Cours/Lecture Series

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LECTURER	:	G. PIZZELLA / University of Rome & CERN-EF
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ABSTRACT

After a brief introduction on gravitational waves in general relativity, a survey of possible cosmic sources is given. Most of the lectures will be devoted to the experimental apparatuses for the search of gravitational waves. Resonant and non-resonant antennas will be discussed and the relative advantages and disadvantages will be indicated. A survey of the experimental activity going on all over the world will be given and the latest experimental results will be shown.

DETECTORS OF GRAVITATIONAL WAVES

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1. Gravitational waves
2. Motion of test bodies in a g.w. field
3. Energy carried by g.w.
4. G.W. sources
5. G.W. detectors
6. Electromechanical transducers
7. Data analysis
8. The resonant transducers
9. The Wiener-Kolmogoroff filter
10. The cross section and the effective temperature
11. The antenna bandwidth
12. The gravitational wave experiments in the world

1. Gravitational waves

The existence of gravitational waves was deduced by Einstein, in 1916 [1], as the weak field solution of the general relativity equation in vacuum

(1.1) $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \qquad (\mu, \nu = 0,1,2,3)$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ the symetric metric tensor and $R = g^{\mu\nu} R_{\mu\nu}$. The weak field approximation consists in assuming

(1.2) $$g_{\mu\nu} = g^{(o)}_{\mu\nu} + h_{\mu\nu}$$

where $g^{(o)}_{\mu\nu}$ is the galileian metric tensor and the symmetric tensor $h_{\mu\nu}$ is very small, that is

(1.3) $$|h_{\mu\nu}| \ll 1 \quad .$$

With this approximation the equation (1.1), which is non linear, can be linearized, and the final result, in the Lorentz gauge

(1.4) $$\frac{\partial h_{\mu\alpha}}{\partial x_\alpha} = 0 \qquad (\mu = 0,1,2,3) \quad ,$$

turns out to be the d'Alembert wave equation

(1.5) $$\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x_\alpha} \equiv \Box h_{\mu\nu} = 0$$

A quadridimensional symmetric tensor has ten independent components. For $h_{\mu\nu}$ we can eliminate eight of them, four by means of the condition (1.4) and four with a proper choice of the reference system. Thus $h_{\mu\nu}$ has only two independent components, that is $h_{\mu\nu}$

has only two distinct polarization states.

Let us now examine the most simple solution: a monochromatic plane wave propagating along the $x^1 \equiv x$ axis,

(1.6) $$h_{\mu\nu} = A_{\mu\nu} \, \mathrm{Re}\, [\exp(i\omega[t - \frac{x}{c}])]$$

Imposing the Lorentz condition (14) we get

$$\frac{\partial h_{\mu 0}}{\partial x_0} + \frac{\partial h_{\mu 1}}{\partial x_1} = A_{\mu 0}\frac{i\omega}{c} - A_{\mu 1}\frac{i\omega}{c} = 0$$

that is we obtain the four conditions

(1.7) $$A_{\mu 0} = A_{\mu 1} \qquad (\mu = 0,1,2,3)$$

We choose now the reference system in such a way to have the additional four conditions

(1.8) $$A_{m0} = 0 \quad (m = 1,2,3) \quad , \quad A^{\alpha}_{\alpha} = 0$$

We are left, in conclusions, with the following non-zero components

(1.9a) $$A_{32} = A_{23} \equiv A_{\times}$$

(1.9b) $$A_{22} = -A_{33} \equiv A_{+}$$

This gauge is called transverse-traceless gauge "TT gauge", and shows that the gravitational waves are transverse waves having the two distinct polarization states $A_{\times}$ and A_{+} .

2. Motion of test bodies in a g.w. field

Let us consider the motion of test bodies in a g.w. field. If we consider just one body we find that its trajectory must satisfy the geodesic equation

(2.1) $$\frac{d^2x^i}{ds^2} + \Gamma^i_{mk}\frac{dx^m}{ds}\frac{dx^k}{ds} = 0$$

Since the Christoffel symbols Γ^i_{mk} do not have tensorial character, the equation (2.1) is not a covariant equation, that is it depends on the reference system, which can be choosen in such a way as to cancel the gravitational field acting on the test body.

The situation changes when considering two test bodies. Let us indicate with ξ^m the vectorial distance between the two bodies in free fall and which no other force acting on them but the given gravitational field. It can be shown that the following equation holds

(2.2) $$\frac{\delta^2\xi^m}{\delta s^2} + u^\mu u^\nu \xi^n R^m_{\mu n \nu} = 0$$

called the geodesic deviation equation. In the above $\delta/\delta s$ indicates the covariant derivative with respect to the curvilinear coordinate s taken along a geodesic line, u^μ indicates the four-velocity of the two geodesic described by the test bodies and $R^m_{\mu n \nu}$ is the Riemann tensor. Equation (2.2) shows how the two geodesic, described by the two test bodies, deviate one with respect to the other one by effect of the gravitational field. It is in a covariant form because it contains only tensorial quantities, and all gravitational wave detectors are based on it. In practice eq.(2.2) is considered for velocities small compared to that of light, thus $u^\mu \simeq (-1,0,0,0)$. Then eq.(2.2) becomes

(2.3) $$\frac{\delta^2\xi^m}{\delta s^2} + \xi^n R^m_{0n0} = 0$$

We now consider small changes of the distance ξ^m with respect to the original distance

$$\xi^m = \xi_0^m + \eta^m \tag{2.4}$$

and we choose a local galileian system, so the covariant derivative becomes the normal derivative. Furthermore we consider that, in presence of a gravitational wave, $g_{00} = 1$ and therefore the proper time and the time t of the reference system in the origin are identical. Eq.(2.3) becomes

$$\frac{d^2\eta^m}{dt^2} = -c^2 R^m_{0n0}\,\xi_0^n \tag{2.4}$$

We put one of the two test bodies in the origin of the reference system, so $\xi_0^n \equiv (ct,x,y,z)$. Considering a gravitational wave in the TT gauge propagating along the x axis, with some calculation, we get from (2.4)

$$\begin{aligned}
&\frac{d^2\eta_x}{dt^2} = 0\\
&\frac{d^2\eta_y}{dt^2} = \frac{1}{2}\left(y\,\frac{d^2h_+}{dt^2} + z\,\frac{d^2h_\times}{dt^2}\right)\\
&\frac{d^2\eta_z}{dt^2} = \frac{1}{2}\left(y\,\frac{d^2h_\times}{dt^2} - z\,\frac{d^2h_+}{dt^2}\right)
\end{aligned} \tag{2.5}$$

If only $h_+ \neq 0$, then the second body (the first one is in the origin) moves as subjected to a force whose lines are equilateral hyperbolas with the y and z axes as asymptodes. If only $h_\times \neq 0$, the lines of force are obtained from the previous ones by means of a rotation of 45° around the x axis.

In all cases we notice that the relative displacement of the test bodies occurs in the plane (y,z), perpendicularly to the g.w. direction. This is due to the transverse character of the g.w.

field.

3. Energy carried by g.w.

Gravitational waves carry energy. For calculating the flux of energy we must use the pseudo-tensor energy-momentum. It can be shown that the energy carried per unit time across the unit area is given by

$$(3.1)\qquad I(t) = \frac{c^3}{16\pi G}\left[\dot{h}_+^2(t) + \dot{h}_\times^2(t)\right] \quad \left[\frac{\text{joule}}{s\cdot m^2}\right]$$

where the dot "." indicates the time derivative. The total energy per unit area is

$$(3.2)\qquad \mathcal{Y} = \int_{-\infty}^{\infty} I(t)\,dt \qquad \left[\frac{\text{joule}}{m^2}\right]$$

For semplicity we consider one polarization status only, $h_\times(t)$ or $h_+(t)$, and indicate it with h(t). We indicate with $H(\omega)$ the Fourier transform of h(t),

$$(3.3)\qquad H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$$

From (3.1) we get

$$(3.4)\qquad \mathcal{Y} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} |\omega H(\omega)|^2\, d\nu$$

were $\nu = \omega/2\pi$ is the frequency. The quantity

$$(3.5)\qquad f(\omega) = \frac{c^3}{16\pi G}\,|\omega H(\omega)|^2 \quad \left[\frac{\text{joule}}{m^2\,Hz}\right]$$

is called "spectral energy density", written here in bilateral form (frequencies from $-\infty$ to $+\infty$).

As a special case, we consider a g.w. burst of duration τ_g

that can be described by a sinusoidal wave with h_0 amplitude and angular frequency ω_0 for $|t|< \tau_g/2$ and zero value for $|t|> \tau_g/2$. Since we have $H(\omega)\simeq h_0 \tau_g/2$, from (3.5) we get

$$(3.6)\quad f(\omega_0) = \frac{c^3}{64\pi G}\,\omega_0^2 h_0^2 \tau_g^2 \qquad \left[\frac{\text{joule}}{m^2\,Hz}\right]$$

Another interesting case is a g.w. burst of the type $h(t)=h_0\,e^{-\beta_w|t|}\cos\omega_0 t$. This wave has duration of the order of $\tau_g = 2/\beta_w$. If $\tau_g \gg 2\pi/\omega_0$ then the Fourier transform is

$$H(\omega) \simeq \frac{h_0}{\beta_w} = \frac{h_0 \tau_g}{2}$$

and we obtain again the result (3.6).

In order to obtain the total amount of energy per unit area we consider that the frequency bandwith for a duration τ_g is $\sim 1/\tau_g$. Multiplying (3.6) for it we get

$$(3.7)\quad \mathcal{J} = f(\omega_0)/\tau_g = \frac{c^3}{64\pi G}\,\omega_0^2 h_0^2 \tau_g \qquad \left[\frac{\text{joule}}{m^2}\right]$$

Finally an interesting case is also a g.w. burst of a δ-type, $h(t)=H_0\,\delta(t)$, $\delta(t)$ being the Dirac function. Since $H(\omega)=H_0$ we obtain from (3.5)

$$(3.8)\quad f(\omega) = \frac{c^3}{16\pi G}\,\omega^2 H_0^2 \qquad \left[\frac{\text{joule}}{m^2\,Hz}\right]$$

In order to compute the value of h_0 on the Earth due to a g.w. burst of duration τ_g that occurs at a distance R, indicating with $M_{GW}\,c^2$ the total g.w. energy, we multiply (3.7) by $4\pi R^2$; we obtain

$$(3.9)\quad h_0 = \left(\frac{16\,G\,M_{GW}\,c^2}{c^3 R^2 \omega_0^2 \tau_g}\right)^{1/2} =$$

$$= 1.38\times 10^{-17}\,\frac{1000\ Hz}{\nu}\,\frac{1000\ pc}{R}\left(\frac{M_{GW}}{10^{-3} M_\odot}\,\frac{10^{-3}\,s}{\tau_g}\right)^{1/2}$$

If we consider a sinusoidal g.w. of angular frequency ω with amplitude h_0 we obtain the average power per unit area from (3.1)

$$(3.10)\qquad I_0 = \frac{c^3}{32\pi G}\omega^2 h_0^2 \qquad \left[\frac{\text{watt}}{m^2}\right]$$

Indicating with W the total power irradiated by the source, at distance R we obtain

$$(3.11)\qquad h_0 = \left(\frac{8G}{c^3}\frac{W}{R^2\omega^2}\right)^{1/2} = 2.29\times10^{-41}\,\frac{1000\,pc}{R}\,\frac{1000\,Hz}{\nu}\sqrt{W}$$

4. <u>Gravitational wave sources</u>

The mathematical solution of the problem regarding the generation of g.w. is similar to that for the electromagnetic (e.m.) waves. From the linearized field equation we find the retarded-potential solution for $h_{\mu\nu}$. In this case the field source is due to bodies having gravitational mass; for the e.m. case the source is due to electric charges.

For the gravitational waves the linearized equations written in the presence of matter are

$$\Box h_i^k = \frac{16G}{c^4}\tau_i^k$$

$$(4.1)\qquad \frac{\partial^2 h_i^k}{\partial x^k} = 0$$

where τ_i^k is a tensor related to the energy-momentum tensor.

For the electromagnetic waves we have, as well known,

(4.2) $$\Box A_i = \frac{4\pi}{c} j^i$$
$$\frac{\partial A^i}{\partial x^i} = 0$$

where A_i is the four-potential vector and j^i the four-current vector. An important difference is that the electromagnetic field is described by the vector A_i, whilst the gravitational field is described by the tensor $h_{\mu\nu}$, that is spin 1 for photons and spin 2 for gravitons.

Between the two equation systems there is another important difference From eqs. (4.1), combining the first equation with the second one, we get

(4.5) $$\frac{\partial \tau_i^k}{\partial x^k} = 0$$

It can be shown that this equation expresses the mechanics conservation laws, in particular the momentum conservation. The equivalent equation for the electromagnetic case, obtained from (4.2), is

(4.4) $$\frac{\partial j^i}{\partial x^i} = 0$$

which expresses the charge invariance.

Equations (4.3) and (4.4) affect in a different way the solution of, respectively, Eqs. (4.1) and (4.2).

In both cases the solution is of the retarded-potential type, that is for eq.(4.1)

(4.5) $$h_i^k = -\frac{4G}{c^4}\int\left(\frac{\tau_i^k}{r}\right)_{t-\frac{r}{c}} dV$$

However, if we consider the expansion in multipole terms for the wave zone, that is at distances from the source very large compared with the source dimension, we find that, due to (4.3), the dipole term is zero.

The first non zero term is the quadrupole term which, averaged over all directions, gives

$$W = \frac{G}{45 c^5} \dddot{D}_{\alpha\beta}^{2} \qquad [\text{watt}] \tag{4.6}$$

for the radiated power, where

$$D_{\alpha\beta} = \int \rho \left(3x_\alpha x_\beta - \delta_{\alpha\beta} x_\gamma^2 \right) dV \tag{4.7}$$

is the quadrupole tensor (ρ being the mass density) of the source.

We notice in eq.(4.6) the factor c^5 in the denominator, which has to be compared to the factor c^3 in the dipole term that is non zero for the electromagnetic waves. For this reason it is necessary to consider, as sources of g.w., very large masses of stellar size.

The sources [2,3] of gravitational waves can be divided into two classes: periodic sources and aperiodic sources due to catastrophic events. Examples of periodic sources are spinning rods, double stars, spinning stars etc. Examples of catastrophic events, in which gravitational waves are produced, the explosion of supernovae, and in general any gravitational collapse of systems deviating from perfect spherical symmetry, ending in non radial oscillations.

The power emitted from periodic sources is, generally speaking, much smaller than that emitted in some subclasses of events belonging to the second class. The fact, however, that the emission from periodic sources has a permanent nature over long periods of time, allows the use of detectors tuned to the source, so that the wide band noise of the detection system is correspondingly reduced.

As indicated by Eq. (4.6), deduced from the Einstein theory, gravitational waves are emitted when the quadrupole mass tensor of a massive body changes with time. The generation of gravitational waves in laboratory appears to be extremely difficult as it can be seen considering the classical example of a spinning rod. Let us consider a rod spinning about an axis perpendicular to its length with angular velocity ω . Using (4.6) we get

$$W = \frac{32}{5} \frac{G}{c^5} I^2 \omega^6 \tag{4.8}$$

where I is the moment of inertia about the axis of spin. As an extreme case we consider a rod of steel of 1 m radius and 20 m length spinning at the maximum rate of 4.4 revolution/s without breaking. We obtain

$$W = 2.2 \times 10^{-29} \text{ watt} \tag{4.9}$$

i.e. a value out of any possibility of detection for many years to come. This example shows that we are forced to turn our attention to cosmical objects, of very large mass, to see if some of them provides a source strong enough to be within the grasp of the present technology. In the following we indicate very briefly a few possible sources of gravitational waves of cosmic origin.

4.1 Spinning star

A spinning star can generate gravitational waves if it has a departure from axial symmetry. In such case we get

$$W = \frac{288\, G I^2 e^2 \omega^6}{45\, c^5} \qquad (4.10)$$

where e indicates the departure from axial symmetry ($e = (a-b)/\sqrt{ab}$, with a and b the two principal axis in the equatorial plane) and I the moment of inertia.

An example is provided by the Crab pulsar PSR 0532, with ν =30 Hz at 2 kps from the Earth usually described as an oblique rotating neutron star, continuously emitting gravitational waves.

The most reasonable estimation, although rather uncertain because of the unknown value for e, gives the value

$$h_0 \sim 10^{-26}$$

which is smaller, by at least 4 orders of magnitude, than that detectable with the present technology at $\nu \simeq 30$ Hz.

Another interesting example is the fast pulsar [4] 1937+214 rotating with ω =4033.8 rad/s at a distance of 2.5 kpc. The frequency of the g.w. emitted is ν_{gw}=1283.856542 Hz. From the observation, the power lost by the pulsar turns out to be $2\mathrm{x}10^{29}$ watt. If we assume that all of it is due to emission of g.w. then, using (3.11), we obtain h_0 =$3\mathrm{x}10^{-27}$ that it is very near to the present possibility of detection.

4.2 Double star systems

Fifty per cent of the stars belongs to multiple systems. Many binaries formed by a normal star and a collapsed body are at present under observation. Considering that their mean life should be of the order of 10^4 years, the estimate is made that in our Galaxy there should be more than 10^6 double stars with a collapsed body. In addition one can speculate on the possible existence of binary systems made of two collapsed bodies. Using (4.6) we obtain, for the power irradiated by such a system, in the case of circular orbits with radius r and angular frequency ω ,

$$(4.11) \qquad W = \frac{32G}{5c^5}\left(\frac{m_1 m_2}{m_1+m_2}\right)^2 r^4 \omega^6$$

For instance the binary β Per (at 30 pc from the Earth with a period of 2.9 years) gives

$$W = 1.4 \times 10^{21} \text{ watt}$$

and, from (3.11), we get $h_0 \simeq 10^{-18}$ at the Earth. Another interesting example [5] is the binary system PSR 1913+16 with period T=7h45m7s at a distance of 5kpc. From (4.11) we estimate $W \simeq 6.4 \times 10^{23}$ watt that gives at the Earth, using (3.11), $h_0 \simeq 1 \times 10^{-22}$.

4.3 Fall into Schwarzschild black hole [3]

If a body of mass m falls into a collapsed body of mass M, a finite amount $\Delta\mathcal{E}$ of gravitational radiation should be emitted

$$(4.12) \qquad \Delta\mathcal{E} \simeq 0.0025 \frac{m^2 c^2}{M}$$

with a frequency spectrum peaked at the characteristic frequency

$$(4.13) \qquad \nu_{peak} \sim 4.9 \times 10^3 \frac{M_\odot}{M} \text{ Hz}$$

As an example let us consider a neutron star with mass $m = 0.68\, M_\odot$ falling into a black hole of mass $M = 10\, M_\odot$. The total amount of gravitational radiation, emitted in about 0.5 ms, will be about 10^{-4} solar masses with a characteristic frequency of 6000 Hz. If the phenomenon occurs at 1000 pc from the Earth we should observe a burst with $h_0 \sim 3 \times 10^{-19}$ (using (3.9)), well within the possibility of the present technology.

4.4 Radiation frm gravitational collapse [3]

If the collapse of a star occurred symmetrically no gravitational radiation would be emitted. The collapse, however, occurs while the star is rotating and various bursts of gravitational radiation might be emitted in a way which depends upon the details of the phenomenon.

The spectrum of the emitted radiation is expected to be continuous from zero to a critical frequency

(4.14) $\nu_g \sim 1/\tau_g$

where the characteristic time is

(4.15) $\tau_g \sim \sqrt{\pi G \rho}$

with ρ the density of the final stage collapse.

The total emitted energy is

(4.16) $$\Delta \mathcal{E} \sim \frac{1}{30\pi} \frac{D_{max}^2}{\tau_g^5}$$

where D_{max} is the quadrupole tensor component along the spin axis.

In the cases of impulsive events, like those due to gravitational collapses, it is useful to consider the spectral energy density (joule/m^2Hz) at the Earth. For instance, in the case of a collapsing star with mass of 6 solar masses we could have $\Delta\mathcal{E} \sim 5 \times 10^{46}$ joule. If the collapse occurs in the center of the Galaxy the spectral energy at the Earth is

$$f(\omega) = 30 \text{ joule/m}^2 \text{ Hz} .$$

For $\nu = 850$ Hz and $\tau_g = 5 \times 10^{-4}$ s we get $h_0 \simeq 3 \times 10^{-17}$, which is well within the present experimental possibilities. But such an event is expected to occur only, at most, a few times per century.

Considering the galaxies of the Virgo Cluter we expect to have one event of this type per week with spectral energy density at the Earth of the order of

$$f(\omega) = 6 \times 10^{-6} \text{ joule/m}^2 \text{ Hz} \quad (h_0 \simeq 1.4 \times 10^{-20})$$

To detect such an event is the goal for the present and the near future experimental activity.

5. G.W. detectors

Gravitational waves can be detected by measuring the changes η^m of the distance between two test bodies as shown by equations (2.4) and (2.5).

This should be done while the two bodies are in free fall, which is possible only in the outer space; one can think of two space probes or two celestial bodies like Earth and Moon, or two test bodies in a Spacelab environment.

In an Earth based laboratory the two bodies must be supported against their own weight. We can classify these detectors in two cathegories: a) the non resonant detectors, where the two supports for the two bodies are independent one from each other as more as possible, b) the resonant detectors, where the two bodies are linked one to each other by means of an elastic support.

5.1 The non resonant detectors

These detectors are, essentially, Michelson or Fabry-Perot interferometers. Suppose that a g.w. travels along the direction of the arm(a) of the interferometer and perpendicularly to the arm(b). Suppose that both arms, in absence of g.w., have lenght l and that the g.w. has polarization h_+. From (2.5) we deduce that, at the g.w. arrival, the lenght of the arm(a) remains unchanged and the lenght of the arm(b) changes with time by the amount

$$\Delta l(t) = \frac{1}{2} l\, h_+(t) \tag{5.1}$$

Therefore we expect a small change in the interference fringes obtained by combining the light (laser) beams that travel along the two arms. From the measurement of Δl we deduce, from (5.1), the value of h_+.

There are several sources of noise which set the limits to this technique. The most fundamental is the shot noise, which simulates a rms change in the distance l

$$\delta l = \sqrt{\frac{\hbar c\, \lambda \Delta\nu}{\pi P}} \tag{5.2}$$

where P is the laser power, λ the wavelenght, $\Delta\nu$ the frequency bandwidth. From (5.1) and (5.2) we deduce the minimum detectable value of h_+

$$h_+ \simeq 2\frac{\Delta l}{l} = 2\sqrt{\frac{\hbar c\, \lambda \Delta\nu}{\pi P l^2}} \tag{5.3}$$

Putting P=1 watt, λ=0.6 μm and $\Delta\nu$=1000 Hz, in order to detect a supernova in the Virgo Cluster with $h_0 \sim 1.4 \times 10^{-20}$ it is necessary to have $l \simeq 300$ km. The plans for realizing so large values of l are to construct interferometers with arms of lenght $l_0 \ll l$ and to reflect the light beams n times so that $l = nl_0$. With $l_0 = 3$ km and $n = 100$ one obtains $l = 300$ km.

It should be noted that the light beam of lenght 300 km spends 10^{-3} second in the apparatus. This gives a bandwidth of 1000 Hz. An increase of l will correspond to a corresponding decrease of the bandwidth.

5.2 The resonant detectors

A detector where two or more test bodies are linked together by an elastic support can be realized by means of a metallic bar [6]. Since the first pioneer experiment by J.Weber, the bar is usually an alluminium cylinder of lenght L and mass M. The bar must be thought in free fall during the interaction with the g.w., althought it is suspended by a wire across the baricentral section in order to support it against the attraction from the Earth. We take a reference system where the center of mass is at rest. If we consider two mass elements dm of the bar located along the cylinder axis symmetrically with respect to the center of mass we notice that their distance tends to change according to eq.(2.5) by effect of the g.w. h_+ and $h_\times$. Elastic and dissipative forces also act on the two mass elements due to the surrounding material. As a consequence the bar tends to dilate and contract and elastic waves propagate in it. For a thin bar these are longitudinal waves and move with velocity

$$v = \sqrt{\frac{Y}{\rho}} \qquad (5.4)$$

where Y is the Young modulus and ρ the bar density. For alluminium at room temperature v $\simeq$ 5100 m/s.

It can be demonstrated that the bar behaves like a system of an infinite number of elementary oscillators with angular resonance frequency

$$\omega_k = (2k+1)\frac{\pi v}{L} \qquad (k = 0, 1, 2, \ldots) \tag{5.5}$$

The fundamental mode oscillator has angular frequency

$$\omega_0 = \frac{\pi v}{L} \tag{5.6}$$

In conclusion the study of the interaction of the bar with a g.w. is semplified if we consider one elementary oscillator at a time with resonant angular frequency ω_k. The displacement of the bar end faces at the angular frequency ω_k can be described by the displacement with respect to it equilibrium position of a point-like mass.

The mass m of this "equivalent" oscillator can be related to the mass M of the bar with the following considerations based on the energy of the process. In the bar, due to the boundary conditions of null stress on the end faces, a system of standing waves is generated. For the fundamental mode

$$\eta(t,z) = \eta_0 \cos \omega_0 t \sin \frac{\pi z}{L}$$

where z is a coordinate along the bar axis with origin in the center of mass and $\eta(t,z)$ the displacement of an elementary mass located at z. The displacement is largest at the bar ends, $z = \pm L/2$. The total kinetic energy of the bar is (S is the bar section)

$$\mathcal{E}_{bar} = \int_M \frac{1}{2} v^2 dm = \int_{-L/2}^{L/2} \frac{1}{2}\rho S \dot{\eta}^2 dz = \frac{1}{4} M \eta_0^2 \omega_0^2 \cos^2 \omega_0 t$$

A point-like oscillator has energy

$$\mathcal{E}_{osc} = \frac{1}{2} m v^2 = \frac{1}{2} m \omega_0^2 \eta_0^2 \cos^2 \omega_0 t$$

Since the two energies must be the same then

(5.7) $$m = \frac{M}{2}$$

Unless otherwise specified we consider only the fundamental mode oscillator at ω_0. It can be shown that its unidimensional displacement $\eta(t)$ follows the equation

(5.8) $$\ddot{\eta} + 2\beta_1 \dot{\eta} + \omega_0^2 \eta = \frac{2}{\pi^2} L \ddot{h}$$

where h indicates h_+ or $h_\times$ and the numerical factor on the right side is obtained by solving the problem of the continuous bar [6]. The quantity $2\beta_1$ expresses the losses and is related to the merit factor Q by

(5.9) $$2\beta_1 = \omega_0 / Q$$

In the cases of interest Q>>1, typically $Q>10^6$. The solution of (5.8) obtained by means of the Fourier transform is

(5.10) $$\eta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2L}{\pi^2} \frac{\omega^2 H(\omega) e^{i\omega t}}{(\omega^2 - \omega_0^2) - 2i\omega\beta_1} d\omega$$

If $H(\omega)$ does not have zeros and the poles of the integrand are

(5.11) $$\omega_{1,2} \simeq i\beta_1 \pm \omega_0$$

for the cases $H(\omega_1) = H(\omega_2)$ we obtain from (5.10)

(5.12) $$\eta(t) = -\frac{2L}{\pi^2} e^{-\beta_1 t} H(\omega_1)\, \omega \sin \omega_0 t$$

having neglected terms of the order Q^{-1}. This formula expresses the two fundamental points of the resonant g.w. antenna functioning: a) the antenna detects the Fourier component $H(\omega)$ of the metric tensor perturbation near its resonance frequency, b) the antenna has a "memory" for a time of the order of β_1^{-1}.

The non resonant antennas, on the other hand, detect, see formula (5.1), the metric tensor perturbation h(t) and do not have any memory.

Let us now consider some interesting cases.

i) G.W. burst of the δ-type, $h(t)=H_0\delta(t)$. From (5.12) we obtain

$$\eta(t) = -\frac{2L}{\pi^2} H_0 \omega_0 e^{-\beta_1 t} \sin\omega_0 t \tag{5.13}$$

ii) G.W. short packet, $h(t)=h_0 \cos\omega_0 t$ for $|t|<\tau_g/2$, $h(t)=0$ for $|t|>\tau_g/2$. We have

$$H(\omega_1) = H(\omega_2) = \frac{h_0\tau_g}{2}\,\frac{e^{+\beta_1\tau_g/2} - e^{-\beta_1\tau_g/2}}{\beta_1\tau_g}$$

From (5.12), using (5.9) and with τ_g small, that is $\beta_1\tau_g \ll 1$, we have for $t>\tau_g/2$

$$\eta(t) = -\frac{L}{\pi^2} h_0 \tau_g \omega_0 e^{-\beta_1 t} \sin\omega_0 t \tag{5.14}$$

iii) Monochromatic g.w., $h(t)=h_0 \cos\omega_0 t$. In this case it is convenient to start from (5.10) since the Fourier transform is $H(\omega)=h_0\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$, a combination of δ-functions. We obtain

$$\eta(t) = -\frac{2}{\pi^2} L Q h_0 \sin \omega_0 t \tag{5.15}$$

6. Electromechanical transducers

The electromechanical transducer is the device that is needed for converting the mechanical vibrations induced by the gravitational waves into an electrical signal.

From a mathematical point of view it can be represented with the components of the Z_{ik} matrix which connects the input variables (force $f(t)$ acting on the transducer and velocity $\dot{x}(t)$ of the transducer mechanical parts) with the output variables (voltage $v(t)$ and current $i(t)$)

$$f(t) = Z_{11}\dot{x}(t) + Z_{12}i(t) \tag{6.1}$$

$$v(t) = Z_{21}\dot{x}(t) + Z_{22}i(t) \tag{6.2}$$

With this representation the transducer is a two port system

In some important cases the Z_{ik} components satisfy the relationships

$$\begin{aligned} Z_{11}Z_{22} &= Z_{12}Z_{21} \\ Z_{12} &= Z_{21} \end{aligned} \tag{6.3}$$

In these cases the transducer behaves as a bipole system. We describe now briefly three interesting transducers.

6.1 Piezoelectric ceramic

This transducer can be mounted, for instance, in a slot cut in the baricentral section of the bar. Its presence modifies the mechanical characteristics of the bar (resonance frequency, merit factor, ecc.)

It is possible, however, to regard the transducer as non perturbative by including all these modifications in the bar itself. In this case the transducers satisfies eq.(6.3) and we obtain

(6.4) $$Z_{12} = Z_{21} = \alpha / j\omega$$

(6.5) $$Z_{22} = 1/j\omega C_2$$

C_2 is the piezoelecric ceramic capacity and α is

(6.6) $$\alpha = \frac{\pi \zeta d}{\epsilon s L}$$

L = length of the bar.

ζ = thickness of the ceramic,

d = piezoelectric modulus,

ϵ = dielectric constant, and

S = elastic constant of the ceramic.

For a typical Gulton G-1408 ceramic (ζ = 1.5 cm)

$$\frac{\pi d \zeta}{s \epsilon} = 7.8 \times 10^{7} \text{ volt}$$

From eq.(6.2) with infinite load (i=0) we get

(6.7) $$v(t) = Z_{21} \dot{x}(t) = \alpha x(t)$$

where x(t) is the displacement of the bar ends corresponding to a strain

(6.8) $$u(t) = \frac{\pi}{L} x(t)$$

at the bar baricentral cross section where the transducer is located.

A very important parameter is the ratio of the energy in the transducer to the total energy in the bar (see (5.7))

(6.9) $$\beta = \frac{\frac{1}{2} C_2 V^2}{\frac{1}{4} M \omega^2 x^2} = \frac{2\alpha^2 C_2}{M\omega^2}$$

6.2 <u>The capacitor</u>

This transducer consists in a metallic plate at a distance d from an end face of the bar. When the bar vibrates the distance d changes with time and so does the capacity between the plate and the bar

(6.10) $$C(t) = \epsilon_0 S/d(t)$$

If we put, for instance, a constant charge on the capacitor, we can determine d(t) by measuring the voltage v(t) across the capacitor plates. In this case also (6.3) is valid and we have

(6.11) $$Z_{12} = Z_{21} = E_0/j\omega$$

where E_0 is the average electrical field in the condenser (typically of the order of 5×10^6 V/m). For the energy coupling parameter we have

(6.12) $$\beta = \frac{\frac{1}{2}\epsilon_0 E_0^2 S d}{\frac{1}{4} M \omega^2 d^2} = \frac{2\epsilon_0 E_0^2 S}{M\omega^2 d}$$

6.3 <u>The inductor</u>

This transducer can be, for instance, of the type developed by the Stanford group. The vibration of a superconducting diaphragm mounted at an end face of the bar near a coil of inductance L, causes a change in the magnetic flux though the inductance L, generating a current i.

It can be shown that for a zero load we get

(6.13) $Z_{11} = 0; \qquad Z_{12} = -Z_{21} = L\, I_0/d; \qquad Z_{22} = j\omega L$

where I_0 is the steady current circulating in the coils with inductance L generating a magnetic induction B_0, and d is the distance between the coils and the diaphragm. In this case eq.(6.3) is not satisfied. We get for the energy coupling parameter

$$\beta = \frac{\frac{1}{2} B_0^2/\mu_0\, S d}{\frac{1}{4} M \omega^2 d^2} = \frac{2 B_0^2/\mu_0\, S}{M \omega^2 d} \qquad (6.14)$$

It can be shown that, in general,[7]

$$\beta = \frac{1}{M\omega} \frac{|Z_{21}|^2}{Z_{22}} \qquad (6.15)$$

7. Data analysis

In order to detect a very small g.w. signal one has to consider all possible sources of noise.

At this energy level noises of all kind are likely to be larger or much larger than the signal: thermal, electrical, acustic, cosmic rays, seismic, electromagnetic noises, all possible phenomena occurring in the laboratory can jeopardize the experiment. We classify the various possible noises in two groups. In the first group we put all those which can be estimated quantitatively employing well established mathematical models. These noises are: the brownian noise of the bar considered as an oscillator at frequency ν_0 and the current and voltage noise of the amplifier. We shall see that it is possible to reduce these noises to a minimum by employing appropriate mathematical algorithms of data analysis. In the second group we put all the remaining noises which are often of unknown origin. In order to get rid of these noises, in addition to take proper precautions, one can only consider coincidences between two or more antennas.

7.1 The brownian noise

Even in absence of signals the bar vibrates because of the brownian motion of the atoms in the thermal bath due to the environment at the temperature T.

We can write

(7.1) $$\ddot{\eta} + 2\beta_1 \dot{\eta} + \omega_0^2 \eta = F(t)/m$$

where F(t) is a stochastic force due to the atoms of the bar and m=M/2 is the reduced mass of the bar.

By means of the energy equipartation principle it is possible to obtain the power spectrum of the force F(t)

(7.2) $$S_F(\omega) = 4\beta_1 mkT$$

From eq.(7.1) we get the power spectrum for $\eta(t)$

(7.3) $$S_\eta(\omega) = \frac{S_F}{m^2} \frac{1}{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta_1^2}$$

and the autocorrelation function

$$(7.4)\quad R_{\eta\eta}(\tau) \cong \frac{S_F e^{-\beta_1|\tau|}\cos\omega_0\tau}{4m^2\omega_0^2\beta_1}$$

The mean square displacement is given by

$$(7.5)\quad x_b^2 \equiv \overline{x^2} = R_{\eta\eta}(0) = \frac{kT}{m\omega_0^2}$$

as expected from the initially imposed equipartition principle.

As a consequence we find a mean square voltage of the transducer output, in the case of a piezoelectric ceramic, given by

$$(7.6)\quad V_b^2 = \frac{\alpha^2 kT}{m\omega_0^2} = \beta\frac{kT}{C_2} = \beta\omega_0|Z_{22}|\,kT$$

7.2 <u>The back action</u>

The transducer is connected to an electrical amplifier whose noise can be characterized by two parameters. The two parameters are, usually, the power spectra of the voltage and current noise, V_n^2 and I_n^2, or the following combinations of them

$$(7.7)\quad T_n = V_n I_n/k$$

$$(7.8)\quad R_n = V_n/I_n ,$$

T_n is called the amplifier noise temperature (V_n^2 and I_n^2 expressed in bilateral form). R_n is the amplifier noise resistance. The transducer also has a noise. An interesting example of this noise is the Johnson noise of the piezoelectric ceramic

(7.9) $$I_{pz}^2 = 2kT \tan\delta \cdot \omega C$$

where δ is the "loss angle" of the ceramic. The analysis is greatly simplified if this noise is included in V_n^2 and I_n^2 and so we do in the following.

It is convenient to represent V_n and I_n by means of two noise generators. For sake of simplicity let us suppose that the amplifier input impedance is infinite, that is we deal with a voltage amplifier with null current. Then from (6.1) we notice that the current noise I_n exerts a force on the bar ("back-action") which is stochastic in nature and, therefore, just of the type of the brownian force. From (6.1) and (6.5) we find the power spectrum of this force

(7.10) $$S_f(\omega) = |Z_{12}|^2 I_n^2 = m\beta\omega_0 |Z_{22}| I_n^2$$

By comparison with the power spectrum (7.2) we find immediately, making use of (7.5), the narrow band noise due to the back action

(7.11) $$x_{ba}^2 = \frac{kT}{m\omega_0^2} \frac{m\beta\omega_0 |Z_{22}| I_n^2}{4\beta_1 m kT} = \frac{\beta |Z_{22}| I_n^2}{4\beta_1 m\omega_0^2}$$

We introduce the dimensionless parameter

(7.12) $$\lambda = \frac{R_n}{|Z_{22}|}$$

and we obtain

(7.13) $$x_{ba}^2 = \frac{\beta Q T_n k}{2m\omega_0^2\lambda} = x_b^2 \frac{\beta Q T_n}{2\lambda T}$$

In total the narrow band noise is then given by

$$(7.4)\quad x_{nb}^2 = x_b^2\left(1+\frac{\beta Q T_n}{2\lambda T}\right) = \frac{KT_e}{m\omega_0^2}$$

where

$$(7.15)\quad T_e = T\left(1+\frac{\beta Q T_n}{2\lambda T}\right)$$

is called the equivalent temperature. Therefore it is possible to ignore the existence of the back action by substituting in the brownian noise expressions T with T_e . In particular from (7.6) we get

$$(7.16)\quad V_{nb}^2 = \frac{\alpha^2 K T_e}{m\omega_0^2} = \frac{\beta K T_e}{C_2} = \beta\omega_0 |Z_{22}| K T_e$$

7.3 The wide band noise, data analysis and optimization

In the case of a voltage signal V(t) there is a wide band noise whose power spectrum is

$$(7.17)\quad S_0 = V_n^2 + I_n^2|Z_{22}|^2 = V_n^2\left(1+\frac{1}{\lambda^2}\right) = KT_n|Z_{22}|\frac{1+\lambda^2}{\lambda}$$

This noise adds to the narrow band noise in a way which depends on the specific algorithms used for the data analysis.

In order to greatly reduce the amount of data recorded on magnetic tape it is necessary to eliminate, in the data, the Fourier component at the resonance frequency ν_0 of the antenna. This is done by means of lock-ins (also called Phase Sensitive Detector, PDS) which give the two components of the signal, in phase and in quadrature.

The autocorrelation of the signal at the PSD output is obtained from (7.4) eliminating $\cos\omega_0\tau$ and adding a term containing the wide band noise integrated by the PSD with time constant $t_0 = \beta_2^{-1}$:

(7.18) $$R(\tau) = V_{nb}^2 e^{-\beta_1|\tau|} + S_0 \beta_2 e^{-\beta_2|\tau|}$$

(see (9.5) for the exact autocorrelation function).

For the data analysis the key idea is to compare the actual value of V(t) with the value that can be predicted from a previous time. The difference between the two values has to be compared with the standard deviation in order to judge whether it is contained within the statistical fluctuations or is due to an innovation, say a gravitational wave.

The mathematical algorithm for this data analysis is the following. We consider the difference

(7.19) $$\rho(t) = V(t) - V(t - \Delta t)$$

between two successive samplings of the voltage at the PSD output, Δt being the sampling time. We get the standard deviation of this filter, which we call ZOP (Zero Order Predictive filter), by

(7.20) $$\sigma^2 = \overline{\rho^2(t)} = 2R(0) - 2R(\Delta t)$$

Making use of (7.18) for $\beta_1 \ll \beta_2$ we get

(7.21) $$\sigma^2 = 2 V_{nb}^2 \Delta t \beta_1 + 2 S_0 \beta_2$$

In order to maximize the amount of information recorded on the magnetic tape it is convenient to take the sampling time equal to the integration constant of the PSD

(7.22) $$\Delta t = \beta_2^{-1}$$

From (7.21) we obtain

$$(7.23)\quad \sigma^2(\Delta t) = 2 V_{nb}^2 \beta_1 \Delta t + \frac{2 S_0}{\Delta t}$$

This standard deviation of the innovation of the voltage signal can be also expressed as standard deviation of the innovation of energy of the antenna oscillations

$$(7.24)\quad \Delta \mathcal{E}\,(\Delta t) = \sigma^2 \frac{m \omega_0^2}{\alpha^2}$$

It is convenient to rewrite (7.24) in terms of the following dimensionless parameter

$$(7.25)\quad \Gamma = \frac{S_0}{V_{nb}^2 / \beta_1} = \frac{T_n\,(\lambda + 1/\lambda)}{2 \beta Q T_e}$$

that gives the ratio of the wide band noise in the resonance bandwidth to the narrowband noise. We have

$$(7.26)\quad \Delta \mathcal{E}(\Delta t) = 2 V_{nb}^2 \frac{m \omega_0^2}{\alpha^2} \left(\beta_1 \Delta t + \frac{\Gamma}{\beta_1 \Delta t} \right)$$

The greatest sensitivity is obtained when $\Delta \mathcal{E}$ has the smallest possible value, which occurs for

$$(7.27)\quad \Delta t_{opt} = \frac{\sqrt{\Gamma}}{\beta_1}$$

With this choice for the sampling time (and time constant of the PSD) we obtain the smallest detectable energy innovation

(7.28) $$\Delta\mathcal{E}_{min} = 4 V_{nb}^{2} \frac{m\omega_0^2}{\alpha^2} \sqrt{\Gamma} = 4 k T_e \sqrt{\Gamma}$$

A convenient way to rewrite this equation is by making use of (7.17), (7.15), (6.15) and (6.4) (that is valid in general for any transducer which generates a voltage signal). We obtain

(7.29) $$\Delta\mathcal{E}_{min} = 2 k T_n \sqrt{\left(1+\frac{1}{\lambda^2}\right)\left(1+\frac{2T\lambda}{\beta Q T_n}\right)}$$

In obtaining this formula we have not considered the effect of the ZOP filter on the signal, but only on the noise. If the effect on the signal is taken into account the factor $2kT_n$ of eq.(7.29) becomes $2.42\ kT_n$.

This expression shows the need to use an amplifier with a noise temperature T_n as small as possible. Suppose we have a very good amplifier with small T_n. In order to detect the smallest possible innovation of energy we must satisfy the two conditions

(7.30) $$\lambda^2 \gg 1$$

(7.31) $$\frac{\beta Q T_n}{2\lambda T} \gg 1$$

These two are called the "matching conditions". If they are satisfied we get [8] from (7.29)

(7.32) $$\Delta\mathcal{E}_{min} = 2kT_n$$

If the sign >> in (7.30) and (7.31) is replaced with = we get

(7.33) $$\Delta\mathcal{E}_{min} = 4kT_n$$

When the two matching conditions are fulfilled, from (7.27) we get

(7.34) $$\Delta t_{opt} = \frac{\sqrt{\lambda^2+1}}{\beta\omega_0}$$

8. The resonant transducer

It is very difficult to fulfill (7.31) because it is hard to obtain large values of β . For example, for a capacitive transducer, using (6.12) with $E = 5\times10^{6}$ V/m, $S = 10^{-2}$ m^{2}, $d = 50\,\mu m$, $m = 1000$ kg, $\omega = 5000$ rad/s we get $\beta \sim 2\times10^{-6}$.

For reaching much larger values of β we can use a resonant capacitive transducer [9] of the type shown in Fig.1. The disk, fixed at is center, vibrates in its flexural symmetrical mode with frequency

$$\nu_t = \frac{(2.1)^2}{4\pi}\,\frac{a}{R^2}\sqrt{\frac{Y}{3\rho(1-\sigma^2)}} \tag{8.1}$$

where a and R are the thickness and radius of the disk, Y, σ and ρ are respectively the Young and Poisson moduli and the density of the material. The thickness and radius of the disk are such that is equal to the resonance frequency ν_0 of the bar.

In this way we obtain a system of two coupled oscillators, bar and transducer, which exhibits two resonances at frequencies

$$\nu_{\mp} = \nu_0\left(1 \mp \frac{\sqrt{\mu}}{2}\right) \tag{8.2}$$

where

$$\mu = \frac{m_t}{m} \tag{8.3}$$

is the ratio of the reduced mass of the transducer to the reduced mass of the bar given by (5.7). The displacements of the bar and of the transducer, indicated respectively with $\eta_b(t)$ and $\eta_t(t)$, neglecting the dissipations, are

(8.4) $$\eta_b(t) = \frac{f_o}{2m\omega_o}(\sin\omega_- t + \sin\omega_+ t) = \frac{f_o}{m\omega_o}\sin\omega_o t\cos\Omega_b t$$

(8.5) $$\eta_t(t) = \frac{f_o}{2m\sqrt{\mu}\,\omega_o}(\sin\omega_+ t - \sin\omega_- t) = \frac{-f_o}{m\sqrt{\mu}\,\omega_o}\cos\omega_o t\sin\Omega_b t$$

where f_o is the Fourier transform of an applied δ force on the bar and

(8.6) $$\Omega_b = \frac{\omega_o\sqrt{\mu}}{2}$$

is the beat angular frequency. We notice that the transducer displacement is $1/\sqrt{\mu}$ larger than the bar displacement, and that the mechanical energy is tranferred back and forth between bar and transducer with the beat period $T_b = 2\pi/\Omega_b$. The transducer displacement $\eta_t(t)$ is the quantity that is measured. In this case we obtain a much larger value of β because in eq.(6.12) we must substitute M/2 with m_t. In the case of the Rome antenna $m_t = 0.348$ kg and therefore we obtain

(8.7) $$\beta_{res} = 5 \times 10^{-3}$$

that is much better for fulfilling (7.31).

From (8.5) we can recognize that the system: antenna + resonant transducer, is equivalent to two sistems: antenna + non resonant transducers, resonating at frequencies ν_- and ν_+. The only difference is that the incident absorbed energy is split between the two modes. Using a non resonating transducer and a δ-force applied to the bar with Fourier transform f_o we obtain, with no dissipation, (8.8) $$\eta_b(t) = \frac{f_0}{m\omega_0} \sin\omega_0 t$$

The comparison of (8.8) with (8.4) shows that the displacement for the case of the resonant transducer is 1/2 of that for the case of a non resonant transducer, for each mode. Therefore, as far as the signal, we find in one mode of the transducer (ω_- or ω_+) 1/4 of the available power (the other energy is in the bar).

The brownian noise is also smaller for the case of a non resonant transducer. In fact, for the energy equipartition principle the thermal noise for one mode (say ω_+) is distributed between bar and transducer

(8.9) $$\frac{1}{2} kT = \frac{1}{2} m \omega_+^2 \overline{\eta_b^2} + \frac{1}{2} m_t \omega_+^2 \overline{\eta_t^2}$$

From (8.4) and (8.5) we get $m\overline{\eta_b^2} = m_t\overline{\eta_t^2}$, thus

(8.10) $$\overline{\eta_t^2} = \frac{kT}{2 m_t \omega_+^2}$$

to be compared with (7.5), showing that the thermal noise on the resonant transducer, for one mode only, is 1/2 of that for the non resonating transducer.

As a consequence the signal to noise ratio (SNR) for one mode is 1/2 of the SNR that we have for the non resonant transducer. However, since a δ-force produces the same signal on both modes ω_+ and ω_- , combining the data from the two modes we have a SNR equal to that obtained for a non resonant transducer. In conclusion, the effect of using a resonant transducer is just to obtain a larger β

value while the sensitivity remains the same.

9. The Wiener-Kolmogoroff filter

The electrical equivalent circuit of the gravitational wave detector consists, essentially, in two low-pass filters [10]. One filter, with time constant $t_{o} = \beta_2^{-1}$, represents the integrating part of the PSD. The other filter is due to the bar plus that part of the PSD which selects the Fourier component of the signal; the time constant of this filter is β_1^{-1}. The last assertion can be proven, for instance, by inspecting formula (7.18) which shows that the autocorrelation of the brownian signal is typical of that due to a low-pass filter with time constant β_1^{-1}. In conclusion the input noise applied to the bar is, first, filtered with the transfer function

$$W_b = \frac{\beta_1}{\beta_1 + i\omega} \tag{9.1}$$

and then, together with the noise arriving from the amplifier, is filtered with the transfer function

$$W_e = \frac{\beta_2}{\beta_2 + i\omega} \tag{9.2}$$

If we have a white noise with power spectrum S_{uu} at the input of the W_b filter we obtain the power spectrum $S_{uu}|W_b|^2$ at the output of the filter, with the autocorrelation function

$$R^{(b)}_{uu}(\tau) = S_{uu}\,\frac{\beta_1}{2}\,e^{-\beta_1|\tau|} \tag{9.3}$$

This shows that the relation between S_{uu} and V_{ub}^2 is

$$V_{ub}^2 = S_{uu}\,\frac{\beta_1}{2} \tag{9.4}$$

This reasoning shows us also that the autocorrelation (7.18) is approximate in the limit (usually well verified) $\beta_1 \ll \beta_2$.

In fact, at the end of the electronic chain, the power spectrum of the brownian noise will be $S_{uu}\,|W_b|^2|W_e|^2$ which provides the exact autocorrelation function

$$R^{(b+e)}_{uu}(\tau) = \frac{S_{uu}\,\beta_1\beta_2}{2}\;\frac{\beta_2 e^{-\beta_1|\tau|} - \beta_1 e^{-\beta_2|\tau|}}{\beta_2^2 - \beta_1^2} \tag{9.5}$$

Similarly if we indicate with S_{ee} the power spectrum of the noise due to the amplifier at the entrance of the PSD we have, at the output, the power spectrum $S_{ee}\,|W_e|^2$ with the autocorrelation

$$R^{(e)}_{ee}(\tau) = S_{ee}\,\frac{\beta_2}{2}\,e^{-\beta_2|\tau|} \tag{9.6}$$

The comparison with (7.18) gives the relation between S_{ee} and S_o

$$S_o = \frac{S_{ee}}{2} \tag{9.7}$$

We proceed now to describe the best estimation of the signal u(t) at the bar entrance, obtained by using all data measurements, past and future, weighted according to the Wiener-Kolmogoroff theory. Let us indicate with p(t) one component (say the real component) at the PSD output recorded on magnetic tape with sampling time Δt. We put

$$(9.8) \qquad \hat{u}(t) = \int_{-\infty}^{\infty} p(t-\tau)w(\tau)d\tau$$

where $\hat{u}(t)$ is the estimation of u(t) (concerning only one component of the signal) and $w(\tau)$ are the weights to be determined. We apply the linear mean square method. It is found that the Fourier transform $W(i\omega)$ of the weights $w(\tau)$ which minimizes the quantity $\overline{(\hat{u}(t)-u(t))^2}$ is

$$(9.9) \qquad W(i\omega) = \frac{S_{up}(\omega)}{S_{pp}(\omega)}$$

with the following meaning for symbols. $S_{pp}(\omega)$ is the power spectrum of p(t) and $S_{up}(\omega)$ is the cross spectrum of u(t) and p(t). We have

$$(9.10) \qquad S_{pp}(\omega) = S_{uu}|W_b|^2 |W_e|^2 + S_{ee}|W_e|^2$$

$$(9.11) \qquad S_{up}(\omega) = S_{uu} W_b^* W_e^*$$

where W_b^* and W_e^* indicate the complex conjugate of W_b and W_e. We get

$$(9.12) \qquad W(j\omega) = \frac{1}{W_b W_e} \frac{1}{1+\Gamma/|W_b|^2}$$

where $\Gamma = \frac{S_{ee}}{S_{uu}}$ has already been introduced by (7.25). In order to obtain $w(\tau)$ we must calculate

$$W(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(i\omega)e^{i\omega\tau}d\omega$$

Performing the integration we find

$$(9.13)\quad W(\tau) = \frac{\beta_1}{\beta_2\Gamma}\left[\delta(t) + \frac{(\beta_2 \mp \beta_3)(\beta_1 \mp \beta_2)e^{\mp\beta_3\tau}}{2\beta_3}\right]$$

where the sign - is for $\tau>0$ and the sign + for $\tau<0$.

Using (9.10) and (9.12) we have power spectrum of the estimation $\hat{u}(t)$

$$(9.14)\quad S_{\hat{u}}(\omega) = S_{pp}|W(i\omega)|^2 = \frac{S_{uu}}{1+\Gamma/|W_0|^2}$$

We notice that the estimation is perfect if $\Gamma=0$, corresponding to the absence of electronic noise. From (9.14) and using (9.1) we get

$$(9.15)\quad R_{\hat{u}\hat{u}}(\tau) = \frac{S_{uu}\beta_1}{2\sqrt{\Gamma(\Gamma+1)}}e^{-\beta_3|\tau|}$$

with

$$(9.16)\quad \beta_3 = \beta_1\sqrt{\frac{\Gamma+1}{\Gamma}}$$

The variance of $\hat{u}(t)$ is obtained by taking $R_{\hat{u}\hat{u}}(0)$.

We shall now compare this variance with the signal expected from a gravitational wave signal.

Suppose we have an excitation producing a bar vibration given by (5.12). The voltage signal is

$$(9.17)\quad v_g(t) = -\alpha \frac{2L}{\pi^2} H(\omega_0)\omega_0 e^{-\beta_1 t} \sin \omega_0 t$$

The output of the end of the PSD turns out to be (with some calculation)

$$(9.18)\quad V_g(t) = -\frac{2\alpha L \omega_0 H(\omega_0)}{\pi^2} \frac{e^{-\beta_1 t} - e^{-\beta_2 t}}{1-\beta_1/\beta_2}$$

Applying to this signal the Wiener-Kolmogoroff filter as derived previously

$$u_g(t) = \int_{-\infty}^{\infty} V_g(t - t') \, w(t')dt'$$

we get, with some calculation using (9.13)

$$(9.19)\quad u_g(t) = -\alpha \frac{2L\omega_0 H(\omega_0)}{\pi^2} \frac{\beta_1}{2\beta_3 \Gamma} e^{-\beta_3 t}$$

We consider now the signal to noise ratio (SNR). This is defined as follows:

$$SNR = \frac{\overline{u_g(t)^2}}{2R_{GG}(0)} = \frac{\left[\alpha \frac{2L\omega_0 H(\omega_0)}{\pi^2}\right]^2}{S_{uu}} \frac{e^{-2\beta_3 t}}{4\beta_1\sqrt{\Gamma(\Gamma+1)}}$$

where the factor of 2 has been introduced because both components from the PSD output contribute to the noise. Making use of (9.4) and (7.16) we get, for t=0 when SNR is largest,

$$(9.20)\quad SNR \equiv (SNR)_{max} = \frac{L^2 \omega_0^4 m H^2(\omega_0)}{2\pi^4 k T_e \sqrt{\Gamma(\Gamma+1)}}$$

The minimum detectable value of $H(\omega_0)$ is obtained for SNR=1.

$$\left[H(\omega_0)\right]_{min} = \left(\frac{2\pi^4 kT_e\sqrt{\Gamma(\Gamma+1)}}{L^2 m\omega_0^4}\right)^{1/2} \qquad (9.21)$$

10. <u>The cross section and the effective temperature</u>

The cross section Σ is defined such that, multiplied by the incident spectral energy density $f(\omega_0)$, gives the energy deposited in the bar

(10.1) $$\mathcal{E} = \Sigma f(\omega_0)$$

The energy $\mathcal{E}$ is calculated from (5.12)

(10.2) $$\mathcal{E} = \frac{1}{4}M\omega_0^2\left[\frac{2L}{\pi^2}\omega_0 H(\omega_0)\right]^2 = \frac{M\omega_0^2 H^2(\omega_0)\, v^2}{\pi^2}$$

where $v = \omega L/\pi$ is the sound velocity in the bar. Making use of (3.5) for the spectral energy density we obtain the cross section

(10.3) $$\Sigma = \frac{16}{\pi}\left(\frac{v}{c}\right)^2 \frac{G}{c} M \qquad [m^2\, Hz]$$

in bilateral form.

We now use the result (9.21), derived in the previous section by means of the W-K filter for the minimum detectable value of $H(\omega_0)$. Introducing (9.21) into (10.2) we get [11]

(10.4) $$\mathcal{E} = 4kTe\sqrt{\Gamma(\Gamma+1)}$$

which is very similar to the energy innovation $\Delta\mathcal{E}_{min}$ given by (7.28) and estimated with the ZOP filter, because $\Gamma \ll 1$ in all cases. The difference, essentially, is that, using W-K, there is no need to satisfy (7.27) for the sampling time; also we gain the factor 2.42/2 which was lost with the ZOP filter. However, it is found that the sampling time must satisfy

$$\Delta t < \beta_s^{-1}$$

otherwise the W-K filter breaks down. We notice that, from (7.27), we have

(10.5) $$\Delta t_{opt} \simeq \beta_s^{-1}$$

Finally the effective temperature is defined by

(10.6) $$T_{eff} = \frac{\mathcal{E}}{k} = 4T_e\sqrt{\Gamma(\Gamma+1)}$$

We derive now another expression [12] which relates $H(\omega_0)$, the quantity measured with a resonant g.w. antenna, to T_{eff} which expresses the sensitivity of the apparatus. From (10.2) and (10.6) we obtain

(10.7) $$\left[H(\omega_0)\right]_{min} = \frac{L}{v^2}\sqrt{\frac{kT_{eff}}{M}} \qquad [Hz^{-1}]$$

In general if we detect an energy innovation $\mathcal{E}$ the corresponding value of $H(\omega_0)$ is

(10.8) $$H(\omega_0) = \frac{L}{v^2}\sqrt{\frac{\mathcal{E}}{M}} \qquad [Hz^{-1}]$$

It must be stressed that with a resonant antenna we measure $H(\omega_0)$ and <u>not</u> $h(t)$. If we want to have a feeling about possible values for $h(t)$ we must make assumptions on the $h(t)$ spectrum. For instance for a flat spectrum from 0 to ν_g, which we can think due to a burst of duration $\tau_g \sim 1/\nu_g$, we can put

(10.9) $$h(t) \simeq H(\omega_0)\,\nu_g = \frac{H(\omega_0)}{\tau_g}$$

11. The antenna bandwidth

The problem of the antenna bandwidth arises in particular when exploring the possibility to use a resonant antenna for detecting monochromatic g.w. radiation. One could, erroneously, think that the antenna is sensitive only to m.g.w. with frequency ν_g near the resonance frequency ν_0 in a bandwidth of the order of ν_0/Q. We shall see now that the antenna bandwidth is, instead, equal to β_3/π and therefore much larger than ν_0/Q. It can indeed, under certain conditions, become so large as ν_0.

In order to prove this [13] we use the signal (5.15) obtained for m.g.w. The noise is of narrow band and wide band types. The narrow band noise is obtained from (7.3) and (7.2)

$$(11.1)\quad S_{nb}(\omega) = \alpha^2 \frac{2kT_e\omega_0/mQ}{(\omega_0^2-\omega^2)^2+\omega^2\omega_0^2/Q^2} \quad \left[\frac{\mathrm{volt}^2}{\mathrm{Hz}}\right]$$

The wide band is obtained from (7.17).

As a first step we consider the case of $\nu_g = \nu_0$. The signal to noise ratio is

$$(11.2)\quad SNR(\nu_0) = \frac{\overline{\alpha^2 \eta(t)^2}}{\left[S_{nb}(\omega_0) + V_n^2\left(1+\frac{1}{\lambda^2}\right)\right]\Delta\nu}$$

where $\Delta\nu$ is the integration band given by the total time of measurement $\Delta\nu = 1/t_m$. Introducing the parameter

(11.3) $$\chi = \frac{\beta Q}{\lambda}$$

we obtain from (11.2)

(11.4) $$SNR(\omega_0) = \frac{L^2}{\pi^4} M t_m h_0^2 \omega_0^3 \frac{Q}{2kT + kT_n(\chi + 1/\chi)}$$

The largest SNR is obtained if the following conditions are satisfied

(11.5) $$\frac{T_n}{2T} \ll \chi \ll \frac{2T}{T_n}$$

In such a case

(11.6) $$SNR(\omega_0) = \frac{\nu^2}{2\pi^2 k} \frac{M t_m h_0^2 \omega_0 Q}{T}$$

For SNR=1 we obtain the minimum detectable value of h_0

(11.7) $$h_0 \geq \sqrt{\frac{2\pi^2 kT}{M\nu^2 t_m Q \omega_0}} = 1.87 \times 10^{-24} \cdot \left[\frac{T}{4.2\,K} \frac{2300\,kg}{M} \frac{10^7}{Q} \frac{900\,Hz}{\nu_0} \frac{1\,day}{t_m}\right]^{1/2}$$

We go now to compute the bandwidth by considering a m.g.w. with $\nu_g \neq \nu_0$. We evaluate SNR for $\omega \neq \omega_0$. With some calculation we obtain

(11.8) $$SNR(\omega_g) = \left(\frac{L h_0}{\pi^2}\right)^2 \frac{\omega_g^4 M Q t_m}{2\omega_0 k T_e} \frac{1}{1 + \Gamma\left[Q^2\left(1 - \frac{\omega_g^2}{\omega_0^2}\right)^2 + \frac{\omega_g^2}{\omega_0^2}\right]}$$

The largest value of SNR is, obviously, for $\omega_g = \omega_0$. We calculate now for which value of ω_g the SNR reduces to one half of its maximum. Putting

$$1+\Gamma\left[Q^2\left(1-\frac{\omega_g^{*2}}{\omega_0^2}\right)^2+\frac{\omega_g^{*2}}{\omega_0^2}\right]=2$$

we find

$$\frac{\omega_g^*}{\omega_0} \simeq 1 \pm \frac{1}{2Q\sqrt{\Gamma}} \tag{11.9}$$

which gives a bandwidth

$$\Delta\nu = \frac{\Delta\omega}{2\pi} = \frac{1}{2\pi}\frac{\omega_0}{Q\sqrt{\Gamma}} \simeq \frac{\beta_3}{\pi} \tag{11.10}$$

We see that the parameter Γ plays a very important role. The mathematical limit for $\Delta\nu$ when Γ tends to zero is infinite. In practice, in order to decrease Γ, one must have a very small T_n and a very large βQ. The increase of T_e increases the bandwidth but decreases the antenna sensitivity. It is interesting to consider under which conditions the bandwidth become equal to ν_0. Putting in (11.10) $\Delta\nu = \nu_0$ and using (7.25) we find the relation

$$\frac{\beta T_e}{Q T_n} = 1 \tag{11.11}$$

For reaching this condition it is necessary to realize an apparatus with the following characteristics:

$Q=10^7$, $T_n=10^{-7}$ K, $\beta=1$, $T_e=1$ K

In order to reach $\beta=1$ it is necessary to use an active transducer (not considered in these notes). However, already with $\beta=10^{-2}$ and

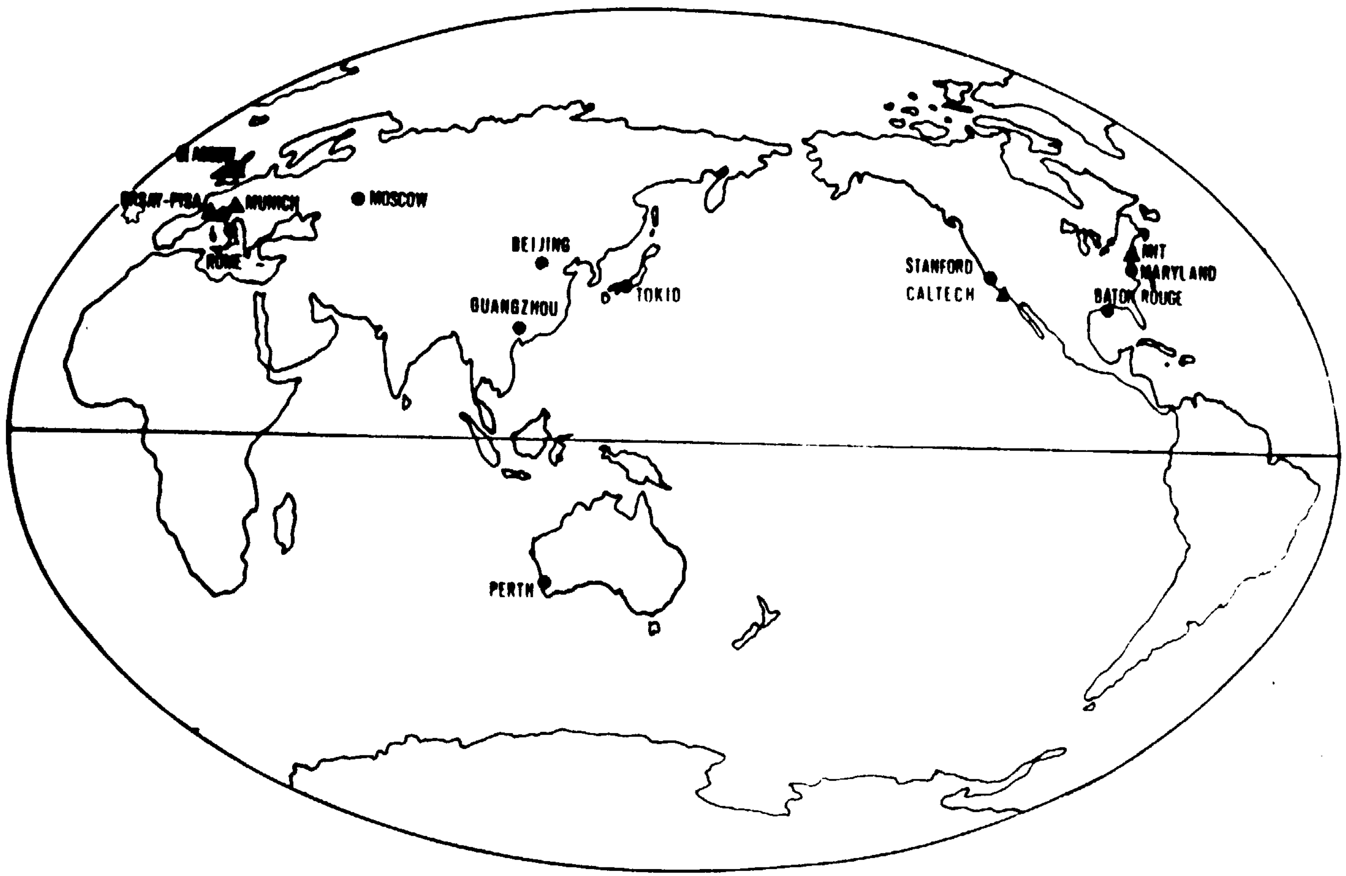

Fig 1 World map of the gravitational wave experiments. Triangles indicate the interferometers. Dots indicate the resonant antennas.

the above values for Q, T_n, T_e it should be possible to obtain

$$\Delta\nu \cong \nu_0/10.$$

12. The gravitational wave experiments in the world

In Fig. 1 we show a world-map of the gravitational wave experiments which are underway. I have indicated with the triangles the places where laser interferometers are being prepared or planned. With the dots I indicate the places where resonant detectors are being developed.

12.1 The laser interferometers

The laser interferometers activity is proceeding in U.S. at the MIT and CALTECH Institutes. Prototypes interferometers have been realized which have reached a sensitivity of the order of

$$(12.1) \qquad \tilde{h} \sim 2\times10^{-19} \frac{1}{\sqrt{Hz}}$$

above 1000 Hz. Below 1000 Hz there is much noise due both to the laser and to the seismic disturbances. Plans have been made for constructing two interferometers with arms that are 5 km long. They plan to reach a sensitivity of the order of

$$(12.2) \qquad \tilde{h} \sim 10^{-23} \frac{1}{\sqrt{Hz}}$$

which, for a bandwidth of 1000 Hz, corresponds to

$$h \sim 3\times10^{-22}$$

In Europe the laser interferometers are being developed in various places. In Germany, at the Max-Planck-Institute in Munich where the work started in 1975, a prototype with arm 30 m long has been constructed with a sensitivity

$$\tilde{h} \sim 10^{-19}/\sqrt{Hz} \tag{12.3}$$

They plan to construct an interferometer with three arms, 3 Km long, along the sides of an equilateral triangle in order to detect gravitational waves of various direction and polarization.

The group at the University of Glasgow has realised a 10 m long interferometer with a sensitivity of the order of

$$\tilde{h} \sim 10^{-19}/\sqrt{Hz} \tag{12.4}$$

Finally a new experiment is being planned as a collaboration between the University of Orsay and the Institute of Nuclear Physics in Pisa. They plan to construct a 3 Km long interferometer with the specific aim to detect gravitational waves at frequencies down to 20 Hz. For this purpose they are developing mechanical filters for eliminating the low frequency seismic noise.

All the various groups have the aim to reach sensitivities of the order of (12.2).

At present, while some noise measurements have been performed, no sistematical use of the developed prototypes has been made for long-term runs.

12.2 The resonant detectors

In U.S this activity takes place at the Universities of Maryland, Louisiana and Stanford. In all places they employ alluminium cylindrical bars, a few tons heavy, cooled with liquid helium. For detecting the mechanical vibrations of the bars they use resonant transducers with dc SQUID or rf SQUID amplifiers. In LSU and Stanford the transducer and the bar constitute a two mode system, in Maryland a three mode system (the transducer itself is a two mode oscillator) with the aim to obtain a larger β value. The LSU and Stanford antennas have operated in 1986 for a few months with a noise effective temperature

$$T_{eff} = 50 \text{ mK} \qquad \text{for Stanford}$$

(12.5)

$$T_{eff} = 140 \text{ mK} \qquad \text{for LSU}$$

At present all these antennas have been cooled to T = 4.2 K but there are plans to reduce the temperature below 100 mK in order to reach

$$T_{eff} < 10^{-6} \text{ K} \tag{12.6}$$

which would allow to obtain

$$h < 10^{-20} \tag{12.7}$$

At Maryland a two room temperature antennas equiped with PZT and FET amplifiers have been in operation during the last 15-20 years. One of them recorded data during the 1987a supernova. These data are now being compared with similar data recorded in Rome.

In Australia an experiment is under way in Perth. It emploies a

Niobium antenna equiped with a parametric transducer. The use of Niobium allows to obtain very large Q values ($Q \simeq 2 \times 10^{8}$).

In Tokyo a resonant antenna is used for detecting the 60 Hz radiation possibly emitted by the Crab pulsar. Experiments have been made at room temperature providing an upper limit

$$h \sim 8.4 \times 10^{-21} \tag{12.8}$$

and there are plans for cooling a large antenna with liquid helium.

In China there are room temperature bars in Peking and Canton and plans are being made for cooling similar bars with liquid helium.

In Moscow they are using a 10 kg silicon crystal bar with a very large Q value ($Q = 2 \times 10^{9}$) and they are developing parametric transducers.

In Europe the Rome group has installed a cryogenic antenna at CERN (Geneva) that operated during 1986 with a sensitivity of the order of

$$T_{eff} \simeq 15 \text{ mK}$$

corresponding to $h \sim 10^{-18}$. Work is in progress to correlate this data with those recorded by LSU and Stanford during the same time. The Rome group has also put in operation during the last four years a room temperature antenna that recorded data at the SN 1987a time. Work is in progress to correlate these data with those recorded at Maryland.

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E.P.

$$\ell_{min} = \sqrt{\frac{\delta P_{EM}}{K \Omega}} \quad \omega$$

$$\Omega = \frac{\omega}{\delta \omega} \qquad K =$$