Solitonic Integrable Perturbations of Parafermionic Theories

Carlos R. Fernández-Pousa [a], Manuel V. Gallas [a],

Timothy J. Hollowood [b] and J. Luis Miramontes [a]

[a] *Departamento de Física de Partículas,*
Facultad de Física,
Universidad de Santiago,
E-15706 Santiago de Compostela, Spain

[b] *Department of Physics,*
University of Wales Swansea,
Singleton Park, Swansea SA2 8PP, U.K.

pousa@gaes.usc.es, gallas@fpaxp1.usc.es

t.hollowood@swansea.ac.uk, and miramont@fpaxp1.usc.es

ABSTRACT

The quantum integrability of a class of massive perturbations of the parafermionic conformal field theories associated to compact Lie groups is established by showing that they have quantum conserved densities of scale dimension 2 and 3. These theories are integrable for any value of a continuous vector coupling constant, and they generalize the perturbation of the minimal parafermionic models by their first thermal operator. The classical equations-of-motion of these perturbed theories are the non-abelian affine Toda equations which admit (charged) soliton solutions whose semi-classical quantization is expected to permit the identification of the exact S-matrix of the theory.

January 1997

1.   Introduction

This paper is concerned with establishing the quantum integrability of certain perturbations of the parafermionic conformal field theories associated to compact simple Lie groups. Classically, these perturbations arise in the context of non-abelian affine Toda equations, and they correspond to the "Homogeneous sine-Gordon theories" of Ref. [1]. At the quantum level, we will show that these theories exhibit conserved densities of (at least) scale dimension 2 and 3, which implies, via the usual folklore, the factorization of their scattering matrices [2] and hence their full quantum integrability.

The non-abelian affine Toda (NAAT) equations are integrable generalizations of the sine-Gordon equations-of-motion for a bosonic field that takes values in some, generally non-abelian, Lie group [3]. In [1], we have determined the class of NAAT equations that can be derived as the classical equations-of-motion of a theory whose action is real, positive-definite, and exhibits a mass-gap. The result is that, besides the usual abelian affine Toda equations with real coupling constant, the NAAT equations give rise to only two series of models, referred to as the Symmetric-Space Sine-Gordon theories and the Homogeneous Sine-Gordon theories. Both are classically integrable and, unlike the abelian affine Toda equations for real values of the coupling constant [4], admit soliton solutions. It is important for us to emphasize that the sine-Gordon theory is the only abelian affine Toda equation with imaginary coupling constant that is included in this class.

In the present paper, we only consider the Homogeneous sine-Gordon theories (HSG), which, at the classical level, are particular examples of the deformed coset models constructed by Park in Ref. [5]. The condition that the theory has a mass-gap constrains the coset to be of the form $G/U(1)^{\times r_g}$ [1], with G a compact simple Lie group of rank r_g in which the field of the theory takes values. At the quantum level, the coupling constant becomes quantized and gives rise to an integer number $k > 0$, known as the "level". It is in this way that the resulting theories are nothing other than perturbations of the G-parafermion theories of level-k [6]. The simplest representative of this class is the HSG associated to the group $SU(2)$, *i.e.* to the coset $SU(2)/U(1)$, whose equation-of-motion is the complex sine-Gordon equation. As shown by Bakas [7], this theory corresponds to the perturbation of the $\mathbb{Z}_k$ parafermions by its first thermal operator, one of the three known integrable perturbations in these models [8]. However, the description of this well-known theory within the framework of NAAT equations has two important advantages that apply

to all the other HSG theories. Firstly, the quantum conserved densities can be obtained through the renormalization of the classical ones. Secondly, since the relevant equations admit soliton solutions, the quantum description of these theories can be elucidated by means of a semi-classical approach, as in the sine-Gordon theory. For the HSG associated to $SU(2)$, the semi-classical analysis leading to the exact spectrum and S-matrix of the theory has been performed in [9], and it agrees with the results of [8]. The study of the soliton spectrum for general HSG theories will be presented elsewhere.

Given a simple compact Lie group G, the construction of a HSG theory involves the choice of two constant elements Λ_+ and Λ_- in some Cartan (maximal abelian) subalgebra of g, the Lie algebra of G. These two elements fix the form of the potential term of the Lagrangian, which is proportional to $\langle \Lambda_+, h^\dagger \Lambda_- h \rangle$, where h is the HSG field taking values in G and $\langle , \rangle$ is the Killing form of g. This potential term specifies both the mass spectrum and the precise form of the perturbation of the G-parafermionic theory. To ensure that the theory has a mass-gap, $\Lambda_\pm$ have to be chosen such that they are regular, which means that they cannot be orthogonal to any root of g; otherwise, their value remains arbitrary. However, different choices of $\Lambda_\pm$ lead to different theories. For instance, the resulting theories are not parity invariant unless $\Lambda_+ = h_0^\dagger \Lambda_- h_0$, where h_0 is the field configuration corresponding to the vacuum [1]. Therefore, $\Lambda_\pm$ have to be considered as continuous (relevant) vector coupling constants of the theory. Moreover, the role of the kinetic term is played by a gauged WZW action associated to the coset $G/U(1)^{\times r_g}$, whose precise form is specified by an automorphism τ of the Cartan subalgebra of g that preserves the Killing form $\langle , \rangle$. Consequently, τ can be (almost) any element of the orthogonal group $O(r_g)$, and hence it provides an additional set of continuous (marginal) coupling constants. Remarkably we shall find that the HSG theories are quantum integrable for any value of $\Lambda_\pm$ and τ. This means that these theories are counter-examples to the generally held belief that integrability is a very special property which requires carefully tuned coupling constants (apart from couplings that can be absorbed into $\hbar$ or an overall mass scale).

The paper is organized as follows. In Section 2, we summarize the main features of the classical HSG theories following [1]. In Section 3, we show that they are classically integrable by exhibiting the existence of an infinite number of classical conserved charges. To be more specific, for each positive integer scale dimension s, there are rank(G) linearly independent conserved densities. Explicit expressions for $s = 2$ and 3 densities are provided in the Appendix. Section 4 is the core of the paper where quantum conserved densities of scale dimension 2 and 3 are constructed. The HSG theories are perturbations of a

coset conformal field theory (CFT) by a relevant primary field, and hence the existence of quantum conserved densities can be investigated by using the method proposed by Zamolodchikov in [10], which has already been used in the context of abelian affine Toda theories [11]. Following the approach of Bais *et al.* [12], the CFT associated to a coset G/H can be described in terms of the Wess-Zumino-Witten (WZW) theory corresponding to the Lie group G. Since, in our case, the underlying ultra-violet CFT is associated to a coset of the form $G/U(1)^{\times r_g}$, the use of this method involves an analysis of operator products in the WZW theory associated to G. Our study is restricted to the first order in perturbation theory; however, at least for values of the level $k \geq$ the dual Coxeter number of G, our results are expected to be exact [10,13]. Finally, in Section 5, we present our conclusions.

2.   The Homogeneous sine-Gordon theories

According to [1], where more details can be found, the construction of the different HSG associated to a given compact Lie group G starts with the choice of two constant elements $\Lambda_{\pm}$ in g, the Lie algebra of G. The condition that the resulting theory has a mass-gap requires $\Lambda_{\pm}$ to be semisimple and regular, which means that their centralizer in g is a Cartan subalgebra. Otherwise, the choice of $\Lambda_{\pm}$ is completely free and therefore they are to be regarded as continuous vector coupling constants of the theory. To comply with the notation of [1], we will refer to the centralizer of Λ_{+} in g and to the corresponding abelian group as $g_0^0 \simeq u(1)^{+r_g}$ and $G_0^0 \simeq U(1)^{\times r_g}$, respectively, where r_g is the rank of G. The HSG is specified by the action

$$S[h, A_{\pm}] = \frac{1}{\beta^2} \left\{ S_{\mathrm{WZW}}[h, A_{\pm}] - \int d^2x \, V(h) \right\}. \tag{2.1}$$

Here, h is a bosonic field that takes values in G, $A_{\pm}$ are (non-dymanical) abelian gauge connections taking values in g_0^0, and $S_{\mathrm{WZW}}[h, A_{\pm}]$ is the gauged Wess-Zumino-Witten action for the coset $G/G_0^0 \simeq G/U(1)^{\times r_g}$,

$$\begin{aligned} S_{\mathrm{WZW}}[h, A_{\pm}] = S_{\mathrm{WZW}}[h] &+ \frac{1}{2\pi} \int d^2x \left(-\langle A_+ , \partial_- h \, h^{\dagger} \rangle \right. \\ &+ \left. \langle \tau(A_-) , h^{\dagger} \, \partial_+ h \rangle + \langle h^{\dagger} \, A_+ \, h , \tau(A_-) \rangle - \langle A_+ , A_- \rangle \right), \end{aligned} \tag{2.2}$$

where $x_{\pm} = t \pm x$ are light-cone variables in $1+1$ Minkowski space. The potential $V(h)$ is

$$V(h) = -\frac{m^2}{2\pi} \langle \Lambda_+ , h^{\dagger} \Lambda_- h \rangle , \tag{2.3}$$

where we have denoted the Killing form on g by $\langle\,,\,\rangle$, normalized such that long roots have square length 2. Finally, $\hbar m$ is a constant with dimensions of mass, and β is a coupling constant that has to be quantized if the quantum theory is to be well defined; namely, $\hbar\beta^2 = 1/k$, where k is a non-negative (dimensionless) integer [14]. The Planck constant is explicitly shown to exhibit that, just as in the sine-Gordon theory, the semi-classical limit is the same as the weak-coupling limit, and that both are recovered when $k \to \infty$. The action (2.1) is invariant with respect to the abelian gauge transformations

$$ h(x,t) \mapsto \mathrm{e}^{\,\alpha}\, h\, \mathrm{e}^{\,-\,\tau(\alpha)}\,, \qquad A_\pm \mapsto A_\pm - \partial_\pm \alpha\,, \tag{2.4} $$

where $\alpha = \alpha(x,t)$ takes values in $g_0^0 \simeq u(1)^{+r_g}$.

It is well known that a coset G/H specifies different conformal field theories associated to the anomaly-free embeddings of H into $G_L \otimes G_R$, the internal chiral symmetry group of the WZW theory corresponding to G, and that all the resulting theories share the same central charge [15]. In our case, the possible anomaly-free embeddings of $G_0^0 \simeq U(1)^{\times r_g}$ into $G_L \otimes G_R$ are specified by τ, which is an arbitrary automorphism of g_0^0 that preserves the restriction of the Killing form $\langle\,,\,\rangle$ to g_0^0. Therefore, since g_0^0 is abelian, τ can be any element of the orthogonal group $O(r_g)$. In particular, τ can always be chosen to be $+I$ or $-I$, which lead to gauge transformations of vector or axial type, respectively, and which are the only possible choices for $G = SU(2)$. However, in order to ensure that the resulting HSG theory has a mass-gap, τ has to be different to the conjugation induced by the field configuration corresponding to the vacuum h_0, which means that $\tau(u) \neq h_0^\dagger u h_0$ for any $u \in g_0^0$ [1]. Nevertheless, the proof of quantum integrability presented in Section 4 is based on the underlying (Kac-Moody) current-algebra structure of the gauged WZW action specified by $G/U(1)^{\times r_g}$, and hence it does not make any reference to τ. Therefore, since the different CFT's associated to the same coset are related by means of deformations induced by marginal operators of the form $\int d^2x\, C_{AB} J^A(z)\, \overline{J^B}(\overline{z})$ involving the Cartan components of the affine chiral currents [16], the automorphism τ provides an additional set of continuous (marginal) coupling constants of the HSG theories.

At the quantum level, the HSG theories can be described as a perturbed CFT of the form

$$ S = S_{\mathrm{CFT}} + \frac{m^2}{2\pi\beta^2} \int d^2x\, \Phi(x,t)\,. \tag{2.5} $$

Here, S_{CFT} is the action of the CFT associated to the coset $G/U(1)^{\times r_g}$ at level k, which is nothing else than the theory of level-k G-parafermions [6], whose central charge is

$$ C_{\mathrm{CFT}}(G,k) = \frac{k \dim(G)}{k + h_g^\vee} - \mathrm{rank}(G)\,, \tag{2.6} $$

$-\,4\,-$

where $h_g^\vee$ is the dual Coxeter number of G.

Following the work of Bais *et al.* [12], the generators of the operator algebra of the coset CFT can be realized as a subset of the generators of the operator algebra of the underlying Wess-Zumino-Witten model associated to G. Those generators include the chiral currents $J^a(z)$ and $\overline{J^a}(\overline{z})$ [17] that satisfy the operator product expansion (OPE)

$$J^a(z)\, J^b(w) \;=\; -\frac{\hbar^2 k}{(z-w)^2}\, \delta^{ab} \;+\; \hbar\, f^{abc}\, \frac{J^c(w)}{z-w} \;+\; \cdots . \tag{2.7}$$

In this paper, we will follow the conventions of [12,18], and we have introduced a notation reminiscent of euclidean space: $z \equiv x_-$ and $\overline{z} \equiv x_+$. The WZW field $h = h(z,\overline{z})$ is a spinless primary field whose conformal dimension

$$\Delta_h \;=\; \overline{\Delta}_h \;=\; \frac{c_h/2}{k + h_g^\vee} \;; \tag{2.8}$$

depends on the chosen representation of G through c_h, the value of the quadratic Casimir of G, $t^a t^a = -c_h\, I$. The relation between the WZW field and the chiral currents is summarized by the "null-equation"

$$\hbar\left(k + h_g^\vee\right) \partial h \;=\; \left(J^a\, t^a h\right), \tag{2.9}$$

where $(AB)(z)$ is the normal ordered product of two operators $A(z)$ and $B(z)$ [18].

In the last equations, the t^a's provide an antihermitian basis for the (compact) Lie algebra g normalized such that $\langle t^a, t^b \rangle = -\delta^{a,b}$. We will distinguish the generators of the Cartan subalgebra g_0^0 as $t^A, t^B, \ldots$ with $A, B, \ldots = 1, \ldots, r_g$, while the other generators will be denoted as $t^i, t^j, t^k, \ldots$. The standard realization of this basis by means of a Chevalley basis for the complexification of g is presented in the appendix.

Then, the operator algebra of the $G/U(1)^{\times r_g}$ coset CFT is generated by those operators whose OPE with the currents $J^A(z)$ and $\overline{J^A}(\overline{z})$ is regular [12]. This condition is the quantum version of gauge invariance with respect to (2.4), as will be shown in the next section.

In eq. (2.5), $\Phi = \langle \Lambda_+, h^\dagger \Lambda_- h \rangle$ will be understood as a matrix element of the WZW field h taken in the adjoint representation, $\Phi = \langle \Lambda_-, h^{\mathrm{ad}} \cdot \Lambda_+ \rangle$. Recall that, with our normalization of $\langle , \rangle$, the quadratic Casimir in the adjoint representation is $c_v = f^{abc} f^{abc} = 2h_g^\vee$. Therefore, the perturbation is given by a spinless primary field with conformal dimension

$$\Delta_\Phi \;=\; \overline{\Delta}_\Phi \;=\; \frac{h_g^\vee}{k + h_g^\vee} \;<\; 1, \tag{2.10}$$

which shows that the perturbation is relevant for any value of k. Correspondingly, m is a dimensionful coupling constant with positive scale dimension

$$[m] = \frac{k}{k + h_g^\vee} > 0 \,. \tag{2.11}$$

All this is equivalent to the statement that the resulting field theory is "super-renormalizable", which means that only a finite number of couterterms are required to renormalize the theory. Moreover, we will assume that $2\,\Delta_\Phi \leq 1$ or, equivalently, $k \geq h_g^\vee$, and, consequently, that no counterterm is actually needed [10,13].

It is worth considering the semi-classical and/or weak coupling $k \to \infty$ limit of the HSG theories. Then, $C_{\mathrm{CFT}}(G, k) \to \dim(G) - \mathrm{rank}(G)$ and $[m] \to 1$, which shows that, in this limit, the theory consists of $\dim(G) - \mathrm{rank}(G)$ free massive bosonic particles whose mass is proportional to $\hbar m$. They are just the "fundamental particles" of the classical theory, which are associated to the roots of the Lie algebra g [1].

3.   Classical integrability

At the classical level, the integrability of the HSG theories is manifested in the zero-curvature form of their equations-of-motion

$$\left[\partial_- + m\Lambda_- - \partial_- h\, h^\dagger + h\tau(A_-)h^\dagger \,,\, \partial_+ + mh\Lambda_+ h^\dagger + A_+\right] = 0 \,, \tag{3.1}$$

along with the constraints that arise from the equations of motion of the gauge field:

$$\begin{aligned}
\langle t^A \,,\, h^\dagger \partial_+ h + h^\dagger A_+ h - \tau(A_+) \rangle &= 0 \,, \\
\langle t^A \,,\, -\partial_- h h^\dagger + h\tau(A_-)h^\dagger - A_- \rangle &= 0 \,,
\end{aligned} \tag{3.2}$$

for all $A = 1, \ldots, r_g$. Eq. (3.1) implies that there exist an infinite number of conserved densities that can be obtained by applying the well known Drinfel'd-Sokolov construction [19]. This requires to express eq. (3.1) as a zero-curvature equation associated to the "homogeneous gradation" of the loop algebra (see the last reference in [19])

$$\mathcal{L}(g) = \mathbb{C}[\lambda, \lambda^{-1}] \otimes g = \bigoplus_{j \in \mathbb{Z}} \mathcal{L}(g)_j \,, \quad \text{where} \quad \mathcal{L}(g)_j = \lambda^j \otimes g \,, \tag{3.3}$$

which is equivalent to the introduction of a "spectral parameter" λ. Since the equations of motion remain unchanged if $\Lambda_\pm \mapsto \lambda^{\mp 1} \otimes \Lambda_\pm$, eq. (3.1) can actually be understood as a zero-curvature equation associated to $\mathcal{L}(g)$ and to the Lax operator $L = \partial_- + \Lambda + q$, where

$$\Lambda \equiv m\,\lambda \otimes \Lambda_- \in \mathcal{L}(g)_1 \quad \text{and} \quad q = -\partial_- h\, h^\dagger + h\tau(A_-)h^\dagger \in \mathcal{L}(g)_0 \,. \tag{3.4}$$

In this case, the Drinfel'd-Sokolov construction goes as follows. First, there is some function y of the form

$$y = \sum_{n>0} y^i(n)\, \lambda^{-n} \otimes t^i \in \mathcal{L}(g)_{<0} \tag{3.5}$$

that "abelianizes" the Lax operator, *i.e.*,

$$e^y \left(\partial_- + \Lambda + q\right) e^{-y} = \partial_- + \Lambda + H\,,$$
$$e^y \left(\partial_+ + m\, h\, (\lambda^{-1} \otimes \Lambda_+)\, h^\dagger + A_+\right) e^{-y} = \partial_+ + \overline{H}\,, \tag{3.6}$$

where H and $\overline{H}$ take values in the centralizer of Λ in $\mathcal{L}(g)_{\leq 0}$:

$$H = \sum_{s\geq 1} I_s^{(0)A}\, \lambda^{-s+1} \otimes t^A \quad \text{and} \quad \overline{H} = \sum_{s\geq 1} \overline{I}_s^{(0)A}\, \lambda^{-s+1} \otimes t^A\,. \tag{3.7}$$

Moreover, y, H, and $\overline{H}$ are local functionals of q, and $I_s^{(0)A}$ and $\overline{I}_s^{(0)A}$ have scale dimension s with respect to the scale transformations $x_\pm \to x_\pm/\rho$. Then, the zero curvature equation (3.1) simplifies to

$$\partial_+ I_s^{(0)A} = \partial_- \overline{I}_s^{(0)A}\,, \tag{3.8}$$

which shows that, for each scale dimension $s \geq 1$, there are $\text{rank}(G)$ classically conserved densities $(I_s^{(0)A}, \overline{I}_s^{(0)A})$.

The transformation of the conserved densities with respect to the gauge transformations (2.4) can be easily derived by realizing that the Lax operator transforms just by conjugation: $L \mapsto e^\alpha L e^{-\alpha}$ or, equivalently,[1]

$$q \mapsto e^\alpha q e^{-\alpha} - \partial_- \alpha\,. \tag{3.9}$$

Then, $e^y \mapsto e^\alpha e^y e^{-\alpha}$, $H \mapsto H - \partial_- \alpha$, and $\overline{H} \mapsto \overline{H} - \partial_+ \alpha$, which means that all the conserved densities with scale dimension $s \geq 2$ are gauge invariant, while the densities with $s = 1$ transform as

$$I_1^{(0)A} \mapsto I_1^{(0)A} - \partial_- \alpha^A \quad \text{and} \quad \overline{I}_1^{(0)A} \mapsto \overline{I}_1^{(0)A} - \partial_+ \alpha^A\,. \tag{3.10}$$

Actually, using eqs. (3.2) and (3.6), one can easily show that $I_1^{(0)A} t^A = A_-$ and $\overline{I}_1^{(0)A} t^A = A_+$, in agreement with eqs. (2.4) and (3.10).

The global symmetries that give rise to the $s = 1$ conserved densities and the associated gauge invariant conserved charges have been discussed in [1]. In the following, we will only be interested in the gauge invariant densities ($s \geq 2$), and we provide explicit expressions for the classically conserved densities with scale dimension $s = 2$ and 3 in the appendix.

---

[1] Notice that the gauge transformation of the Lax operator is independent of τ, *i.e.*, it does not depend on the the precise form of the group of gauge transformations (2.4) of the theory. Actually, this is the reason why the proof of integrability does not make any reference to the automorphism τ.

4.   Quantum conserved densities

In this section we examine the fate of the classical conserved quantities of spin 2 and 3 in the quantum theory. The proof that, after an appropriate renormalization, the r_g currents of spin 2 and 3 remain conserved is quite involved and uses conformal perturbation theory.

Since the quantum HSG theories can be described as perturbed conformal field theories, the existence of quantum conserved charges can be investigated by using the method of Zamolodchikov [10] (see also [13]). In the presence of the perturbation (2.5) any local chiral field $\mathcal{I}(z)$, which in the unperturbed CFT satisfies $\overline{\partial}\mathcal{I}(z) = 0$, adquires a $\overline{z}$ dependence:

$$\overline{\partial}\mathcal{I}(z,\overline{z}) = -k\,m^2\,\oint_z \frac{dw}{2\pi i}\,\Phi(w,\overline{z})\,\mathcal{I}(z)\,. \tag{4.1}$$

This is the contribution at the lowest order in perturbation theory; however, if $k \geq h_g^\vee$, or equivalently $2\Delta_\Phi \leq 1$, this expression is expected to be exact [13] (see the comments below eq. (2.11)).

Consequently, in the perturbed theory, $\mathcal{I}(z,\overline{z})$ will be a conserved density if the right-hand-side of (4.1) is a total ∂ derivative, i.e., if (4.1) can be written as $\overline{\partial}\mathcal{I}(z,\overline{z}) = \partial\overline{\mathcal{I}}(z,\overline{z})$ where $\overline{\mathcal{I}}(z,\overline{z})$ is some field of the original CFT. For this to be true, it is enough that the residue of the simple pole in the OPE between $\mathcal{I}(z)$ and $\Phi(w,\overline{z})$ is a total derivative,

$$\Phi(w,\overline{z})\,\mathcal{I}(z) = \sum_{n>1} \frac{\{\Phi\mathcal{I}\}_n}{(w-z)^n} + \frac{1}{w-z}\,\partial\overline{\mathcal{I}}(z,\overline{z}) + \cdots\,. \tag{4.2}$$

Recall that the residues of the simple pole in the OPE's $\Phi(w,\overline{z})\,\mathcal{I}(z)$ and $\mathcal{I}(z)\,\Phi(w,\overline{w})$ differ only in a total ∂ derivative and, in fact, we will always consider the latter because its expression is usually simpler.

Since the classical theory exhibits gauge invariant conserved densities for any scale dimension $s \geq 2$, we will study local composite operators $\mathcal{I}_s$ of the chiral currents $J^a(z)$ and their derivatives of conformal spin $s \geq 2$. However, since the unperturbed CFT is a coset CFT, these operators have to be constrained by the condition of having a regular OPE with the currents $J^A(z)$, for $A = 1,\ldots,r_g$. Actually, this condition is the quantum version of gauge invariance in the classical theory, as shown by the OPE's

$$\begin{aligned} J^A(z)\,J^B(w) &= -\hbar^2\,k\frac{\delta^{AB}}{(z-w)^2} + \cdots\,, \\ J^A(z)\,J^i(w) &= \hbar\,\frac{f^{Ai\,\overline{\imath}}\,J^{\overline{\imath}}(w)}{z-w} + \cdots\,, \end{aligned} \tag{4.3}$$

compared with the infinitesimal gauge transformation (3.9) of the components of q,

$$\delta q^A(z,\overline{z}) \; = \; -\partial \alpha^A(z,\overline{z}) \; = \; - \oint_z \frac{dw}{2\pi i} \, \alpha^B(w,\overline{z}) \, \frac{\delta^{AB}}{(w-z)^2}$$

$$\delta q^i(z,\overline{z}) \; = \; - \, \alpha^A(z,\overline{z}) \, f^{Ai\,\overline{\imath}} \, q^{\overline{\imath}}(z,\overline{z}) \; = \; - \oint_z \frac{dw}{2\pi i} \, \alpha^A(w,\overline{z}) \, \frac{f^{Ai\,\overline{\imath}} \, q^{\overline{\imath}}(w,\overline{z})}{w-z} \, . \tag{4.4}$$

At this point it is convenient to recall that classical expressions can be recovered from quantum expressions through (see eqs.(2.9) and (3.4))

$$J^a t^a \; = \; -(\hbar k)\, q \, , \quad (\hbar k) \to \frac{1}{\beta^2} \, , \quad \text{and} \quad k \to \infty \, . \tag{4.5}$$

4.1 Quantum spin-2 conserved densities

We consider a generic (normal ordered) spin-2 operator

$$\mathcal{I}_2(z) \; = \; \mathcal{D}_{ab} \left(J^a J^b \right)(z) \, , \tag{4.6}$$

with a c-number tensor $\mathcal{D}_{ab} = \mathcal{D}_{ba}$. The condition that the OPE $J^A(z)\,\mathcal{I}_2(w)$ is regular implies that the only non vanishing components of $\mathcal{D}_{ab}$ are $\mathcal{D}_{\alpha\alpha} = \mathcal{D}_{\overline{\alpha}\,\overline{\alpha}}$ for any positive root $\boldsymbol{\alpha}$,[2] and

$$\mathcal{D}_{AB} \; = \; -\frac{1}{k} \sum_{\boldsymbol{\alpha}>0} \mathcal{D}_{\alpha\alpha} \, \alpha^A \, \alpha^B \, , \tag{4.7}$$

which shows that $\mathcal{D}_{AB}$ vanishes in the $k \to \infty$ limit and therefore is a quantum correction.

To investigate the condition that $\mathcal{I}_2$ is a classical density, we use the OPE

$$J^a(z)\, h(w,\overline{w}) \; = \; - \, \hbar \, \frac{t^a \, h(w,\overline{w})}{z-w} + \cdots \, , \tag{4.8}$$

which is satisfied in an arbitrary representation of G. In particular, by considering the adjoint representation, eqs. (2.9) and (4.8) lead to

$$\hbar \left(k + h_g^{\vee} \right) \partial \, \langle v, P \rangle \; = \; \left(J^a \, \langle [v, t^a] \, , \, P \rangle \right) , \tag{4.9}$$

$$J^a(z) \, \langle v \, , \, P(w,\overline{w}) \rangle \; = \; - \, \hbar \, \frac{\langle [v, t^a] \, , \, P(w,\overline{w}) \rangle}{z-w} + \cdots \, , \tag{4.10}$$

where v is an arbitrary element in g, $P = \left(h\Lambda_+ h^{\dagger} \right) \equiv h^{\mathrm{ad}} \cdot \Lambda_+$, and the perturbing operator is $\Phi = \langle \Lambda_-, P \rangle$. Using this, it is easy to show that the residue of the simple pole in the OPE $\mathcal{I}_2(z)\Phi(w,\overline{w})$ is

$$\mathrm{Res}\Big(\mathcal{I}_2(z)\Phi(w,\overline{w})\Big) \; = \; - \, 2\,\hbar \, \left(J^a \, \langle \mathcal{D}_{ab} \, [\Lambda_-, t^b] \, , \, P \rangle \right)(w,\overline{w}) \, . \tag{4.11}$$

---

[2] $\mathcal{D}_{\alpha\alpha}$, $\mathcal{D}_{\overline{\alpha}\,\overline{\alpha}}$, and $\mathcal{D}_{AB}$ are components of D_{ab} with respect to the basis (A.2), which will be used throughout this Section.

Therefore, using (4.9), $\mathcal{I}_2$ gives rise to a quantum conserved density if $\mathcal{D}_{ab}$ is chosen such that [3]

$$(\hbar k)^2 \, \mathcal{D}_{ab} \, [\Lambda_-, t^b] \;=\; \frac{1}{2} \, [\boldsymbol{\mu} \cdot \boldsymbol{t}, t^a] \,, \tag{4.12}$$

for any r_g-component vector $\boldsymbol{\mu}$, which leads to

$$(\hbar k)^2 \, \mathcal{D}_{\alpha\alpha}(\boldsymbol{\mu}) \;=\; (\hbar k)^2 \, \mathcal{D}_{\overline{\alpha}\,\overline{\alpha}}(\boldsymbol{\mu}) \;=\; \frac{1}{2} \frac{\boldsymbol{\mu} \cdot \boldsymbol{\alpha}}{\boldsymbol{\lambda} \cdot \boldsymbol{\alpha}} \;=\; m \, D_{\alpha\alpha}^{(0)}(\boldsymbol{\mu}) \,,$$

$$(\hbar k)^2 \, \mathcal{D}_{AB}(\boldsymbol{\mu}) \;=\; -\frac{1}{2k} \sum_{\boldsymbol{\alpha} > 0} \frac{\boldsymbol{\mu} \cdot \boldsymbol{\alpha}}{\boldsymbol{\lambda} \cdot \boldsymbol{\alpha}} \, \alpha^A \, \alpha^B \,, \tag{4.13}$$

where $D_{\alpha\alpha}^{(0)} = D_{\overline{\alpha}\,\overline{\alpha}}^{(0)}$ gives the classical conserved density, $\Lambda_- = \boldsymbol{\lambda} \cdot \boldsymbol{t}$ (see the appendix), and the value of $\mathcal{D}_{AB}$ follows from eq. (4.7).

Therefore, we conclude that there exists $\mathrm{rank}(G)$ quantum conserved densities of scale dimension $s = 2$ labelled by the arbitrary vector $\boldsymbol{\mu}$:

$$(\hbar k)^2 \, \mathcal{I}_2(\boldsymbol{\mu}) \;=\; \frac{1}{2} \sum_{\boldsymbol{\alpha} > 0} \frac{\boldsymbol{\mu} \cdot \boldsymbol{\alpha}}{\boldsymbol{\lambda} \cdot \boldsymbol{\alpha}} \left[\left(J^\alpha J^\alpha + J^{\overline{\alpha}} J^{\overline{\alpha}} \right) - \frac{1}{k} \left(J^A J^B \right) \alpha^A \alpha^B \right] \,,$$

$$(\hbar k)^2 \, \overline{\mathcal{I}}_2(\boldsymbol{\mu}) \;=\; -\, m^2 \, (\hbar k)^2 \, \langle \boldsymbol{\mu} \cdot \boldsymbol{t} \,, P \rangle \,. \tag{4.14}$$

The relation between quantum and classical conserved densities of scale dimension 2 is obtained by considering (4.5),

$$\mathcal{I}_2(\boldsymbol{\mu}) \;=\; m \, \boldsymbol{\mu} \cdot \left(\boldsymbol{I}_2^{(0)} + \frac{1}{k} \, \boldsymbol{I}_2^{(1)} \right) \,,$$

$$\overline{\mathcal{I}}_2(\boldsymbol{\mu}) \;=\; m \, \boldsymbol{\mu} \cdot \overline{\boldsymbol{I}}_2^{(0)} \,. \tag{4.15}$$

Finally, let us exhibit the relation between the quantum conserved density $\mathcal{I}_2(\boldsymbol{\lambda})$ and the energy-momentum tensor. First, recall the algebraic relation

$$\sum_{\alpha > 0} \alpha^A \alpha^B \;=\; \frac{c_v}{2} \, \delta^{A,B} \;=\; h_g^{\vee} \, \delta^{A,B} \,. \tag{4.16}$$

This implies that eq (4.14) becomes

$$(\hbar k)^2 \, \mathcal{I}_2(\boldsymbol{\lambda}) \;=\; -(k + h_g^{\vee}) \left\{ \frac{-1}{2(k + h_g^{\vee})} \left(J^a J^a \right) + \frac{1}{2k} \left(J^A J^A \right) \right\} \;\equiv\; -(k + h_g^{\vee}) \, T_{z,z} \,, \tag{4.17}$$

in agreement with the Sugawara construction of the energy-momentum tensor of the coset CFT, and, consequently,

$$(\hbar k)^2 \, \overline{\mathcal{I}}_2(\boldsymbol{\lambda}) \;=\; -\, m^2 \, (\hbar k)^2 \, \langle \Lambda_- \,, P \rangle \;\equiv\; -(k + h_g^{\vee}) \, T_{z,\overline{z}} \,. \tag{4.18}$$

---

[3] The normalization is chosen to simplify the comparison with the classical results. Explicit factors of $\hbar k$ arise because of the relation $J^a = -(\hbar k) \, q^a$ (see eq. (4.5)).

4.2   Quantum spin-3 conserved densities

Let us consider a (normal ordered) spin-3 operator of the form

$$\mathcal{I}_3(z) \;=\; \mathcal{P}_{ijk}\left(J^i\left(J^j J^k\right)\right)(z) \;+\; \mathcal{P}_{Aij}\left(J^A\left(J^i J^j\right)\right)(z) \;+\; \mathcal{Q}_{ij}\left(J^i \partial J^j\right)(z)$$

$$+\; \mathcal{P}_{ABC}\left(J^A\left(J^B J^C\right)\right)(z) \;+\; \mathcal{Q}_{AB}\left(J^A \partial J^B\right)(z) \;+\; \mathcal{R}_A\,\partial^2 J^A(z)\,, \tag{4.19}$$

where $\mathcal{P}_{ijk}$ and $\mathcal{P}_{ABC}$ are totally symmetric, $\mathcal{P}_{Aij} = \mathcal{P}_{Aji}$, and $\mathcal{Q}_{ij}$ and $\mathcal{Q}_{AB}$ are antisymmetric c-number tensors.

The requirement of gauge invariance, $i.e.$ that the OPE's $J^A(z)\,\mathcal{I}_3(w)$ are regular implies the following constraints. Firstly,

$$\mathcal{P}_{ABC} \;=\; -\,\frac{1}{6k}\left(f^{Aik}\,f^{Bjk}\,\mathcal{P}_{Cij} \;+\; f^{Aik}\,f^{Cjk}\,\mathcal{P}_{Bij}\right)\,,$$

$$\mathcal{Q}_{AB} \;=\; -\,\frac{\hbar}{2k}\,f^{Ail}\,f^{jlm}\,f^{Bkm}\,\mathcal{P}_{ijk}\,, \quad \text{and} \quad \mathcal{R}_A \;=\; \frac{\hbar}{6}\,f^{Aij}\,\mathcal{Q}_{ij}\,, \tag{4.20}$$

which, taking into account the relation between quantum and classical variables, $J^a = -(\hbar k)q^a$, can be written as

$$[-(\hbar k)^3 \mathcal{P}_{ABC}] \;=\; -\,\frac{1}{6k}\left(f^{Aik}\,f^{Bjk}\,[-(\hbar k)^3 \mathcal{P}_{Cij}] \;+\; f^{Aik}\,f^{Cjk}\,[-(\hbar k)^3 \mathcal{P}_{Bij}]\right)\,,$$

$$[(\hbar k)^2 \mathcal{Q}_{AB}] \;=\; +\,\frac{1}{2k^2}\,f^{Ail}\,f^{jlm}\,f^{Bkm}\,[-(\hbar k)^3 \mathcal{P}_{ijk}]\,,$$

$$[-(\hbar k)\mathcal{R}_A] \;=\; -\,\frac{1}{6k}\,f^{Aij}\,[(\hbar k)^2 \mathcal{Q}_{ij}]\,, \tag{4.21}$$

which shows that $\mathcal{P}_{ABC}$, $\mathcal{Q}_{AB}$, and $\mathcal{R}_A$ arise as quantum corrections. Secondly, the only non-vanishing components of $\mathcal{P}_{Aij}$ and $\mathcal{Q}_{ij}$ are $\mathcal{P}_{A\alpha\alpha} = \mathcal{P}_{A\overline{\alpha}\,\overline{\alpha}}$ and $\mathcal{Q}_{\alpha\overline{\alpha}} = -\mathcal{Q}_{\overline{\alpha}\alpha}$, for any positive root $\boldsymbol{\alpha}$. Finally,

$$\alpha^A\,\mathcal{Q}_{\alpha\overline{\alpha}} \;+\; (\hbar k)\,\mathcal{P}_{A\alpha\alpha} \;=\; -\,3\,\hbar\,f^{Aik}\,f^{jk\alpha}\,\mathcal{P}_{ij\alpha}\,, \tag{4.22}$$

as can be checked by means of an involved but straightforward calculation.

To study the quantum conservation of (4.19), it is more convenient to write the spin-3 density as

$$\mathcal{I}_3(z) \;=\; \widehat{\mathcal{P}}_{abc}\left(J^a\left(J^b J^c\right)\right)(z) \;+\; \widehat{\mathcal{Q}}_{ab}\left(J^a \partial J^b\right)(z) \;+\; \widehat{\mathcal{R}}_A\,\partial^2 J^A(z)\,, \tag{4.23}$$

where $\widehat{\mathcal{P}}_{abc}$ is totally symmetric and $\widehat{\mathcal{Q}}_{ab}$ is antisymmetric, and we have the following identities (recall that we are using the conventions of [18] for normal ordered products)

$$\mathcal{P}_{ijk} \;=\; \widehat{\mathcal{P}}_{ijk}\,, \qquad \mathcal{P}_{Aij} \;=\; 3\,\widehat{\mathcal{P}}_{Aij}\,,$$

$$\mathcal{Q}_{ij} \;=\; \widehat{\mathcal{Q}}_{ij} \;-\; \frac{3\hbar}{2}\left(f^{Aik}\,\widehat{\mathcal{P}}_{Ajk} \;-\; f^{Ajk}\,\widehat{\mathcal{P}}_{Aik}\right)\,. \tag{4.24}$$

In this way, it is easy to show that

$$\text{Res}\Big(\mathcal{I}_3(z)\Phi(w,\overline{w})\Big) \;=\; \hbar\Big\{\big(J^a\,(J^b\,\langle X_{ab}\,,\,P\rangle)\big) \;+\; \big(\partial J^a\,\langle \Omega_a\,,\,P\rangle\big)\Big\}(w,\overline{w})\,, \tag{4.25}$$

where the algebraic coefficients are

$$\begin{aligned}
X_{ab} &\;=\; -\,3\,\widehat{\mathcal{P}}_{abc}\,[\Lambda_-\,,\,t^c]\,, \\
\Omega_a &\;=\; 2\,\widehat{\mathcal{Q}}_{ab}\,[\Lambda_-\,,\,t^b] \;+\; 3\,\hbar\,\widehat{\mathcal{P}}_{abc}\,[[\Lambda_-\,,\,t^b]\,,\,t^c]\,.
\end{aligned} \tag{4.26}$$

Therefore, taking into account (4.9), the condition to have a spin-3 quantum conserved density is that

$$\begin{aligned}
X_{ab} &\;=\; [F_a\,,\,t^b] \;+\; [F_b\,,\,t^a]\,, \\
\Omega_a &\;=\; 2\,\hbar\,(k+h_g^\vee)\,F_a \;+\; \hbar\,f^{abc}\,[F_b\,,\,t^c]\,,
\end{aligned} \tag{4.27}$$

for some elements F_a in g, which ensures that

$$\text{Res}\Big(\mathcal{I}_3(z)\Phi(w,\overline{w})\Big) \;=\; 2\,\hbar^2\,(k+h_g^\vee)\,\partial\big(J^a\,\langle F_a\,,\,P\rangle\big)(w,\overline{w})\,. \tag{4.28}$$

Eqs. (4.27) can be solved for $\widehat{\mathcal{P}}_{abc}$ and $\widehat{\mathcal{Q}}_{ab}$ as functions of F_a:

$$3\,\widehat{\mathcal{P}}_{abc}\,[\Lambda_-\,,\,t^c] \;=\; -\,[F_a\,,\,t^b] \;-\; [F_b\,,\,t^a]\,, \tag{4.29}$$

$$2\,\widehat{\mathcal{Q}}_{ab}\,[\Lambda_-\,,\,t^b] \;=\; 2(\hbar k)\,F_a \;+\; \hbar\Big\{[F_b\,,\,[t^a\,,\,t^b]] \;+\; [t^b\,,\,[t^a\,,\,F_b]]\Big\}\,. \tag{4.30}$$

The condition to have a classical conserved density of scale dimension 3 is recovered by removing the last term in eq. (4.30), which is an explicit quantum correction. Then, the resulting classical equations have $\text{rank}(G)$ solutions

$$(\hbar k)^3\,F_a^{(0)} \;=\; m^2\,Q_{ab}^{(0)}(\boldsymbol{\mu})\,[\Lambda_-,t^b]\,, \tag{4.31}$$

which give rise to the $\text{rank}(G)$ spin-3 classically conserved densities presented in the appendix. Notice that, for these classical solutions, $F_A \equiv F_A^{(0)} = 0$.

The quantum corrections to these classical solutions are induced by the last term in eq. (4.30). In fact, this equation with $a = A$ becomes a constraint for F_a

$$2(\hbar k)\,F_A \;+\; \hbar\Big\{[F_b\,,\,[t^A\,,\,t^b]] \;+\; [t^b\,,\,[t^A\,,\,F_b]]\Big\} \;=\; 0\,, \tag{4.32}$$

which admits the exact solution

$$F_i \;=\; F_i^{(0)}\,, \qquad F_A \;=\; -\,\frac{1}{2k}\,\Big\{[F_i^{(0)}\,,\,[t^A\,,\,t^i]] \;+\; [t^i\,,\,[t^A\,,\,F_i^{(0)}]]\Big\}\,, \tag{4.33}$$

which, therefore, can be understood as the "renormalization" of the classical solution $F_a = F_a^{(0)}$.

Finally, eqs. (4.29), (4.30), and (4.24) provide the precise value of the tensors that specify the conserved spin-3 density $\mathcal{I}_3$, namely

$$-(\hbar k)^3 \, \mathcal{P}_{ijk} \;=\; m^2 \, P^{(0)}_{ijk}(\boldsymbol{\mu}) \,, \tag{4.34}$$

$$-(\hbar k)^3 \, \mathcal{P}_{Aij} \;=\; m^2 \left(P^{(0)}_{Aij}(\boldsymbol{\mu}) + \frac{1}{k} \, P^{(1)}_{Aij}(\boldsymbol{\mu}) \right) \,, \tag{4.35}$$

$$(\hbar k)^2 \, \mathcal{Q}_{ij} \;=\; m^2 \left(Q^{(0)}_{ij}(\boldsymbol{\mu}) + \frac{1}{k} \, Q^{(1)}_{ij}(\boldsymbol{\mu}) \right) \,, \tag{4.36}$$

where the classical contributions can be found in the appendix. The quantum corrections are given by

$$P^{(1)}_{A\alpha\alpha}(\boldsymbol{\mu}) \;=\; 2 \sum_{\boldsymbol{\beta}>0} \frac{(\boldsymbol{\lambda}\cdot\boldsymbol{\beta})\,(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})}{(\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})} \, \beta^A \, Q^{(0)}_{\beta\,\overline{\beta}}(\boldsymbol{\mu}) \,, \tag{4.37}$$

$$Q^{(1)}_{\alpha\,\overline{\alpha}}(\boldsymbol{\mu}) \;=\; \alpha^A \, P^{(0)}_{A\alpha\alpha} + \sum_{\boldsymbol{\beta}>0} \frac{(\boldsymbol{\lambda}\cdot\boldsymbol{\beta})}{(\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})} \, D(\boldsymbol{\beta},\boldsymbol{\alpha}) \, Q^{(0)}_{\beta\,\overline{\beta}}(\boldsymbol{\mu}) \,, \tag{4.38}$$

and we have introduced the notation

$$D(\boldsymbol{\beta},\boldsymbol{\alpha}) \;=\; \sum_i \left(f^{\alpha\beta i} \, f^{\alpha\beta i} + f^{\alpha\overline{\beta}i} \, f^{\alpha\overline{\beta}i} \right) \,, \tag{4.39}$$

such that

$$[t^\beta, [t^\alpha, t^\beta]] + [t^{\overline{\beta}}, [t^\alpha, t^{\overline{\beta}}]] \;=\; D(\boldsymbol{\beta},\boldsymbol{\alpha}) \, t^\alpha \,. \tag{4.40}$$

By means of a tedious but straightforward calculation, it can be shown that the coefficients in eqs. (4.34)-(4.36) satisfy the constraints (4.22), as required to ensure that $\mathcal{I}_3$ is a generator of the coset operator algebra. Correspondingly, eq. (4.28) leads to

$$\begin{aligned} \overline{\mathcal{I}}_3(\boldsymbol{\mu}) &= -(\hbar k)\, m^2 \left(J^a \langle \widetilde{F}_a \,,\, P \rangle \right), \\ \widetilde{F}_a &= -2\hbar\,(k+h_g^\vee)\, F_a + \widehat{\mathcal{Q}}_{ab}\,[\Lambda_-\,,\,t^b] + 3\hbar\,\widehat{\mathcal{P}}_{abc}\,[[\Lambda_-\,,\,t^b]\,,\,t^c] \,. \end{aligned} \tag{4.41}$$

Therefore, for any simple compact Lie group G, we conclude that the classically conserved densities give rise to precisely $\mathrm{rank}(G)$ quantum conserved densities

$$\begin{aligned} \mathcal{I}_3(\boldsymbol{\mu}) &= m^2\, \boldsymbol{\mu}\cdot\left(\boldsymbol{I}_3^{(0)} + \frac{1}{k}\,\boldsymbol{I}_3^{(1)} + \frac{1}{k^2}\,\boldsymbol{I}_3^{(2)} \right), \\ \overline{\mathcal{I}}_3(\boldsymbol{\mu}) &= m^2\, \boldsymbol{\mu}\cdot\left(\overline{\boldsymbol{I}}_3^{(0)} + \frac{1}{k}\,\overline{\boldsymbol{I}}_3^{(1)} + \frac{1}{k^2}\,\overline{\boldsymbol{I}}_3^{(2)} \right), \end{aligned} \tag{4.42}$$

of scale dimension 3 (see eqs. (4.21), and (4.34)-(4.36)).

5.   Conclusions and Discussion

We have shown that the Homogeneous sine-Gordon theories of [1] exhibit r_g quantum conserved densities of scale dimension 2 and 3, which implies that these theories have a factorized S-matrix [2] and, hence, that they are quantum integrable. The quantum conserved currents are related to the classical ones by a non-trivial renormalization. These theories are perturbations of the level-k theories of G-parafermions, which are well-known conformal field theories [6], where G is a compact simple Lie group. They are characterized by an anomaly-free embedding of the maximal abelian subgroup $U(1)^{\times r_g}$ into $G_L \otimes G_R$, which is specified by an automorfism τ of the Cartan subalgebra that preserves the Killing form $\langle\,,\,\rangle$. Consequently, τ can be (almost) any element of the orthogonal group $O(r_g)$, and hence it provides a set of continuous (marginal) coupling constants. The perturbation is given by a relevant primary field specified by two constant elements $\Lambda_\pm$ in the Lie algebra of G, which play the role of continuous vector coupling constants of the theory. In contrast, the other coupling constant in the theory β^2 has to be quantized, giving rise to the integer level k, if the quantum theory is to be well defined [14]. Our results show that the resulting theories are quantum integrable for any compact simple Lie group G, and for any value of the level k (at least if $k \geq$ the dual Coxeter number of G). More remarkably the theories are integrable for any value of the vector coupling constants $\Lambda_\pm$ and for any choice of the automorphism τ, a fact which appears to be in contradiction to the commonly held belief that integrability requires careful fine-tuning of coupling constants.

An important property of these theories is that they have soliton solutions, like the sine-Gordon theory. As explained in [1], the solitons and fundamental particles of the HSG theories are associated to the positive roots of g, the Lie algebra of G. Recall that the simplest theory of this class is the complex sine-Gordon theory (CSG) [5,7,9], which has periodic time-dependent soliton solutions (see [9] and references therein). For the more general theory, the solitons are constructed by embeddings of the CSG soliton associated to a particular $su(2)$ subalgebra generated by $E_{\pm\boldsymbol{\alpha}}$.

The quantum integrability of these theories implies that they should admit a factorizable S-matrix and the next stage of analysis consists in establishing its form. We expect that it should be possible to infer the form of the exact S-matrix through the semi-classical quantization of the solitons. Actually, these solitons carry abelian charges and the expectation is that they give rise to a tower of massive states where the lightest ones correspond

to the fundamental particles appearing in the Lagrangian, as in the complex sine-Gordon theory. More particularly, the semi-classical limit of the S-matrix can be compared with the time-delays that occur in the classical scattering [20]. The classical and semi-classical analysis of the soliton spectrum, together with the expressions for the time-delays, will be presented in a subsequent publication.

Finally, let us recall that all these theories can be viewed as generalizations of the perturbation of the simplest $\mathbb{Z}_k$ parafermions by their first thermal operator [7], which is only one of their three known series of massive integrable perturbations [8]. It would be interesting, therefore, to investigate the existence of other massive integrable perturbations of the theory of G-parafermions for a given compact Lie group G. In this respect, let us recall that, according to [1], the non-abelian affine Toda equations give rise to a second family of classical massive solitonic integrable theories associated to (compact) Symmetric-Spaces. The theory associated to a symmetric space G/G_0 describes a perturbation of the CFT's corresponding to a coset of the form $G_0/U(1)^{\times p}$, where p is always $< \mathrm{rank}(G)$ and $\geq \mathrm{rank}(G) - \mathrm{rank}(G/G_0)$. Thus, we expect that the analysis of this second class of models along the lines of this paper could lead to different solitonic integrable perturbations of more general coset CFT's and, as particular examples, of parafermionic ($p = \mathrm{rank}(G_0)$) as well as WZW ($p = 0$) theories.

Acknowledgements

Appendix: The classically conserved densities

Below, we give the explicit expressions for the the classical conserved densities with

scale dimension $s = 2$ and 3, which can be obtained by solving eqs. (3.6).

First, let us consider an explicit realization of the basis $\{t^a\}$ in terms of a Chevalley basis of the complexification of the compact Lie algebra g. It consists of a Cartan subalgebra with generators, H^A, $A = 1, \ldots, r_g$, and step operators E_α normalized such that

$$[H^A, E_\alpha] = \alpha^A E_\alpha, \quad \text{and} \quad [E_\alpha, E_{-\alpha}] = \frac{2}{\alpha^2}\, \boldsymbol{\alpha} \cdot \boldsymbol{H} \equiv \frac{2}{\alpha^2}\, \alpha^A H^A\,. \tag{A.1}$$

Then, the antihermitian basis for the compact Lie algebra g is

$$t^A = iH^A, \quad A = 1, \ldots, r_g\,,$$
$$t^\alpha = \sqrt{\frac{\boldsymbol{\alpha}^2}{4}}\,(E_\alpha - E_{-\alpha}), \quad \text{and} \quad t^{\overline{\alpha}} = i\sqrt{\frac{\boldsymbol{\alpha}^2}{4}}(E_\alpha + E_{-\alpha})\,, \tag{A.2}$$

for any positive root $\boldsymbol{\alpha}$ of g. This way, in particular, $f^{\overline{\alpha}\,\overline{\beta}\,\overline{\gamma}} = f^{\alpha\beta\overline{\gamma}} = 0$ for any three positive roots $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and $[t^\alpha, t^{\overline{\alpha}}] = \boldsymbol{\alpha} \cdot \boldsymbol{t} \equiv \alpha^A t^A$, which means that $f^{A\alpha\overline{\alpha}} = \alpha^A$. With respect to this basis, $\Lambda_- = \boldsymbol{\lambda} \cdot \boldsymbol{t}$ (recall that $\Lambda = m\, z \otimes \Lambda_-$), and we will also use the following notations. First, $t^{\overline{i}}$ indicates $t^{\overline{\alpha}}$ when $i = \alpha$, or t^α if $i = \overline{\alpha}$, and, second, for any r_g-component vector $\boldsymbol{\mu}$, $f^{\mu ij} \equiv \mu^A f^{Aij}$.

Introducing $q = q^a t^a$, the expression for the $s = 2$ classical conserved densities is

$$\mu^A I_2^{(0)A} \equiv \boldsymbol{\mu} \cdot \boldsymbol{I}_2^{(0)} = D_{i,j}^{(0)}(\boldsymbol{\mu})\, q^i q^j\,,$$
$$\mu^A \overline{I}_2^{(0)A} \equiv \boldsymbol{\mu} \cdot \overline{\boldsymbol{I}}_2^{(0)} = -\,m\, \langle \boldsymbol{\mu} \cdot \boldsymbol{t}\,, h\Lambda_+ h^\dagger \rangle\,, \tag{A.3}$$

where the only non-vanishing components of $D_{i,j}^{(0)}(\boldsymbol{\mu})$ are

$$D_{\alpha\alpha}^{(0)}(\boldsymbol{\mu}) = D_{\overline{\alpha}\,\overline{\alpha}}^{(0)}(\boldsymbol{\mu}) = \frac{1}{2m}\frac{\boldsymbol{\mu} \cdot \boldsymbol{\alpha}}{\boldsymbol{\lambda} \cdot \boldsymbol{\alpha}}\,. \tag{A.4}$$

Here, $\boldsymbol{\mu}$ is an arbitrary vector which manifests the existence of $\mathrm{rank}(G)$ conserved densities for each integer scale dimension s.

As expected, one of the $s = 2$ densities is related to the classical energy-momentum tensor, which is recovered for $\boldsymbol{\mu} = m\boldsymbol{\lambda}$ [1]

$$m\,\boldsymbol{\lambda} \cdot \boldsymbol{I}_2^{(0)} = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\left(q^\alpha q^\alpha + q^{\overline{\alpha}} q^{\overline{\alpha}}\right)$$
$$= -\frac{1}{2}\langle q - q^A t^A\,, q - q^A t^A \rangle \equiv 4\pi\beta^2\, T_{--}\,,$$

$$m\,\boldsymbol{\lambda} \cdot \overline{\boldsymbol{I}}_2^{(0)} = -\,m^2\, \langle \Lambda_-\,, h\Lambda_+ h^\dagger \rangle \equiv 4\pi\beta^2\, T_{+-}\,. \tag{A.5}$$

For scaling dimension $s = 3$, the $\mathrm{rank}(G)$ classically conserved densities are of the form

$$\boldsymbol{\mu} \cdot \boldsymbol{I}_3^{(0)} = P_{ijk}^{(0)}(\boldsymbol{\mu})\, q^i q^j q^k + P_{Aij}^{(0)}(\boldsymbol{\mu})\, q^A q^i q^j + Q_{ij}^{(0)}(\boldsymbol{\mu}) q^i \partial_- q^j\,, \tag{A.6}$$

where the only non-vanishing coefficients are

$$P_{ijk}^{(0)}(\boldsymbol{\mu}) \;=\; \frac{f^{\mu i\bar{\imath}}\,f^{\lambda j\bar{\jmath}} - f^{\mu j\bar{\jmath}}\,f^{\lambda i\bar{\imath}}}{6m^2\,f^{\lambda i\bar{\imath}} f^{\lambda j\bar{\jmath}} f^{\lambda k\bar{k}}}\;f^{ij\bar{k}}\;, \tag{A.7}$$

$$P_{A\alpha\alpha}^{(0)}(\boldsymbol{\mu}) \;=\; P_{A\overline{\alpha}\,\alpha}^{(0)}(\boldsymbol{\mu}) \;=\; -\frac{1}{2m^2}\,\frac{\boldsymbol{\mu}\cdot\boldsymbol{\alpha}}{(\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})^2}\,\alpha^A\;, \tag{A.8}$$

$$Q_{\alpha\overline{\alpha}}^{(0)}(\boldsymbol{\mu}) \;=\; -Q_{\overline{\alpha}\alpha}^{(0)}(\boldsymbol{\mu}) \;=\; -\frac{1}{2m^2}\,\frac{\boldsymbol{\mu}\cdot\boldsymbol{\alpha}}{(\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})^2}\;. \tag{A.9}$$

Notice that $P_{A\alpha\alpha}^{(0)}(\boldsymbol{\mu}) = \alpha^A Q_{\alpha\overline{\alpha}}^{(0)}(\boldsymbol{\mu})$, which, according to (4.22), is a manifestation of the gauge invariance of $\boldsymbol{\mu}\cdot\boldsymbol{I}_3^{(0)}$. Correspondingly,

$$\boldsymbol{\mu}\cdot\overline{\boldsymbol{I}}_3^{(0)} \;=\; -\frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\frac{(\boldsymbol{\mu}\cdot\boldsymbol{\alpha})}{(\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})}\,\langle q^{\alpha}t^{\alpha} \,+\, q^{\overline{\alpha}}t^{\overline{\alpha}}\,,\, h\Lambda_+ h^{\dagger}\rangle\;. \tag{A.10}$$

References

[1] C.R. Fernández-Pousa, M.V. Gallas, T.J. Hollowood, and J.L. Miramontes, *"The Symmetric Space and Homogeneous Sine-Gordon Theories"*, hep-th/9606032 to appear in Nucl. Phys. **B**.

[2] S. Parke, Nucl. Phys. **B 174** (1980) 166.

[3] A.N. Leznov and M.V. Saveliev, Commun. Math. Phys. **89** (1983) 59; *Group theoretical methods for integration of non-linear dynamical systems*, Prog. Phys. 15 (Birkhauser, Basel, 1992);
J. Underwood, *Aspects of Non-Abelian Toda Theories*, Imperial/TP/92-93/30, hep-th/9304156;
L.A. Ferreira, J.L. Miramontes, and J. Sánchez Guillén, Nucl. Phys. **B 449** (1995) 631.

[4] T.J. Hollowood, Nucl. Phys. **B 384** (1992) 523.

[5] Q-H. Park, Phys. Lett. **B 328** (1994) 329.

[6] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP **62** (1985) 215;
D. Gepner, Nucl. Phys. **B 290** (1987) 10;
D. Gepner and Z. Qiu, Nucl. Phys. **B 285** (1987) 423.

[7] I. Bakas, Int. J. Mod. Phys. **A 9** (1994) 3443.

[8] V.A. Fateev, Int. J. Mod. Phys. **A 6** (1991) 2109;
V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. **B 271** (1991) 91;
L. Palla, Phys. Lett. **B 253** (1991) 342.

[9] N. Dorey and T.J. Hollowood, Nucl. Phys. **B 440** (1995) 215.

[10] A.B. Zamolodchikov, Adv. Stud. Pure Math. **19** (1989) 641; Int. J. Mod. Phys. **A3** (1988) 743; JETP Lett. **46** (1987) 160.

[11] G.W. Delius, M.T. Grisaru, and D. Zanon, Nucl. Phys. **B 385** (1992) 307.

[12] F.A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, Nucl. Phys. **B 304** (1988) 371.

[13] J.L. Cardy, *"Conformal invariance and statistical mechanics"*, in *"Fields, Strings and Critical Phenomena"*, eds. E. Brézin and J. Zinn-Justin, Les Houches 1988, Session XLIX, North-Holland, Amsterdam (1990) 169-245.

[14] E. Witten, Commun. Math. Phys. **92** (1984) 455.

[15] E. Witten, Commun. Math. Phys. **144** (1992) 189;
K. Gawedzki and A. Kupianen, Nucl. Phys. **B 320** (1989) 625.

[16] M. Henningson and C.R. Nappi, Phys. Rev. **D 48** (1993) 861;
S.F. Hassan and A. Sen, Nucl. Phys. **B 405** (1993) 143;
E.B. Kiritsis, Mod. Phys. Lett. **A 6** (1991) 2871; Nucl. Phys. **B 405** (1993) 109.

[17] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. **B 247** (1984) 83.

[18] F.A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, Nucl. Phys. **B 304** (1988) 348.

[19] V.G. Drinfel'd and V.V. Sokolov, J. Sov. Math. **30** (1985) 1975; Soviet. Math. Dokl. **23** (1981) 457;
G.W. Wilson, Ergod. Theor. & Dyn. Sist. **1** (1981) 361;
M.F. de Groot, T.J. Hollowood and J.L. Miramontes, Commun. Math. Phys. **145** (1992) 57.

[20] R. Jackiw and G. Woo, Phys. Rev. **D12** (1975) 1643