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Dynamical Topology Change, Compactification and Waves in String Cosmology

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ABSTRACT

Exact string solutions are presented, where moduli fields are varying with time. They provide examples where a dynamical change of the topology of space is occurring. Some other solutions give cosmological examples where some dimensions are compactified dynamically or simulate pre-big bang type scenarios. Some lessons are drawn concerning the region of validity of effective theories and how they can be glued together, using stringy information in the region where the geometry and topology are not well defined from the low energy point of view. Other time dependent solutions are presented where a hierarchy of scales is absent. Such solutions have dynamics which is qualitatively different and resemble plane gravitational waves.

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1 Introduction, Results and Conclusions

One of the strongest motivations which made (super) string theory popular is the fact that it provides (at least perturbatively) a consistent and maybe even UV finite theory of quantum gravity. Moreover it unifies gravity with other interactions including interactions of the gauge and Yukawa type. It is thus appropriate to try to investigate the behaviour of string dynamics associated with gravitational phenomena. There are several problems in gravity where the classical, and even worse the semiclassical treatment have perplexed physicists for decades. We are referring here to questions concerning the behaviour in regions of strong (or infinite) curvature with both astrophysical (black holes) and cosmological (big-bang, wormholes) interest. It is only appropriate to try to elucidate such questions in the context of stringy gravity. There has been progress towards this direction, and by now we have at least some ideas on how different string gravity can be from general relativity in regions of spacetime with strong curvatures. We need however exact (in α') classical solutions of string theory in order to have more quantitative control on phenomena that are characteristic of stringy gravity.

In this lecture we shall present some exact solutions to string theory in 3+1 dimensions with interesting interpretations like *dynamical topology change*, *dynamical compactification* or *dynamical restoration of gauge symmetries* [1] as well as wavelike solutions [2, 3, 4, 5]. The number of exact solutions to string theory where the background fields indicate some interesting behaviour is not large, but we can however try to get to some conclusions from studying the exact solutions we possess. This is the first time we have a model where we can study for example dynamical topology change in a region where the curvature is strong (of the order of the Planck scale) and where the α' expansion (or organization) of the effective field theory breaks down. However we can extend at the string level our description past the strong curvature region (where the topology change occurs) towards another asymptotic region where we have a different low energy field theory. The topology change is described by a modulus field that varies with time.

Studies on string theory have so far given hints for the presence of several interesting phenomena, like the existence of a minimal distance [6], finiteness at short distances [7], smooth topology change [8, 9], spacetime duality symmetries [10],[11]-[15],[9], variable dimensionality of spacetime [16], existence of maximal (Hagedorn) temperature and subsequent phase transitions [17] etc. The lessons we learn from the exact solutions we are going to describe essentially corroborate some items on the list above and we would like to present them in a somewhat general context.*

We will start from the simple case of a compactification on a flat torus to indicate the idea and we will eventually translate for non-flat backgrounds. The spectrum of string physical states in a given compactification can be generically separated in three kinds of sub-spectra.

i) The first one consists of Kaluza-Klein-like effective field theory modes (or momentum modes). The masses of such modes are always proportional to the typical compactification

*Ideas of a similar form have already been presented in [18].

scale $M_c = 1/\sqrt{\alpha'}R$ where R is a typical radius[†].

ii) Another set contains the winding modes which exist here because of two reasons. The first is that the string is an extended object. The second is that the target space has a non-trivial π_1 so the string can wind around in a topologically non-contractible way. In a large volume compact space, these modes are always super-heavy, since their mass is inversely proportional to the compactification volume M_c , (more precisely, it is proportional to $1/\alpha'M_c$).

iii) The third class of states are purely stringy states constructed from the string oscillator operators. Their masses are proportional to the string scale $M_{str} = 1/\sqrt{\alpha'}$.

This separation of the spectrum is strictly correct in toroidal backgrounds. However, further analysis indicates that one can extend the notions of Kaluza-Klein (KK) type modes and winding modes to at least non-flat backgrounds with some Killing symmetries. This generalization comes with the help of the duality symmetries present in such background fields [11]-[15],[9]. However, the “winding” states are not associated with non-contractible circles of the manifold. They appear as winding configurations in a (usually) contractible circle associated with a Killing coordinate [14]. An example would be given by a winding in the Cartan torus of a group, which is however contractible inside the full group manifold. Because of the above, such modes are special combinations of zero modes as well as oscillator excitations. There may be dual versions of the model where such states are in fact associated with non-contractible circles of the dual manifold. We will see such a case in section 2. Although the above seems to apply to backgrounds with Killing symmetries, we feel that it might be more general. There are indications [12, 14, 19] that one can have duality symmetries without the presence of isometries.

We will illustrate better the issues above in two cases. The first corresponds to Cartan deformations in a WZW model. There, although the Cartan torus is contractible, the role of windings and momenta are played by the eigenvalues of the left and right Cartan generators. The duality map here involves also the oscillators (that is, the currents). We will analyse an example of this sort in the rest of the paper. The second kind of behaviour concerns models of the $SL(2, R)_k/U(1)$ type. There, one has zero modes and oscillator excitations (here we mean the non-compact parafermions), however, unlike the standard case, the energies associated to such oscillators are k -dependent. Duality interchanges zero modes with energies $1/k$ with high lying modes with energies $\sim k$ [14].

Once we have the picture above, concerning different types of string excitations we can state that the notion of a string effective field theory makes sense when the “winding” states as well as the oscillator states are much heavier than the field theory-like KK states. Here, we will assume the presence of a single scale (except α'). If there are more such scales then one has to investigate the different regimes. In any such regime, our discussion below is applicable.

Using the α' -expansion one finds the effective theory which is applicable to low energy processes ($E_t < M_{st}$) among the massless states, as well as the lowest lying KK states,

[†]When there are more than one compact dimensions one can still define the concept of a scale [20].

provided, that the typical mass scale M_c , (which could be a compactification scale, gravitational curvature scale etc) is much below the string scale M_{st} . In the last few years, a lot of activity was devoted in understanding the effective theories of strings at genus zero, [21] and in some cases the genus one corrections were included [22]. The output of this study confirms that the winding and the $\mathcal{O}(M_{st})$ string superheavy modes can be integrated out and one can define consistently the string low energy theory in terms of the massless and lowest massive KK states. Thus, the perturbative string solutions are well described by classical gravity coupled to some gauge and matter fields with unified gauge, Yukawa and self interactions. As long as we stay in the regime where $E_t, \tilde{M}_c < M_{st}$, our description of the physics in terms of the effective theory is good with well defined and calculable $\mathcal{O}(\frac{E_t}{M_{st}}), \mathcal{O}(\frac{M_c}{M_{st}})$ corrections.

The situation above changes drastically once the mass of "winding" type states becomes smaller than the string scale and when (usually at the same time) the KK modes have masses above the string scale. This is the case when the typical M_c scale is larger than M_{st} . When this occurs, the relevant modes are not any more the KK ones but rather the winding ones. Thanks to the well known by now generalized string duality [11]-[15],[9] (e.g. the generalization of the well known R to 1/R torroidal duality) it is possible to find an alternative effective field theory description by means of $\mathcal{O}(\frac{E_t, M_c}{M_{st}})$ dual expansion, where one uses the dual background which is characterized by \tilde{M}_c instead of the initial one characterized by a high mass scale. The dual mass scale $\tilde{M}_c = \frac{M_{st}^2}{M_c}$ is small when the M_c becomes big and vice-versa. We observe that in both extreme cases, either (i) $M_c < M_{st} < \tilde{M}_c$ or (ii) $M_c > M_{st} > \tilde{M}_c$, a field theory description exists in terms of the original curved background metric in the first case or in terms of its dual in the second case. This observation is of main importance since it extends the notion of the effective field theories in backgrounds with associated high mass scales (due to size or curvature).

Then, a general string solution would give rise to a set of effective field theories defined in restricted regions of space-time $(x^\mu)_I, I=1,2,3\dots$ with M_I smaller or of the same order as M_{st} ; If $T_{I,J}$ is the boundary region among $(x^\mu)_I$ and $(x^\mu)_J$, then on $T_{I,J}$ we have almost degenerate effective characteristic scales $M_I \sim M_{st} \sim M_J$. In such regions, the effective field theory description of regions I, J break down (individually), and the full string theory is needed in order to have a smooth transition between the two. A goal in that direction would be to establish some simple rules that would provide the extra (stringy)[‡] information that would patch the field theoretic regions together. This effectively amounts to a reorganization of the α' expansion.[§] The models we are proposing in this paper could very well serve as a laboratory towards answering this question. This is certainly important, since many interesting phenomena happen precisely at such regions in string theory. We can mention, the gluing of dual solutions in cosmological contexts, [25] and global effective theories for large regions of internal moduli spaces.

A related issue here is, that with each region, one has an associated geometry and

[‡]Here by stringy we mean exact to all orders in α' , or equivalently the full CFT description.

[§]In simple backgrounds, like torroidal ones this patching can be effectively done [23] by constructing an effective action, containing an infinite number of fields.

spatial topology, as dictated by the effective field theory. It turns out that moving from one region to another not only the geometry can change but also the topology. Examples in the context of Calabi-Yau compactifications [8] and more simple models, [9] have been given. There is another important point about topology change that we would like to stress here. Where topology change happens, depends crucially on the values of some of the parameters in the background. One such parameter is always α' , but usually, in string backgrounds, there are others, like various different levels for non-simple WZW and their descendant CFTs, various radii or related moduli etc. The absolute judge concerning topology change is the effective field theory.

As we will see in later sections, the solutions we will describe can be viewed as a time dependent background, where a modulus of space (or internal space) changes with time[¶]. For the solution that describes topology change we have at $t = 0$ a manifold with the topology of a disk times a line which evolves at $t = \infty$ to a manifold with the topology of a cylinder times a line. The topology changes in the intermediate (stringy) region. Of course, the backgrounds we are describing are time-dependent and thus, strictly speaking, we loose the meaning of energy (or mass in the compactified case). However, we will show that there are regimes where the time dependence is adiabatic, and where it makes sense to use an approximate concept of energy or mass. Solutions with time dependent radii have been found in the past, solving the string equations to leading order in α' [26]. The advantage of the solutions we present is that they are exact string solutions (to all orders in α').

In this context, we will be able to demonstrate our general picture described above. One of the models considered, is $SL(2, R)_k$ or $SU(2)_k$ with its Cartan deformed. This deformation is parametrized by a new continuous parameter, R such that at $R = 1$ we have the $SU(2)$ or $SL(2, R)$ model. The $SL(2, R)$ family of models was considered in [9] as a simple example of topology change. The topology change happens only at the boundary $R = 0, \infty$.^{||} Before we consider any time variation of R , let us illustrate a few features of the (compact) static model. The spectrum has the form [27]:

$$\begin{aligned}
L_0 + \bar{L}_0 = & 2\frac{j(j+1)}{k+2} - \frac{m^2 + \bar{m}^2}{k} + \\
& + \frac{1}{2k} \left[(m - \bar{m} + kM)^2 R^2 + \frac{(m + \bar{m} + kN)^2}{R^2} \right] + \\
& + N' + \bar{N}'
\end{aligned} \tag{1.1}$$

where $m, \bar{m} \in [0, 2k]$, $M, N, N', \bar{N}' \in Z$. For $R \sim 1$ the low-lying spectrum is close to that of S^3 . The effective field theory contains only the low-lying spectrum and the geometry is certainly that of S^3 . However, for $R \gg 1$, the low lying spectrum is dominated from the Cartan contributions and the geometry, from the effective field theory point of view is that of S^1 . The leftover piece, which is that of $SU(2)_k/U(1)$ is strongly curved (for $k \sim \mathcal{O}(1)$)

[¶]Some solutions of that type were recently described in [24].

^{||}In [9] a variation was given where the topology change happens in the interior of the σ -model moduli space.

and thus not visible at low energy. Thus, we see that the change of effective field theory happens in the $R \gg 1$ or $R \ll 1$ regions and is certainly not describable in the language field theory. In the half line $0 < R < \infty$, we need one effective field theory to describe the region for $R \ll 1$ and $R \gg 1$ and another in-between.

The modulus R can be made to be time dependent without destroying the exactness of the solution. Moreover as we show in the main text there are regions where this dependence is adiabatic. Thus, in the (euclidean) $SL(2, R)_k$ case, we can make the topology change picture described above, dynamical.

We also find solutions in which spacetime starts out (at $t = 0$) as 5 (or more) dimensional but the extra dimension becomes compact and settles at $t = \infty$ to the *self-dual* point $R = 1$, of Plankian size. In such a case the early universe has at late times an extra $SU(2)$ local gauge symmetry generated by the dynamical compactification.

Another application of such solutions could be in considering strings at finite temperature. They would describe a string ensemble with temperature that varies (adiabatically) in space. It might be interesting to entertain such an idea in more detail, in order to investigate temperature gradients in string theory. We will comment on this possibility in the last section.

There are on the other hand exact string solutions, with non-trivial background fields, which behave as flat space, in the sense that there is only a unique scale, and this is α' . Such exact backgrounds have been analysed recently [2, 3, 4, 5] and resemble plane gravitational waves. We will analyse the simplest of these models in order to show the qualitative differences from the models described above and point out some interesting dynamics that might arise in such stringy backgrounds.

2 The time-independent model

We would like to construct exact classical solutions to string theory where some moduli vary with time and their variation can induce geometry or topology change, in the sense presented in the introduction. In order to do that, we will start from a solution where the modulus is time independent, but arbitrary (alternatively speaking, there is no potential for it, in the effective action). This will be the subject of this section where we will use the results of [9]. In the next section we will proceed to include time dependent moduli.

The models we will analyse are related with the Cartan deformations of $SU(2)_k$ or $SL(2, R)_k$ and its euclidean continuation H_+^3 . We will briefly describe the σ -model action for this radius deformation. For more details we refer the reader to [9].

Although the deformation of $SU(2)_k$ was originally found using $O(2, 2, R)$ deformations, [28]* we will present a different approach [9], which has the advantage of showing that the σ -model action and dilaton we will thus obtain, are exact to all orders in the α' expansion

*The fact that $O(2, 2, R)$ transformations can produce marginal current-current perturbations infinitesimally was observed in [14].

(in some scheme).

Conformal perturbation theory indicates that there is a line of theories obtained by perturbing around the $SU(2)$ or $SL(2, R)$ WZW model by $\int J^3 \bar{J}^3$. The theories along the line have a $U(1)_L \times U(1)_R$ chiral symmetry. Thus the σ -model action of these theories must satisfy at least the following four properties:

- 1) It should have $U(1)_L \times U(1)_R$ chiral symmetry along the line.
- 2) It should have the group property: $\delta S \sim \int J^3 \bar{J}^3$ at *any* point of the line.
- 3) At a specific point it should reduce to the known action of the $SU(2)$ or $SL(2, R)$ WZW model.
- 4) The σ -model should be conformally invariant.

In [9] it was shown that properties (1-3) above imply that the σ -model action is:

$$S(R) = \frac{k}{2\pi} \int d^2z \left[\frac{(1 - \Sigma)\partial\theta\bar{\partial}\theta + R^2(1 + \Sigma)\partial\tilde{\theta}\bar{\partial}\tilde{\theta}}{1 + \Sigma + R^2(1 - \Sigma)} + \frac{(1 + \Sigma)(\partial\theta\bar{\partial}\tilde{\theta} - \partial\tilde{\theta}\bar{\partial}\theta)}{1 + \Sigma + R^2(1 - \Sigma)} + \partial x \bar{\partial} x \right] \quad (2.1)$$

where the angles $\theta, \tilde{\theta}$ take values in $[0, 2\pi]$. For $SU(2)$, $\Sigma = \cos 2x$ with $x \in [0, \pi/2] \cup [\pi, 3\pi/2]$ whereas for H_3^+ , $\Sigma = \cosh 2x$ with x real. Finally we can go to $SL(2, R)$ by changing the sign of the $\partial x \bar{\partial} x$ term in (2.1). R is the parameter which varies along the line, $R \in [0, \infty)$.

So far we have not imposed conformal invariance. At one-loop the β -function equations determine the dilaton, which has not been fixed yet:

$$\phi(x, R) = \log\left[1 + \frac{1 - R^2}{1 + R^2}\Sigma(x)\right] + f(R). \quad (2.2)$$

where $f(R)$ is an arbitrary R -dependent constant. The only way (2.1) can change consistent with our requirements (1-3) is by a redefinition of R , which implies that there is a scheme in σ -model perturbation theory where the metric and the antisymmetric tensor receive no higher order corrections in α' . In such a case also the dilaton receives no higher order corrections.[†]

The R -dependent constant in (2.2) can be fixed by the requirements that $\sqrt{G(R)}e^{\phi(R)}$ (which represents the physical string coupling) is invariant along the line. This implies that $f(R) = \log(R + 1/R)$ and that

$$\phi(x, R) = \log[(1 - \Sigma(x))R + (1 + \Sigma(x))/R] + \phi_0. \quad (2.3)$$

[†]The dilaton β -function (central charge) does get corrections. This is what is happening also in the WZW model. However, like in that case, one can replace k with $k + 2$ in front of the action. Then the central charge is given by the classical and 1-loop piece only, without spoiling the vanishing of the other β -functions.

The constant ϕ_0 in (2.3) is R -independent. When $R \rightarrow 0, \infty$, the dilaton has the appropriate asymptotic behaviour. This is an independent way of fixing the constant R -dependent part of the dilaton and the result coincides with the previous method.

Several observations are in order here, [9].

- For the line of theories above one has $R \rightarrow 1/R$ duality. If we parametrize R around the WZW point $R = 1$ as $R = 1 + \epsilon + \mathcal{O}(\epsilon^2)$ then the duality transformation becomes $\epsilon \rightarrow -\epsilon + \mathcal{O}(\epsilon^2)$. This is equivalent to the Weyl invariance ($J^3 \bar{J}^3 \rightarrow -J^3 \bar{J}^3$) of the theory at $R = 1$. The fact that the remnants of the Weyl invariance can be retained far away is guaranteed by the fact the perturbation theory converges and it can also be organized in order to preserve duality order by order. Non-perturbative effects (in α') exist only for $SU(2)$ and there a glimpse at the exact partition function [27] indicates that they don't spoil the symmetry.

- At $R = 0$ the model becomes a direct product of a non-compact boson and the vector coset $SU(2)_k/U(1)_V$ (or $SL(2, R)_k/U(1)_V$). Strictly speaking, the radius of $\tilde{\theta}$ becomes zero, but this is equivalent to a non-compact boson (as can be verified from the exact partition function in the $SU(2)$ case). There is also a constant antisymmetric tensor piece, but in the limit it can be safely dropped since it couples to a non-compact coordinate.

- At $R = \infty$ the model becomes a direct product of a non-compact boson (θ , this time) and the axial coset $SU(2)_k/U(1)_A$ (or $SL(2, R)_k/U(1)_A$).

- The three observations above imply that the axial and the vector cosets are equivalent CFTs.

We will look now at the geometry along the line, [9]. The σ -model metric in the case of deformed $SU(2)$ (in the coordinates $\theta, \tilde{\theta}, x$) is given by

$$G \sim k \begin{pmatrix} \frac{\sin^2 x}{\cos^2 x + R^2 \sin^2 x} & 0 & 0 \\ 0 & \frac{R^2 \cos^2 x}{\cos^2 x + R^2 \sin^2 x} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.4)$$

The scalar curvature \hat{R} is

$$\hat{R} = -\frac{2}{k} \frac{2 - 5R^2 + 2(R^4 - 1) \sin^2 x}{(1 + (R^2 - 1) \sin^2 x)^2}. \quad (2.5)$$

The manifold is regular except at the end-points where

$$\hat{R}(R = 0) = -\frac{4}{k \cos^2 x}, \quad \hat{R}(R = \infty) = -\frac{4}{k \sin^2 x}. \quad (2.6)$$

At $R = 1$ we get the constant curvature of S^3 , $\hat{R} = 6/k$.

It should be noted that the geometric data (metric, curvature, etc.) are invariant under $R \rightarrow 1/R$ and $x \rightarrow \pi/2 - x$. To put it differently the two (dual) branches are diffeomorphic. Another interesting object is the volume of the manifold as a function of R that can be computed to be

$$V(R) \sim \frac{R \log R}{R^2 - 1} \quad (2.7)$$

satisfying $V(R) = V(1/R)$. The volume becomes singular only at the boundaries of moduli space, $R = 0, \infty$. To be more precise, it vanishes there, reflecting the vanishing of the radius of one of the angles (either θ or $\tilde{\theta}$). It is interesting to note though that at the limits, the model factorizes to a theory that has zero volume (corresponding to one of the angles) and another with infinite volume (the coset, this is due to the semiclassical singularity). The zero volume space dominates the infinite volume one. The “string” volume, $\int e^\phi \sqrt{G}$ is constant along the line.

The topology of the manifold is S^3 except at the end-points. Since one of the circles there shrinks to a point we have really a degeneration of the manifold to a disk times a point.

For H_3^+ , the trigonometric functions in (2.5) are replaced by the corresponding hyperbolic functions. Here the manifold has a curvature singularity for $0 \leq R < 1$. It should be noted that here the two dual branches are not diffeomorphic to each other. This is obvious, since one branch is singular and the other is not. The transition point is at $R = 1$, where a singularity “sneaks in” from infinity. In this case the manifold has always infinite volume.

The R marginal deformation generates a continuous family of CFTs that interpolate between two manifolds with different topology: the “cigar” shape ($R = \infty$) with the topology of the disk and the “trumpet” shape ($R = 0$) with the topology of a cylinder. The essential transition happens at $R = 1$. In the $SL(2, R)$ case similar things happen as above.

In [9] we have given also a continuous family of theories, where the topology change happens in the interior of moduli space. There, for example, an S^3 would evolve into a disk times a finite radius circle. We will not pursue further these backgrounds here.

3 Dynamical Topology Change

We would like to make the static picture described in the previous section dynamical, ie. evolving in real time. In order to do that, we would need to add time into the σ -model (2.1). Thus, we will consider the following 4-d σ -model action:

$$\begin{aligned}
S_{3+1} = & S(R(t)) - \frac{1}{2\pi} \int d^2z \partial t \bar{\partial} t - \frac{1}{8\pi} \int d^2z R^{(2)} \Phi = \\
& \frac{k}{2\pi} \int d^2z \left[\frac{(1 - \Sigma) \partial \theta \bar{\partial} \theta + R(t)^2 (1 + \Sigma) \partial \tilde{\theta} \bar{\partial} \tilde{\theta}}{1 + \Sigma + R(t)^2 (1 - \Sigma)} + \right. \\
& \left. + \frac{(1 + \Sigma) (\partial \theta \bar{\partial} \tilde{\theta} - \partial \tilde{\theta} \bar{\partial} \theta)}{1 + \Sigma + R(t)^2 (1 - \Sigma)} + \partial x \bar{\partial} x \right] - \\
& - \frac{1}{2\pi} \int d^2z \partial t \bar{\partial} t - \frac{1}{8\pi} \int d^2z R^{(2)} \Phi(x, t)
\end{aligned} \tag{3.1}$$

where $S_{3+1}(R(t))$ is the action in (2.1) but where the radius has (an unspecified for the moment) time dependence. We would like to find exact string solutions of the form (3.1).

Let us consider the following exact CFT described by the action $\tilde{S} = \tilde{S}_1 + \tilde{S}_2$ with

$$\begin{aligned} \tilde{S}_1 = & \frac{k'}{2\pi} \int d^2z \left[-\partial t \bar{\partial} t + \frac{\tan^2 t}{k} \partial \theta \bar{\partial} \theta \right] - \\ & - \frac{1}{8\pi} \int d^2z R^{(2)} \log[\cos^2 t] \end{aligned} \quad (3.2)$$

$$\begin{aligned} \tilde{S}_2 = & \frac{k}{2\pi} \int d^2z \left[\partial x \bar{\partial} x + \frac{1 + \Sigma(x)}{1 - \Sigma(x)} \partial \phi \bar{\partial} \phi \right] - \\ & - \frac{1}{8\pi} \int d^2z R^{(2)} \log[1 - \Sigma(x)] \end{aligned} \quad (3.3)$$

with $\phi = \tilde{\theta} + \theta/k$. This is (almost) a product of the (minkowskian analytic continuation*) of the $SU(2)_{k'}/U(1)$ coset model (described by action \tilde{S}_1) and of the $SU(2)_k/U(1)$ or the $SL(2, R)_k/U(1)$ coset model (this depends on the form of $\Sigma(x)$). The coupling between the two models will be discussed in detail below.

Let us do a duality transformation on the action \tilde{S} with respect to the angle θ . Then we obtain precisely the σ -model action (3.1) with

$$R(t) = \sqrt{k} \tan(t/\sqrt{k'}) \quad (3.4)$$

and

$$\begin{aligned} \Phi(x, t) = & \log \left[(1 + \Sigma(x)) \cos^2(t/\sqrt{k'}) + \right. \\ & \left. + k(1 - \Sigma(x)) \sin^2(t/\sqrt{k'}) \right] + \text{constant} \end{aligned} \quad (3.5)$$

We have succeeded to find an exact CFT whose (dual) σ -model interpretation is that of a CFT with a modulus varying with time. By considering other tensor products using flat space or the $SU(2)/U(1)$ coset we can construct more exact solutions with different types of time dependence. We will now show that this is all there concerning the time dependence of $R(t)$.

To see this we will solve the (1-loop) β -function equations for the σ -model action (3.1). The one-loop β -function equations imply

$$\frac{R'''}{R''} = \frac{R''}{R'} + \frac{R'}{R} \quad (3.6)$$

for $R(t)$

$$\Phi(x, t) = \log \left[\frac{1 + \Sigma(x) + R^2(t)(1 - \Sigma(x))}{R'(t)} \right] \quad (3.7)$$

for the dilaton (up to a constant) and

$$\delta^{(1)}c = -\frac{6}{k} + \frac{3}{2} \frac{R''}{R'} \left(2 \frac{R'}{R} - \frac{R''}{R'} \right) \quad (3.8)$$

for the one-loop correction to the central charge. It should be noted that (3.6) is invariant under $R(t) \rightarrow 1/R(t)$.

*This is done by rotating the killing coordinate on the imaginary axis and changing the sign of k'

Equation (3.6) has three classes of solutions corresponding to flat space, $SU(2)/U(1)$ or $SL(2,R)/U(1)$ coset models and their duals. In more detail, (3.6) can be integrated to

$$R' = C_1 R^2 + C_2 \quad (3.9)$$

$1/R(t)$ is a solution of the same equation with $(C_1, C_2) \rightarrow (-C_2, -C_1)$. The solutions are (up to shifts in t):

(ia) $C_1 = 0$:

$$R^2(t) = C_2^2 t^2 \quad (3.10)$$

For $t \in [0, \infty)$, $R^2 \in [0, \infty)$.

(ib) $C_2 = 0$:

$$R^2(t) = \frac{1}{C_1^2 t^2} \quad (3.11)$$

For $t \in [0, \infty)$, $R^2 \in [0, \infty)$.

(ii) $C_1 C_2 > 0$:

$$R(t) = \sqrt{\frac{C_2}{C_1}} \tan(\sqrt{C_1 C_2} t) \quad (3.12)$$

For $t \in [0, \pi/2\sqrt{C_1 C_2}]$, $R^2 \in [0, \infty)$.

(iii) $C_1 C_2 < 0$:

$$R(t) = \sqrt{-\frac{C_2}{C_1}} \tanh(\sqrt{|C_1 C_2|} t) \quad (3.13)$$

and

$$R(t) = \sqrt{-\frac{C_2}{C_1}} \coth(\sqrt{|C_1 C_2|} t) \quad (3.14)$$

Here, for the *tanh* solution, $R^2 \in [0, 1]$ whereas for the *coth* solution $R^2 \in [1, \infty)$.

For all of the above $\delta^{(1)}c = \mp \frac{6}{k} + 4C_1 C_2$, where the $-$ and $+$ corresponds to $SU(2)$ and $SL(2,R)$.

We can also discuss here the adiabaticity conditions for the solutions above. The backgrounds we describe change with time so, strictly speaking there is no conserved energy. However if there is an adiabatic region where masses change slowly then one could still use the concept of energy to a good approximation. The adiabaticity condition, as we will show later on is $|R'/R| \ll 1$ which using (3.9) becomes $|C_1 R + C_2/R| \ll 1$.

For (i) the adiabatic region is at $t \gg 1$.

In case (ii) which corresponds to $SU(2)/U(1)$ in the dual version, we obtain that $|R'/R| \geq 2\sqrt{C_1 C_2}$. Thus, there is always a lower bound in adiabaticity which tends to zero only if $k \rightarrow \infty$, where k is the central element of the $SU(2)_k/U(1)$ theory.

Finally in case (iii) the adiabatic region is again $t \gg 1$. In cases (ii) and (iii) the “size” of the adiabatic region is governed by $|C_1 C_2|$, in the sense that the adiabaticity condition implies $\sqrt{C_1 C_2} t \gg 1$. We can blow-up that region by making $|C_1 C_2|$ arbitrarily small. In fact, we can even arrange for the total system to have $c = 4$ and still be able to blow up the adiabatic region.

As we will now show, the presence of the solutions (i-iii) is not accidental. If we do a duality transformation in (3.1) with respect to the θ angle we obtain $\tilde{S} = \tilde{S}_1 + \tilde{S}_2$ with

$$\begin{aligned} \tilde{S}_1 = & \frac{1}{2\pi} \int d^2z \left[-\partial t \bar{\partial} t + \frac{R^2(t)}{k} \partial \theta \bar{\partial} \theta \right] + \\ & + \frac{1}{8\pi} \int d^2z R^{(2)} \log[R'(t)] \end{aligned} \quad (3.15)$$

$$\begin{aligned} \tilde{S}_2 = & \frac{k}{2\pi} \int d^2z \left[\partial x \bar{\partial} x + \frac{1 + \Sigma(x)}{1 - \Sigma(x)} \partial \phi \bar{\partial} \phi \right] - \\ & - \frac{1}{8\pi} \int d^2z R^{(2)} \log[1 - \Sigma(x)] \end{aligned} \quad (3.16)$$

with $\phi = \tilde{\theta} + \theta/k$. Using this redefinition ϕ is an independent angle, so one might think that the two models are decoupled. This is not the case though due to the presence of $1/k$ in the redefinition. We will discuss this coupling in more detail below. \tilde{S}_2 describes the $SU(2)/U(1)_V$ or $SL(2, R)/U(1)_V$ coset.

Of course, in this “dual” formulation, we know the exact conformal field theory associated to \tilde{S}_1 corresponding to the three types of solutions found above. Case (i) is flat 2-d space (we could also have a linear dilaton but we will not entertain this further). Case (ii) corresponds to the Minkowski version of the $SU(2)_{k'}/U(1)$ coset model, (with $k' \sim 1/C_1 C_2$). Finally case (iii) corresponds to the 2-d black-hole [30].

Another way to state what was said above is that there is an $O(2,2)$ transformation [32, 33] which maps the model (3.1) to a product of two 2-d target spaces.* Start from the action (3.1) and perform the following $O(2,2)$ transformation: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.17)$$

$$c = \frac{1}{k} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.18)$$

We thus obtain (3.15,3.16) with an extra duality transformation on $\tilde{\theta}$ which gives instead of \tilde{S}_1 :

$$\begin{aligned} \tilde{S}'_1 = & \frac{1}{2\pi} \int d^2z \left[-\partial t \bar{\partial} t + \frac{k}{R^2(t)} \partial \tilde{\theta} \bar{\partial} \tilde{\theta} \right] - \\ & - \frac{1}{8\pi} \int d^2z R^{(2)} \log[R^2/R'] \end{aligned} \quad (3.19)$$

The coupling between the original models is hidden here in the fact that the $O(2,2)$ matrix has fractional elements.

To summarize: in the dual formulation, the original theory corresponds to the product of two conformal fields theories, one being $SU(2)_k/U(1)$ or $SL(2, R)_k/U(1)$ (depending

*In fact the solutions we describe are special lines in the general $O(2,2)$ moduli space described in [33].

on what $\Sigma(x)$ is) and the other (giving the time dependence) is one of the $SU(2)_k/U(1)$, $SL(2, R)_k/U(1)$, flat 2-d space or its dual.

As it is obvious from (3.16), the two CFTs corresponding to \tilde{S}_1 and \tilde{S}_2 are coupled. The type of coupling that exists is precisely the one which transforms a parafermionic theory times a free boson to an $SU(2)_k$ theory. This is a direct coupling of the parafermionic Z_k charge to the translational Z_k charge of the compact boson. In terms of the σ -model angles, a translation of $\tilde{\theta}$ by 2π induces a translation of ϕ by $2\pi/k$. This implies that the operators in the spectrum of the theory must have the mod- k residues of their angular momenta (corresponding to $\tilde{\theta}$ and ϕ) the same. For scattering amplitudes, this just affects only the way one couples the operators of the two theories.

We can also write now the exact central charge of the model:

$$c = 2\frac{k+1}{k-2} + 2\frac{3+2C_1C_2}{3-4C_1C_2} \quad (3.20)$$

Observe that we can choose $C_1C_2 = 3/2(4-k)$ so that the total central charge is exactly 4. This is important because, if the central charge differs substantially from four, we lose the clear four-dimensional interpretation, whereas if it is close to four then the extra compactified theory will have many low lying states which again will interfere with those coming from the four dimensional part. When $c = 4$, the background admits N=4 superconformal symmetry [31]. Observe also that we can have this extra symmetry, and at the same time, by choosing $k \rightarrow \infty$, be in the semiclassical regime, where time evolution is adiabatic for a long time.

There are also some global possibilities to be addressed. Let us first start from the Euclidean case ($t \rightarrow it$). There, we have two options: $SU(2)$ or $SL(2, R)$ (and the Euclidean version, H_3^+). We will focus first $SU(2)_k$ in the sequel although using also $SL(2, R)$ gives some interesting cosmological models with a non-compact spatial slice and dynamical topology change. Concerning the time dependence of the radius $R(t)$, we have the three different cases, (i,ii,iii).

Once we go to the Minkowski case, there are several possibilities. We will consider first the case where t is the time. If we look at the dual action (3.15) then it is obvious that there are two distinct possibilities for the coordinate $\tilde{\theta}$: that it has a finite radius, or that it is a non-compact coordinate. The former case is a suitable orbifold of the latter with respect to a discrete infinite abelian group.

There is another possibility in which x is taken as time, by doing the $x \rightarrow ix$ continuation in the Euclidean case. For the possibilities (ii,iii) this model has a similar interpretation as before with $t \leftrightarrow x$.

We will study here the semiclassical picture of these solutions. If we consider the time-independent case, for the deformed $SU(2)_k$ theory at radius R , it is not difficult to show that the eigenfunctions of the Laplacian (specified by the metric (2.4) and the dilaton (2.3) as $\square_3 \equiv \frac{e^{-\Phi}}{G} \partial_\mu e^\Phi G^{\mu\nu} \partial_\nu$) are the same as those for $SU(2)$, namely the standard D-functions, $D_{m,\bar{m}}^j(x, \theta, \tilde{\theta}) = e^{i(m-\bar{m})\theta/2+i(m+\bar{m})\tilde{\theta}/2} d_{m,\bar{m}}^j(x)$. The energy ($L_0 + \bar{L}_0$) eigenvalues

change though:

$$E_3(R) = \frac{1}{k} \left[2j(j+1) - m^2 - \bar{m}^2 + \frac{1}{2}(m - \bar{m})^2 R^2 + \frac{1}{2} \frac{(m + \bar{m})^2}{R^2} \right] + \mathcal{O}(1/k^2) \quad (3.21)$$

in agreement with the exact formula, (1.1) with $M, N, N', \bar{N}' = 0$. If we now consider the four-dimensional case then we obtain that the 4-d stringy Laplacian is given as

$$\square_4 = \frac{\partial^2}{\partial t^2} + (R'/R - R''/R') \frac{\partial}{\partial t} + \square_3 \quad (3.22)$$

Changing variables from $t \rightarrow R(t)$, and separating variables we obtain that the wavefunctions can be written in the form $F_{j,m,\bar{m}}(R) D_{m,\bar{m}}^j(x, \theta, \tilde{\theta})$ where $F(R)$ satisfies the following second order differential equation:

$$F'' + \frac{1}{R} F' + \frac{E_4 - E_3(R)}{(C_1 R^2 + C_2)^2} F = 0 \quad (3.23)$$

where $E_3(R)$ is given in (3.21) and we used (3.9). The behaviour of solutions varies for cases (i-iii). In case (i) it is easy to see from (3.23) that the spectrum has a continuum part (when either $m \pm \bar{m} = 0$ is valid) whose wavefunctions are Bessel functions as well as a discrete part whose wavefunctions are localized at $t = 0$.

For cases (ii,iii) the wavefunctions are hypergeometric function in the variable $x = -C_1 R^2 / C_2$. $x \geq 0$ corresponds to case (iii) while $x \leq 0$ to case (ii). The spectrum in both cases has continuous and discrete parts. These wavefunctions correspond to specific oscillator states in the dual coset model (3.15).

The semiclassical wavefunctions described above give a picture of the zero mode geometry which however does not reflect the behaviour of the whole spectrum. This can be seen by looking at the zero mode spectrum in the dual formulation (3.15,3.16). The Laplacian here factorizes into a sum of

$$\frac{\partial^2}{\partial t^2} + (R'/R + R''/R') \frac{\partial}{\partial t} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} \quad (3.24)$$

and the laplacian of the $SU(2)/U(1)$ coset. Their wavefunctions though, are coupled as was mentioned earlier. Observe that although (3.22) is invariant under $R(t) \rightarrow 1/R(t)$, (3.24) is certainly not.

In case (i), for example, we have almost the same wavefunctions on the spatial slice (although the d -functions are vector ($m = \bar{m}$)) but the time dependent part of the wavefunction has continuous spectrum only and behaves as a plane wave for large t . This case has been analysed in detail in [14]. For case (ii) we obtain similar results as in the original version while in case (iii) again we have different zero mode spectrum.

For fixed time, the transverse spectrum of energies (or masses) behaves as $E \sim m^2/R^2 + n^2 R^2$. The adiabaticity condition is essentially $|\dot{E}/E| \ll 1$ which translates to $|\dot{R}(m^2/R^2 - n^2 R^2)/R(m^2/R^2 + n^2 R^2)| \ll 1$ and eventually to $|\dot{R}/R| \ll 1$ as advocated earlier.

4 The Physical Interpretation

In this section we will discuss in more detail the physical interpretation of some of the models presented in the previous section.

A) Case (iii) time dependence associated with deformed $SU(2)$ ($\Sigma(x) = \cos 2x$).

We have chosen $C_1 = -C_2 = \sqrt{3/2k'}$ in order to have the standard $SL(2, \mathbb{R})$ model.* Here in terms of the radial time dependence we have two branches (see (??)). However both of them participate in this Minkowski signature solution. The best way to see that is to write the dual model in light-cone like coordinates. The action (3.15) is precisely that of the 2-d black hole [30], with metric $ds^2 = k'dudv/(1-uv)$ and the two branches in (iii) are realized in regions I and V while there is case (ii), that is the Minkowski version of $SU(2)/U(1)$ that appears in between, [34]. Thus the evolution of the radius has three patches. In the first (corresponding to region I), $R(t) \sim \tanh t$. At $t \rightarrow \infty$, $R(t) \equiv 1$. The effective string coupling is given by

$$g_{eff}^2 = e^{-\phi} = \frac{e^{-\phi_0} R'(t)}{1 + \Sigma(x) + R^2(t)(1 - \Sigma(x))} \quad (4.1)$$

where we used (3.7) and where ϕ_0 is the arbitrary constant part of the dilaton. Thus, at $t \rightarrow \infty$, $g_{eff} \rightarrow 0$ exponentially fast. At $t \rightarrow 0$, $R(t) \sim t$ and the string coupling is finite and given by the (anisotropic) $SU(2)/U(1)$ dilaton. At this point the time dependence is matched to case (ii) with $t = 0$ and continues till $t \rightarrow \pi\sqrt{k'/6}$ where $R(t) \sim \infty$ and g_{eff} is finite. The adiabatic period in this model is around the "horizon" in dual model. Finally, from that point on, the t -dependence is matched into the \coth branch at $t = 0$ with $R(t) \sim 1/t$ and evolves till $t \rightarrow \infty$, where $R(t) \sim 1$ and the coupling is again exponentially small.

As argued in the previous section, we will adjust $k + k' = 4$ so that the central charge is $c = 4$. If $k \gg 1$ then S^3 KK states are always low lying although the lowest part of the spectrum is dominated by the Cartan if $R \gg 1$ or $R \ll 1$. Thus, as we argued in the introduction, in the periods of time where the radius is either large or small the low energy spectrum is dominated by the Cartan, whereas in between there is a stringy region. There is no topology change here.

In order to give a physical interpretation for these solutions we have to use the σ -model picture very carefully. Consider a period in time where $R(t) \rightarrow 0, \infty$. As we have seen in section 2, one of the radii in the σ -model goes to zero and the spatial volume also goes to zero (qf. (2.7)). One would say that during the period the "universe" heads for a "big crunch". However, from the low energy point of view things seem different. The inhabitants of this world see a large circular direction which expands (and becomes non-compact in the limit) but the excitations (momenta) of the rest of the spatial dimensions are large so that they effectively disappear at low energy. Thus it is a "big squeeze" but for reasons completely different from those coming naively from the σ -model (see also [35]). That is,

*Since C_2/C_1 determines the radius of $\tilde{\theta}$ we can go to other rational values by orbifolding.

directions we thought were tiny, are not, and those we thought are finite, are not visible. The effective string coupling in such periods becomes exponentially small.

Once now we are in the regions where $R(t) \sim \mathcal{O}(1)$ there is a unique low energy scale set by α' (or equivalently k) and the low energy effective spectrum is that of a (deformed) S^3 . The string coupling here is finite, space dependent and can be made almost everywhere arbitrarily small by adjusting ϕ_0 .

Thus we see that although the volume and σ -model geometry of the “universe” implies that we have an expanding or contracting universe, the low energy effective theory has a very different picture of it. These issues are very important when we try to construct stringy realistic cosmological models.

B) Instead, we could have the same dependence as above for $R(t)$ but deforming H_+^3 now. Here we have an example of dynamical topology change. The change happens in the region around $R \sim 1$ as discussed in section 2. Similar remarks apply here as for the case discussed above.

C) Case (ii) time dependence associated with deformed $SU(2)$ ($\Sigma(x) = \cos 2x$).

In this model the radius goes from $R = 0$ at $t = 0$ to $R = \infty$ at $t = \pi\sqrt{k'/6}$.[†] Similar remarks like in case A apply here. The only difference is that here we have an oscillating “universe”. At $t \ll \sqrt{k'}$, $R(t) \sim t$, while around $t = \pi\sqrt{k'/6}$, $R(t) \sim 1/(t - \pi\sqrt{k'/6})$. Like (A), at the boundaries $R = 0, \infty$ the effective string coupling is finite and given by the dilaton of the $SU(2)/U(1)$ model. In between the model has an effective S^3 topology, but unlike model (A) the effective string coupling does not vanish at $R = 1$ but is finite and constant there.

Although this model appears (from the σ -model geometry point of view) to correspond to an eternally expanding and recontracting universe, the real story is different, as in (A). The model starts out as a large S^1 while the rest of space is curled up and turns into a continuously deforming squashed S^3 until it comes back to its original squeezed state.

D) Case (ia,ib) time dependence associated with deformed $SU(2)$ ($\Sigma(x) = \cos 2x$). This case is qualitatively similar to (B) in the sense that the radius evolves from 0 to ∞ . There is no periodicity in time however.

Another interpretation of the models above can be given. If one considers the $SU(2)_k$ as a part of the compactified dimensions of string theory, the model then provides a time-variation of the masses (and other data like couplings) of the particle spectrum in the adiabatic region. This would describe a higher dimensional universe at $t = 0$, where some of its dimensions compactify as time goes by, until their scale becomes of the order of the Planck scale (this would correspond to either the *tanh* or the *coth* solution (case (iii))). In the context of the heterotic string, this would lead to late time $SU(2)$ gauge symmetry. This is a very interesting cosmological solution which in our opinion deserves further study.

The stringy cosmological models presented above seem to have some features which

[†]It is very similar to the model presented in [36], and is in fact in the same $O(2, 2, R)$ -moduli space of the σ -model [33].

are not compatible with standard cosmological scenarios (due to anisotropy etc). It may be though, that isotropy is something that is achieved later in the history of the universe[‡]. In particular, consider the product of flat 2-d space times the $SL(2, R)_k/O(1, 1)$ coset. In the first of ref. [36] it was shown that for large times the spacetime was isotropic. The dual version of this model would give a universe where the Cartan radius R of $SL(2, R)$ changes with t as $R \sim t$. This would mean that there are particles in this theory which see asymptotically an isotropic space.

The solutions we presented have some extra interesting features, apart from the fact that being exact string solutions they can give us a chance to understand stringy phenomena in the context of cosmology. In particular, although anisotropic inflation has not been analysed so far we can remark that what plays here the role of the scale factor is $R(t)$ and there are regimes where it satisfies the usual conditions for inflation, $R', R'' > 0$, namely, cases (ib,ii) and the *tanh* branch of (iii). It remains however to be seen if such conditions remain valid in the presence of anisotropy. Many interesting isotropic σ -model solutions exist, however we don't know at present if they can be made exact solutions. This is a serious drawback since some of the interesting physics is supposed to happen at strong curvatures where our confidence on the solution breaks down, [25].

As was mentioned in the introduction, another interesting application of the Euclidean version of (3.15,3.16), is the identification of the angle θ with Euclidean time. In this case $R(t)$ [§] plays the role of the inverse temperature (Matsubara interpretation) which varies spatially. In this context, we have a background describing a thermal ensemble of strings with a spatially non-uniform temperature. More precisely there is a temperature gradient in the t direction. Studying further this kind of solutions might give us useful information about Hagedorn-like instabilities and phase transitions in some regions of space.

Finally, some generalizations of the solutions above can be envisaged using the same ideas as those discussed in this paper. One would be higher dimensional generalizations of the above providing interesting cosmological solutions where some non-compact dimensions compactify. The other, would be to find all paths inside $O(d, d)$ which provide exact string solutions. Looking at the boundaries of $O(d, d)$ moduli space may be the hint for finding such solutions.

5 Exact wavelike solutions in string theory

In this section we will discuss another class of exact solutions to string theory which are time dependent and contrary to the solutions presented above possess only a single scale, α' . They are closer in this respect to flat space. We will focus in a particular solution of this kind [2] which has high symmetry (it is a WZW model for a non-compact non-semisimple group). This solution can be interpreted as a plane gravitational (as well as $B_{\mu\nu}$) wave. It has seven Killing symmetries and the corresponding 2-d σ -model is the WZW model for

[‡]Some exact CFT models of that sort seem to be possible and will be presented elsewhere.

[§]Remember that now t is a spatial coordinate.

the group, E_2^c , the central extension of the 2-d Euclidean group. This model is interesting because it provides us with an exact classical solution to string theory which has at the same time:

- a simple and clear physical interpretation,
- a non-trivial spacetime,
- it can be solved,
- it has unusual features, in particular the spacetime is nowhere asymptotically flat.

Thus, a priori, it is not obvious that a sensible scattering matrix exists in such a background.

In [3, 4] we began solving the model by developing the representation theory of the E_2^c current algebra. In particular, among other things, we mapped the current algebra and the representation theory to (almost) free fields.

We are interested in considering this background as an exact classical solution to superstring theory. Thus we will supersymmetrize the current algebra and we will again map the model to free bosons and fermions. By studying the σ -model we will show that the model has a $N = 4$ worldsheet supersymmetry. The associated string theory has a large unbroken spacetime supersymmetry. For the type II string this is N=4 spacetime supersymmetry. In fact, this type of 4-d background turns out to be a special case of the class of solutions found recently in [37]. Moreover, we will explicitly construct the N=4 superconformal algebra out of the (super) current algebra.

Some of the features though of the model like the potential spectrum and tree scattering of bosonic states have little difference between the bosonic and the supersymmetric model. Thus we will find the exact spectrum of the associated bosonic string theory and we will compute the (modular invariant) vacuum energy. There are two types of states in the theory, corresponding to different kinds of representations of the current algebra. We will qualitatively describe their scattering.

We will start our discussion by presenting the current algebra symmetry of the background [2]. This is specified by the OPEs

$$J_a(z)J_b(w) = \frac{G_{ab}}{(z-w)^2} + f_{ab}^c \frac{J_c(w)}{(z-w)} + \text{regular}, \quad (5.2)$$

where $(J_1, J_2, J_3, J_4) \sim (P_1, P_2, J, T)$, the P_i generating the translations and J being the generator of rotations and T being the central element. The only non-zero structure constants are $f_{31}^2 = 1$ G_{ab} is an invariant bilinear form (metric) of the algebra. G is a symmetric matrix and the Jacobi identities constrain it so that $f_{abc} \equiv f_{ab}^d G_{dc}$ is completely antisymmetric. The most general invariant bilinear form for E_2^c is

$$G = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (5.3)$$

where we can assume without loss of generality that $k > 0, b > 0$ [3]. For \hat{E}_2^c there is a unique solution to the Master Equation [38], which has all the properties of the Affine Sugawara construction. It is given by:

$$\begin{aligned}
L_{AS} &= \frac{1}{2k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -b + \frac{1}{k} \end{pmatrix} = \\
&= \frac{1}{2}G^{-1} + \frac{1}{2k^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.4}
\end{aligned}$$

The σ -model realizing the current algebra above can be constructed in a straightforward fashion, and provides the tool to study string propagation. Its action is [2]

$$\begin{aligned}
S = \frac{k}{2\pi} \int d^2z \left[\partial a_i \bar{\partial} a_i + \partial u \bar{\partial} v + \bar{\partial} u \partial v + b \partial u \bar{\partial} u - \right. \\
\left. - \epsilon^{ij} a_i \bar{\partial} a_j \partial u \right] \tag{5.5}
\end{aligned}$$

From now on we will set $b = 0$ by an appropriate shift in v . As was shown in [3], from the exact current algebra point of view, once b is finite it can always be rescaled away (assuming that the light-cone coordinates are non-compact). We can also scale $k \rightarrow 1$ by appropriate scaling of a_i, v^* . In this coordinate system, the nature of the background is not obvious, however as we will show below, it is in this background that the free field representation of the algebra described in [3] is manifest. It should be also noted that in (4) the last term describes the departure of the background from flat Minkowski space, and its coupling can be made arbitrarily small by a boost of the u, v coordinates. However, this does not imply that the perturbation is insignificant since the perturbing operator has bad IR behaviour. It should also be mentioned here that the action (4) can be obtained by an $O(3, 3, R)$ transformation of flat space.

The coordinate system where the nature of the background is more transparent is given by [2]

$$a_1 = x_1 + \cos u x_2, \quad a_2 = \sin u x_2, \tag{5.6}$$

$$v \rightarrow v + \frac{1}{2} x_1 x_2 \sin u \tag{5.7}$$

where the action (5.5) becomes

$$\begin{aligned}
S = \frac{1}{2\pi} \int d^2z \left[\partial x_i \bar{\partial} x_i + 2 \cos u \partial x_2 \bar{\partial} x_1 + \right. \\
\left. + \partial u \bar{\partial} v + \bar{\partial} u \partial v \right]. \tag{5.8}
\end{aligned}$$

* Effectively we are setting the parameter Q of [3] to one.

In this form we can immediately identify the perturbation from flat space with a graviton+antisymmetric tensor mode given by $\cos u \partial x_2 \bar{\partial} x_1$. This is an operator with no IR problems, however its coupling is of order one and cannot be rescaled at will. One can however look at the structure of perturbation theory around the flat background, and verify that indeed the current algebra structure described in [3] indeed emerges.

Before continuing further into the quantum theory, we should point out that the string geodesics (solutions to the classical equations of motion of the σ -model can be automatically obtained, since the model is just a WZW model. The solutions are obtained by writing the group element as a matrix product of a left moving and a right moving group element.

In [3] we described a resolution of the \hat{E}_2^c current algebra in terms of free fields and studied its irreducible representations with unitary base. We will focus for the moment on one copy of the current algebra, and return to the full theory in due time. We introduce four free fields, x^μ with

$$\langle x^\mu(z) x^\nu(w) \rangle = \eta^{\mu\nu} \log(z-w),$$

$$\eta^{\mu\nu} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (5.9)$$

and define the light-cone $x^\pm = x^0 \pm x^3$ and transverse space $x = x^1 + ix^2, \bar{x} = x^1 - ix^2$ coordinates. Then, the current algebra generators can be uniquely (up to Lorentz transformations) written as

$$J = \frac{1}{2} \partial x^+ \quad , \quad T = \partial x^- \quad (5.10)$$

$$P^+ = P^1 + iP^2 = ie^{-ix^-} \partial x \quad (5.11)$$

$$P^- = P^1 - iP^2 = ie^{ix^-} \partial \bar{x} \quad (5.12)$$

while the affine-Sugawara stress tensor (5.4) becomes the free field stress tensor

$$T_{AS} = \frac{1}{2} \eta_{\mu\nu} \partial x^\mu \partial x^\nu. \quad (5.13)$$

The current algebra representations can be built by starting with unitary representations of the \hat{E}_2^c algebra as a base and then acting on them by the negative modes of the currents. They fall into two classes:

(I) Representations with neither highest nor lowest weight state.[†] The base is generated by vertex operators with $p_- = 0$. The representation is characterized by $p_+ \in [0, 1)$ and transverse momentum \vec{p}_T . The base operators are

$$V_{p_+, n}^I = e^{i(p_+ + n)x^- + i\vec{p}_T \cdot \vec{x}}. \quad (5.14)$$

[†] Except when $\vec{p} = 0$ in which case the representations are one dimensional.

(II) Representations with a highest weight state in the base. Their conjugates have a lowest weight. They have $p_- > 0$ (negative for the conjugates). The highest weight operator can be written as

$$V^{II} \sim e^{ip_+x^- + ip_-x^+} H_{p_-} \quad (5.15)$$

where H_{p_-} is the generating twist field in the transverse space corresponding to the transformation

$$x(e^{2\pi i} z) = e^{-4\pi i p_-} x(z), \quad (5.16)$$

$$\bar{x}(e^{2\pi i} z) = e^{4\pi i p_-} \bar{x}(z). \quad (5.17)$$

The conformal weight of the operators at the base is $\Delta = -2p_+p_- + p_-(1 - 2p_-)$.

As we will see later on all the representations participate in the modular invariant vacuum energy. Thus we can easily describe the spectrum of physical states. We will assume that the extra 22 dimensions are non-compact although the compact case is as easy to handle.

- For type I states ($p_- = 0$) we have the physical state conditions $L_0 = \bar{L}_0 = 1$ which translate to

$$N = \bar{N} \quad \text{and} \quad \vec{p}_T^2 = 2 - 2N \quad N = 0, 1, 2, \dots \quad (5.18)$$

where, N, \bar{N} are the contribution of the transverse currents or oscillators (from the left or right) and \vec{p}_T is the 24-d transverse momentum. It is obvious that for $N = 0$ we have $\vec{p}_T^2 = 2$ which is a component of the tachyon, while for $N = 1$ we have $\vec{p} = 0$ and we obtain some modes of the massless fields (graviton, antisymmetric tensor and dilaton). which propagate along the wave. For $N > 1$ there are no physical states.

- For type II states the situation is like in usual string theory, it is just their dispersion relation that changes:

$$-4p_+p_- + 2p_-(1 - 2p_-) + \vec{p}_T^2 = 2 - 2N \quad (5.19)$$

and \vec{p}_T now stands for the 22 transverse dimensions. Eq. (5.19) can also be written after $p_- \rightarrow p_-/4, p_+ \rightarrow p_+ - p_-/4 + 1/2$ as

$$p_+p_- - \vec{p}_T^2 = 2(N - 1) \quad (5.20)$$

which is the flat space spectrum but in 24-d Minkowski space. Thus for almost all states the wave reduces the effective dimensionality by two. As we will see below, this happens because the states are localized in the extra two dimensions.

Eventually we would like to compute the partition function (vacuum energy). In order to do this we can guess the modular invariant sowing of representations by looking at the wavefunctions on the group. These wavefunctions can be computed and we easily deduce that we have the standard diagonal modular invariant.

From these wavefunctions we can see the qualitative behaviour of the fluctuations in this background. Type I states are just plane waves. As for type II states, if we go to the x_i coordinates (5.7) where the wave nature is manifest, then we see that the states are localized where $x_1^2 + x_2^2 + 2x_1x_2 \cos(u) = 0$. So for fixed time the state is localized on the

two lines $x_1 = \pm x_2$, $x_3 = \text{constant}$ where $u = t - x_3$. This disturbance travels to the left in the x_3 direction with the speed of light (in flat space).

We can now try to construct the vacuum energy by putting together the representations. What can happen at most is the truncation phenomenon as in the compact case. In trying to built the modular invariant partition function we have to overcome the problem that the representations are infinite dimensional and since all the states in the base have the same energy the partition function diverges. We will have to regulate this divergence and ensure that upon the removal of the regulator we will obtain a modular invariant answer. We will start by computing the character formulae for the representations above. In fact what we are interested in is the so called signature character which keeps track of the positive and negative norms. We will not worry much about the type I representations (although they are the easiest to deal with) since their contribution is of measure zero. For the type II representations we can easily compute the signature character

$$\chi_{p_+, p_-}^{II}(q, w) \equiv Tr[(-1)^\epsilon q^{L_0 - \frac{c}{24}} w^{J_0}] \quad (5.21)$$

where $\epsilon = 0, 1$ for positive (respectively negative) norm states. If $2p_- \neq \text{integer}^\ddagger$ then the only non-trivial null vector is the one at the base responsible for the fact that we have a highest (or lowest) weight representation. Thus the character is [39]

$$\begin{aligned} \chi_{p_+, p_-}^{II}(q, w) &= \\ &= \frac{q^{-2p_+ p_- + p_-(1-2p_-) - \frac{1}{6}} w^{p_+}}{(1-w) \prod_{n=1}^{\infty} (1 - q^n w)(1 - q^n w^{-1})} = \\ &= i q^{-2p_+ p_- + p_-(1-2p_-) - \frac{1}{12}} w^{p_+ - \frac{1}{2}} \frac{\eta(q)}{\vartheta_1(v, q)} \end{aligned} \quad (5.22)$$

where ϑ_1 is the standard Jacobi ϑ -function and $w = e^{2\pi i v}$. The infinity we mentioned before can be seen as $w \rightarrow 1$ as the pole in the ϑ -function. In order to see how we must treat this infinity we will calculate this partition function in the quantum mechanical case (that is keeping track only of the zero modes). We will introduce a twisted version of the (Minkowski) amplitude for the particle to go from x to x' in time τ :

$$\langle x|x'; \tau, v, \bar{v} \rangle = \langle x|e^{i\tau H} e^{\zeta J_0 + \bar{\zeta} \bar{J}_0}|x' \rangle \quad (5.23)$$

The quantum mechanical partition function can be obtained by setting $x = x'$, $\zeta, \bar{\zeta} \rightarrow 0$ and integrating over x . This last integral gives the overall volume of space that we will drop since we are interested in the vacuum energy per unit volume. The amplitude (5.23) in the coordinates u, v and $a_1 = r \cos \theta$, $a_2 = r \sin \theta$, can be calculated with the result

$$\begin{aligned} \langle x|x'; \tau, \zeta, \bar{\zeta} \rangle &\sim \frac{1}{\tau^2} \frac{(u - u' + \bar{\zeta} - \zeta)}{\sin\left(\frac{u - u' + \bar{\zeta} - \zeta}{2}\right)} \times \\ &\times \exp \left[i \frac{(u - u' + \bar{\zeta} - \zeta)^2}{4\tau} - i \frac{(u - u' + \bar{\zeta} - \zeta)(v - v')}{2\tau} \right] \end{aligned}$$

[‡]We can also compute the character when $2p_- \in \mathbb{Z}$ but this is not needed again for the partition functions since it is of measure zero.

$$-\frac{i}{8\tau}(u-u'+\bar{\zeta}-\zeta)\cot\left(\frac{u-u'+\bar{\zeta}-\zeta}{2}\right) \quad (5.24)$$

$$\left(r^2+r'^2-2rr'\frac{\cos\left(\theta-\theta'-\frac{u-u'+\zeta+\bar{\zeta}}{2}\right)}{\cos\left(\frac{u-u'+\bar{\zeta}-\zeta}{2}\right)}\right)]$$

We can then perform the limits to obtain the *finite* result

$$Z(\tau) = \langle x|x; \tau, \zeta = 0, \bar{\zeta} = 0 \rangle \sim \frac{1}{\tau^2} \quad (5.25)$$

This result may seem surprising since we have only two continuous components of the momentum, namely p_+ and p_- but no transverse momentum. However the result is not as surprising as it looks at first, and to persuade the reader to that we will provide with an analogous situation where the answer is obvious. Consider the case of a flat 2-d plane. The zero mode spectrum are the usual plane waves, $e^{i\vec{p}\cdot\vec{x}}$, but we will work in the rotational basis, where the appropriate eigenfunctions corresponding to energy p^2 are $e^{im\theta} J_{|m|}(pr)$. In such a basis, we will have precisely the same problem in calculating the partition function as we had above. Namely, for a given p there are an infinite number of states (numbered by $m \in \mathbb{Z}$) with the same energy. Thus the partition function seems to be infinite. However here we know what to do: go to the (good) plane wave basis, or calculate the propagator and then take the points to coincide in order to get the partition function. This will also give the right result. This is precisely the prescription we have applied above and we found that the quantum mechanical partition function of the gravitational wave zero modes is precisely that of flat Minkowski space. Having found the way to sum the zero mode spectrum, it is not difficult to show using (5.25) that the vacuum energy of the associated bosonic string theory is equal to the flat case, namely

$$F = \int \frac{d\tau d\bar{\tau}}{Im\tau^2} (\sqrt{Im\tau\eta\bar{\eta}})^{-24} \quad (5.26)$$

where we have added another 22 flat non-compact dimensions[§].

We would add here a few comments about scattering. We hope to present the full picture in the future. States corresponding to type I representations scatter among themselves, this is already obvious from their vertex operator expressions [3]. This is dangerous however, since we might have trouble with unitarity. We will check here that in a 4-point amplitude of type I tachyons only physical states appear as intermediate states. Remember that type I tachyons have $p_- = 0$. Thus the four type I tachyons are characterized by their p_+ and \vec{p}_T . The 4-point amplitude is then given by the standard δ -functions of p_+ and \vec{p}_T multiplied by the Shapiro-Virasoro amplitude with one difference: instead of the invariants $p_i \cdot p_j$ we now have them restricted to transverse space, $\vec{p}_T^i \cdot \vec{p}_T^j$. This is similar to the Shapiro-Virasoro amplitude in 24 Euclidean dimensions (modulo the delta functions) and its analytic structure is different. It can be easily checked that instead of the infinite sequence of poles of the usual amplitude here we have just two poles, one corresponding

[§] The extra 22 dimensions could be compactified with no additional effort

to intermediate on-shell tachyons and the other to intermediate on-shell massless type I particles. This is in agreement with the fusion rules of the current algebra.*

The scattering of the type II states is more complicated and will be dealt with elsewhere.

The bosonic string vacuum as it stands is ill defined in the quantum theory due to the presence of the tachyon in its spectrum. We will construct now the N=1 extension of this background, in order to make a superstring out of it.

In order to do this we will add four free fermions (with Minkowski signature), $\psi_{1,2}, \psi_J, \psi_T$, normalized as

$$\psi_a(z)\psi_b(w) = \frac{G_{ab}}{(z-w)} + \text{regular} \quad (5.27)$$

The N=1 supercurrent is given by

$$G = E^{ab} J_a \phi_b - \frac{1}{6} f^{abc} \phi_a \phi_b \phi_c \quad (5.28)$$

where

$$E = \frac{1}{k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -b + 1/2k \end{pmatrix} \quad (5.29)$$

and the group indices are always raised and lowered with the invariant metric G_{ab} . The only non-zero component of f^{abc} is $f^{124} = 1/k^2$. It can be verified that (5.29) satisfies the superconformal ME [41] and preserves the current algebra structure. In particular

$$G(z)G(w) = \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{(z-w)} + \text{regular} \quad (5.30)$$

with $c = 6$, $T = T^b + T^f$, with T^b being the affine Sugawara stress tensor and T^f the free fermionic stress tensor

$$T^f = -\frac{1}{2} G^{ab} \phi_a \partial \phi_b \quad (5.31)$$

G is also primary with conformal weight $3/2$ with respect to T . The supersymmetric currents in the theory are the superpartners of the free fermions

$$G(z)\phi_a(w) = \frac{\hat{J}_a(w)}{(z-w)} + \text{regular} \quad (5.32)$$

It is easy to find that

$$\hat{P}_1 = P_1 - \frac{1}{k} \phi_2 \phi_T, \quad \hat{P}_2 = P_2 + \frac{1}{k} \phi_1 \phi_T \quad (5.33)$$

$$\hat{J} = J + \frac{1}{2k} T - \frac{1}{k} \phi_1 \phi_2, \quad \hat{T} = T \quad (5.34)$$

The supersymmetric currents \hat{J}_a satisfy the same algebra (5.2) as the bosonic ones J_a . This should be contrasted with the compact case where the level is shifted by the dual

*Scattering for type I states for arbitrary plane waves has been considered in [26] with similar results.

Coxeter number. The supercurrent G has a similar expression in terms of \hat{J}_a as in the compact case

$$G = G^{ab} \hat{J}_a \phi_b - \frac{1}{3} f^{abc} \phi_a \phi_b \phi_c \quad (5.35)$$

As in the bosonic case, the supersymmetric current algebra can be written in terms of free bosons, x^\pm, x, \bar{x} and fermions $\phi^\pm, \phi, \bar{\phi}$ with

$$\langle x^+(z)x^-(w) \rangle = -\langle x(z)\bar{x}(w) \rangle = 2 \log(z-w) \quad (5.36)$$

$$\langle \phi(z)\bar{\phi}(w) \rangle = -\langle \phi^+(z)\phi^-(w) \rangle = \frac{2}{(z-w)} \quad (5.37)$$

all others being zero. Explicitly,

$$P_1 + iP_2 = i\sqrt{k}e^{-ix^-/Q}\partial x, \quad (5.38)$$

$$P_1 - iP_2 = i\sqrt{k}e^{ix^-/Q}\partial \bar{x} \quad (5.39)$$

$$J = \frac{Q}{2}\partial x^+ + \frac{1}{2}\left(Q + \frac{1}{Q}\right)\partial x^- + \frac{i}{2}\phi\bar{\phi} \quad (5.40)$$

$$T = \frac{Q}{b}\partial x^- \quad (5.41)$$

$$\phi_1 + i\phi_2 = \sqrt{k}e^{-ix^-/Q}\phi, \quad \phi_1 - i\phi_2 = \sqrt{k}e^{ix^-/Q}\bar{\phi} \quad (5.42)$$

$$\phi_J = i\frac{Q}{2}(\phi^+ + \phi^-), \quad \phi_T = i\frac{Q}{b}\phi^- \quad (5.43)$$

where $Q = \sqrt{kb}$. The fermionic admixture to J is added to guarantee that J_a and ϕ_a commute.

It is not difficult to see that the stress tensor and supercurrent are free in the free-field basis

$$T = \frac{1}{2}[\partial x^+\partial x^- - \partial x\partial \bar{x}] + \frac{1}{4}[\phi^+\partial\phi^- + \phi^-\partial\phi^+ - \phi\partial\bar{\phi} - \bar{\phi}\partial\phi] \quad (5.44)$$

$$G = \frac{i}{2}[\partial x\bar{\phi} + \partial \bar{x}\phi + \partial x^-\phi^+ + \partial x^+\phi^-] \quad (5.45)$$

In the free field basis it can be easily seen that the theory has an N=2 superconformal algebra. The conventionally normalized U(1) current that determines the "complex structure" is given by

$$\tilde{J} = \frac{1}{2}[\phi\bar{\phi} - \phi^+\phi^-] \quad (5.46)$$

whereas the second supercurrent is

$$G^2 = -\frac{1}{2}[\partial x\bar{\phi} - \partial \bar{x}\phi + \partial x^+\phi^- - \partial x^-\phi^+] \quad (5.47)$$

In conformity with our σ -model discussion later, we will also consider the presence of the dilaton, which here is reflected as background charge in the lightcone directions. We have the following modifications for the N=2 generators

$$\delta\tilde{J} = -i(Q^+\partial x^- + Q^-\partial x^+) \quad (5.48)$$

$$\delta T = \frac{i}{2} (Q^+ \partial^2 x^- - Q^- \partial^2 x^+) \quad (5.49)$$

$$\delta G^1 = Q^+ \partial \phi^- - Q^- \partial \phi^+ \quad (5.50)$$

$$\delta G^2 = i (Q^+ \partial \phi^- + Q^- \partial \phi^+) \quad (5.51)$$

and $\delta c = -8Q^+Q^-$.

The final step is the realization that when $Q^- = 0$ the theory has even larger superconformal symmetry, namely $N=4$. The $N=4$ superconformal algebra in question contains the stress tensor, $SU(2)_k$ currents J^a and four supercurrents that transform as conjugate doublets under the $SU(2)$. It is defined in terms of the OPEs

$$J^a(z)J^b(w) = \frac{k}{2} \frac{\delta^{ab}}{(z-w)^2} + i\epsilon^{abc} \frac{J^c(w)}{(z-w)} + \text{regular} \quad (5.52)$$

$$J^a(z)G^i(w) = \frac{1}{2} \sigma_{ji}^a \frac{G^j(w)}{(z-w)} + \text{regular} \quad (5.53)$$

$$J^a(z)\bar{G}^i(w) = -\frac{1}{2} \sigma_{ij}^a \frac{\bar{G}^j(w)}{(z-w)} + \text{regular} \quad (5.54)$$

$$\begin{aligned} G^i(z)\bar{G}^j(w) &= \frac{4k\delta^{ij}}{(z-w)^3} + 2\sigma_{ji}^a \left[\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} \right] + \\ &+ 2\delta^{ij} \frac{T(w)}{(z-w)} + \text{regular} \end{aligned} \quad (5.55)$$

the rest being regular. J^a , G^i , \bar{G}^i are primary with the appropriate conformal weight and $c = 6k$. In our case the $SU(2)$ current algebra has level $k = 1$. The $N=4$ generators, in the free field basis are

$$G^1 = \frac{i}{\sqrt{2}} [\partial \bar{x} \phi + \partial x^- \phi^+] \quad (5.56)$$

$$G^2 = -\frac{i}{\sqrt{2}} [\partial \bar{x} \phi^- + \partial x^- \bar{\phi}] e^{iQ^+ x^-} \quad (5.57)$$

$$\bar{G}^1 = \frac{i}{\sqrt{2}} [\partial x \bar{\phi} + \partial x^+ \phi^- - 2iQ^+ \partial \phi^-], \quad (5.58)$$

$$\bar{G}^2 = \frac{i}{\sqrt{2}} [\partial x \phi^+ + \partial x^+ \phi - iQ^+ \phi \phi^+ \phi^-] e^{-iQ^+ x^-} \quad (5.59)$$

$$J^1 + iJ^2 = \frac{1}{2} \phi \phi^+ e^{-iQ^+ x^-} \quad (5.60)$$

$$J^1 - iJ^2 = \frac{1}{2} \bar{\phi} \phi^- e^{iQ^+ x^-} \quad (5.61)$$

$$J^3 = \frac{1}{4} [\phi \bar{\phi} - \phi^+ \phi^- - 2iQ^+ \partial x^-] \quad (5.62)$$

and T is given in (5.44).

One can invert the map to free fields and write these operators in terms of the original (σ -model) currents. Care should be exercised though in normal ordered expressions. What

we need here is the appearance, in the N=4 generators, of terms which are non-local in the original currents. These terms are of the form $\exp\left[\pm\frac{i}{k}(QQ^+ - 1)\int Tdz\right]$. However, in the sigma model, the current T is given as a total derivative of a free field, $T = \partial u$, with $\partial\bar{\partial}u = 0$. Thus the exponential factors are just $\exp\left[\pm\frac{i}{k}(QQ^+ - 1)u(z)\right]$. Notice also that they disappear at a specific value of the background charge, $Q^+ = 1/Q$.

To conclude this section, exact solutions as the one above, have very interesting dynamics, and they will sharpen our understanding of strings propagating in non-trivial background fields. Several issues remain open however, the most interesting in our opinion being to compute the scattering of the bulk of the spectrum in the supersymmetric case.

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