Part 5
12.7.67

FUNDAMENTALS OF QUANTUM MECHANICS
AND PARTICLE PHYSICS

O.R. Frisch

(Cambridge University)

Lectures given for the
Academic Training Programme
1 9 6 7

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15. ORBITAL ANGULAR MOMENTUM

We have seen that the solutions of Schrödinger's equation for a central field, where the potential depends only on the distance from the centre of force, are not themselves spherically symmetrical. They have a dependence on ϑ and φ which is given by the Legendre polynomials and are characterized by two quantum numbers, m and ℓ, where these have the meaning that ℓ refers to the total angular momentum and m refers to the component of the angular momentum in the z direction. Thus the total angular momentum is

$$\mathcal{M} = \ell(\ell + 1)\hbar^2 \tag{97}$$

and its z component is

$$\mathcal{M}_z = m\hbar \ .$$

I should now like to give a better explanation for this, which can be done by using operators. From classical physics the angular momentum in the z direction, for example, is given by

$$\mathcal{M}_z = x \cdot p_y - y \cdot p_x \ . \tag{98}$$

I have already shown you that results can be obtained in quantum mechanics by replacing the momentum in the classical expression by the corresponding operator,

$$p_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \ . \tag{73}$$

Applying this to Eq. (98) we obtain

$$\mathcal{M}_z = x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x} \tag{99}$$

or, in spherical polar coordinates,

$$\mathcal{M}_z = \frac{\hbar}{i}\frac{\partial}{\partial \varphi} \ . \tag{100}$$

If we now apply this very simple operator to a wave function of the simple form

$$\psi = R\,\Theta\,e^{im\varphi} \tag{101}$$

then we obtain*)

$$\mathcal{M}_z\psi = \frac{\hbar}{i}\, im\, \psi = \hbar m\, \psi \; . \qquad (102)$$

Thus the operator $\mathcal{M}_z$ acting on ψ is equivalent to multiplying ψ by a number $\hbar m$. This number gives the value of the physical quantity corresponding to the operator $\mathcal{M}_z$, and we call this an eigenvalue. The meaning of that word is that if you have the right sort of wave function, operating on it with a given operator will merely multiply the wave function by a constant. We interpret this physically to mean that if the wave function is of the form of Eq. (101) the particle in the central field carries a component of angular momentum in the z direction, $\mathcal{M}_z$, equal to $\hbar m$.

This is all very formal, and perhaps I should bring in a little more physics. Let us consider a charged particle, an electron say, in a central field, and let us apply a magnetic field H. Strictly speaking we should rewrite the Schrödinger equation, putting in an extra term to represent the interaction between the particle and the magnetic field, and then solve it. Fortunately, this is one of the cases, of which there are several in physics, where you can use quantum mechanics without actually applying the mechanism, but where you can use the classical analogue suitably applied. Classically, if you consider a charged particle moving in a circular path, then the magnetic moment is

$$\mu = \text{current} \times \text{area}$$

$$= \frac{e\Omega}{2\pi c} \cdot \pi r^2$$

or

$$\mu = \frac{e\Omega r^2}{2c} \; , \qquad (103)$$

where Ω = the angular momentum of the electron.

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*) A bad habit quantum mechanicians have is to write operators in a way indistinguishable from ordinary algebraic quantities, leaving it to the intelligence of the reader to guess whether he means that ψ is multiplied by a number $\mathcal{M}_z$, or operated on by an operator $\mathcal{M}_z$.

Now the angular momentum in the z-direction is

$$\mathcal{M}_z = M r^2 \Omega , \tag{104}$$

so that

$$\mu = \mathcal{M}_z \cdot \frac{e}{2Mc} . \tag{105}$$

Now we can argue that since we know that the angular momentum is quantized, we can obtain the quantum mechanical form of this equation as

$$\mu = m \left(\frac{\hbar e}{2Mc}\right) \tag{106}$$

and this can indeed be confirmed by proper quantum mechanical calculation. These would then be the values of the magnetic moment of our particle in the z-direction. The quantity in brackets is a kind of 'universal' magnetic moment and is often called the Bohr magneton; it has a value of

$$\mu_0 = 0.9 \times 10^{-20} \text{ cgsu} .$$

Thus if we have a magnetic field H, the energy can be written as

$$W = W_{H=0} + W_H ,$$

where the component due to the magnetic field is

$$W_H = \underline{H} \cdot \underline{\mu} = H_z \mu_z = H \mu_0 m ,$$

if we choose our z-direction in the direction of the magnetic field. So if we switch on a magnetic field then the degeneracy between all these different m-values disappears and the values will line themselves up with certain magnetic energy values, 0, $\pm\hbar\mu_0$, $\pm 2\hbar\mu_0$ and so on, depending on the value of ℓ.

It is sometimes difficult to realize that for whichever axis you choose, the angular momentum around that axis will be quantized. One has the feeling that if you have chosen one particular direction of angular momentum, then if you change your magnetic field direction slightly, surely that ought to mean a different amount of angular momentum about the new axis. But the point is similar to that in the polarization of light. If you take light polarized in a certain direction (Fig. 42), and pass it through a Polaroid screen aligned in a different direction, the transmitted beam will be weakened by a certain factor, $\cos^2 \alpha$. But what does this mean to a single photon? A single photon either goes through or it does not; what is affected is the probability that it should go through. If the Polaroid is parallel to the plane of polarization of the light, and if it is an ideal Polaroid, then every photon will go through, and the probability of transmission is unity. If there is an angle between Polaroid and light then the probability becomes smaller. You can express this by saying that if you take light polarized in any direction and you determine with the aid of a Polaroid screen the polarization in that direction you will always again find that the light is either polarized in that direction or it is polarized crosswise. The second type does not go through, the first does. This problem should really be discussed with the nature of measurements, how they are performed and what they do to wave functions, but for the moment I shall just urge you to remember this similarity with a photon and say that if you choose any given axis then your wave function can be decomposed, in the same way as polarized light can be decomposed according to a chosen system of axes, into different wave functions corresponding to the different values of m, from $-\ell$ to $+\ell$.

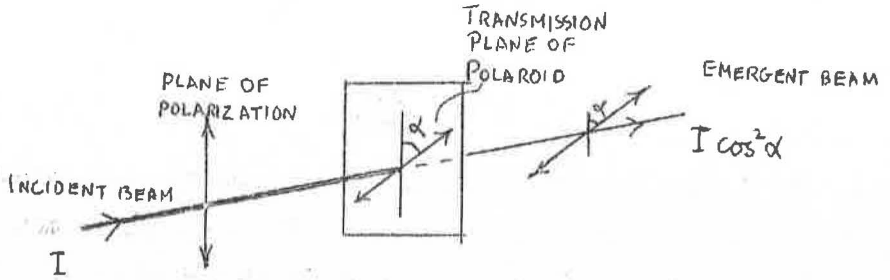

Figure 42

One way of making this whole thing more clearly visible is to draw the so-called vector model (Fig. 43). Here we represent the angular momenta as vectors, as one often does, and there we have the particular case when $\ell = 2$. In this case the total angular momentum is of course $\sqrt{\ell(\ell + 1)} = 2.4$, so I have drawn a semicircle with a radius of

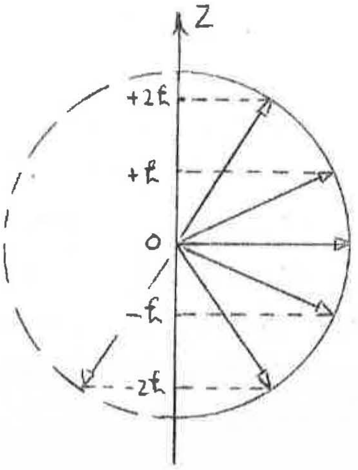

Figure 43

2.4 units. On the other hand I have marked on the z-axis those points which correspond to 0, ±ħ and ±2ħ. Now how do you combine total angular momentum of 2.4 with a momentum component of, shall we say, -2? You can draw an arrow out to the semicircle and you might imagine that if the particle is in the state -2ħ for its z-component of angular momentum, then the vector representing it will be pointing in approximately that direction given by the diagram. Strictly speaking, of course, the semicircle of Fig. 43 is only a section of a sphere and the vector can lie anywhere on the cone. This is a way of visualizing angular momentum, but as is usual with such models one has to be careful not to take it too literally. If one ascribes a precisely defined direction to an angular momentum then difficulties arise and one has to remind oneself that the only quantities which really have meaning in a given situation are the total angular momentum $\mathcal{M}$ and the component in one arbitrarily chosen direction z.

16. HYDROGENIC ATOM

So much at present for the angular momentum. Now I have already indicated how the Schrödinger equation can be solved with respect to its variation along the radius, and here I shall write down the answers which you get if you carry out this solution for what I call the 'hydrogenic atom'. Now the difference between a hydrogenic atom and a hydrogen atom is that in a hydrogenic atom I do not assume that the charge of the nucleus is unity, nor do I assume that the masses of all the particles are the same as in hydrogen. In fact anything is a hydrogenic atom if it consists of two opposite charges which attract each other more or less according to a Coulomb field and which may have any mass and charge whatsoever. Let us consider first a nucleus with charge +Ze, and an electron with charge -e, which then at a distance r will have a potential

$$V = \frac{-Ze^2}{r} \qquad (107)$$

In this case the calculation gives the following simple result. Since for great distances V approaches the value zero, obviously for a particle energy greater than zero all values are permitted because the particle is not held in the field, but remains essentially free, with a wave function only temporarily modified as it runs through the field. Thus we have

$$\left.\begin{array}{ll} \text{for } W > 0, & \text{all values} \\ \text{for } W < 0, & W = \dfrac{1}{n^2} W_0 \end{array}\right\} \qquad (108)$$

where

$$W_0 = - \frac{M(Ze^2)^2}{2\hbar^2} \qquad (109)$$

This is exactly the same formula which Bohr gave in 1913, on the basis of a totally different model*). This is certainly striking, because Bohr had assumed that the particle runs in circles, whereas here no such limiting assumption is made. In fact all these values come out with $\ell = 0$, where if you want to make any kind of model you would have to imagine the electron making a straight dive at the nucleus, missing it by a small distance, being flung back and flying out again, tracing infinitely narrow ellipses around the nucleus. Yet the energies come out the same and I have sketched here (Fig. 44)

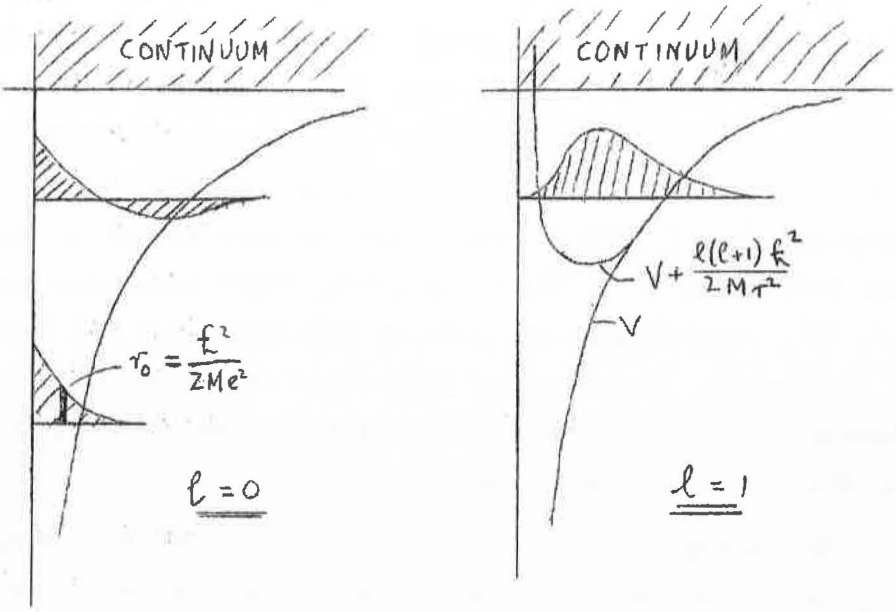

Figure 44

what the wave functions look like. The interesting thing is that if you now look for other values, for instance with $\ell = 1$, then you get quite different wave functions.

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*) This is Schrödinger's result of 1926. In 1928 Dirac wrote down what seemed a very different wave equation from Schrödinger's and again the same values came out, at least in a first approximation. At that time I think it was Weisskopf who said that "the hydrogen atom is invariant against theories".

For the case $\ell = 0$, as one might expect, the wave function is a maximum at the centre, and has in fact the unique property of hitting the centre at an angle. (No other wave function does that, no sensible wave function, that is!) This one does so because the potential drops to minus infinity. This of course is not strictly true; one ought to ascribe a finite size to the nucleus, thus 'rounding off' the lower end of the trough so that it behaves in a normal way. At any rate it becomes largest near the centre.

If you go to the case $\ell = 1$, then you have orbits which do go around the nucleus in some fashion, rather than head straight for it. Therefore, since classically they always miss the nucleus by a considerable margin, you would expect the wave function to decrease in the vicinity of the nucleus. The reason that it does so is that, as you remember, when ℓ differs from 0 then we have to add to the potential a term containing r^2 in the denominator, which increases very rapidly, more than compensating the Coulomb function, when you approach closely to the centre. The electron thus finds itself in a field containing a minimum at a certain definite value of r, and as a result the wave functions are small near the centre.

The oddity of the Coulomb field is that for this field alone, and only in the non-relativistic approximation, the two energies $n = 2$ indicated in Fig.44 are exactly alike, and the same is true of higher levels not shown. You get in fact a spectrum of levels as shown at the right of the diagram. For $\ell = 0$ you have levels given by Eq. (109) for all values of n. For $\ell = 1$ you have a similar pattern, but with the lowest level missing. For $\ell = 2$, the two bottom ones are missing and so on. In the ideal Schrödinger Coulomb field solution the values of energy are all represented in the so-called s-states ($\ell = 0$); the p ($\ell = 1$), d ($\ell = 2$), etc. states do not produce any new energy levels, but merely reproduce the old ones.

Now this is in the first place not true, as I have already stressed, in the relativistic case. (Of course the Schrödinger equation just is not relativistic.) One obtains excellent agreement with the measurements in the equation by Dirac, which I cannot speak about at the moment, except to say that this takes account of the increase of mass of the electron as it travels very fast near the nucleus, and the correction to the hydrogen spectrum is then of the order of

$$\left(\frac{e^2}{\hbar c}\right)^2 = \left(\frac{1}{137}\right)^2 ,$$

For that reason this constant within the brackets is usually called the fine structure constant*).

This constant is a useful thing to remember. For instance, once you have obtained the formula for the energy states of the hydrogenic atom

$$W_1 = -\frac{M(Ze^2)^2}{2\hbar^2} , \qquad (109)$$

if you require to know this quantity in energy units, you can of course slog it out, inserting known values for the constants, and if you do not lose any powers of ten on the way, which is very unlikely, then you will obtain the answer. But it is often possible to change

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*) There has been a great deal of thought about why the constant should have this value, which is non-dimensional and does not depend on the system of units employed, but so far no theory exists that produces that value at all well. There was a theory by Eddington, but unfortunately nobody understood it. There is now, I am told, a new theory by Heisenberg which gives at least a rough value, and which at least Heisenberg understands.

the formula in such a way that you do not have to put quite so many constants in. Here, for instance, we can write:

$$W_0 = - Z^2 \cdot \frac{1}{2} \cdot \left(\frac{e^2}{\hbar c}\right)^2 \cdot Mc^2$$

$$= - Z^2 \cdot \frac{1}{2} \cdot \left(\frac{1}{137}\right)^2 \cdot 511 \text{ keV}$$

$$= - Z^2 \times 13.6 \text{ eV} .$$

where

$$Mc^2 = 511 \text{ keV}$$

is the rest mass of the electron. (This is another quantity which it is worth learning by heart.) If you carry out this much simpler calculation you are much more likely to arrive at the correct answer. The value of 13.6 eV is simply then the energy value of the lowest state, or in other words it is the energy which you have to give to a hydrogen atom in the ground state in order to take the electron to the edge of the continuum. It is, in other words, the ionization energy.

As I said, this equality of the levels with different ℓ is no longer true in the relativistic case. People sometimes call it an 'accidental' degeneracy, one which happens to occur in a particular approximation (which is not really valid). But there are other reasons why it should not be a genuine degeneracy, one of them being that in many atoms you can apply this sort of calculation approximately. For instance, if you consider a sodium atom this can be regarded as containing a nucleus surrounded by many electrons fairly close by. Then the wave function of the last electron is much more widely spread out, and can be represented fairly well by an orbit lying outside the main electron cloud. This is of course not true, because it is an s orbit and an s-wave function is always densest near the centre. Much of it, however, lies outside the others. Therefore, since the ten inner electrons surround the nucleus tightly, the outer electron sees only a single nuclear charge. Thus it moves in a field that is very steep at the centre, but flattens out where the screening effect of the other electrons come into play (Fig. 45). The result is that if the electron is in a p or d or f state, with a high value of ℓ then most of its wave function is in the shielded region and the electron 'knows' nothing of the strong nuclear charge. But when it is in an s-state then it experiences the effect of that charge, at least for those short periods when it finds itself near the centre, and therefore its energy is lowered. The result is that the p-states are slightly higher in energy than the s-states (Fig. 45).

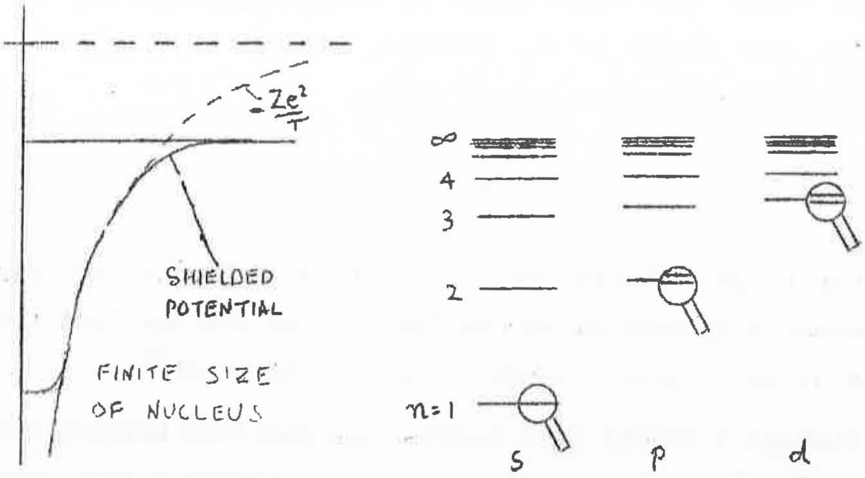

Figure 45

There is an opposite effect in principle, due to the finite size of the nucleus, rounding off the lower end of the Coulomb potential curve (Fig. 45). This effect is extremely small in the case of ordinary electron spectra, and it is only by the extraordinary accuracy of which spectroscopists are capable that it has in fact been observed. It can be used to give an estimate of the nuclear size, but not very much more. On the other hand, we have today a variety of other hydrogenic atoms available, for instance one consisting of a nucleus with charge +Ze around which circulates a muon, instead of an electron. Now a muon is really a 'heavy' electron, which has 206.7 times the electron mass, but otherwise its properties are identical. Thus, considering Eq. (109) you will see that it is bound with energies two hundred times greater, of the order of kilo-electron volts rather than a few electron volts, particularly if Z is large. The other point is that a nuon will on the whole spend its time much closer to the nucleus, the characteristic distance (the so-called Bohr electron radius) being

$$r_0 = \frac{\hbar^2}{ZMe^2} \ . \tag{110}$$

Thus for large Z (heavy nuclei) and large M (muons) the particle approaches very much closer to the nucleus and the fact that the latter is not a point charge becomes significant*).

Perhaps I should just mention the relation between all this and spectroscopy. In spectroscopy you take something like sodium vapour and excite it, for instance by passing an electric current through it, causing collisions between atoms and electrons. Thus you produce atoms in excited states. Such an atom will return to the ground state either directly or possibly via an intermediate state. In these transitions light is emitted with a frequency following from Planck's relation. This is all very elementary atomic physics, and I am sure that you already know all about it. What we really do is to measure these frequencies and from them reconstruct (quite a laborious process) the levels by which we can account for the observed frequencies.

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*) Measurements are being carried out on this and similar systems in order to determine the sizes of nuclei. The circulating particle may also be a π-meson or a K-meson, having a still larger mass but these are very short-lived, thus making the measurements more difficult and less accurate.

One interesting fact which was discovered experimentally is that you only get transitions between levels in adjacent rows (Fig. 46),

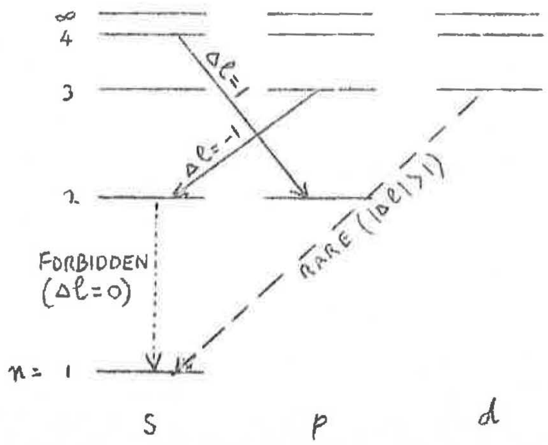

Figure 46

that is, between states whose ℓ-values differ by one. Other transitions do not occur. An electron that is in the lowest excited s-level is trapped. We say that the atom is in a metastable state. It cannot emit light, and it usually hangs around for quite a while until it can lose its energy as kinetic energy in a collision with another atom. The reason for this so-called selection rule comes out in the solution of the time-dependent Schrödinger equation, but at the moment I shall merely remind you of the diagrams of the wave function of an oscillator, superimposing two wave functions of adjacent quantum number, and pointing out that the result is a wave function which alternates from side to side. Such a system would be liable to emit light if the particle in question was charged. Translated into proper quantum mechanical language this means that if you emit light by going from one quantum state to another there exists temporarily a state where both quantum states are present together, and this superposition must have the necessary property of emitting light. This condition is fulfilled in the case where you superimpose on s and a p wave function, or else a p and a d function, but any such oscillation dipole would be absent if you superimposed two s wave functions, which have the same angular distribution. Similarly a transition from a d to an s state would give a quadrupole radiation, and would therefore be a very unlikely transition, although not completely forbidden.

17. SPIN

We now come to consider the role of spin. One of the puzzles, historically speaking, was that all levels in this sort of alkali system are double, with the exception of the s levels (Fig. 45). This was not understood, and people made a variety of theories, but in the meantime other clues accumulated. One of the best clues was the Stern-Gerlach experiment (1921), illustrated in Fig. 47.

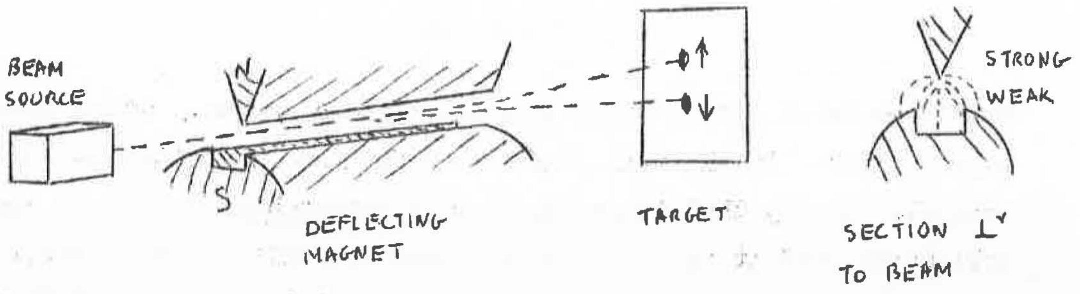

Figure 47

This experiment was performed before the formulation of quantum mechanics in 1926, and at the time the result was still pretty mysterious, although it had been foreseen by some forms of the old quantum theory and in fact the whole experiment was done in order to check this.

The experiment is most simply explained if you assume that we fire a beam of hydrogen atoms (for technical reasons Stern and Gerlach used silver atoms, but the properties involved are the same) into a magnetic field formed by the pole pieces of a specially designed magnet, in which the lines of force are strongly divergent in a plane perpendicular to the atomic beam. It is found that two beams emerge from the field. If we look at this result in the light of what we have done so far, we would certainly expect a deflection, because in a magnetic field, depending on the orientation of the system we can have different energy values. Those values which are positive will be the more positive the more the atoms are in the strong field, and therefore these will be repulsed from the knife-edge down towards the groove, where the field is weaker. Conversely those having a negative energy will be deflected towards the knife edge.

There are two things that are wrong with this result. One is that the hydrogen atom in its ground state is supposed to have $\ell = 0$, and no angular momentum, therefore no magnetic moment and no force from the magnetic field. The other point, which is perhaps more serious, is that even if we were wrong, and the hydrogen atom for some reason did have an angular momentum with $\ell \neq 0$, the number of possible orientations and therefore the number of beams emerging should always be odd, being equal to $2\ell + 1$, where ℓ is an integer. So the observed result could not be explained at all on the basis of the theory of angular momentum as I have presented it, which, however, did not exist at the time. This result was just one more puzzle, among many. But it was at least a very important clue and when the idea of angular momentum and its quantization was properly formulated it became clear that in order to get an even number of beams, and in particular two, you must have something like

$$'\ell' = \frac{1}{2} \tag{111}$$

This of course does not make sense in connection with spatial wave functions and spherical harmonics, which are described by integral values of ℓ. On the other hand it turns out that the kind of operator description we have already used can be generalized in such a way that we no longer use the differential symbol, but only the algebraic properties of such an operator, the so-called commutation relations. When that is done it turns out that values of a half quantum number are possible. This was a belated justification of the proposal made in 1925 by Goudsmit and Uhlenbeck that the electron should possess an angular momentum of one half a quantum unit, irrespective of its spatial wave function. It would therefore also possess it in the ground state of the hydrogen atom.

If you place such an electron in a magnetic field it would then be capable of possessing only two energy values. The proposal of Goudsmit and Uhlenbeck amounted to saying that the z-oomponent of spin should be capable only of the values

$$s_z \hbar = \pm \frac{1}{2} \hbar^{*)}. \qquad (112)$$

The energy, however, is given by

$$W_H = \pm s_z \mu_0 H g \qquad (113)$$

The factor g, called the gyromagnetic ratio, Goudsmit and Uhlenbeck assumed to have the value 2, simply because that accounted for a number of spectroscopic observations existing at the time.

The next development was that Dirac two years later formulated a relativistic wave equation for the electron and to everybody's amazement, including I am sure his own, when he worked out the consequences one of them was that such an electron would possess an angular momentum and a magnetic momentum exactly as Goudsmit and Uhlenbeck had deduced from the experimental facts. From 1928 onwards it was in principle possible to ignore all that we have discussed above and say that the electron must have a spin because it obeys Dirac's equation.

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*) This fulfils the condition that we had previously, namely that the step between successive values of angular momentum is always unity. It turns out that this is an essential condition and it just allows values of one half. There is no fear that one day we shall be faced with yet finer subdivisions of the angular momentum.

On the other hand, if the experiments had not already existed Dirac's equation probably would not have been seized on with so much joy. This equation was later improved and carried to a further approximation, and it turns out that the factor g takes the slightly greater value of 2,0023, the difference being of the order of the fine structure constant. There has been very great interest in these refinements, because naturally one always wants to know how accurate a given theory is, and therefore this quantity has been measured with very great care both for the electron and particularly here in CERN for the muon. It turns out that the value is exactly what Dirac's theory demands, so it can be said to have been verified to an accuracy of about one part in a million.

One more thing which I want to discuss is the way in which we can describe this spin phenomenon. We obviously cannot describe it, in the way of spherical harmonics, as something that comes out of a wave equation, except in a much more abstract way out of Dirac's equation. What we say rather is that the electron has not only the three degrees of freedom characterized by its spatial coordinates, but also a fourth, in that it can point up or down. (As I said before, we can arbitrarily choose our direction, but once we have chosen it there are only these two states possible.) So if you want to describe an electron completely you would have to write something of the form

$$\psi(x,y,z)\chi(s_z).$$

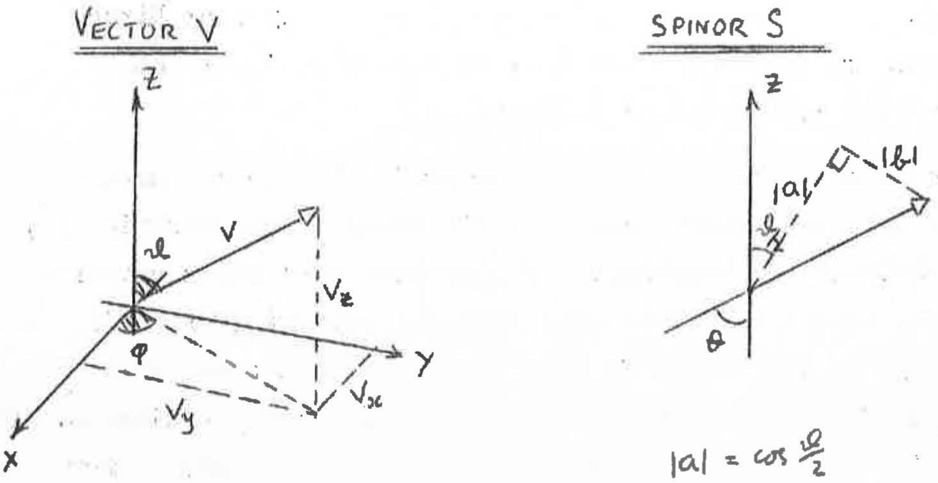
The function χ simply stands for a pair of numbers, one value belonging to the direction 'up' and another to the direction 'down'. This is often written as

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

where you have to fix some convention that, shall we say, a corresponds to the amplitude of the wave function for the 'upward-pointing' electron and b for the 'downward-pointing' one, and further we demand that, since the electron must be in either one state or the other,

$$|a|^2 + |b|^2 = 1. \qquad (114)$$

In Fig. 48 I have shown how this sort of quantity can be written down with a little algebra, and in particular how such a two-number symbol, which is called a spinor, transforms when you change the coordinate system.



$$V_x = \sin\vartheta \cos\varphi$$
$$V_y = \sin\vartheta \sin\varphi$$
$$V_z = \cos\vartheta$$
$$V = (V_x, V_y, V_z)$$

$$|a| = \cos\frac{\vartheta}{2}$$
$$|b| = \sin\vartheta/2$$
$$a = e^{-i\varphi/2} \cos\frac{\vartheta}{2}$$
$$b = e^{+i\varphi/2} \sin\vartheta/2$$
$$S = \begin{pmatrix} a \\ b \end{pmatrix}$$

Figure 48

You all know how a vector is commonly described in a coordinate system by its three projections V_x, V_y and V_z. Since for a unit vector

$$V_x^2 + V_y^2 + V_z^2 = 1, \qquad (115)$$

one of these is redundant, and can be obtained from the other two. Now if you want to represent that same oriented quantity as a spinor, you want to describe it by only two components. The prescription for that, which I give you without derivation, is the following. You take half the angle between it and the z-axis, giving, for a unit spinor/vector,

$$\left.\begin{aligned} |a| &= \cos\frac{\vartheta}{2}\,, \\ \text{and} \qquad |b| &= \sin\frac{\vartheta}{2}\,. \end{aligned}\right\} \tag{116}$$

This guarantees that the condition

$$|a|^2 + |b|^2 = 1 \tag{114}$$

is fulfilled. If a and b are given, then you know ϑ, and can determine what angle the spinor makes with the axis. Moreover the reason for the choice of the half-angles can also be explained as follows. If you have an ordinary vector, as for example polarized light, pointing at 45°, shall we say, then its projections on two perpendicular axes are equal, corresponding to the fact that if such light is passed through a Nicol prism we find half the intensity in one or, if we turn the prism through 90°, half the intensity in the other beam. We could in principle do the same experiment on a beam emerging from the Stern- Gerlach experiment. Let us assume that the electron comes out of the magnetic field with its spin pointing upwards in the upper beam and downwards in the lower beam (Fig. 49). Now stop one of these beams and let the other pass through a second Stern- Gerlach experiment which has been rotated by 90° with respect to the first.

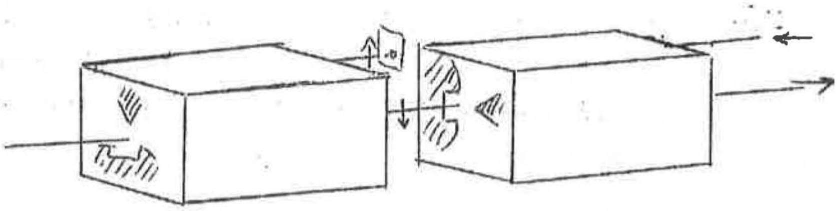


Figure 49

We are trying to decompose a beam, all of whose electrons are spinning vertically upwards, into two components with spins in a horizontal direction. Then, simply for reasons of symmetry the two must be equal, so that we have the completely analogous behaviour here, if we turn the field through 90°, as we have with polarized light if we turn our original direction by 45°, because the two relevant directions for polarized light are perpendicular to one another and therefore a rotation of 45° bisects that angle. The bisector of 180° is 90°, and that shows why we have used half-angles in the definition of the spinor, in order to get this behaviour. If we turn the beam at right angles to the z-axis then a and b both become equal to cos 45°.

This much gives us no information about φ, but since the quantities we write down can be complex, analogous to the complex wave function which we used in the Schrödinger equation, we can squeeze the information in by writing

$$\left.\begin{aligned} a &= e^{-i\varphi/2} \cos\frac{\vartheta}{2}\,, \\ b &= e^{+i\varphi/2} \sin\frac{\vartheta}{2}\,. \end{aligned}\right\} \qquad (117)$$

It turns out that if you put it into this form, then any rotation of the coordinate system does not upset the correlation which we had established. I am afraid that I have not found any way of making this sound plausible with the aid of simple mathematics, so I must just request you to believe me.

So what we have found is that the electron has a fourth degree of freedom, but a freedom limited to either pointing parallel to a given direction or anti-parallel, and that this is associated again with a magnetic moment, but that this magnetic moment is twice as much per angular momentum (the gyromagnetic ratio is twice that associated with ordinary motion). If this hypothesis is adopted we first of all can account for the Stern- Gerlach effect on hydrogen and similar atoms in the ground state, for the doubling of all the states with $\ell > 0$, and various other things. You may ask why do not the s-states split? The point is that the doubling consists of an interaction between the electron spin and the magnetic field caused by the orbital motion. In the s-state there is no angular momentum and hence no orbital motion that can cause a magnetic field to which the spin could orient itself.