section 6, finally summarizes our results and gives an outlook. The appendix contains some useful technical details.

2 The relation between M_{Planck} and M_{string} at one–loop

As a starting point we write the N=1 effective action for the heterotic string through a superconformal Lagrangian as D–density with a gauge invariant linear multiplet[1] $\hat{L}$ (without superpotential) [10, 11, 12]:

$$\mathcal{L}_{linear} = \left\{ S_0 \overline{S}_0 \Phi \left(\frac{\hat{L}}{S_0 \overline{S}_0}, \Sigma, \overline{\Sigma} \right) \right\}_D \tag{2.1}$$

and

$$\Phi \left(\frac{\hat{L}}{S_0 \overline{S}_0}, \Sigma, \overline{\Sigma} \right) = -\frac{1}{\sqrt{2}} \left(\frac{\hat{L}}{S_0 \overline{S}_0} \right)^{-\frac{1}{2}} e^{-G^{(0)}/2} - \frac{1}{2} \frac{\hat{L}}{S_0 \overline{S}_0} \frac{G^{(1)}}{16\pi^2} \ , \tag{2.2}$$

where the chiral superfield S_0 denotes the compensator and the chiral superfields $\Sigma = (z, \psi_z, f_z)$ refer to the moduli fields $z = T, U, C, \ldots$ describing the vacuum of the underlying string model. The moduli dependent functions $G^{(0)}$ and $G^{(1)}$ will be specified later. Note, that all non–holomorphic gauge dependence is encoded in the D–density [11]. When fixing dilatation symmetry, we may arrive at different actions depending on how we choose the vev of the scalar component z_0 of the compensator. For the Einstein frame we take [13]

$$(z_0 \overline{z}_0)^{3/2} = \sqrt{2} \kappa^{-2} c^{\frac{1}{2}} e^{G^{(0)}/2} \tag{2.3}$$

and the resulting bosonic Lagrangian at one–loop becomes:

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{Einstein}} &= -\frac{1}{2\kappa^2} R - \frac{1}{4\kappa^2} \frac{1}{c^2} \partial_\mu c \partial^\mu c + \frac{1}{4\kappa^2} \frac{1}{c^2} v_\mu v^\mu \\ &\quad - \kappa^{-2} \left[G_{z\overline{z}}^{(0)} - \frac{\kappa^2 c}{16\pi^2} G_{z\overline{z}}^{(1)} \right] \partial_\mu z \partial^\mu \overline{z} - \frac{1}{4} \left[\frac{1}{\kappa^2 c} - \frac{G^{(1)}}{16\pi^2} \right] F_{\mu\nu}^a F^{a\mu\nu} \ . \end{aligned} \tag{2.4}$$

Here c is the lowest component of the linear multiplet. The vector field v_μ obeys the constraint $\partial_\mu v^\mu = 0$. There are no one–loop corrections to the Einstein term [14, 15], to the kinetic terms for field c and the antisymmetric tensor $b_{\mu\nu}$ which is contained in v_μ via $v_\mu \sim \epsilon_{\mu\nu\rho\sigma} \partial^\nu b^{\rho\sigma}$ [16]. Notice that $\kappa^2 c$ is the string–loop counting parameter, $G^{(0)}$ is the tree–level part of the Kähler potential for the fields z and $G^{(1)}$ denotes the one–loop corrections to it [11, 12, 17, 18, 19].

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[1]For more details see [13]. This reference contains a complete list of references.

Before we extract from this Lagrangian model independent relations, let us perform a duality transformation on (2.1) to eliminate the linear multiplet $\hat{L}$ by introducing an additional chiral multiplet S. Then (2.1) involves only chiral fields:

$$\mathcal{L}_{chiral} = -\frac{3}{2} \left\{ S_0 \overline{S}_0 \left[e^{-G^{(0)}/3} \left(\frac{-iS + i\overline{S}}{4} + \frac{1}{2} \frac{G^{(1)}}{16\pi^2} \right)^{1/3} \right] \right\}_D + \frac{1}{4} \{ \frac{-iS}{2} W^a W^a \}_F . \quad (2.5)$$

The scalar kinetic terms of the action (2.5) with the fixing

$$z_0 \overline{z}_0 = \kappa^{-2} \left(\frac{-iS + i\overline{S}}{4} + \frac{1}{2} \frac{G^{(1)}}{16\pi^2} \right)^{-1/3} e^{G^{(0)}/3} , \quad (2.6)$$

corresponding to (2.3) follow then from the Kähler potential:

$$K = -\kappa^{-2} \ln \left[-iS + i\overline{S} + \frac{1}{8\pi^2} G^{(1)}(z, \overline{z}) \right] - \kappa^{-2} G^{(0)}(z, \overline{z}) . \quad (2.7)$$

The bosonic terms are

$$
\begin{aligned}
e^{-1} \mathcal{L}_{\text{Einstein}} &= -\frac{1}{2\kappa^2} R + \frac{\kappa^{-2}}{(-iS + i\overline{S} + \frac{1}{8\pi^2} G^{(1)})^2} \partial_\mu S \partial^\mu \overline{S} - \frac{1}{4} \left[\frac{-iS + i\overline{S}}{2} \right] F^a_{\mu\nu} F^{a\mu\nu} \\
&\quad - \kappa^{-2} \left[G^{(0)}_{z\overline{z}} + \frac{G^{(1)}_{z\overline{z}}/8\pi^2}{-iS + i\overline{S} + \frac{1}{8\pi^2} G^{(1)}} - \frac{G^{(1)}_z G^{(1)}_{\overline{z}}/(8\pi^2)^2}{(-iS + i\overline{S} + \frac{1}{8\pi^2} G^{(1)})^2} \right] \partial_\mu z \partial^\mu \overline{z} , \\
&\quad - \kappa^{-2} \left[\frac{iG^{(1)}_{\overline{z}}/8\pi^2}{(-iS + i\overline{S} + \frac{1}{8\pi^2} G^{(1)})^2} \partial_\mu S \partial^\mu \overline{z} - \frac{iG^{(1)}_z/8\pi^2}{(-iS + i\overline{S} + \frac{1}{8\pi^2} G^{(1)})^2} \partial_\mu z \partial^\mu \overline{S} \right] .
\end{aligned}
\quad (2.8)
$$

As we see from (2.7) the function $G^{(1)}$ amounts to an –in general– non–holomorphic shift of the dilaton field S. We shall return to this shift (and possible further holomorphic shifts originating from the F–density) in section 3.1. Note that the linear formalism (2.4) is the most natural one to discuss non–harmonic gauge couplings given by $G^{(1)}$ [11].

Now we can compare eq. (2.4) with (2.8). Let us first extract the tree–level relation

$$g^{-2}_{string} \Big|_{bare} = \frac{1}{\kappa^2 c} \overset{duality}{=} \frac{-iS + i\overline{S}}{2} , \quad (2.9)$$

by looking at the kinetic term for c or S and setting $G^{(1)} = 0$. Inspection of the gauge terms in (2.8) leads to the convention:

$$S = \frac{\theta_a}{8\pi^2} + i \frac{1}{g_a^2} . \quad (2.10)$$

With the identification $M^2_{string} = \langle c \rangle$ and $\kappa^{-1} = M_{\text{Planck}}$ we may cast (2.9) into

$$M^2_{\text{Planck}} = \text{Im}(S) \, M^2_{string} , \quad (2.11)$$

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or:

$$M^2_{\text{Planck}} = g^{-2}_{\text{string}}\Big|_{bare}\ M^2_{\text{string}}\ . \tag{2.12}$$

This is the well–known relation for the heterotic string [20] at tree–level. At one–loop one obtains the following string–coupling:

$$g^{-2}_{\text{string}}\Big|_{one-loop} = \frac{1}{\kappa^2 c} \overset{duality}{=} \frac{-iS + i\overline{S}}{2} + \frac{1}{16\pi^2}G^{(1)}\ . \tag{2.13}$$

Again, the last equality follows from the duality transformation. Here the field S appears as a Lagrange multiplier and has to be defined order by order in perturbation theory. The linear multiplet has a fixed relation to string vertices and therefore the left hand side of (2.13) stays invariant under all perturbative symmetries.

We want to elaborate whether the relation (2.12) is stable against one–loop corrections. As a consequence of eq. (2.4) the form[2] of the coupling of the graviton to the dilaton energy momentum tensor (with $\kappa^2 c = e^{2D}$)

$$\kappa h_{\mu\nu}(\partial^\mu D \partial^\nu D - \eta^{\mu\nu}\partial_\alpha D\partial^\alpha D) \tag{2.14}$$

remains unchanged at the one–loop level, since the kinetic terms of the field c and the graviton do not change. To clarify whether (2.12) receives one–loop corrections we have to extract from the string one–loop amplitude[3] $\mathcal{A}[D(k_1), G(h_{\mu\nu}), D(k_3)]$ a possible contribution to the term

$$g_{\text{string}}\big|_{bare}\ \epsilon^2_{\mu\nu}(k_1^\mu k_3^\nu + k_1^\nu k_3^\mu)\ . \tag{2.15}$$

Here $\epsilon^2_{\mu\nu}$ is the polarization tensor of the graviton. The coupling $g_{\text{string}}\big|_{bare}$ appears as normalization of the graviton vertex operator, which is given by the non–linear sigma–model action. Therefore it should be taken at tree–level. Since it is the linear multiplet, which has a fixed relation to the dilaton vertex operator, we are really determing the one–loop correction to (2.14) rather than to $\kappa h_{\mu\nu}(\partial^\mu S\partial^\nu S - \eta^{\mu\nu}\partial_\alpha S\partial^\alpha S)$. This term can be worked out easily by using [21]. There it was shown that the $O(k^2)$ part of a three graviton amplitude vanishes. We have only to replace two graviton polarization tensors with those of two dilatons and work out the contribution corresponding to (2.15) to see that it vanishes for the same reasons. However, this shows that the relation (2.12) is unchanged at one–loop.

To summarize, from (2.4) one deduces that the form for the coupling of the graviton to the dilaton energy momentum tensor does not change at one–loop and by doing an explicit string one–loop calculation one verifies that (2.12) is *not* changed[4] at one–loop. Nevertheless, the meaning of S does change at one–loop when going from (2.9) to (2.13) and the right hand side of (2.11) refers to quantities defined at tree–level. Therefore, at

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[2]We rescale fields so that we arrive at canonical kinetic energy terms, e.g.: $g_{\mu\nu} = 2\kappa h_{\mu\nu}$.

[3]In [20], a comparison of (2.14) and the tree–level string amplitude (2.15) lead to the conclusion (2.12).

[4]Related discussions may be found in [22].

one–loop, the right hand side of eqs. (2.11), (2.12) should better be written entirely in terms of one–loop quantities:

$$M_{\text{Planck}}^2 = \left[\text{Im}(S) + \frac{1}{16\pi^2}G^{(1)}\right] M_{\text{string}}^2 . \tag{2.16}$$

3 String–unification and low–energy predictions

3.1 String threshold corrections

At string tree level the gauge couplings, encoded in (2.10), are related to the string coupling (2.9) via [23]

$$g_a^{-2} = k_a \left. g_{\text{string}}^{-2}\right|_{bare} = k_a\text{Im}(S) . \tag{3.1}$$

Here k_a is the Kac–Moody level of the group factor labeled by a. At one–loop, the gauge coupling in eq. (2.4), which follows from the D–density (2.1) receives additional harmonic contributions $\triangle_{\text{harmonic}}$ from integrating out heavy string states. These corrections are usually summarized in the F–density. Besides non–harmonic pieces $\triangle_{\text{triangle}}$ arising from triangle graphs involving massless fields and some constants c_a. The effective gauge coupling at the scale $\mu = M_{\text{string}}$ then reads:

$$
\begin{aligned}
g_{a,eff.}^{-2} &= \frac{1}{\kappa^2 c} - \frac{k_a}{16\pi^2}G^{(1)} + \frac{1}{16\pi^2}(\triangle_{\text{triangle}} + \triangle_{\text{harmonic}}) + c_a \\
&= \left. g_{\text{string}}^{-2}\right|_{one-loop} + \frac{1}{16\pi^2}\triangle_a + c_a .
\end{aligned}
\tag{3.2}
$$

In (2.4) $g_{a,eff.}^{-2}$ then appears as the bare coupling in front of F^2. The one–loop corrections to the physical gauge coupling are denoted by $\triangle_a$.

In the following we want to discuss toroidal orbifold models with $d = 4$, N=1 space–time supersymmetry [24]. They provide an interesting class of realistic string models and their moduli–space is rich enough to discuss various aspects of string theory. Their spectrum may contain a subsector which can be arranged in N=2 multiplets. For this subsector all perturbative corrections like $\triangle_a, G^{(1)}$ can be calculated. In these models $\triangle_a$ can be split into three pieces:

$$\triangle_a = \alpha_a\triangle - k_aG^{(1)} + k_a\sigma . \tag{3.3}$$

This splitting makes sense, since we know from (2.4), that the one–loop correction $\triangle_a$ contains a non–holomorphic piece $G^{(1)}$ which has to cancel the mixing between the dilaton and the moduli fields given in (2.13). The latter contains so–called GS–corrections[5] $G_{N=1}^{(1)}$ referring to twisted planes and therefore involving only moduli of these planes [11].

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[5]These terms cancel target–space anomalies arising from triangle graphs with a coupling to the Kähler– and sigma–model connections in four dimensions. These corrections should not be confused with the anomaly cancelation terms in ten dimensions, which we will discuss in sect. 5. Their job is to cancel gauge and gravitational anomalies.

Furthermore, one has GS–corrections $G_{N=2}^{(1)}$ coming from the untwisted planes. These corrections give rise to IR–divergent wave function renormalizations due to singularities associated with additional massless particles at subvarieties of the moduli space. The generic gauge–group dependent moduli–dependent threshold–corrections are contained in $\triangle$. They have been first calculated in [25] for orbifold models without Wilson lines and in [26, 27] for orbifold models with one Wilson line. Generalization of the latter to more Wilson lines have been performed in [28, 29]. The coefficient α_a is the total anomaly coefficient for the sigma–model and Kähler connection anomaly. Only matter which fits in N=2 multiplets contributes to $\triangle$. The following splitting seems to be convenient [30]

$$\frac{b_a^{j,N=2}}{|D|/|D_j|} = \alpha_a^j - k_a \delta_{GS}^j \ , \tag{3.4}$$

(with $b_a^{j,N=2}$ being the N=2 β–function coefficient), since the anomaly α_a^j with respect to the orbifold plane j is partially cancelled by the string thresholds $b_a^{j,N=2}\triangle$ and a GS–term $k_a \delta_{GS}^j \triangle$, respectively [11]. Here D, D_j refer to the orbifold group and to a subgroup generating the N=2 sector, respectively.

To illustrate this mechanism, let us give an example. For orbifolds with untwisted planes j the equation (3.4) turns out to be fulfilled in such a way that $\delta_{GS}^j = 0$ [30]. Therefore, we choose the standard Z_3–orbifold with gauge group $SU(3) \times E_6 \times E_8'$. In this case one has no fixed orbifold plane. Thus, we have $b_a^{j,N=2} = 0$ and we need a GS–term

$$G_{N=1}^{(1)} = \sum_{j=1,2,3} \delta_{GS}^j \ln(-iT^j + i\overline{T}^j) \ , \tag{3.5}$$

to cancel the total anomaly, given by $\alpha_a^j \ln(-iT^j + i\overline{T}^j)$ with $\alpha_a^j = -30$ for the plane j and $\delta_{GS}^j = -30$. Although in (3.2) the whole moduli–dependence disappears for these sectors, i.e. $\triangle_a = 0$, in the combination of (3.3) the correction

$$\sigma_{N=1}(T^k, U^k) = - \sum_{j=1,2,3} \delta_{GS}^j \ln |\eta(T^j)|^4 |\eta(U^j)|^4 \ . \tag{3.6}$$

may be thought as an universal correction. To keep the string–coupling (2.13) invariant, the dilaton field S has to transform properly. We may use the universal correction σ to perform a holomorphic field redefinition of the dilaton field S

$$-iS_{\text{inv.}} := -iS - \frac{1}{8\pi^2} \sum_{j=1,2,3} \delta_{GS}^j \ln \eta^2(T)\eta^2(U) \tag{3.7}$$

to obtain an invariant field $S_{\text{inv.}}$ [11]. This is the only possible source for moduli dependent universal corrections $\sigma_{N=1}$ appearing from twisted planes.

Let us now turn to the GS–type corrections $G_{N=2}^{(1)}$ arising from an untwisted plane T^2 with moduli T, U and possible Wilson line modulus C. These corrections can be expressed by the N=2 prepotential: thanks to special geometry it can be entirely written as derivatives of the prepotential [31, 18, 28]

$$G_{N=2}^{(1)} = -\frac{32\pi^2}{Y}\text{Re}\left[h^{(1)} - \frac{1}{2}(T-\bar{T})\partial_T h^{(1)} - \frac{1}{2}(U-\bar{U})\partial_U h^{(1)} - \frac{1}{2}(C-\bar{C})\partial_C h^{(1)}\right] , \tag{3.8}$$

with [cf. the Kähler potential in (2.7)]:

$$Y = e^{G^{(0)}} = -(T-\overline{T})(U-\overline{U}) + \frac{1}{4}(C-\overline{C})^2 . \tag{3.9}$$

The group–independent corrections $\sigma_{N=2}$, which do not contain the one–loop Kähler correction $G^{(1)}$, are related to the charge insertion which appears in the threshold calculation and the gravitational back reaction to the background gauge fields in [32]. They have been first derived in [15], further developed in [33, 22] and will be generalized to more moduli in section 4. In general, there are also moduli–independent corrections c_a from N=1 sectors and from the different IR–regularization of field– and string–theory. But these are small and will be neglected in the following. They are discussed in [32, 34].

3.2 Perturbative string–unification

We go to a string model with N=1, $d = 4$ space–time supersymmetry and the gauge group of the Standard Model $SU(3) \times SU(2) \times U(1)$. Such models can be constructed as toroidal orbifolds with Wilson lines. The Wilson lines break the gauge groups and may also reduce their rank. For more details see refs. [35, 36]. The corrections in (3.2) spoil the string tree–level result (3.1) and split the one–loop gauge couplings at M_{string}. This splitting could allow for an effective unification at a scale $M_{\text{GUT}} < M_{\text{string}}$ or destroy the unification. The identities (3.2) serve as boundary conditions for the running field–theoretical couplings valid below M_{string}. The evolution equations below M_{string}

$$\frac{1}{g_a^2(\mu)} = k_a\text{Im}(S) + \frac{b_a}{16\pi^2}\ln\frac{M_{\text{string}}^2}{\mu^2} + \frac{1}{16\pi^2}\alpha_a\triangle + \frac{k_a}{16\pi^2}\sigma , \tag{3.10}$$

allow us to determine $\sin^2\theta_W$ and α_S at M_Z. After eliminating $\text{Im}(S)$ in the second and third equation one obtains

$$\begin{aligned}
\sin^2\theta_W(M_Z) &= \frac{k_2}{k_1+k_2} - \frac{k_1}{k_1+k_2}\frac{\alpha_{em}(M_Z)}{4\pi}\left[\mathcal{A}\ln\left(\frac{M_{\text{string}}^2}{M_Z^2}\right) + \mathcal{A}'\triangle\right] , \\
\alpha_S^{-1}(M_Z) &= \frac{k_3}{k_1+k_2}\left[\alpha_{em}^{-1}(M_Z) - \frac{1}{4\pi}\mathcal{B}\ln\left(\frac{M_{\text{string}}^2}{M_Z^2}\right) - \frac{1}{4\pi}\mathcal{B}'\triangle\right] ,
\end{aligned} \tag{3.11}$$

with $\mathcal{A} = \frac{k_2}{k_1}b_1 - b_2, \mathcal{B} = b_1 + b_2 - \frac{k_1+k_2}{k_3}b_3$ and $\mathcal{A}' = \frac{k_2}{k_1}\alpha_1 - \alpha_2$ and $\mathcal{B}' = \alpha_1 + \alpha_2 - \frac{k_1+k_2}{k_3}\alpha_3$. For the MSSM one has $\mathcal{A} = \frac{28}{5}, \mathcal{B} = 20$. The coefficients $\mathcal{A}', \mathcal{B}'$ depend on the string model under consideration, i.e. how its relevant particle content enters the anomaly coefficients in (3.4) [37]. These two equations determine the gauge group dependent

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threshold corrections $\triangle$ and M_{string} that are necessary to obtain the correct experimental low–energy data $\sin^2_W(M_Z)$ and $\alpha_S(M_Z)$. The three couplings meet at the single point

$$M_{\text{GUT}} = M_{\text{string}}\, e^{\frac{\mathcal{A}'}{2\mathcal{A}}\triangle} \sim 2 \cdot 10^{16}\text{GeV} \ , \tag{3.12}$$

if the following relation holds:

$$\mathcal{A}'\mathcal{B} = \mathcal{A}\mathcal{B}' \ . \tag{3.13}$$

Note, that the universal correction σ does not play any rôle when considering the low–energy implications (3.11) and (3.12). However it does influence the unification coupling constant e.g.: $k_2^{-1}g_2^{-2}(M_{\text{GUT}})$:

$$k_2^{-1}g_2^{-2}(M_{\text{GUT}}) = \text{Im}(S) + \frac{1}{16\pi^2}\sigma + \frac{1}{16\pi^2 k_2}\left(\alpha_2 - b_2\frac{\mathcal{A}'}{\mathcal{A}}\right)\triangle \ . \tag{3.14}$$

The last term vanishes[6] for $\alpha_a = b_a$. Therefore we define:

$$g_{\text{GUT}}^{-2} := \text{Im}(S) + \frac{1}{16\pi^2}\sigma \ . \tag{3.15}$$

Thus, we expect $\text{Im}(S) + \frac{1}{16\pi^2}\sigma \sim 2.0$. There is yet another reason to see this: Combine the first and second eqs. of (3.10) to:

$$g_{\text{GUT}}^{-2} \equiv \text{Im}(S) + \frac{1}{16\pi^2}\sigma = \frac{1}{k_1 + k_2}\left[\frac{1}{4\pi}\alpha_{em}^{-1}(M_Z) - \frac{1}{16\pi^2}(b_1 + b_2)\ln\frac{M_{\text{string}}^2}{M_Z^2} - \frac{\alpha_1 + \alpha_2}{16\pi^2}\triangle\right] \ . \tag{3.16}$$

For the model $(\alpha_1 + \alpha_2 = 10)$ considered in [2] one immediately obtains for the right hand side the value 2.1. On the other hand, looking at the solutions of (3.16) one realizes:

$$1.9 \leq \text{Im}(S) + \frac{1}{16\pi^2}\sigma \leq 2.1 \ . \tag{3.17}$$

arising from the two cases $\triangle = 0$; $M_{\text{string}} = 2 \cdot 10^{16}\text{GeV}$ and $\triangle = -16$; $M_{\text{string}} = 4 \cdot 10^{17}\text{GeV}$ which both solve eqs. (3.11). Therefore we conclude:

$$g_{\text{GUT}}^{-2} \sim \text{Im}(S) + \frac{1}{16\pi^2}\sigma \sim 2 \ . \tag{3.18}$$

After a rearrangement of some scheme dependent parts in (3.10) and one obtains[7] from (2.16)

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[6]In the model $(\alpha_2 = 5/2)$ considered in [2]: $\mathcal{A}'\triangle \sim 2(-16)$, i.e. $-\frac{1}{16\pi^2}\frac{b_2}{k_2}\frac{\mathcal{A}'}{\mathcal{A}}\triangle + \frac{1}{16\pi^2}\alpha_2\triangle \sim -0.2$ leading to $g_2^{-2}(M_{\text{GUT}}) = 1.9$.

[7]The relation between the quantities $\sigma, G^{(1)}$ and Y defined in [15, 33, 22] is: $-Y = -G^{(1)}_{N=2} + \sigma$. In the next sect. we will see that the correction $G^{(1)}$ is rather small compared to σ. Actually, $G^{(1)}_{N=2} \sim 1/R^2$ with R being the compactification radius. Therefore in the following we will concentrate our discussion on the correction σ.

$$M_{\text{string}} = 0.527 \cdot g_{\text{GUT}} \cdot 10^{18} \text{GeV} \frac{1}{\sqrt{1 - \frac{g_{\text{GUT}}^2}{16\pi^2}[\sigma - G^{(1)}]}} \ . \tag{3.19}$$

Therefore the determination of the correct solution $(\Delta, M_{\text{string}})$ of (3.11) requires the knowledge of either σ or the vev of the dilaton S. Let us present some solutions to (3.11) for the model discussed in [2].

$\triangle$	0	-5	-10	-16.75	-20	-30
$M_{\text{string}}[10^{17}Gev]$	0.2	0.44	1.1	3.6	6.4	38.5
σ	-133121	-22108	-3474	0	237	347

Tab.1 – Solutions $(\triangle, M_{\text{string}})$ of eq. (3.11) which require after (3.19) a specific σ.

From eq. (3.19) one learns that only huge values $\sigma \sim O(-10^5)$ would influence this equation, i.e. may considerably lower the string–scale down to the GUT–scale (3.12). It is the smallness of $g_{\text{GUT}}^2/16\pi^2 \sim 10^{-3}$ entering the formula (3.19) that is responsible for this fact. Since both σ and Δ are moduli–dependent functions it is the vev of the moduli fields, which finally selects one of the above solutions. In [2] we investigated the moduli dependences of $\triangle$ and its influence to string unification. We assumed the term $\sigma/16\pi^2$ to be small and found that the solution $(\triangle = -16.75, \sigma = 0)$ can be achieved with rather small vevs of the moduli fields: $T/2\alpha' = 4i = U$ and $C = \sqrt{2}/4$. In the next section we will elaborate the full moduli dependence of $\sigma(T, U, C)$. We shall see that in the perturbative regime the correction σ is always negative, i.e. the effect of σ results in a lowering of the string–scale. As a consequence of (3.18) this drives the string coupling to smaller values. A huge negative σ requires a huge vev of T which on the other hand pushes $\triangle$ into the opposite direction. In the next section, we will see, that with $\sigma(T, U, C)$ we cannot reach such large values as required from Tab.1. At $T = 4i = U$, $C = \sqrt{2}/4$, for example, we have $\Delta \sim -17$ and $\sigma \sim -50$. Although both are of comparable size[8], only Δ is relevant for unification. In general, the preferred solution to (3.11) is $\triangle = -16.75$, achieved with $T/2\alpha' = 4i = U$ and $C = \sqrt{2}/4$. The universal corrections σ do not play an important rôle in string unification. Nevertheless, in section 5 we will see that they become important when moving to stronger coupling.

4   Universal one–loop corrections

In this section we want to derive the Wilson line dependence of the universal one–loop correction σ. Its moduli dependence is completely given by the N=2 subsectors of the full N=1 $d = 4$ heterotic string theories we have discussed in the previous section. For concreteness we will do this calculation for an $N = 1$ $d = 4$ toroidal orbifold which has an

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[8]With $\mathcal{A}' \sim 2$, i.e. $\mathcal{A}'\Delta \sim -36$ these two values enter (3.10) as corrections of the same order.

N=2 subsector, generated by a Z_2 orbifold twist which leaves one torus T^2 fixed and leads to the N=2 gauge group $SU(2) \times E_7 \times E_8'$. In addition we introduce a Wilson line (C) with respect to to the torus T^2, which may break E_7 down to $SO(12)$ for nontrivial vevs of C. The torus is described by the two moduli T and U. This example should then allow us to study all the relevant properties of the N=1 models introduced above, including the Higgs mechanism for the N=1 gauge group. For more details see [36].

4.1 Without Wilson line

To warm up let us first consider the case without a Wilson line. This case has already been discussed in quite detail in [15, 33, 22]. Nonetheless we would like to repeat the calculation here because of the following two reasons. By using [28] we find an alternative way to obtain these results by means of a differential equation for the prepotential. Secondly, this method seems to be more convenient for cases involving an arbitrary number of Wilson lines of which we will make use it in section 4.2. In this model we may consider the two physical gauge groups $G_a = E_7$, E_8'. For them we do not expect any singularities in the (T, U)–moduli space and the gauge group dependent part $\triangle$ of their threshold corrections $\triangle_a$ (3.3) are expressed by the well–known formula [25]:

$$\triangle = -\ln\left[(-iT + i\overline{T})(-iU + i\overline{U})|\eta(T)|^4|\eta(U)|^4\right] . \tag{4.1}$$

The full correction $\triangle_a$ enters in a second order differential equation for the one–loop correction $h^{(1)}$ to the prepotential of the underlying N=2 theory [28]

$$\mathrm{Re}\left[8\pi^2\partial_T\partial_U h^{(1)}(T, U)\right] - G^{(1)}_{N=2} = \triangle_a(T, U) + 4\mathrm{Re}\left[\ln\Psi_a(T, U)\right] + b_a^{N=2}K_0 , \tag{4.2}$$

with:

$$\Psi_a(T, U) = [\eta(T)\eta(U)]^{b_a^{N=2}}[j(T) - j(U)]^{1/2} , \quad b_{E_7}^{N=2} = 84, \ b_{E_8'}^{N=2} = -60 . \tag{4.3}$$

Using (4.2) and (3.3) we are able to extract $\sigma(T, U)$ [31]

$$\sigma(T, U) = \mathrm{Re}\left\{8\pi^2\partial_T\partial_U h^{(1)}(T, U) - 2\ln[j(T) - j(U)]\right\} , \tag{4.4}$$

which gives for the prepotential in the chamber $T_2 > U_2$ [28]

$$h^{(1)}(T, U) = -\frac{i}{12\pi}U^3 - \frac{1}{(2\pi)^4}\sum_{(k,l)>0} c_1(kl)Li_3\left[e^{2\pi i(kT+lU)}\right] + const. \tag{4.5}$$

with $Li_3(x) = \sum_{k=1}^{\infty} x^k/k^3$ and $\frac{E_4 E_6}{\eta^{24}} = \sum_{k=-1}^{\infty} c_1(k)q^k$ finally resulting in

$$\sigma(T, U) = \mathrm{Re}\left\{-2\ln[j(T) - j(U)] - 2\sum_{(k,l)>0} c_1(kl) \, kl \ln\left[1 - e^{2\pi i(kT+lU)}\right]\right\} . \tag{4.6}$$

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In fact, the large T–limit leads to

$$\sigma(T,U) \longrightarrow -4\pi\mathrm{Im}(T) \quad , \quad T \longrightarrow i\infty \; , \tag{4.7}$$

in agreement[9] with the limit of [33]. As expected for the threshold–corrections under consideration, $\sigma(T,U)$ stays finite throughout the moduli space. In particular in the limit $T \to U$ the contribution of $k = 1$, $l = -1$ in the sum cancels precisely the first term in (4.6). The combination $\mathrm{Im}(S) + \frac{1}{16\pi^2}\sigma(T,U)$ is invariant under the perturbative duality group $SL(2,Z)_T \times SL(2,Z)_U \times Z_2^{T\leftrightarrow U}$. However, σ alone, receives shifts from $Z_2^{T\leftrightarrow U}$. On the other hand, that behaviour precisely compensates the non–invariance of S arising from its multi–valuedness.

4.2   Wilson–line dependence

Now we go to the case with non–trivial gauge background fields of an E_7 subgroup. More concretely, we turn on a Wilson line in an $SU(2)$ subgroup of E_7. For a generic vev of the Wilson line modulus C the gauge group E_7 is broken to $SO(12)$. This then leads to the phenomenological interesting case of physical gauge couplings which develop logarithmic singularities in certain regions of the moduli space (T, U, C), when particles charged with respect to the gauge group under consideration, i.e. $SO(12)$, become massless. The gauge group–dependent part $b_a^{N=2}\Delta$ of one–loop corrections with a Wilson line modulus were first derived in [26] by looking at their perturbative modular symmetries and their singularity structure in the moduli space. Two cases of physical gauge couplings are discussed. In the first case, referring to E_8', no particles become massless for $C \to 0$ and the form of these thresholds is given by[10] [26]:

$$\Delta^{II} = -\frac{1}{12}\ln(\kappa Y^{12})|\chi_{12}(\Omega)|^2 \; . \tag{4.8}$$

In the second case some particles, charged under both the $SU(2)$ and $SO(12)$, become massless for $C \to 0$. This means that the effective one–loop correction develops a logarithmic singularity in this limit since these particles run in the loop. The form of these thresholds is given by:

$$\Delta^{I} = -\frac{2}{5}\ln\left|\frac{\chi_{10}^{1/2}(\Omega)}{\chi_{12}^{5/12}(\Omega)}\right|^2 - \frac{1}{12}\ln(\kappa Y^{12})|\chi_{12}(\Omega)|^2 \; . \tag{4.9}$$

The quantity Y has been defined in (3.9). The number $2/5$ refers to the additional contribution (relative to $b_{SO(12)}^{N=2}$) to the β–function arising from the particles becoming massless. No universal contributions are included in these functions, as they refer to $\triangle$ in (3.3). This means that they can be determined by considering the difference of two gauge groups [38].

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[9]One has to multiply by a factor of $-3/2$ to take into account the three orbifold planes and the N=1 structure of their $Z_2 \times Z_2$ orbifold. The minus sign arises from the different conventions for σ in (3.3).

[10]In appendix A some basics of Siegel modular forms are presented. The relevance of Siegel modular forms in the context of string one–loop corrections was originally observed in [26].

Also here we can now derive a differential equation for the one loop correction $h^{(1)}$ to the prepotential h

$$h(S, T, U, C) = -iS(TU - C^2) + h^{(1)}(T, U, C) + O(e^{2\pi iS}) , \qquad (4.10)$$

with [39]

$$h^{(1)}(T, U, C) = -\frac{i}{4\pi}d(T, U, C) - \frac{1}{(2\pi)^4} \sum_{(k,l,b)>0} c_1\left(kl - \frac{1}{4}b^2\right) Li_3[e^{2\pi i(kT+lU+bC)}] + const. , \qquad (4.11)$$

$\frac{E_4\hat{E}_{6,1}}{\eta^{24}} = \sum_{k\in\mathbf{Z}, \, k\in\mathbf{Z}+3/4} c_1(k)q^k$ and $(k, l, b) > 0$ is composed by the three orbits (i) $k > 0, l \in \mathbf{Z}, b \in \mathbf{Z}$, (ii) $k = 0, l > 0, b \in \mathbf{Z}$, (iii) $k = 0 = l, b < 0$. Moreover,

$$d(T, U, C) = \frac{1}{3}U^3 + \frac{40}{3}C^3 - 7UC^2 - 6TC^2 . \qquad (4.12)$$

There are ambiguities for the cubic polynomial (4.12) due to the fact that the holomorphic prepotential is only fixed up to quadratic pieces in the homogeneous coordinates $\hat{X}^I$. These quadratic pieces include e.g. cubics in C. On the other hand, this ambiguity can be fixed when comparing the prepotenial with the corresponding one of the typeII theory, which leads to the form (4.12) [40, 39, 41]. For the differential equation one has [38]:

$$\mathrm{Re}\left[\frac{32\pi^2}{5}(\partial_T\partial_U - \frac{1}{4}\partial_C^2)h^{(1)}\right] - G_{N=2}^{(1)} = \triangle_a + \frac{16}{5}\mathrm{Re}\left[\ln \Psi_a(T, U, C)\right] + b_a^{N=2}K_0 . \qquad (4.13)$$

This differential equation holds for both types of gauge groups $G_a = E_8', \, SO(12)$ with

$$\begin{aligned} b_{SO(12)}^{N=2} &= 60, \\ b_{E_8'}^{N=2} &= -60 \end{aligned} \qquad (4.14)$$

and the functions

$$\begin{aligned} \Psi_{SO(12)}(T, U, C) &= [4i\chi_{35}(\Omega)]^{1/2}[-4\chi_{10}(\Omega)]^2 , \\ \Psi_{E_8'}(T, U, C) &= \frac{[4i\chi_{35}(\Omega)]^{1/2}}{[-4\chi_{10}(\Omega)]^{11/2}} , \end{aligned} \qquad (4.15)$$

respectively. Using (4.13) and (3.3) we can extract $\sigma(T, U, C)$

$$\sigma(T, U, C) = \mathrm{Re}\left\{\frac{32\pi^2}{5}(\partial_T\partial_U - \frac{1}{4}\partial_C^2)h^{(1)}(T, U, C) - \frac{4}{5}\ln\left(\frac{\Delta_{35}^2(\Omega)}{\Delta_{10}^{22}(\Omega)}\chi_{12}^{25/2}(\Omega)\right)\right\} , \qquad (4.16)$$

with $\Delta_{35} = 4i\chi_{35}$ and $\Delta_{10} = -4\chi_{10}$ which gives for the chamber $T_2 > U_2 > 2C_2$:

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$$\sigma(T,U,C) = \text{Re}\left\{ -\frac{8\pi i}{5}(3T + \frac{7}{2}U - 20C) - \frac{4}{5}\ln\left(\frac{\Delta_{35}^2(\Omega)}{\Delta_{10}^{22}(\Omega)}\chi_{12}^{25/2}(\Omega)\right)\right.$$
$$\left. - \frac{4}{5}\sum_{\alpha=(k,l,b)>0}\alpha^2 c_1\left(\frac{\alpha^2}{2}\right)\ln\left(1 - e^{2\pi i(kT+lU+bC)}\right)\right\} \, . \tag{4.17}$$

It can be checked that this expression stays finite for the two limits $T \to U$ and $C \to 0$. In the first case the state with $k = 1, l = -1$ has to be extracted from the instanton sum to cancel the same logarithmic singularity arising from the Siegel forms. In the second case these are the states $k = 0 = l, b = -1$ and $k = 0 = l, b = -2$ which cause the logarithmic singularity $-168/5 \ln C$, which is cancelled by a same term from the Siegel forms. Moreover, it is not too difficult to show that (4.17) becomes (4.6) for $C \to 0$. Again, for the large T–limit we obtain:

$$\sigma(T,U,C) \longrightarrow -4\pi\text{Im}(T) \quad , \quad T \longrightarrow i\infty \, . \tag{4.18}$$

The one–loop threshold correction $\triangle_a$ to the physical, i.e. effective gauge couplings (3.2), must be a regular and duality invariant quantity. This can also be inferred from the world–sheet torus integral. The holomorphic functions $\Psi_a^{16/5}$ in (4.13) have weight $+120$ and -120, respectively. With the corresponding b_a from (4.14) the left hand side of (4.13) becomes a duality invariant expression up to shifts caused e.g. by $T \leftrightarrow U$ and $C \to -C$. Comparing the left hand side of (4.13) with (4.16) tells us, that the transformation behaviour of σ is the same as that of $G_{N=2}^{(1)}$. On the other hand, when we look at equations (3.2) and (3.3) we conclude that $-iS + i\overline{S} + \frac{1}{8\pi^2}G_{N=2}^{(1)} = inv.$. This is precisely the object entering the Kähler potential [cf. after eq. (3.8)] or eq. (2.16). We therefore write

$$\frac{-iS_{\text{inv}} + i\overline{S}_{\text{inv}}}{2} := \frac{-iS + i\overline{S}}{2} + \frac{1}{16\pi^2}\sigma(T,U,C) = inv. \tag{4.19}$$

with:

$$-iS_{\text{inv}} := -iS + \left[\frac{2}{5}(\partial_T\partial_U - \frac{1}{4}\partial_C^2)h^{(1)}(T,U,C)\right] - \frac{1}{5\pi^2}\ln\left(\frac{\Delta_{35}^2}{\Delta_{10}^{22}}\chi_{12}^{25/2}\right)^{1/4} . \tag{4.20}$$

Whereas the quantities S and $\sigma(T,U,C)$ are shifted under $Z_2^{T\leftrightarrow U}$ and $C \to e^{\pi i}C, C \to e^{2\pi i}C$, the field $S_{\text{inv.}}$ is completely invariant and it is that combination which enters the physical gauge couplings (3.10).

Let us now turn to the GS–correction (3.8). With (4.10) it becomes

$$G_{N=2}^{(1)} = -\frac{8}{Y}\pi\left(\frac{2}{3}U_2^3 + \frac{80}{3}C_2^3 - 14C_2^2U_2 - 12C_2^2T_2 + const.\right) +$$
$$+ \frac{4}{Y\pi}\text{Re}\left\{\sum_{(k,l,b)>0}c_1\left(kl - \frac{1}{4}b^2\right)\mathcal{P}\left[e^{2\pi i(kT+lU+bC)}\right]\right\} , \tag{4.21}$$

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with:

$$\mathcal{P}\left(e^{2\pi ix}\right) = \frac{1}{2\pi}Li_3\left(e^{2\pi ix}\right) + \mathrm{Im}(x)Li_2\left(e^{2\pi ix}\right) \ . \tag{4.22}$$

We give two plots, one[11] for $\sigma(T,U,C)$ and a second for $G_{N=2}^{(1)}(T,U,C)$:
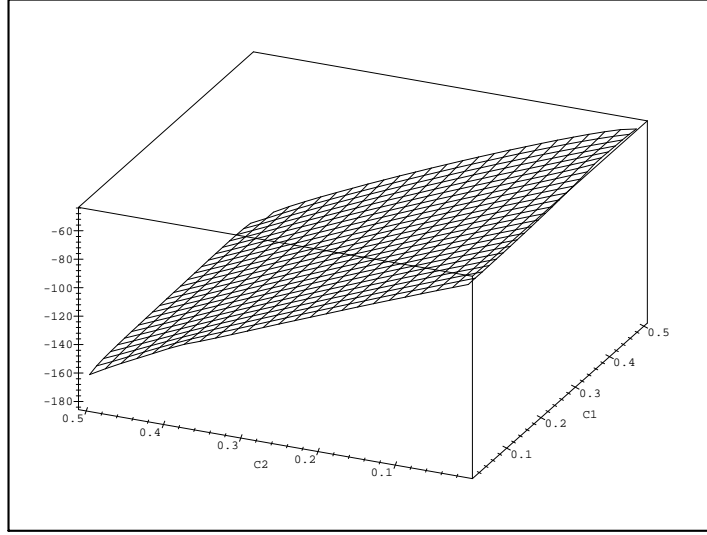

Fig.1 – Dependence of the universal one–loop correction σ on the Wilson line modulus C for fixed $T = 4i, \ U = i$.
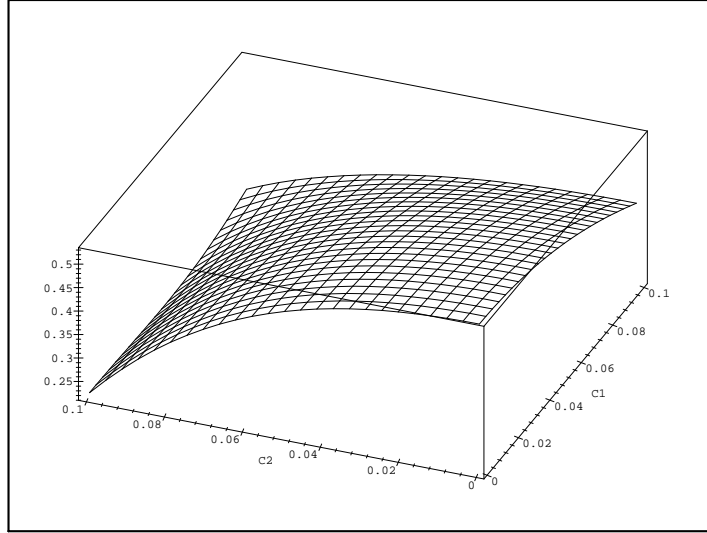

Fig.2 – Dependence of the GS–correction $G_{N=2}^{(1)}$ on the Wilson line modulus C for fixed $T = 4i, \ U = i$.

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[11]The 'instanton contributions' given by the last term of (4.17) are taken into account up to a certain order. This is necessary, e.g. in order to obtain the finite value for $C \to 0$.

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5 Towards stronger coupling

Let us now discuss the results of the previous threshold calculation from a more general point of view. First, we summarize (3.2) and (3.3) in the holomorphic gauge kinetic function ($k_a = 1$)

$$f_a(S, T, U, C) = -iS + \frac{2}{5}(\partial_T \partial_U - \frac{1}{4}\partial_C^2)h^{(1)} - \frac{1}{5\pi^2} \ln \left(\frac{\Delta_{35}^2}{\Delta_{10}^{22}} \chi_{12}^{25/2} \right)^{1/4} + \frac{b_a^{N=2}}{8\pi^2}\Delta_{holom.}^X . \quad (5.1)$$

Here $\Delta_{holom.}$ is just the holomorphic part of (4.9) and (4.8), for $X = I, II$ and $G_a = SO(12), E_8'$, respectively, i.e.:

$$\Delta_{holom.}^X = \begin{cases} -\frac{2}{5} \ln \frac{\chi_{10}^{1/2}(\Omega)}{\chi_{12}^{5/12}(\Omega)} - \frac{1}{12} \ln \chi_{12}(\Omega) , & X = I \\ \\ -\frac{1}{12} \ln \chi_{12}(\Omega) , & X = II \end{cases} \quad (5.2)$$

Using identities of the previous section allows us to write alternatively

$$f_a(S, T, U, C) = -i\tilde{S} - \frac{1}{5\pi^2} \ln \Psi_a(T, U, C) , \quad (5.3)$$

with Ψ_a defined in (4.15) and the pseudo–invariant dilaton

$$-i\tilde{S} = -iS + \frac{2}{5}(\partial_T \partial_U - \frac{1}{4}\partial_C^2)h^{(1)}(T, U, C) , \quad (5.4)$$

which is invariant under the perturbative duality group up to shifts [31]. Before we proceed, let us introduce a different normalization of the dilaton field S in (2.10) and the f functions (5.1) by $S \to 4\pi S$ and $f \to 4\pi f$. Thus S becomes:

$$S = \frac{\theta_a}{2\pi} + i\frac{4\pi}{g_a^2} . \quad (5.5)$$

This form of S is more convenient for the study of duality symmetries like e.g. $S \leftrightarrow T$–exchange symmetry.

We want to investigate how much basic information is encoded in the results given in (5.1). For this let us take a step back and consider the whole situation from a field theoretic point of view. There we know that the existence of nontrivial threshold corrections can be deduced from the explicit anomaly cancelation mechanism in the low energy $d = 10$ field theory. For completeness we repeat this line of arguments. The anomaly cancelation mechanism discovered by Green and Schwarz requires additional terms in the action of the low energy $d = 10$ field theory: there are Chern–Simons terms appearing in the definition in the 3-index field strength H as well as specific one loop counter terms [3]. It is these counterterms that are of special interest for us, e.g.

$$\epsilon^{KLMNOPQRST} B_{KL} Tr F_{MNOP}^2 Tr F_{QRST}^2 \quad (5.6)$$

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where upper case latin indices $K, L = 0 \ldots 9$ denote the components of spacetime, B is the two index antisymmetric tensor and F is the field strength of the Yang–Mills interactions. We are interested in the compactification of this $d = 10$ theory to a $d = 4$ theory with $N = 1$ supersymmetry. Curvature terms TrR^2 as well as field strengths TrF^2 will have nontrivial vacuum expectation values in the extra six dimensions, fulfilling a consistency condition in order for the three-form field strength H with

$$dH = TrF^2 - TrR^2 \tag{5.7}$$

to be well defined. Let us assume now that TrF^2_{klmn} is nonzero. Lower case latin indices will represent the components of the compactified 6 dimensions while greek indices will denote the uncompactified four dimensions. The Green-Schwarz term given above (5.6) will then reduce to

$$\epsilon^{mn} B_{mn} \epsilon^{\mu\nu\rho\sigma} TrF_{\mu\nu}F_{\rho\sigma} \tag{5.8}$$

in the $d = 4$ theory. The gauge kinetic terms in that theory will be given by the usual term $f(\phi)W^\alpha W_\alpha$ where at tree level $f = -iS$ with S being the dilaton superfield. An explicit inspection of the $d = 4$ action in components tells us that $\epsilon^{mn} B_{mn}$ is the pseudoscalar axion that belongs to the T-superfield [42, 43, 4]. Upon supersymmetrization the term in (5.8) will then lead to a one loop correction to the holomorphic f–function that is proportional to T and its coefficient being fixed by the anomaly [4, 5]. This is, of course, nothing else than a threshold correction. In the simple case of the standard embedding leading to a $d = 4$ theory with gauge group $E_6 \times E_8$ one obtains [5]

$$f_{E_6} = -iS - \epsilon iT \quad \text{and} \quad f'_{E_8} = -iS + \epsilon iT \tag{5.9}$$

respectively, where ϵ is a constant fixed by the anomaly. This should then be compared to the results given in eq. (5.1) Of course, the results in (5.9) are obtained in the field theory limit, i.e. the large radius limit of string theory and should therefore correspond to a threshold calculation in the large T–limit.

We now consider the decompactification limit $T \longrightarrow i\infty$ of (5.1). Using the limits

$$\begin{aligned}
\frac{\Delta^2_{35}}{\Delta^7_{10}} &\longrightarrow e^{2\pi i(-3T)} \\
\Delta_{10} &\longrightarrow e^{2\pi i T} \\
\chi_{12} &\longrightarrow e^{2\pi i T} ,
\end{aligned} \tag{5.10}$$

we obtain from (5.3) and (5.4) with the changes made before eq. (5.5)

$$\begin{aligned}
f_{SO(12)} &\overset{T \to i\infty}{\Longrightarrow} -iS - 6iT = -iS - i\left(\frac{b^{N=2}_{E_7}}{12} - 1\right) T . \\
f_{E'_8} &\overset{T \to i\infty}{\Longrightarrow} -iS + 6iT = -iS - i\left(\frac{b^{N=2}_{E'_8}}{12} - 1\right) T
\end{aligned} \tag{5.11}$$

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Note, how in the second limit the E_7 β–function appears. In the limit $T \to i\infty$ all $W-$ bosons with masses $m^2 \sim |C|^2/\text{Im}(T)$ become light and all states can be arranged in E_7 gauge multiplets. To make contact with (5.9) we have to clarify how our N=2 results (5.11) can be taken over to the N=1 theory. The moduli dependence of all our results arises from an N=2 subsector of our N=1 theory and therefore the β–functions are those of the N=2 theory. As a consequence of (3.4) they can be related to the N=1 $d = 4$ anomaly coefficients

$$
\begin{aligned}
b_{E_8'}^{N=2} &= 2\alpha_{E_8'}^{N=1} = -60 \\
b_{E_7}^{N=2} &= 2\alpha_{E_6}^{N=1} = 84 \ ,
\end{aligned}
\tag{5.12}
$$

if e.g. the N=2 subsector of our N=1 $d = 4$ theory represents a standard Z_2 orbifold and $f_{SO(12)}$ becomes f_{E_6}.

We have thus seen that the results of section 4 correspond exactly to the anomaly cancelation terms in the large T–limit (5.9), and are thus to a large extent determined by the mechanism of anomaly cancelation. This result gains further importance because of the fact that there exists a nonrenormalization theorem for the holomorphic f–function. No further correction do appear at higher loops, i.e. the above results give the full perturbative result to all loops. This can be verified by an inspection of the symmetries of the theory [7] and general theorems of N=1 supersymmetry [6].

Let us now summarize what we have learned up to now. We have seen

- that the holomorphic threshold corrections in the large T–limit are just a reflection of the anomaly in ten dimensions.

- that both the gauge dependent (Δ) and the gauge independent threshold corrections (σ) are important for the holomorphic f-function. Although σ is not of great importance for the question of unification it is crucial for the comparison of the coefficients in the large–T limit: Δ alone would lead to the wrong result.

- that the actual result of (5.9) reveals a surprise that was not appreciated enough when it was first obtained. The limit of large T will always lead to a situation where one of the gauge groups becomes strongly coupled, independent of the size of the original string coupling. The other coupling will become small at the same time, a situation that has meanwhile been seen in various other considerations [44, 45] that are, of course, based on the same argument as the above, namely anomaly cancelation. This might be a first hint for a conjecture that the results of the perturbative theory might carry over to the strong coupling regime.

From the practical point of view it would be interesting to know how well the large T–limit approximates the exact results. Although this is model dependent we find in many cases that this approximation holds even for smaller T–values. Let us just consider a few examples. The gauge dependent thresholds Δ (see (4.1)) are proportional to $\ln \eta(T)$ and it turns out that this function can be approximated very well by the linear function $i\pi T/12$. Even for values as small as $T = 2i$ the linear approximation is better than 10^{-3}.

In the gauge group–independent corrections σ (see (4.6)) there is a term $\ln j(T)$ that is relevant in the large–T limit. Again a linear approximation is very accurate for values of $T > 2i$. Thus the results obtained in the large–T limit hold even for rather small values of T. In some cases one might even deduce the full threshold corrections using symmetry arguments like T–duality $SL(2,Z)$ from the results obtained in the large–T limit. However, this mechanism works only in the simplest cases [25]. Already for the examples considered in [46] it does not lead to an unique answer.

As a side remark, let us add some comments about the question of unification. We have seen in the previous sections that with vacuum expectation values of the moduli fields T, U and the Wilson line C not too far away from the string scale, the unification of the coupling constants can be achieved. This is in contrast to the case without a Wilson line, where one has to choose rather large values of T and/or U to achieve unification [37]. In this case it could be argued (see e.g. [47]) that this leads in general to strong string coupling since the coupling constants evolve very fast between the compactification scale and the string scale. But there exist models where such a fast evolution can be avoided [48]. Of course, we do not understand why S should be large in the first place, so we maybe would prefer stronger coupling for aesthetic reasons. At the moment, however, we can conclude that the requirement of unification does not necessarily lead to strong coupling. The problem with the strong coupling regime is the the lack of ability to do reliable calculations and the study of unification might have to rely on wishful thinking.

To improve this situation let us now investigate to what extent our exact calculations in the weak coupling regime could extend to the region of strong coupling. Our main result is the fact that because of the anomaly and holomorphicity the complete f-function is:

$$f_{\text{complete}} = f_{\text{tree}} + f_{\text{one-loop}} + f_{\text{nonpert.}}, \tag{5.13}$$

where f_{tree} and $f_{\text{one-loop}}$ are known in perturbation theory [see e.g. (5.1)].

Of course, we cannot say very much about the last term, although it might be important for the question of the size of the coupling constant [49]. The first term is given by the anomaly and we believe that this should also be relevant in the region of stronger coupling. The strong coupling limit of the $E_8 \times E_8$ heterotic string is believed to be the M–theory orbifold S_1/Z_2 as discussed by Horava and Witten [9]. The eleventh dimension is an interval whose length ρ becomes large in the strong coupling limit. The gauge fields live on the ten–dimensional boundaries of this interval, one E_8 on each side. Witten has investigated the question of unification of gauge and gravitational couplings in the framework of this theory. He pointed out that one could, in principle, push the Planck mass to arbitrarily large values (by making ρ large) and thus adjust the gravitational coupling to any desired value while keeping the gauge coupling fixed [1]. However, in many cases, there is an obstruction to this adjustment once quantum effects are incorporated. To see that we can consider a $d = 4$, $N = 1$ theory with gauge group $E_6 \times E_8$ obtained via compactification on a Calabi-Yau manifold (X). Viewed from the eleven dimensional theory, the compactification is not a direct product of X and S_1/Z_2 as one might have naively guessed. The reason for it is a nontrivial vacuum expectation value of the four index field strength G_{klmn} of the eleven dimensional theory, that appears because of a

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consistency condition similar to (5.7). In the ten dimensional theory we can satisfy (5.7) by the standard embedding

$$Tr F_6^2 = Tr R^2 \tag{5.14}$$

with $Tr F_8^2 = 0$. In M–theory there is, however, a contribution $Tr F_i^2 - \frac{1}{2} Tr R^2$ at each boundary. Via supersymmetry this then induces nontrivial values for the G–field at the boundaries. Witten has solved the conditions for unbroken supersymmetry in a linearized approximation to first order [1]. This leads to a solution where the size of the Calabi-Yau manifold varies with the eleven dimensional coordinate. The limiting case where the size of X vanishes at the E_8 boundary then leads to an upper limit on the possible size of the eleven-dimensional interval, which in turn leads to an upper limit on the Planck mass. In fact the radii at the different boundaries are given in the linearized approximation by

$$R_8 \sim S - \epsilon T \quad \text{and} \quad R_6 \sim S + \epsilon T \tag{5.15}$$

where the coefficient ϵ is the same as in (5.9). The size of X vanishes on the E_8 side at the same point at which f_8 vanishes in the calculation done in the weak coupling regime (where the coupling of E_8 will become large). The similarity of (5.9) and (5.15) is, of course, not an accident and should be no surprise. In both cases the reason for the result comes from the mechanism of the cancelation of anomalies, and our large-T limit in the weakly coupled heterotic string [5] coincides with Witten's linearized approximation in the strongly coupled region[12][1]. This shows that the result of (5.9) is very robust and carries over to the strong coupling regime, the reason being the holomorphicity of f and the relation to the anomaly. (5.1) allows us to go beyond the large T limit of (5.9) and the linearized approximation of (5.15). In the strongly coupled theory it is still valid, but has a different (now geometrical) interpretation. Moving the Planck scale to a large value in M-theory is nothing else than a threshold calculation in the weakly coupled heterotic theory. At large coupling we should, however, be aware of the fact that nonperturbative effects like $\exp(-S)$ might become important. There is the hope that such terms could be determined with the help of string dualities but at the moment we have to rely on simplified assumptions. Such corrections might be important for the relation of S to the fundamental string coupling constant and might be the reason why g_{string} is small even if $S \sim 1$ [49].

So far we have only concentrated on the discussion of T (the case of U being equivalent) and not yet the Wilson line. In the last section, we have seen the exact results for the thresholds in the perturbative theory including the Wilson lines. Are these results also valid in the limit of stronger coupling. This question is not so easy to answer, because the relation of the Wilson lines to the mechanism of anomaly cancelation is more complicated. Still they give a contribution to the holomorphic f–function. We can have a look at the anomaly cancelation terms as in (5.6), which contain the pseudoscalar axion that is related to the T–multiplet as well as to the Wilson line. In the weakly coupled theory the kinetic terms are determined by the Kähler potential. In general with a Wilson A:

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[12]The similarity between these calculations has also been observed in [47]

$$K = -\log(S + \bar{S}) - 3\log(T + \bar{T} - 2A\bar{A}) \ . \tag{5.16}$$

Thus the mixing of T and A which determines the scalar partner of the axion $\epsilon^{mn}B_{mn}$ is controlled by the Kähler potential. As of now, we do not know any useful nonrenormalization theorems for the Kähler potential in N=1 supersymmetric theories. Corrections might appear at any loop level and thus the weak coupling results are not likely to carry over to the strong coupling regime. Of course, the form of the f–function at one loop will still remain the complete perturbative result. But there is no easy way to deduce it from first principle, *it has to be computed in the framework of string theory.* More exact information about the Kähler potential might be obtained in theories with extended supersymmetry, where there is a relation to the holomorphic prepotential. So the question how the threshold results with Wilson line carry over to strong coupling is still open. It is certainly worthwhile to have a closer look at this question in the future.

6 Conclusions

The main technical achievement presented in this paper is a full calculation of the gauge independent thresholds in the presence of a continuous Wilson line (4.17). Equipped with these results we can draw important conclusions for the following two questions:

- Is the (perturbative) heterotic string able to describe the unification of gauge and gravitational coupling constants?

- Does the perturbative result give us information about the limit of strong coupling?

The first question can be answered with yes, as we pointed out earlier already [2]. It is not necessary to go to strong coupling to achieve unification. With moderate values of the T, U and C moduli unification can be achieved. The role of the Wilson lines is very important. *It shows that a discussion of string unification without the consideration of such Wilson lines is not very meaningful.* Let us also stress that the gauge group dependent thresholds Δ are of great importance for unification. The universal terms σ although quite large in some cases are not as important in that respect. The question, why the size of the string coupling constant is small compared to unity is not yet understood in this scenario, but this question will strongly depend on the size of nonperturbative effects.

The answer to the second question is positive as well. We have shown that the perturbative calculation for the f-function is even reliable in the regime of stronger coupling. The reason for that is the nonrenormalization of f beyond one loop due to its holomorphic structure as well as the close connection of the thresholds to anomaly cancelation. Here both Δ and σ play an important role. The calculation of the thresholds at weak coupling in the large T limit coincide with those of strongly coupled M-theory in the linearized approximation (although with a different geometrical interpretation). *The full result (5.1) allows us to go beyond the linearized approximation of M–theory.* The separate role of T,

U and C depends, however, on the loop corrections to the Kähler potential that cannot be easily controlled in theories with $N = 1$ supersymmetry. Therefore the dependence of f on the Wilson line cannot be easily obtained from field theoretic arguments and has to be computed in string theory. It remains an open question how to obtain reliable results for the Kähler potential in the strong coupling limit of $N = 1$ theories. Certainly extended supersymmetry could give more restrictions. In addition, any information about possible nonperturbative contribution to the f-function will be extremely useful both from a theoretical and phenomenological point of view. It might ultimately lead to a situation where the value of S is fixed, implying an understanding of the size of the string coupling. It might also be decisive for an understanding of the dynamical breakdown of supersymmetry. Hopefully string theory dualities will eventually give some information in that direction.

Appendix

This appendix is taken from [50].

A Siegel forms of weight k and degree $g = 2$

Siegel modular forms are a natural generalization of elliptic modular forms to the higher genus case.

Definition 1: A *Siegel modular form* $f(\Omega)$ of degree g and weight k is defined by the following two conditions:

1. For every element $M \in Sp(2g, \mathbb{Z})$, $f(\Omega)$ satisfies:

$$f(M\Omega) = \det(\mathcal{C}\Omega + \mathcal{D})^k f(\Omega) \ , \ k \in \mathbb{Z} \ ,$$

   with $M \cong \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in Sp(2g, \mathbb{Z})$, i.e.: $\mathcal{A}^t\mathcal{C} = \mathcal{C}^t\mathcal{A}$, $\mathcal{D}^t\mathcal{A} - \mathcal{B}^t\mathcal{C} = \mathbf{1}_g$ and $M\Omega := (\mathcal{A}\Omega + \mathcal{B})(\mathcal{C}\Omega + \mathcal{D})^{-1}$.

2. It is holomorphic for $\Omega = \begin{pmatrix} T & C \\ C & U \end{pmatrix} \in \mathcal{F}_g$, *Siegels fundamental region* for genus g.

Examples of them are the *Eisenstein series*:

$$\mathcal{E}_k(\Omega) = \sum_{\mathcal{C},\mathcal{D}} \frac{1}{\det(\mathcal{C}\Omega + \mathcal{D})^k} \ , \tag{A.1}$$

where the summation runs over all inequivalent bottom rows of elements of $Sp(2g, \mathbb{Z})$ with respect to left multiplication by unimodular integer matrices of degree g.

The graded ring of the modular forms is generated for $g = 1$ by the two Eisenstein series E_4 and E_6. Igusa has proven an analogous result for $g = 2$ [51]: The Eisenstein series of $g = 2$: $\mathcal{E}_4, \mathcal{E}_6, \mathcal{E}_{10}$ and $\mathcal{E}_{12}$ are algebraically independent over $\mathbb{C}$ and they generate the graded ring of the modular forms for $g = 2$ and even weight. As in the $g = 1$ case one is interested in the cusp forms. They are constructed out of $\mathcal{E}_4, \mathcal{E}_6, \mathcal{E}_{10}, \mathcal{E}_{12}$. There are *two cusp forms* in the $g = 2$ case, one of weight 10 and the other of weight 12, respectively [51]:

$$\begin{aligned}
\chi_{10}(\Omega) &= c_1 \left[\mathcal{E}_4(\Omega)\mathcal{E}_6(\Omega) - \mathcal{E}_{10}(\Omega) \right] \\
\chi_{12}(\Omega) &= c_2 \left[441\mathcal{E}_4^3(\Omega) + 250\mathcal{E}_6^2(\Omega) - 691\mathcal{E}_{12}(\Omega) \right] \ .
\end{aligned} \tag{A.2}$$

with the numerical constants $c_1 = -43867/(371 \ 2^{12}3^55^2)$ and $c_2 = 77683/(2^{13}3^75^37^2337)$. For arbitrary weight $k \in \mathbb{Z}$ one has in addition a cusp form χ_{35} of weight 35. Alternatively the graded ring of of modular forms is generated by $\mathcal{E}_4, \mathcal{E}_6$ and the three cusp forms $\chi_{10}, \chi_{12}, \chi_{35}$.

All the cusp forms introduced before may be also expressed in terms of the genus–two theta functions. The most general $g = 2$ theta–function is defined by:

$$\vartheta \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} (\Omega, \boldsymbol{z}) = \sum_{n_1, n_2 \in \mathbb{Z}} e^{\pi i (\boldsymbol{n} + \boldsymbol{a})^t \Omega (\boldsymbol{n} + \boldsymbol{a}) + 2\pi i (\boldsymbol{n} + \boldsymbol{a})^t (\boldsymbol{z} + \boldsymbol{b})} \ . \tag{A.3}$$

There are sixteen of them: ten of even characteristics and six of odd characteristics [52]. The ten even characteristics are:

$$\begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} \in \frac{1}{2} \left\{ \begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{matrix} \right\} \ .$$

We can represent our cusp forms by ϑ–characteristics:

$$\begin{aligned}
\chi_{10}(\Omega) &= -\frac{1}{2^{14}} \prod_{even} \vartheta_a^2(\Omega, 0) \\
\chi_{35}(\Omega) &= -\frac{i}{2^{39}5^3} \left[\prod_{even} \vartheta_a(\Omega, 0) \right] \sum_{(a,b,c) \in \ azygous} \delta_{abc} \ \vartheta_a^{20}(\Omega, 0)\vartheta_b^{20}(\Omega, 0)\vartheta_b^{20}(\Omega, 0) \ .
\end{aligned} \tag{A.4}$$

There are 60 *azygous* triples (a, b, c) denoting a certain combination of ϑ–characteristics and $\delta_{abc} = \pm 1$ to ensure the correct behaviour under $Sp(4, \mathbb{Z})$.

To obtain an expression for χ_{12} in terms of ϑ–characteristics we first consider two theorems, which allow one to represent the limit $B \to 0$ of a modular form of genus 2 in a product of well–known genus one functions [52]:

Theorem 2: If F is a modular form of $g = 2$ and weight $k = 0 \bmod 2$, then $F \begin{pmatrix} T & 0 \\ 0 & U \end{pmatrix}$ can be represented as isobar polynomial in the functions:

$$E_4(T)E_4(U) \quad , \quad E_6(T)E_6(U) \quad , \quad E_4^3(T)E_6^2(U) + E_4^3(U)E_6^2(T) \ .$$

Theorem 3: There are modular forms F_k of $g = 2$ with degree $k = 4, 6, 12$, respectively and:

$$F_k \begin{pmatrix} T & 0 \\ 0 & U \end{pmatrix} = \begin{cases} E_4(T)E_4(U) \ , & k = 4 \\ E_6(T)E_6(U) \ , & k = 6 \\ E_4^3(T)E_6^2(U) + E_4^3(U)E_6^2(T) \ , & k = 12 \ . \end{cases} \tag{A.5}$$

The modular functions

$$
\begin{aligned}
F_4(\Omega) &= \frac{1}{4} \sum_{even} \vartheta_a^8(\Omega, 0) \\
F_6(\Omega) &= \frac{1}{4} \sum_{(a,b,c)\in syzygous} \delta_{abc}\, \vartheta_a^4(\Omega, 0)\vartheta_b^4(\Omega, 0)\vartheta_b^4(\Omega, 0) \\
F_{12}(\Omega) &= -\frac{81}{44}\frac{1}{2} \sum_{even} \vartheta_a^{24}(\Omega, 0) + \frac{11}{2} F_4^3(\Omega) + \frac{2}{11} F_6^2(\Omega)
\end{aligned}
\tag{A.6}
$$

are constructed to have this behavior for $B = 0$. Again there are 60 *syzygous* triples (a, b, c) denoting a certain combination of ϑ–characteristics and $\delta_{abc} = \pm 1$. The modular function F_{35} consists of 60 *azygous* combinations. All these functions can also be expressed by the $g = 2$ Eisenstein functions (A.1). In fact we have $F_4(\Omega) = E_4(\Omega)$, $F_6(\Omega) = E_6(\Omega)$. With a $F_{10} \sim \chi_{10}$ every $g = 2$ modular form of even weight can be represented as isobar polynomial in F_4, F_6, F_{10} and F_{12} [52]. Finally with (A.6) we obtain an expression for χ_{12} in the sense of (A.4):

$$\chi_{12}(\Omega) = \frac{1}{1728^2} \left[F_4^3(\Omega) + F_6^2(\Omega) - F_{12}(\Omega) \right] \tag{A.7}$$

The factorization properties in the theorems 2 and 3 originate from the factorization properties of ϑ–characteristics [51]:

$$\vartheta \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} \begin{pmatrix} T & C \\ C & U \end{pmatrix} = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} \frac{d^n}{dT^n} \theta \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}(T) \frac{d^n}{dU^n} \theta \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}(U) C^{2n} \ . \tag{A.8}$$

References

[1] E. Witten, *Nucl. Phys.* **B 471** (1996) 135

[2] H.P. Nilles and S. Stieberger, *Phys. Lett.* **B 367** (1996) 126

[3] M. Green and J. Schwarz, *Phys. Lett.* **B 149** (1984) 117

[4] J.P. Derendinger, L. Ibanez and H.P. Nilles, *Nucl. Phys.* **B 267** (1986) 365

[5] L. Ibanez and H.P. Nilles, *Phys. Lett.* **B 169** (1986) 354

[6] M. Shifman and A. Vainshtein, *Nucl. Phys.* **B 277** (1986) 456

[7] H. P. Nilles, *Phys. Lett.* **B 180** (1986) 240

[8] E. Witten, *Nucl. Phys.* **B 443** (1995) 85

[9] P. Horava and E. Witten, *Nucl. Phys.* **B 460** (1996) 506; *Nucl. Phys.* **B 475** (1996) 94

[10] S. Cecotti, S. Ferrara and M. Villasante, *Int. J. Mod. Phys.* **A 2** (1987) 1839

[11] J.P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, *Nucl. Phys.* **B 372** (1992) 145

[12] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Nucl. Phys.* **B 407** (1993) 706

[13] J.P. Derendinger, F. Quevedo and M. Quirós, *Nucl. Phys.* **B 428** (1994) 282

[14] I. Antoniadis, K.S. Narain and T.R. Taylor, *Phys. Lett.* **B 267** (1991) 37;
I. Antoniadis, E. Gava and K. S. Narain, *Nucl. Phys.* **B 383** (1992) 93;
Phys. Lett. **B 283** (1992) 209

[15] E. Kiritsis and C. Kounnas, *Nucl. Phys. Proc. Suppl.* **B 41** (1995) 331; *Nucl. Phys.* **B 442** (1995) 472; hep-th/9507051; *Nucl. Phys.* **B 456** (1995) 699; *Nucl. Phys. Proc. Suppl.* **B 45** (1996) 207

[16] P. Mayr and S. Stieberger, *Nucl. Phys.* **B 412** (1994) 502

[17] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Nucl. Phys.* **B 432** (1994) 187

[18] I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, *Nucl. Phys.* **B 447** (1995) 35

[19] P.M. Petropoulos, hep-th/9605012

[20] D. Gross, J. Harvey, E. Martinec and R. Rohm, *Nucl. Phys.* **B 267** (1986) 75

[21] K. Förger, B.A Ovrut, S. Theisen and D. Waldram, hep–th/9505145

[22] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, hep-th/9605011; hep-th/9608034

[23] P. Ginsparg, *Phys. Lett.* **B 197** (1987) 139

[24] L. Dixon, J. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* **B 261** (1985) 678; **B 274** (1986) 285;
L.E. Ibáñez, J. Mas, H.P. Nilles and F. Quevedo, *Nucl. Phys.* **B 301** (1988) 157

[25] L. Dixon, V. Kaplunovsky and J. Louis *Nucl. Phys.* **B 355** (1991) 649

[26] P. Mayr and S. Stieberger, *Phys. Lett.* **B 355** (1995) 107

[27] T. Kawai, *Phys. Lett.* **B 371** (1996) 59

[28] J.A. Harvey and G. Moore, *Nucl. Phys.* **B 463** (1996) 315

[29] M. Henningson and G. Moore, hep–th/9608145

[30] V. Kaplunovsky and J. Louis, *Nucl. Phys.* **B 444** (1995) 191

[31] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Nucl. Phys.* **B 451** (1995) 53

[32] V. Kaplunovsky, *Nucl. Phys.* **B 307** (1988) 145, Erratum: *Nucl. Phys.* **B 382** (1992) 436

[33] P.M. Petropoulos and J. Rizos, *Phys. Lett.* **B 374** (1996) 49

[34] I. Antoniadis, J. Ellis, R. Lacaze and D. V. Nanopoulos, *Phys. Lett.* **B 268** (1991) 188;
L. Dolan and J.T. Liu, *Nucl. Phys.* **B 387** (1992) 86;
P. Mayr, H.P. Nilles and S. Stieberger, *Phys. Lett.* **B 317** (1993) 53;
M. Chemtob, *Phys. Rev.* **D 53** (1996) 3920;
K.R. Dienes and A.E. Faraggi, *Phys. Rev. Lett.* **75** (1995) 2646; *Nucl. Phys.* **B 457** (1995) 409;
C. Bachas, C. Fabre and T. Yanagida, *Phys. Lett.* **B 370** (1996) 49;
Last reference of [22]

[35] L.E. Ibáñez, H.P. Nilles and F. Quevedo, *Phys. Lett.* **B 187** (1987) 25;
L.E. Ibáñez, J.E. Kim, H.P. Nilles and F. Quevedo, *Phys. Lett.* **B 191** (1987) 282;
L.E. Ibáñez, H.P. Nilles and F. Quevedo, *Phys. Lett.* **B 192** (1987) 332

[36] For a recent review see: P. Mayr and S. Stieberger, Proceedings *28th International Symposium on Particle Theory*, p. 72–79, Wendisch–Rietz (1994) (hep–th/9412196, DESY 95–027)

[37] L.E. Ibáñez and D. Lüst, *Nucl. Phys.* **B 382** (1992) 305

[38] S. Stieberger, NEIP–001/97 to appear

[39] T. Kawai, hep–th/9607078

[40] P. Berglund, S. Katz, A. Klemm and P. Mayr, hep–th/9605154;
J. Louis, J. Sonnenschein, S. Theisen and S. Yankielowicz, hep–th/9606049

[41] G.L. Cardoso, G. Curio and D. Lüst, hep–th/9608154

[42] E. Witten, *Phys. Lett.* **B 155** (1985) 151

[43] J.P. Derendinger, L. Ibanez and H.P. Nilles, *Phys. Lett.* **B 155** (1985) 65

[44] M. Duff, R. Minasian and E. Witten, *Nucl. Phys.* **B 465** (1996) 413

[45] G. Aldazabal, A. Font, L. Ibanez and F. Quevedo, *Phys. Lett.* **B 380** (1996) 33

[46] P. Mayr and S. Stieberger, *Nucl. Phys.* **B 407** (1993) 725;
D. Bailin, A. Love, W.A. Sabra and S. Thomas, *Mod. Phys. Lett.* **A9** (1994) 67; **A10** (1995) 337

[47] T. Banks and M. Dine, hep–th/9605136

[48] I. Antoniadis, C. Bachas, D. Llewellen and T. Tomaras, *Phys. Lett.* **B 207** (1988) 441;
C. Kounnas and M. Porrati, *Nucl. Phys.* **B 310** (1988) 355;
S. Ferrara, C. Kounnas, M. Porrati and F. Zwirner, *Nucl. Phys.* **B 318** (1989) 75:
H. P. Nilles and M. Spalinski, hep–th/9606145;
E. Kiritsis, C. Kounnas, P. Petropoulos and J. Rizos, *Phys. Lett.* **385** (1996) 87

[49] Z. Lalak, A. Niemeyer and H. P. Nilles, *Phys. Lett.* **B 349** (1994) 99; *Nucl. Phys.* **B 453** (1995) 100

[50] S. Stieberger, *One–loop corrections and gauge coupling unification in superstring theory*, Ph.D. thesis Munich May 1995, TUM–HEP–220/95

[51] J.I. Igusa, *Amer. J. Math.* **84** (1962) 175; **86** (1964) 392; **89** (1967) 817

[52] E. Freitag, *Siegelsche Modulfunktionen*, A series of comprehensive studies in mathematics No. 254 (1983), Springer