Hybrid Renormalization of Penguins and Dimension-5 Heavy–Light Operators

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Abstract

We discuss the renormalization of local, scalar and pseudoscalar dimension-5 operators containing a heavy and a light quark field at scales below the heavy-quark mass, using the formalism of the heavy-quark effective theory (HQET). We calculate the anomalous dimensions of these operators and their mixing to one-loop order. We also perform the one-loop matching of gluon and photon penguin operators onto operators of the HQET. We discuss applications of our results to the mixing of gluon and photon penguin operators at low renormalization scales, and to the calculation of $1/m_Q^2$ corrections to meson decays constants.

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1 Introduction

Local operators containing both heavy and light quark fields exhibit an interesting behaviour under renormalization at scales below the heavy-quark mass m_Q. Then there arise large logarithms of the type $\alpha_s \ln(m_Q/\mu)$, characterizing the exchange of gluons that are "hard" with respect to the light quark but "soft" with respect to the heavy quark. To leading order in an expansion in $1/m_Q$, such gluons see the heavy quark as a static colour source. The large logarithms can thus be summed to all orders in perturbation theory using an effective theory for static heavy quarks, the so-called heavy-quark effective theory (HQET) [1]–[3]. In the HQET, the 4-component heavy-quark field $Q(x)$ is replaced by a velocity-dependent 2-component field $h_v(x)$ satisfying $\not{v} h_v = h_v$, where v is the velocity of the hadron containing the heavy quark. Because of the particular hierarchy of mass scales involved, the renormalization of heavy–light operators in the HQET is called "hybrid" renormalization. Operators in the effective theory have a different evolution than in usual QCD. For instance, whereas the vector current $\bar{q}\gamma^\mu Q$ is conserved in QCD (i.e. its anomalous dimension vanishes), the corresponding current $\bar{q}\gamma^\mu h_v$ in the HQET has a non-trivial anomalous dimension [4]–[6], which governs the evolution for scales below the heavy-quark mass.

In the literature, hybrid renormalization has been discussed extensively for local current operators of dimension 3 [4]–[9] and 4 [10, 11], as well as for 4-quark operators such as the ones governing B–$\bar{B}$ mixing [12]–[14]. Here we shall consider the renormalization of local dimension-5 operators in the HQET. The matrix elements of such operators appear, for instance, at order $1/m_Q^2$ in the heavy-quark expansion of weak transition form factors. In particular, they contribute to the decay constants of heavy mesons, which have been explored already in great detail at leading and next-to-leading order in $1/m_Q$ [15]. The same operator matrix elements also determine certain moments of meson wave functions [16], which play an important role in the heavy-flavour phenomenology. The theoretical predictions for weak decay form factors involve operator matrix elements renormalized at the scale m_Q. Our results can then be used to rewrite these matrix elements in terms of ones renormalized at a scale $\mu \ll m_Q$, which may be identified with the scale at which a non-perturbative evaluation of these matrix elements is performed (such as the inverse lattice spacing in lattice field theory, or the Borel parameter in QCD sum rules).

Our results also apply to the hybrid renormalization of genuine dimension-5 operators in QCD, such as the gluon and photon penguin operators, which appear in the weak effective Hamiltonian renormalized at the scale m_b [17]:[1]

$$\mathcal{H}_{\text{eff}} = c_g(m_b)\, g_s\, \bar{s}\,(1+\gamma_5)\sigma_{\mu\nu}G^{\mu\nu}b + c_\gamma(m_b)\, e\, \bar{s}\,(1+\gamma_5)\sigma_{\mu\nu}F^{\mu\nu}b + \dots. \tag{1}$$

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[1]Our definition of the coefficients c_g and c_γ contains a power of the b-quark mass, which is usually included in the definition of the operators.

We will derive the effective Hamiltonian at a lower scale $\mu < m_b$, which is relevant to the calculation of hadronic matrix elements of the penguin operators. Lattice calculations of such matrix elements, in particular, are usually performed in the static theory (HQET), since the b quark is too heavy to be described as a dynamical field on present-day lattices. It must be stressed, however, that the validity of the effective theory is restricted to the kinematic region where, in the rest frame of the heavy hadron, the light quarks and gluons carry momenta much smaller than m_Q. For two-body decays such as $B \to K^* \gamma$, one thus needs to evaluate the hadronic matrix elements for unphysical particle momenta (i.e., $|\mathbf{p}_{K^*}| < O(1 \text{ GeV})$ rather than the physical value $|\mathbf{p}_{K^*}| = \frac{1}{2}(m_B^2 - m_{K^*}^2)/m_B \simeq 2.56 \text{ GeV}$) and then continue the results to the physical region. In more complicated processes such as $B \to X_s \gamma^* \to X_s \ell^+ \ell^-$ this restriction no longer applies, and our results are of direct relevance to the region where the strange particle carries a small momentum.

In Sec. 2, we calculate the one-loop anomalous dimension matrix of local dimension-5 heavy–light operators carrying zero total momentum. Using the equations of motion, the problem is reduced to the mixing of two operators containing the gluon field-strength tensor. We solve the renormalization-group equation (RGE) for the scale dependence of these operators in the leading logarithmic approximation. In Sec. 3, we extend the basis to the general case where the operators carry non-zero momentum. Then there appear four additional operators, which can be written as the total derivatives of some lower-dimensional operators. We show that, with a suitable choice of the basis, there is no mixing between these new operators and the ones considered in Sec. 2. In Sec. 4, we extend the basis further by including operators containing the photon field, and we calculate the mixing between gluon and photon operators under hybrid renormalization. This extends the analysis of the mixing of gluon and photon penguin operators, which has been discussed previously for scales larger than the b-quark mass [18]–[20], to low renormalization scales. In Sec. 5, we calculate the one-loop matching of the QCD penguin operators onto their HQET counterparts. This provides a test of our results for the hybrid anomalous dimensions. In addition, the exact one-loop expressions for the Wilson coefficient functions may be more appropriate to use than the leading-order renormalization-group improved results in cases where $\ln(m_Q/\mu)$ is not a particularly large parameter; at least, they provide an estimate of the importance of next-to-leading corrections. Also, the one-loop matching conditions at the scale m_Q will eventually be part of a full next-to-leading order calculation, once the two-loop anomalous dimensions of the operators will have been calculated. In Sec. 6, we apply our results to the analysis of higher-order corrections to meson decay constants, and to the discussion of moments of meson wave functions. Section 7 contains the conclusions.

2   Anomalous dimensions

We start by considering local, Lorentz-scalar operators of dimension 5, carrying zero total momentum. A basis of such operators in the HQET can be constructed by considering operators containing two covariant derivatives acting on the heavy-quark field. It is convenient to adopt the background-field formalism [21], so that it suffices to consider gauge-invariant operators. They are of the form $\bar{q}\,\Gamma_{\mu\nu}iD^\mu iD^\nu h_v$, where $\Gamma_{\mu\nu} \in \{v_\mu v_\nu, g_{\mu\nu}, \gamma_\mu v_\nu, \gamma_\nu v_\mu, [\gamma_\mu, \gamma_\nu]\}$. We define:

$$
\begin{aligned}
O_1 &= g_s\,\bar{q}\,\sigma_{\mu\nu}G^{\mu\nu}h_v\,,\\
O_2 &= g_s\,\bar{q}\,\gamma_\mu v_\nu\,iG^{\mu\nu}h_v\,,\\
O_3 &= \bar{q}\,(iv{\cdot}D)^2 h_v\,,\\
O_4 &= \bar{q}\,(iD)^2 h_v\,,\\
O_5 &= \bar{q}\,[i\slashed{D}\,iv{\cdot}D + iv{\cdot}D\,i\slashed{D}]\,h_v\,,
\end{aligned}
\tag{2}
$$

where $ig_s G^{\mu\nu} = [iD^\mu, iD^\nu]$ is the gluon field-strength tensor, and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. A basis of pseudoscalar operators $O_i^{(5)}$ can be obtained by replacing $\bar{q} \to \bar{q}\,\gamma_5$. As long as we neglect the mass of the light quark and work in a regularization scheme with anticommuting γ_5, nothing changes by this replacement. Below, we shall thus consider scalar operators, but it is understood that the light-quark field q could be replaced by $\gamma_5 q$ or, equivalently, by a left- or right-handed field, q_L or q_R.

Not all of the operators in (2) are independent when the equations of motion, $\bar{q}\,(i\slashed{D})^\dagger = 0$ and $iv D\,h_v = 0$, are used.[2] They give the relations $O_3 \mathrel{\widehat{=}} 0$, $O_5 \mathrel{\widehat{=}} -O_2$, and

$$
\begin{aligned}
O_4 &= \bar{q}\left([(i\slashed{D})^\dagger]^2 - \tfrac{1}{2}g_s\sigma_{\mu\nu}G^{\mu\nu}\right)h_v - (i\partial)^2(\bar{q}\,h_v) + 2i\partial_\mu(\bar{q}\,iD^\mu h_v)\\
&\mathrel{\widehat{=}} -\frac{1}{2}\,O_1 + \text{total derivatives},
\end{aligned}
\tag{3}
$$

implying that for zero total momentum all operators can be reduced to the operators O_1 and O_2. That it is legitimate to use the equations of motion to reduce the operator basis has been established in Ref. [22].

The scale dependence of the renormalized operators O_1 and O_2 is governed by a 2×2 anomalous dimension matrix, which can be obtained by calculating the $1/\varepsilon$ poles of the matrix elements of the bare operators in dimensional regularization (i.e. in $d = 4 - 2\varepsilon$ space-time dimensions). The relevant one-loop diagrams are shown in Fig. 1. Z_h and Z_q are the wave-function renormalization constants for the quark fields. In a covariant gauge ($a = 1$ corresponds to the Feynman gauge, $a = 0$ corresponds to the Landau gauge), they are given by

$$
Z_h = 1 + C_F\,(3 - a)\,\frac{\alpha_s}{4\pi\varepsilon}\,, \qquad Z_q = 1 - C_F\,a\,\frac{\alpha_s}{4\pi\varepsilon}\,.
\tag{4}
$$

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[2] We use the notation $iD = i\partial + g_s A$ and $(iD)^\dagger = -i\overleftarrow{\partial} + g_s A$.
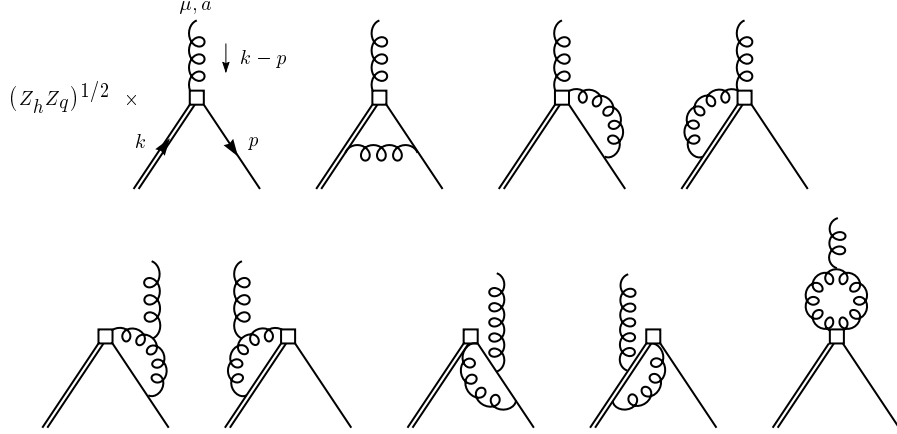
Figure 1: One-loop diagrams contributing to the calculation of the anomalous dimensions of the operators O_1 and O_2, denoted by the square. Heavy-quark propagators in the HQET are drawn as double lines.

A virtue of the background-field formalism is that the gluon field is not renormalized, since $Z_g Z_A^{1/2} = 1$ [21]. We have calculated the diagrams in Fig. 1 in an arbitrary covariant gauge, and with arbitrary momentum assignments (however, for zero total momentum). The sum of all diagrams is gauge independent, and the dependence on the external momenta combines in such a way that the result can be expressed in terms of the matrix elements of the basis operators in (2). We find

$$\langle O_1^{\text{bare}} \rangle = \left\{ 1 + \frac{\alpha_s}{4\pi\varepsilon} \left(C_F - \frac{N}{2} \right) \right\} \langle O_1 \rangle + \frac{\alpha_s}{4\pi\varepsilon} \left(N \langle O_2 \rangle + 3 C_F \langle O_4 \rangle \right) ,$$
$$\langle O_2^{\text{bare}} \rangle = \left\{ 1 + \frac{\alpha_s}{4\pi\varepsilon} \left(C_F - \frac{3N}{2} \right) \right\} \langle O_2 \rangle + \frac{\alpha_s}{4\pi\varepsilon} \left(\frac{3N}{8} \langle O_1 \rangle + \frac{2C_F}{3} \langle O_4 \rangle + \frac{C_F}{6} \langle O_5 \rangle \right)$$

(5)

where N is the number of colours, and $C_F = \frac{1}{2}(N - 1/N)$ is the eigenvalue of the quadratic Casimir operator in the fundamental representation. Eliminating the operators O_4 and O_5 by means of the relations $\langle O_4 \rangle = -\frac{1}{2}\langle O_1 \rangle$ and $\langle O_5 \rangle = -\langle O_2 \rangle$, we obtain

$$\langle O_1^{\text{bare}} \rangle = \left\{ 1 + \frac{\alpha_s}{4\pi\varepsilon} \left(-\frac{N}{2} - \frac{C_F}{2} \right) \right\} \langle O_1 \rangle + \frac{\alpha_s}{4\pi\varepsilon} N \langle O_2 \rangle ,$$
$$\langle O_2^{\text{bare}} \rangle = \left\{ 1 + \frac{\alpha_s}{4\pi\varepsilon} \left(-\frac{3N}{2} + \frac{5C_F}{6} \right) \right\} \langle O_2 \rangle + \frac{\alpha_s}{4\pi\varepsilon} \left(\frac{3N}{8} - \frac{C_F}{3} \right) \langle O_1 \rangle .$$

(6)

These results define a matrix Z_{ij} of renormalization constants through $\langle O_i^{\text{bare}} \rangle = Z_{ij} \langle O_j \rangle$. Denoting by $Z_{ij}^{(1)}$ the coefficient of the $1/\varepsilon$ pole in this matrix and using the relation [23]

$$\gamma_{ij} = -2\alpha_s \frac{\partial}{\partial \alpha_s} Z_{ij}^{(1)} ,$$

(7)

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we obtain the anomalous dimensions appearing in the RGE for the renormalized operators:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} O_i + \gamma_{ij} O_j = 0 \,. \tag{8}$$

At the one-loop order, the result reads

$$\hat{\gamma} = \frac{\alpha_s}{4\pi} \begin{pmatrix} N + C_F & -2N \\ -\frac{3}{4}N + \frac{2}{3}C_F & 3N - \frac{5}{3}C_F \end{pmatrix} \,. \tag{9}$$

It is remarkable that the eigenvalues of the one-loop anomalous dimension matrix are given by irrational numbers (in units of $\alpha_s/4\pi$):

$$\gamma_\pm = \left(2N - \frac{C_F}{3} \right) \pm \sqrt{ \frac{5N^2}{2} - 4NC_F + \frac{16C_F^2}{9} } = \frac{1}{9}\left(50 \pm \sqrt{\frac{1565}{2}} \right) \,. \tag{10}$$

We know of no other case where this happens at the one-loop order. In the leading logarithmic approximation, the solution of the RGE (8) is given by

$$O_i(m_Q) = U_{ij}(m_Q, \mu) O_j(\mu) \,, \tag{11}$$

where

$$\hat{U}(m_Q, \mu) = \hat{V} \begin{pmatrix} r_+ & 0 \\ 0 & r_- \end{pmatrix} \hat{V}^{-1} \,, \qquad r_\pm = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{\gamma_\pm/2\beta_0} \,. \tag{12}$$

Here $\beta_0 = \frac{11}{3}N - \frac{2}{3}n_f$ is the first coefficient of the β function, and $\hat{V}$ is the matrix that diagonalizes the anomalous dimension matrix:

$$\hat{V}^{-1}\hat{\gamma}\hat{V} = \frac{\alpha_s}{4\pi} \begin{pmatrix} \gamma_+ & 0 \\ 0 & \gamma_- \end{pmatrix} \,. \tag{13}$$

The explicit form of the evolution matrix is

$$\hat{U}(m_Q, \mu) = \begin{pmatrix} \frac{1}{2}(r_+ + r_-) - (N - \frac{4}{3}C_F)\Delta & -2N\Delta \\ (-\frac{3}{4}N + \frac{2}{3}C_F)\Delta & \frac{1}{2}(r_+ + r_-) + (N - \frac{4}{3}C_F)\Delta \end{pmatrix} \,, \tag{14}$$

where $\Delta = (r_+ - r_-)/(\gamma_+ - \gamma_-)$.

3 Complete operator basis

We now consider the general case where the total momentum carried by the operators does not vanish. The purpose of this section is to show that even then the operator basis (O_1, O_2) closes under renormalization. The reader not concerned about this issue can proceed directly with Sec. 4.

To obtain a complete basis of local, scalar (or pseudoscalar) dimension-5 operators, we have to consider, in addition to (2), operators with one or two derivatives acting on the light-quark field. These differ from the operators considered so far by total derivatives. The number of such operators is reduced significantly when the equations of motion are used. We find that a complete basis contains only four new operators in addition to O_1 and O_2, and choose them in the following form:

$$
\begin{aligned}
T_1 &= i\partial_\mu(\bar{q}\, iD^\mu h_v)\,, \\
T_2 &= (iv{\cdot}\partial)^2(\bar{q}\, h_v)\,, \\
T_3 &= (i\partial)^2(\bar{q}\, h_v)\,, \\
T_4 &= i\partial_\mu\, iv{\cdot}\partial\, (\bar{q}\,\gamma^\mu h_v)\,.
\end{aligned}
\tag{15}
$$

In momentum space, the total derivative of a local operator corresponds to the total external momentum carried by that operator and thus does not affect the behaviour under renormalization. Therefore, as far as the calculation of ultraviolet divergences is concerned, T_1 behaves like a dimension-4 operator, while T_2, T_3 and T_4 behave like dimension-3 operators. Under renormalization, operators of lower "effective dimension" cannot mix into higher-dimensional operators, but the contrary is, in general, not true. As a consequence, the Wilson coefficients of the operators O_1 and O_2 could, in principle, be modified by the presence of the lower-dimensional operators.

It follows from this discussion that the anomalous dimension matrix governing the mixing of the operators O_i and T_i is of the general form

$$
\hat{\Gamma} = \begin{pmatrix} \hat{\gamma} & \hat{A} \\ \hat{0} & \hat{B} \end{pmatrix}\,,
\tag{16}
$$

where $\hat{\gamma}$ is the 2×2 matrix given (at the one-loop order) in (9). The 4×4 matrix $\hat{B}$ describing the mixing of the operators T_i among themselves has been calculated in Ref. [11]. To all orders in perturbation theory, it has the simple form

$$
\hat{B} = \gamma_{\mathrm{hl}}\,\hat{1} + \gamma^a \begin{pmatrix} 1 & \frac{2}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,,
\tag{17}
$$

where γ_{hl} is the anomalous dimension of heavy–light current operators of dimension 3 [4]–[9], and γ^a has been defined in Ref. [11]. At the one-loop order, one finds $\gamma_{\mathrm{hl}} = -\gamma^a = \gamma_0(\alpha_s/4\pi)$, where

$$
\gamma_0 = -3C_F\,.
\tag{18}
$$

It remains to find the 2×4 matrix $\hat{A}$ describing the mixing of O_1 and O_2 into the operators T_i. Since these operators contain total derivatives, the calculation
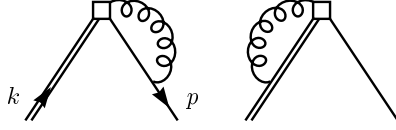
Figure 2: One-loop diagrams contributing to the calculation of the anomalous dimension matrix $\hat{A}$.

must be performed with non-zero total momentum. However, a simplification is that the operators T_i have non-vanishing quark matrix elements at tree level. The matrix $\hat{A}$ can thus be calculated by considering a two-point (rather than three-point) vertex function. The relevant one-loop diagrams are shown in Fig. 2. For arbitrary quark momenta, we find the pole terms:

$$\langle O_1^{\text{bare}} \rangle = C_F \frac{\alpha_s}{4\pi\varepsilon} \bar{u}_q(p) \, (18p^2) \, u_h(v,k) + \dots ,$$
$$\langle O_2^{\text{bare}} \rangle = C_F \frac{\alpha_s}{4\pi\varepsilon} \bar{u}_q(p) \, (4p^2 + 2v \cdot p \, \not{p}) \, u_h(v,k) + \dots . \quad (19)$$

Both matrix elements vanish on-shell, implying that there is no mixing of the dimension-5 operators O_1 and O_2 into the operators T_i, i.e.

$$\hat{A} = 0 \,, \qquad \hat{\Gamma} = \begin{pmatrix} \hat{\gamma} & \hat{0} \\ \hat{0} & \hat{B} \end{pmatrix} . \quad (20)$$

In other words, the evolution of the operators T_i is disconnected from that of the operators O_i. Moreover, we shall see below that (at least to next-to-leading order) the operators T_i are not induced in the matching of QCD operators onto HQET operators. This is a virtue of our particular choice of the operators in (2) containing derivatives acting only on the heavy-quark field. Hence, from now on the operators T_i can be omitted from our discussion.

4 Hybrid renormalization of penguins

We now include in our discussion local operators containing the photon field. This is most conveniently done by extending the definition of the covariant derivative $(D^\mu = \partial^\mu - ig_s t_a A_a^\mu - ie\mathcal{A}^\mu)$, so that $[iD^\mu, iD^\nu] = ig_s G^{\mu\nu} + ieF^{\mu\nu}$, where $F^{\mu\nu}$ is the electromagnetic field. Then the two gluon operators O_1 and O_2 in (2) are supplemented by their photon counterparts:

$$O_1^\gamma = e \, \bar{q} \, \sigma_{\mu\nu} F^{\mu\nu} h_v \,,$$
$$O_2^\gamma = e \, \bar{q} \, \gamma_\mu v_\nu \, iF^{\mu\nu} h_v \,. \quad (21)$$

The problem is to find the mixing of these four operators under renormalization.
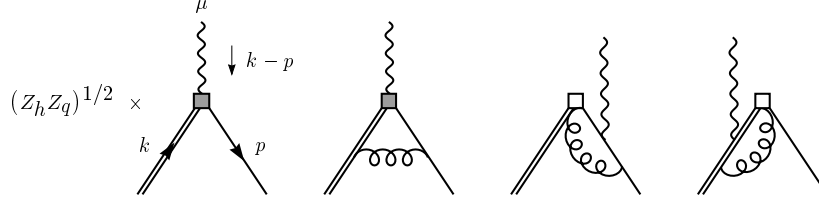
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Figure 3: Additional one-loop diagrams contributing to the calculation of the anomalous dimensions in the presence of photon penguin operators, denoted by the gray square.

At leading order in the fine-structure constant α, the gluon operators can mix into the photon operators, but not vice versa. Thus, we write the corresponding 4×4 anomalous dimension matrix in the form

$$\hat{\Gamma} = \begin{pmatrix} \hat{\gamma} & \hat{X} \\ \hat{0} & \hat{Y} \end{pmatrix} . \tag{22}$$

The 2×2 submatrices $\hat{X}$ and $\hat{Y}$ can be obtained from the calculation of the diagrams shown in Fig. 3. It is straightforward to see that the values of these diagrams can be obtained from the results for the corresponding diagrams in Fig. 1 by performing a simple replacement of colour factors, namely by taking the limit $N \to 0$ keeping C_F fixed. The first two diagrams in Fig. 3 determine the matrix $\hat{Y}$ and yield

$$\hat{Y} = \gamma_{\text{hl}} \, \hat{1} = \frac{\alpha_s}{4\pi} \left(-3C_F\right) \hat{1} . \tag{23}$$

Since QCD is blind to the photon field, the operators O_3 and O_4 renormalize as dimension-3 heavy–light current operators, i.e. their anomalous dimension is given by γ_{hl}. The submatrix $\hat{X}$ is obtained from the third diagram in Fig. 3, whereas the last diagram vanishes. The result is

$$\hat{X} = \frac{\alpha_s}{4\pi} \, Q_q C_F \begin{pmatrix} 4 & 0 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} , \tag{24}$$

where Q_q is the electric charge of the light quark in units of e. Note that the sum of these two matrices reproduces the "abelian part" of the matrix $\hat{\gamma}$ in (9), i.e. $\hat{X}/Q_q + \hat{Y} = \lim_{N \to 0} \hat{\gamma}$.

The three eigenvalues of the anomalous dimension matrix $\hat{\Gamma}$ in units of $\alpha_s/4\pi$ are $\gamma_\pm$ in (10) and γ_0 in (18). The solution of the RGE for the four operators O_i can be obtained either by diagonalizing the 4×4 matrix $\hat{\Gamma}$ directly, or by solving first the homogeneous equation for the two operators O_1^γ and O_2^γ, and then inserting the solution for these operators into the inhomogeneous equation for the operators O_1 and O_2. In leading logarithmic order, we find that the photon operators renormalize multiplicatively:

$$O_i^\gamma(m_Q) = r_0 \, O_i^\gamma(\mu) ; \quad i = 1, 2, \tag{25}$$

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with
$$r_0 = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{\gamma_0/2\beta_0} . \tag{26}$$

The evolution of the gluon operators is given by
$$\begin{pmatrix} O_1 \\ O_2 \end{pmatrix}_{m_Q} = \hat{U}(m_Q, \mu) \begin{pmatrix} O_1 \\ O_2 \end{pmatrix}_\mu + \hat{W}(m_Q, \mu) \, \hat{X}_{(0)} \begin{pmatrix} O_1^\gamma \\ O_2^\gamma \end{pmatrix}_\mu , \tag{27}$$

where $\hat{X}_{(0)}$ denotes the matrix $\hat{X}$ in units of $\alpha_s/4\pi$, the evolution matrix $\hat{U}$ has been given in (12) and (14), and
$$\hat{W}(m_Q, \mu) = \hat{V} \begin{pmatrix} \dfrac{r_+ - r_0}{\gamma_+ - \gamma_0} & 0 \\ 0 & \dfrac{r_- - r_0}{\gamma_- - \gamma_0} \end{pmatrix} \hat{V}^{-1} . \tag{28}$$

Consider now the effective Hamiltonian in (1). At the scale m_Q, it contains the gluon and photon penguin operators
$$\mathcal{Q}_g = g_s \, \bar{q} \, \sigma_{\mu\nu} G^{\mu\nu} Q , \qquad \mathcal{Q}_\gamma = e \, \bar{q} \, \sigma_{\mu\nu} F^{\mu\nu} Q , \tag{29}$$

with known coefficients $c_g(m_Q)$ and $c_\gamma(m_Q)$. Once again q may be replaced by q_L or q_R. Our goal is to evolve the Hamiltonian down to a low renormalization scale $\mu \ll m_Q$. Each of the two penguin operators (renormalized at the scale m_Q) has an expansion in terms of the HQET operators renormalized at a scale $\mu < m_Q$. We define a set of coefficient functions by:
$$\mathcal{Q}_g(m_Q) \to O_1(m_Q) = \sum_{i=1,2} \left[C_i^g(\mu) \, O_i(\mu) + C_i^\gamma(\mu) \, O_i^\gamma \right] ,$$
$$\mathcal{Q}_\gamma(m_Q) \to O_1^\gamma(m_Q) = \sum_{i=1,2} D_i^\gamma(\mu) \, O_i^\gamma(\mu) . \tag{30}$$

At leading logarithmic order, the initial values of the coefficients at the scale m_Q are determined by a tree-level comparison of operators matrix elements in QCD and in the HQET and are thus simply given by $C_1^g(m_Q) = D_1^\gamma(m_Q) = 1$; all other coefficients vanish at the scale m_Q. The values of the coefficients at a lower scale $\mu < m_Q$ can be read of from the solution of the RGE in (25) and (27). We obtain
$$D_1^\gamma(\mu) = r_0 , \qquad D_2^\gamma(\mu) = 0 , \tag{31}$$

and
$$C_1^g(\mu) = \frac{r_+ + r_-}{2} + \frac{1}{2} \left(1 - \frac{4C_F}{3N} \right) C_2^g(\mu) ,$$
$$C_2^g(\mu) = -2N \frac{r_+ - r_-}{\gamma_+ - \gamma_-} ,$$
$$C_1^\gamma(\mu) = 2Q_q C_F \left\{ \frac{r_+ - r_0}{\gamma_+ - \gamma_0} + \frac{r_- - r_0}{\gamma_- - \gamma_0} \right\} + 2 \left(1 - \frac{C_F}{N} \right) C_2^\gamma(\mu) ,$$
$$C_2^\gamma(\mu) = -\frac{8}{3} Q_q \frac{NC_F}{\gamma_+ - \gamma_-} \left\{ \frac{r_+ - r_0}{\gamma_+ - \gamma_0} - \frac{r_- - r_0}{\gamma_- - \gamma_0} \right\} . \tag{32}$$

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With $N = 3$ and $C_F = 4/3$, this yields:

$$
\begin{aligned}
C_1^g(\mu) &\simeq 0.6966\,r_- + 0.3034\,r_+\,, \\
C_2^g(\mu) &\simeq 0.9652\,(r_- - r_+)\,, \\
C_1^\gamma(\mu) &\simeq Q_q\,(0.7093\,r_- + 0.0600\,r_+ - 0.7693\,r_0)\,, \\
C_2^\gamma(\mu) &\simeq Q_q\,(0.2661\,r_- - 0.1355\,r_+ - 0.1306\,r_0)\,.
\end{aligned}
\tag{33}
$$

As an example, we evaluate the coefficients for the b quark at the scale $\mu = 1$ GeV, using $n_f = 4$ flavours (i.e. $\beta_0 = 25/3$), $\alpha_s(m_b) = 0.210$ and $\alpha_s(\mu) = 0.458$. This gives $D_1^\gamma \simeq 1.21$, as well as $C_1^g \simeq 0.82$, $C_2^g \simeq 0.22$, $C_1^\gamma \simeq -0.26\,Q_q$, and $C_2^\gamma \simeq -0.01\,Q_q$. We observe that the effects of hybrid renormalization are typically of order 20%, except for the coefficient C_2^γ, which is of order $[\alpha_s \ln(m_Q/\mu)]^2$.

5   One-loop matching

In the above discussion of operator evolution, we have combined the tree-level matching of the QCD penguin operators onto the HQET penguin operators at the scale m_b with the evolution equations solved in the leading logarithmic approximation. This approach is justified if $\ln(m_Q/\mu) \gg 1$. As an alternative, we shall now discuss the full one-loop matching of the QCD operators onto the HQET operators. Our calculation will be equivalent to that performed by Eichten and Hill for the chromo-magnetic operator containing two heavy-quark fields [1]. The advantage of this approach is that we will be able to include non-logarithmic terms of $O(\alpha_s)$. On the other hand, we will not be able to resum logarithmic terms to higher orders in perturbation theory. A consistent combination of one-loop matching with renormalization-group improvement would require to calculate the operator anomalous dimensions to two-loop order, which is beyond the scope of our work. However, once this calculation will have been done, the one-loop matching conditions at the scale m_Q derived here will be part of the full next-to-leading order analysis.

To derive expressions for the Wilson coefficients, we must compare the matrix elements of the QCD operators $\mathcal{Q}_g$ and $\mathcal{Q}_\gamma$ in (29) with the corresponding matrix elements of the HQET operators O_i and O_i^γ. Order by order in perturbation theory, the comparison ("matching") of these matrix elements defines the coefficient functions. By construction, the differences between matrix elements in the two theories are insensitive to any long-distance properties, such as the nature of the infrared regulator or of the external states. Therefore, it is legitimate to perform the matching calculation using (on-shell) quark and gluon states, and working with any infrared regularization scheme that is convenient. Following Refs. [1, 11], we choose to regulate both ultraviolet and infrared divergences using dimensional regularization. Moreover, we expand the resulting expressions for the Feynman amplitudes in powers of the external momenta, and then set

the external momenta to zero inside the loop integrals. Then the only mass scale remaining in the QCD calculation is the heavy-quark mass, and hence only diagrams containing a heavy-quark propagator in a loop contribute. Likewise, in the HQET there is no mass scale left after the external momenta are set to zero, and hence all loop integrals vanish on dimensional grounds. So only the tree-level matrix elements of the HQET operators multiplied by their Wilson coefficient functions remain. That is why this particular regularization scheme is most economic for our purpose.

In the matching calculation, we have to consider all operators in the HQET that have the same quantum numbers as the QCD operator in (29). In addition to the genuine dimension-5 operators O_i and O_i^γ and the total derivatives T_i encountered so far, this set includes in principle also operators proportional to the heavy-quark mass m_Q. They are:

$$
\begin{aligned}
S_1 &= m_Q \, iv \cdot \partial \left(\bar{q} \, h_v \right), \\
S_2 &= m_Q \, i\partial_\mu (\bar{q} \, \gamma^\mu h_v), \\
S_3 &= m_Q \, \bar{q} \, iv \cdot D \, h_v, \\
S_4 &= m_Q \, \bar{q} \, i\slashed{D} h_v, \\
S_5 &= m_Q^2 \, \bar{q} \, h_v.
\end{aligned}
\tag{34}
$$

The matrix elements of the operators S_3 and $S_4 - S_2$ evaluated between physical states vanish by the equations of motion. Nevertheless, these operators may be induced when the matching calculation is performed with unphysical states, such as on-shell quarks and gluons.
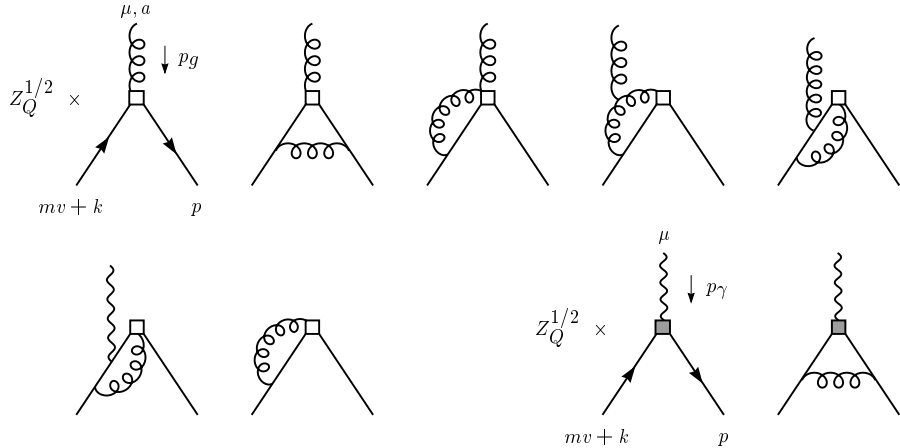

Figure 4: One-loop diagrams contributing to the matching calculation for the gluon and photon penguin operators, $\mathcal{Q}_g$ and $\mathcal{Q}_\gamma$, denoted by the white and gray squares.

Since some operators (T_2, T_3 and T_4, as well as S_1, S_2 and S_5) do not have gluon matrix elements at tree level, we have to perform the matching calculation

by considering both the quark–quark and the quark–quark–gluon vertex functions, with arbitrary (but on-shell) external momenta. The relevant diagrams are shown in Fig. 4. We have performed the calculation of these diagrams in an arbitrary covariant gauge, and find that the results for the coefficient functions are gauge independent. The on-shell wave-function renormalization constant for the heavy-quark field in our regularization scheme is

$$Z_Q = 1 - C_F \frac{\alpha_s}{4\pi} \left(\frac{m_Q^2}{4\pi\mu^2} \right)^{d/2-2} \frac{(d-1)\,\Gamma(2-d/2)}{(d-3)}\,, \tag{35}$$

which is also gauge independent. The light-quark field is not renormalized in this scheme (i.e. $Z_q = 1$). Note that only the second diagram in Fig. 4 involves a light-quark propagator in a loop. It is easily seen that the contribution of this diagram is linear in the external gluon momentum, but independent of the momentum of the light quark. Hence, it follows that no contributions proportional to the light-quark momentum appear in the matching calculation. This means that the operators T_i and S_1, S_2 are not induced by radiative corrections, at least not to $O(\alpha_s)$.

Let us now present the result of the evaluation of the diagrams involving the gluon penguin operator $\mathcal{Q}_g$. For the sum of all contributions, we find

$$\begin{aligned} \langle \mathcal{Q}_g \rangle &= C_1^g \langle O_1 \rangle + C_2^g \langle O_2 \rangle + C_1^\gamma \langle O_1^\gamma \rangle + C_2^\gamma \langle O_2^\gamma \rangle \\ &\quad + C_3^g \langle O_3 \rangle + C_S \left(-2\langle S_3 \rangle + \langle S_4 \rangle + 2\langle S_5 \rangle \right), \end{aligned} \tag{36}$$

where the exact expressions for the coefficients in an arbitrary space-time dimension d are

$$C_1^g = 1 + \frac{K}{2} \left[(2d^3 - 23d^2 + 71d - 62)C_F - (d^3 - 10d^2 + 25d - 14)N \right],$$

$$C_2^g = -K \left[2(d-2)(d-3)(d-4)C_F - (d^3 - 9d^2 + 23d - 14)N \right],$$

$$C_1^\gamma = KQ_qC_F\,(d-3)(d^2 - 9d + 16)\,,$$

$$C_2^\gamma = -2KQ_qC_F\,(d-4)(d^2 - 6d + 10)\,, \tag{37}$$

as well as

$$\begin{aligned} C_3^g &= -4KC_F\,(d-1)(d-4)\,, \\ C_S &= -2KC_F\,(d-1)(d-3)\,, \end{aligned} \tag{38}$$

where we have abbreviated

$$K = \frac{\alpha_s}{4\pi} \left(\frac{m_Q^2}{4\pi\mu^2} \right)^{d/2-2} \frac{\Gamma(2-d/2)}{(d-2)(d-3)}\,. \tag{39}$$

Note that the operators O_3 and S_3 induced by one-loop matching in (36) vanish by the equations of motion. These operators appear only because the matching calculation is (legitimately) being performed using unphysical quark and gluon states, rather than physical hadron states. The quark states satisfy the on-shell conditions $v \cdot k = 0$ and $\not{p} = p^2 = 0$, which differ from the true equations of motion at $O(g_s)$. The presence of operators that vanish by the equations of motion in (36) is, however, without physical significance. For all practical applications these operators can be ignored, since their physical matrix elements vanish. A similar statement obtains for the operators S_4 and S_5. That fact that they appear in (36) in the combination $S_5 + \frac{1}{2}S_4$ is a consequence of the so-called reparametrization invariance of the HQET [24]. Indeed, this combination is nothing but the HQET counterpart of the QCD operator $m_Q^2 \bar{q} Q$. But this operator has the form of an off-diagonal mass term, which has no observable effect as it can be removed by a redefinition of the quark fields. This simply shifts the physical quark masses by an amount of $O(G_F \alpha_s m_Q^3)$. This discussion shows *a posteriori* that, as shown in (30), the matching of the QCD operator $\mathcal{Q}_g$ onto the HQET operators involves only genuine dimension-5 operators, preserving thus the structure of an effective field theory, in which operators enhanced by powers of the large scale m_Q do not contribute to physical matrix elements.

A similar calculation for the matrix element of the photon penguin operator gives, omitting now unphysical operators,

$$\langle \mathcal{Q}_\gamma \rangle = D_1^\gamma \langle O_1^\gamma \rangle + D_2^\gamma \langle O_2^\gamma \rangle, \tag{40}$$

where

$$D_1^\gamma = 1 + \frac{K}{2} C_F \left(d^2 - 15d + 34 \right),$$
$$D_2^\gamma = -2K C_F \left(d - 4 \right)^2. \tag{41}$$

Let us now set $d = 4 - 2\varepsilon$, take the limit $\varepsilon \to 0$ and remove the poles in $1/\varepsilon$ using the $\overline{\text{MS}}$ subtraction prescription. In doing this, we have to take account of the fact that not only the matrix elements of the HQET operators are ultraviolet divergent, but also the matrix elements of the QCD operators $\mathcal{Q}_g$ and $\mathcal{Q}_\gamma$ themselves. If we are interested in the evolution of the operators below the scale m_Q, taking their values at $\mu = m_Q$ as given, we have to remove this second type of divergence by renormalizing the bare QCD operators at the scale $\mu = m_Q$. This is accomplished by a 2×2 matrix $\hat{Z}^{-1}$, which is given by [18, 19]:[3]

$$\hat{Z}^{-1} = \hat{1} + \frac{\alpha_s}{8\pi} \left(\frac{m_Q^2}{4\pi\mu^2} \right)^{d/2-2} \Gamma(2 - d/2) \begin{pmatrix} 2C_F & 0 \\ 8Q_q C_F & 10C_F - 4N \end{pmatrix}, \tag{42}$$

_____

[3]The matrix in parenthesis contains the one-loop anomalous dimensions of the QCD penguin operators, after a factor of the heavy-quark mass has been removed.

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which multiplies the 2×4 matrix of coefficient functions

$$\hat{C} = \begin{pmatrix} C_1^g & C_2^g & C_1^\gamma & C_2^\gamma \\ 0 & 0 & D_1^\gamma & D_2^\gamma \end{pmatrix} \tag{43}$$

from the left. The remaining divergences are removed by subtracting the $1/\varepsilon$ poles using the $\overline{\mathrm{MS}}$ scheme. The result is:

$$D_1^\gamma(\mu) = 1 - C_F \frac{\alpha_s}{\pi} \left(\frac{3}{8} \ln \frac{\mu^2}{m_Q^2} + 1 \right), \qquad D_2^\gamma(\mu) = 0, \tag{44}$$

and

$$C_1^g(\mu) = 1 + (N + C_F) \frac{\alpha_s}{8\pi} \ln \frac{\mu^2}{m_Q^2} + \left(N - \frac{5}{4} C_F \right) \frac{\alpha_s}{\pi},$$

$$C_2^g(\mu) = -N \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{m_Q^2} - (N - 2C_F) \frac{\alpha_s}{2\pi},$$

$$C_1^\gamma(\mu) = Q_q C_F \frac{\alpha_s}{2\pi} \left(\ln \frac{\mu^2}{m_Q^2} - \frac{1}{2} \right),$$

$$C_2^\gamma(\mu) = Q_q C_F \frac{\alpha_s}{\pi}. \tag{45}$$

The Wilson coefficients obey the RGE

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \hat{C}(\mu) = \hat{C}(\mu) \hat{\Gamma}, \tag{46}$$

where $\hat{\Gamma}$ has been given in (22)–(24). The fact that our explicit results for the coefficient functions satisfy this equation is a strong check of our results.

As at the end of Sec. 4, we now evaluate as an example the coefficients for the b quark at the scale $\mu = 1$ GeV, using $m_b = 4.8$ GeV and $\alpha_s = \alpha_s(\sqrt{\mu\, m_b}) = 0.282$. This gives $D_1^\gamma \simeq 1.02$, as well as $C_1^g \simeq 0.97$, $C_2^g \simeq 0.20$, $C_1^\gamma \simeq -0.22\, Q_q$, and $C_2^\gamma \simeq 0.12\, Q_q$. The comparison with the leading-order renormalization-group improved results presented after (33) gives an idea of the importance of next-to-leading corrections. In some cases, such as D_1^γ and C_2^γ, the non-logarithmic terms of $O(\alpha_s)$ in (45) are quite important. For those coefficients a full next-to-leading order calculation would be desirable.

6 Meson decay constants and wave functions

Another application of our results is the calculation of higher-order corrections in the heavy-quark expansion of current matrix elements. As an example, we discuss the matrix elements of heavy–light currents between a ground-state meson and

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the vacuum, which define the meson decay constants f_M. The heavy-quark expansion for heavy-meson decay constants has been discussed at next-to-leading order in $1/m_Q$ in Ref. [15]. In general, at order $1/m_Q^n$ in the expansion there appear contributions from the matrix elements of local current operators of dimension $3 + n$, and (for $n \geq 1$) from the matrix elements of non-local operators containing the time-ordered products of lower-dimensional current operators with terms from the effective Lagrangian of the HQET. Our results are relevant to the discussion of the local corrections of order $1/m_Q^2$. They can be expressed in terms of the matrix elements of operators of the type[4] $\bar{q}\,\Gamma\,iD^\mu iD^\nu h_v$. Unlike the case considered so far in this paper, these operators transform as vectors and axial vectors under the Lorentz group. However, our results still apply because of heavy-quark spin symmetry. In fact, the most general matrix element of operators of the type shown above can be written in the form

$$\langle\,0\,|\,\bar{q}\,\Gamma\,iD^\mu iD^\nu h_v\,|M(v)\rangle_\mu = \frac{1}{2}\sqrt{m_M}\,F(\mu)\,\mathrm{Tr}\left\{\Theta^{\mu\nu}\,\Gamma\,\mathcal{M}(v)\right\}, \qquad (47)$$

where $F \approx f_M\sqrt{m_M}$ is the leading contribution to the meson decay constant, and

$$\mathcal{M}(v) = \frac{1 + \not{v}}{2}\begin{cases} -i\gamma_5 & ;\text{ pseudoscalar meson } M(v), \\ \not{e} & ;\text{ vector meson } M^*(e, v), \end{cases} \qquad (48)$$

is a Dirac matrix representing the spin wave function of the ground-state mesons in the HQET [25]. Here v is the meson velocity, and e is the polarization vector of the vector meson ($e \cdot v = 0$). The matrix $\mathcal{M}(v)$ satisfies $\mathcal{M}(v)\not{v} = -\mathcal{M}(v)$. The most general decomposition of the tensor form factor $\Theta^{\mu\nu}$ consistent with Lorentz covariance, heavy-quark symmetry and the equations of motion involves the "binding energy" $\bar{\Lambda} = m_M - m_Q$, which is a scale-independent mass parameter of the HQET, as well as two new parameters λ_E^2 and λ_H^2 [16]:

$$\Theta^{\mu\nu} = 2\left[\bar{\Lambda}^2 + \lambda_H^2(\mu) + \lambda_E^2(\mu)\right](v^\mu v^\nu - g^{\mu\nu})$$
$$+ 2\lambda_E^2(\mu)\,v^\mu(\gamma^\nu + v^\nu) + \frac{i}{2}\lambda_H^2(\mu)\,\sigma^{\mu\nu}\,(1 - \not{v}). \qquad (49)$$

Heavy-quark spin symmetry ensures that the relation (47) holds for any matrix Γ. Thus, we are free to set $\Gamma = \gamma_5(\gamma_\mu v_\nu - \gamma_\nu v_\mu)$ or $\Gamma = i\gamma_5\sigma_{\mu\nu}$, in which case we find relations between the parameters λ_E^2 and λ_H^2 and the matrix elements of the pseudoscalar operators $O_1^{(5)}$ and $O_2^{(5)}$ evaluated between a pseudoscalar meson and the vacuum:

$$\langle\,0\,|\,O_2^{(5)}\,|M(v)\rangle = -i\sqrt{m_M}\,F(\mu)\lambda_E^2(\mu),$$
$$\langle\,0\,|\,\tfrac{1}{2}O_1^{(5)} - O_2^{(5)}\,|M(v)\rangle = -i\sqrt{m_M}\,F(\mu)\lambda_H^2(\mu). \qquad (50)$$

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[4]There are also contributions from operators that are total derivatives; however, their matrix elements can be expressed in terms of parameters already encountered at order $1/m_Q$ in the expansion.

In the rest frame of the hadron, these are matrix elements of purely chromo-electric and chromo-magnetic operators, respectively.

The scale dependence of these matrix elements is determined by the scale dependence of the operators O_1 and O_2, which is described by the evolution matrix given in (14). We obtain

$$r_0 \lambda_E^2(m_Q) = \left(\frac{r_+ + r_-}{2} - \frac{N}{2} \frac{r_+ - r_-}{\gamma_+ - \gamma_-} \right) \lambda_E^2(\mu) - \left(\frac{3}{2} N - \frac{4}{3} C_F \right) \frac{r_+ - r_-}{\gamma_+ - \gamma_-} \lambda_H^2(\mu)$$

$$\simeq (0.7413\, r_- + 0.2587\, r_+)\, \lambda_E^2(\mu) + 0.4379\, (r_- - r_+)\, \lambda_H^2(\mu)\,,$$

$$r_0 \lambda_H^2(m_Q) = \left(\frac{r_+ + r_-}{2} + \frac{N}{2} \frac{r_+ - r_-}{\gamma_+ - \gamma_-} \right) \lambda_H^2(\mu) - \left(\frac{3}{2} N - \frac{4}{3} C_F \right) \frac{r_+ - r_-}{\gamma_+ - \gamma_-} \lambda_E^2(\mu)$$

$$\simeq (0.7413\, r_+ + 0.2587\, r_-)\, \lambda_H^2(\mu) + 0.4379\, (r_- - r_+)\, \lambda_E^2(\mu)\,, \qquad (51)$$

where the factor r_0 on the left-hand side compensates for the scale dependence of the parameter $F(\mu)$, according to the equation $F(m_Q) = r_0 F(\mu)$, with r_0 given in (26).

Let us now discuss the specific example of meson decay constants to leading logarithmic order. The local corrections in the heavy-quark expansion are obtained from the expansion of the flavour-changing current $\bar{q}\,\gamma^\alpha(1 - \gamma_5)Q$, using [3]

$$Q \to h_v + \frac{i\slashed{D} - iv\cdot D}{2m_Q}\, h_v + \frac{g_s}{4m_Q^2}\, \gamma_\mu v_\nu\, iG^{\mu\nu} h_v + \dots \,. \qquad (52)$$

The corresponding expressions for the meson decay constants are

$$f_M \sqrt{m_M} = F(m_Q) \left\{ 1 - d_M \left(\frac{\bar{\Lambda}}{6m_Q} + \frac{\lambda_E^2(m_Q)}{12m_Q^2} + \dots \right) + \text{non-local terms} \right\}\,, \qquad (53)$$

where the spin-dependent coefficient d_M takes the values 3 and -1 for pseudoscalar and vector mesons, respectively. The non-local matrix elements of order $1/m_Q$ are discussed in Ref. [15]. The new ingredient in (53) is the local correction of order $1/m_Q^2$, which is determined by the single parameter λ_E^2 renormalized at the scale m_Q. Using our result (51), this parameter can be related to the values of λ_E^2 and λ_H^2 renormalized at some lower scale μ, which is typically the scale intrinsic to some non-perturbative calculation of these hadronic parameters. For instance, QCD sum rules have been used to predict that $\lambda_E^2(\mu) \approx$ 0.11 GeV^2 and $\lambda_H^2(\mu) \approx 0.18$ GeV^2 at the scale $\mu \approx 1$ GeV [16]. For the b quark, e.g., using the numbers given at the end of Sec. 4, this translates to $\lambda_E^2(m_b) \approx 0.69\,\lambda_E^2(\mu) + 0.08\,\lambda_H^2(\mu) \approx 0.09$ GeV^2, which is the value to be used in (53). The corresponding correction to the B-meson decay constants is clearly very small, indicating a good convergence of the local terms in the heavy-quark expansion. Before a meaningful prediction for the decay constants at order $1/m_Q^2$

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can be obtained, it would however be necessary to include the non-local terms as well.

For completeness, we note that the hadronic parameters λ_E^2 and λ_H^2 are of a more general interest, since they are related to the second moments of meson wave functions $\varphi_\pm(\omega, \mu)$. Introducing a vector z on the light-cone ($z^2 = 0$, $v \cdot z \equiv t$) and working in light-cone gauge ($A_+ = 0$), we define

$$\frac{1}{2\pi} \int \mathrm{d}t \, e^{i\omega t} \, \langle \, 0 \, | \, \bar{q}(0) \, \gamma_\pm \Gamma \, h_v(z) \, | M(v) \rangle_\mu = \frac{1}{2} \sqrt{m_M} \, F(\mu) \, \varphi_\pm(\omega, \mu) \operatorname{Tr} \{ \gamma_\pm \Gamma \, \mathcal{M}(v) \} \,, \tag{54}$$

where $\gamma_\pm$ are the light-cone projections of the Dirac matrices. Defining the moments

$$\langle \omega^n \rangle_\pm = \int\limits_0^\infty \mathrm{d}\omega \, \varphi_\pm(\omega, \mu) \, \omega^n \,, \tag{55}$$

which are normalized such that $\langle \omega^0 \rangle_\pm = 1$, one can show that [16][5]

$$\langle \omega \rangle_\pm = \left(1 \pm \frac{1}{3} \right) \bar{\Lambda} \,,$$
$$\langle \omega^2 \rangle_\pm = \left(\frac{4}{3} \pm \frac{2}{3} \right) \bar{\Lambda}^2 + \left(\frac{1}{3} \pm \frac{1}{3} \right) \lambda_E^2(\mu) + \frac{1}{3} \lambda_H^2(\mu) \,. \tag{56}$$

Our results in (51) can be used to control the scale dependence of the second moments.

7   Conclusions

We have discussed the hybrid renormalization of local, scalar and pseudoscalar dimension-5 operators containing a heavy and a light quark field. Our results determine the scale dependence and mixing of such operators at scales below the heavy-quark mass. We have calculated the corresponding anomalous dimensions at the one-loop order, which is sufficient to obtain the renormalization-group evolution of the operators in the leading logarithmic approximation.

Two applications of our results have been discussed in detail. The first one concerns the renormalization of genuine dimension-5 QCD operators at low renormalization scales. Important examples are the gluon and photon penguin operators, which appear in the effective weak Hamiltonian renormalized at the scale m_b, and whose evolution at high scales $\mu \gg m_b$ is well known. We have discussed the mixing of these penguin operators at a low scale $\mu \ll m_b$ and derived the effective Hamiltonian for this case. This is relevant for the calculation of hadronic matrix elements of the penguin operators performed, e.g., using lattice

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[5]These results are valid in a regularization scheme without a dimensionful regulator, such as dimensional regularization.

simulations. In addition to the evolution at leading logarithmic order, we have also performed the full one-loop matching of gluon and photon penguin operators onto their effective-theory counterparts. Besides providing a test of our results for the operator anomalous dimensions, this calculation will eventually be part of a full next-to-leading order renormalization-group improved analysis, once the two-loop anomalous dimensions of the operators will have been calculated.

The second application concerns the calculation of certain higher-order corrections in the heavy-quark expansion of current matrix elements, such as they appear in the description of weak decay form factors. In particular, local dimension-5 operators appear at order $1/m_Q^2$ in the heavy-quark expansion of meson decay constants. We have discussed, as an example, the local $1/m_Q^2$ corrections to the decay constants of heavy mesons at the scale m_Q, at which a single new hadronic parameter appears. Our results can then be used to evolve the result down to a lower scale μ, at which non-perturbative evaluations of the relevant hadronic matrix elements may be performed. A similar analysis could be performed for semileptonic transition form factors describing, e.g., semileptonic decays such as $B \to \pi \, \ell \, \nu$.

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