Perturbative gravitational couplings and Siegel modular forms in $D = 4, N = 2$ string models

G. L. Cardoso,[a] [*]

[a]Theory Division, CERN, CH-1211 Geneva 23, Switzerland

We consider four-parameter $D = 4, N = 2$ string models with Hodge numbers $(4, 214 - 12n)$ and $(4, 148)$, and we express their perturbative Wilsonian gravitational coupling F_1 in terms of Siegel modular forms.

1. Introduction

String-string dualities between heterotic strings on $K3 \times T_2$ and corresponding type II strings on suitably chosen Calabi–Yau threefolds in four space-time dimensions have been successfully tested in [1]–[11].

One interesting aspect in the study of these $N = 2$ string-string dualities in four space-time dimensions is the appearence [6] of certain modular forms in the low-energy effective action of these theories. Here, we will focus on $D = 4, N = 2$ four-parameter string models with Hodge numbers $(4, 214 - 12n)$ and $(4, 148)$, and we will express their perturbative Wilsonian gravitational coupling F_1 in terms of Siegel modular forms. For the models with Hodge numbers $(4, 214 - 12n)$, this has been achieved in [10], and the results presented below for these models represent a short summary of the ones contained in [10].

2. $D = 4, N = 2$ four-parameter string models

The four-parameter models we will be considering in the following have a perturbative heterotic description in terms of compactifications of the $E_8^{(1)} \times E_8^{(2)}$ heterotic string on $K3 \times T_2$. They also have a dual type II description in terms of compactifications of the type II string on Calabi–Yau threefolds which are $K3$ fibrations [3,12].

We will consider two classes of models. The first class of models can be obtained, in the type II description, by compactifying on Calabi–Yau threefolds with Hodge numbers $(h^{1,1}, h^{2,1}) = (4, 214 - 12n)$ and, consequently, Euler number $\chi = 24n - 420$. We will restrict the discussion to $n = 0, 1, 2$. These models have a perturbative heterotic description in terms of compactifications of the heterotic $E_8^{(1)} \times E_8^{(2)}$ string on $K3 \times T_2$ with $SU(2)$ gauge bundles on the $K3$ with instanton numbers $(d_1, d_2) = (12 - n, 12 + n)$ [1,13,14]. In the heterotic description, the four moduli comprise the dilaton S, the two toridal moduli T and U as well as a Wilson line modulus V. The massless spectrum thus contains $N_V = 5$ vector multiplets (here we also count the graviphoton) as well as $N_H = 215 - 12n$ neutral hyper multiplets. At the transition surface $V = 0$, the $U(1)_V$ associated with the Wilson line modulus becomes enhanced to an $SU(2)$. Let $N'_V = 2$ and $N'_H = 12n + 32$ denote the additional vector and hyper multiplets becoming massless at $V = 0$. Then, as the $SU(2)$ gauge bosons get swallowed, the transition to an S-T-U model with $N_V = 4$ vector and $N_H = 244$ neutral hyper multiplets takes place [15].

The second class of models can be obtained, in the type II description, by compactifying on Calabi–Yau threefolds with Hodge numbers $(h^{1,1}, h^{2,1}) = (4, 148)$ and, consequently, Euler number $\chi = -288$. These Calabi–Yau threefolds are actually also elliptic, that is, they are elliptically fibred with base $\mathbf{F}_n$ ($n = 0, 1, 2$), and their elliptic fibre is of the E_7 type [16,17]. These models also have a perturbative heterotic description, apparently in terms of compactifications of the heterotic $E_8^{(1)} \times E_8^{(2)}$ string on $K3 \times T_2$ with

$U(1)$ backgrounds on $K3$ [18,16]. Again, the heterotic moduli will comprise the dilaton S, the two toroidal moduli T and U as well as a fourth modulus V. The massless spectrum thus contains, for $V \neq 0$, $N_V = 5$ vector multiplets as well as $N_H = 149$ neutral hyper multiplets. There are also a total of 96 hyper multiplets charged under the $U(1)_V$ associated with the modulus V [18,16]. They are massive at generic points in the moduli space ($V \neq 0$). At $V = 0$, there is no additional gauge symmetry enhancement. Instead, these 96 hyper multiplets become massless. The $U(1)_V$ gauge boson is swallowed, and again the transition to an S-T-U model takes place ($\chi = -288 - 2 \times 96 = -480$).

We will, in the following, collectively denote all these four-parameter models by S-T-U-V models.

3. The perturbative prepotential F

In this section we briefly discuss the relation between the type II and the heterotic prepotentials for the S-T-U-V models. It will turn out to be useful to introduce a parameter t in order to describe the perturbative corrections to the heterotic prepotential for the models described aboved. For the models with Hodge numbers $(4, 214 - 12n)$, the parameter t will take the value $t = 1$, whereas for the models with Hodge numbers $(4, 148)$ it will take the value $t = 2$.

In the following, the dilaton S is defined to be $S = \frac{4\pi}{g^2} - i\frac{\theta}{2\pi}$.

For an S-T-U-V model, the perturbative heterotic prepotential is given by

$$F^{\text{het}} = -S(TU - tV^2) + h(T, U, V) \ ,$$

$$h(T, U, V) = p(T, U, V) - \frac{c(0)\zeta(3)}{8\pi^3} \qquad (1)$$

$$-\frac{1}{4\pi^3} \sum_{\substack{k,l,b\in\mathbf{Z} \\ (k,l,b)>0}} c(4tkl - b^2)Li_3(e^{-2\pi(kT+lU+bV)}).$$

The condition $(k, l, b) > 0$ means that [6]: either $k > 0, l, b \in \mathbf{Z}$ or $k = 0, l > 0, b \in \mathbf{Z}$ or $k = l = 0, b > 0$.

The term $-S(TU - tV^2)$ denotes the tree-level prepotential associated with the classical vector moduli space $\frac{SU(1,1)}{U(1)} \times \mathcal{H}_t^+$, where $\mathcal{H}_t^+$ has a representation as a three-dimensional tube domain

of type IV [19]. $\mathcal{H}_t^+$ is isomorphic to the Siegel upper half plane $\mathcal{H}_2$ of complex dimension 3.

The term $p(T, U, V)$ denotes a cubic polynomial, which is generated at one-loop, and which depends on the particular instanton embedding.

The coefficients $c(4tkl - b^2)$ appearing in the prepotential are, presumably, the expansion coefficients of a Jacobi form $f(\tau, z)$ of weight -2 and index t

$$f(\tau, z) = \sum_{n \geq -1, l \in \mathbf{Z}} c(4tm - l^2)q^m r^l \ , \qquad (2)$$

where $q = \exp 2\pi i\tau, r = \exp 2\pi iz$. For the case $t = 1$, the appropriate Jacobi form of weight -2 and index 1 is given by [9,10]

$$f(\tau, z) = \frac{1}{\eta^{24}} \left(\frac{12 - n}{24} E_6 E_{4,1} + \frac{12 + n}{24} E_4 E_{6,1} \right). \quad (3)$$

It would be interesting to find the appropriate Jacobi form for the $t = 2$ case.

It can be checked, by comparison with the corresponding type II prepotential, that the expansion coefficients $c(4tkl - b^2)$ appearing in (1) indeed only depend on the combination $4tkl - b^2$. In order to do so, one has to consider the contributions of the world sheet instantons to the type II prepotential which, for four-parameter models, are genericallly given by

$$F^{II}_{\text{inst}} = -\frac{1}{(2\pi)^3} \sum_{d_1,d_2,d_3,d_4} n^r_{d_1,d_2,d_3,d_4} Li_3(\prod_{i=1}^{4} q_i^{d_i}), \quad (4)$$

where $q_i = \exp(-2\pi t_i)$. The $n^r_{d_1,d_2,d_3,d_4}$ denote the rational instanton numbers. In order to match (4) with (1), we will perform the following identification of the type II Kähler class moduli t_i with the heterotic moduli S, T, U and V [10]

$$t_1 = U - 2V, \qquad t_2 = S - \frac{n}{2}T - (1 - \frac{n}{2})U,$$

$$t_3 = T - U, \qquad t_4 = V \ , \qquad (5)$$

which is valid in the chamber $T > U > 2V$. Then, the heterotic weak coupling limit $S \to \infty$ corresponds to taking the large Kähler class limit $t_2 \to \infty$. In this limit, only the instanton numbers with $d_2 = 0$ contribute in the above sum (4). Using the identification $kT + lU + bV = d_1t_1 + d_3t_3 + d_4t_4$,

it follows that (independently of n)

$$k = d_3 ,$$
$$l = d_1 - d_3 ,$$
$$b = d_4 - 2d_1 . \qquad (6)$$

Then, (4) turns into

$$F_{\text{inst}}^{II} = -\frac{1}{(2\pi)^3} \sum_{k,l,b} n_{k,l,b}^r Li_3(e^{-2\pi(kT+lU+bV)}). \quad (7)$$

Comparison with (1) shows that the rational instanton numbers should be related to the expansion coefficients $c(4tkl - b^2)$ by

$$n_{k,l,b}^r = -2c(4tkl - b^2) , \qquad (8)$$

in which case they would have to satisfy the following non-trivial constraint

$$n_{k,l,b}^r = n^r(4tkl - b^2) . \qquad (9)$$

It can now be checked that the constraint (9) is indeed satisfied for the models discussed here. This, in particular, also holds for the model[2] with Hodge numbers $(4, 148)$ based on the elliptically fibred Calabi–Yau threefold with E_7 fibre $\mathbf{P}^{1,1,2}[4]$ and base $\mathbf{F}_1$ (denoted by $X^{(4)}$ in [17]). By inspection of the rational instanton numbers for some of the four-parameter models under consideration, it can also be inferred that

$$c(N) = 0 , \quad N > -4t \qquad (10)$$
$$c(-4t) = 1 , \quad c(-1) = -\frac{1}{2}N_H' , \quad c(0) = \frac{1}{2}\chi .$$

For the $X^{(4)}$ model ($t = 2$), for instance, one finds that

$$c(-8) = 1 , \quad c(-4) = 0$$
$$c(-1) = -48 , \quad c(0) = -144 . \qquad (11)$$

The truncation to a three-parameter Calabi–Yau model is made by setting $t_4 = 0$. The instanton numbers $n_{k,l}^r$ of the three-parameter (S-T-U) model are then given by [15]

$$n_{k,l}^r = \sum_b n^r(4tkl - b^2) , \qquad (12)$$

where the summation range over b is finite. For example, when considering the transition from the four-parameter model $X^{(4)}$ to the three-parameter Calabi–Yau model, $n_{0,1}^r = 96 + 288 + 96 = 480$.

The heterotic cubic polynomial $p(T, U, V)$ can also be obtained from the cubic terms of the associated type II polynomial, by making use of the map (5). Consider, for instance, the models with Hodge numbers $(4, 214 - 12n)$, where $n = 0, 1, 2$. The cubic parts of the type II prepotentials are given in [15,20] and can be written as follows [10]:

$$
\begin{aligned}
F_{\text{cubic}}^{II} = & \ t_2(t_1^2 + t_1t_3 + 4t_1t_4 + 2t_3t_4 + 3t_4^2) \\
& + \frac{4}{3}t_1^3 + 8t_1^2t_4 + \frac{n}{2}t_1t_3^2 + (1 + \frac{n}{2})t_1^2t_3 \\
& + 2(n + 2)t_1t_3t_4 + nt_3^2t_4 + (14 - n)t_1t_4^2 \\
& + (4 + n)t_3t_4^2 + (8 - n)t_4^3.
\end{aligned}
\qquad (13)
$$

Then, inserting (5) into (13) yields

$$
\begin{aligned}
F_{\text{cubic}}^{II} = & \ -F_{\text{cubic}}^{\text{het}} = S(TU - V^2) + \frac{1}{3}U^3 \quad (14) \\
& + (\frac{4}{3} + n)V^3 - (1 + \frac{n}{2})UV^2 - \frac{n}{2}TV^2 .
\end{aligned}
$$

For $V = 0$, $F_{\text{cubic}}^{\text{het}}$ precisely reduces to the cubic prepotential of the S-T-U models [21,20].

Next, consider the models with Hodge numbers $(4, 148)$. For the model with base $\mathbf{F}_1$, the cubic part of the type II prepotential is, in the $K3$ phase ($c_2(J) = (92, 24, 36, 88)$), given by

$$
\begin{aligned}
F_{\text{cubic}}^{II} = & \ t_2(t_1^2 + t_1t_3 + 4t_1t_4 + 2t_4^2 + 2t_3t_4) \\
& + \frac{4}{3}t_1^3 + 8t_1^2t_4 + 8t_1t_4^2 + \frac{8}{3}t_4^3 \\
& + \frac{3}{2}t_1^2t_3 + 6t_1t_3t_4 + 3t_3t_4^2 + \frac{1}{2}t_1t_3^2 \\
& + t_3^2t_4 .
\end{aligned}
\qquad (15)
$$

It can be checked that, for $t_2 = 0$, the above turns into the prepotential of the three-parameter model with base $\mathbf{F}_1$ [20]. Inserting (5) (with $n = 1$) into (15) yields

$$
\begin{aligned}
F_{\text{cubic}}^{II} = & \ -F_{\text{cubic}}^{\text{het}} = S(TU - 2V^2) + \frac{1}{3}U^3 \\
& + 8V^3 - 4UV^2 - 2TV^2 .
\end{aligned}
\qquad (16)
$$

4. BPS orbits

An important role in the computation of the Wilsonian gravitational coupling F_1 is played by BPS states [22,23,6,7],

$$F_1 \propto \log \mathcal{M} \ , \tag{17}$$

where $\mathcal{M}$ denotes the moduli-dependent holomorphic mass of an $N = 2$ BPS state. For the heterotic S-T-U-V models under consideration, the tree-level mass $\mathcal{M}$ is given by [24]–[26]

$$\begin{aligned}
\mathcal{M} &= m_2 - im_1 U \\
&+ in_1 T + n_2(-UT + tV^2) + ibV \ .
\end{aligned} \tag{18}$$

Here, $l = (n_1, m_1, n_2, m_2, b)$ denotes the set of integral quantum numbers carried by the BPS state. The level matching condition for a BPS state reads

$$p_L^2 - p_R^2 = 2n^T m + \frac{1}{2t}b^2 \ , \tag{19}$$

where

$$p_R^2 = \frac{|\mathcal{M}|^2}{2Y} \ , \quad Y = \mathrm{Re}T\mathrm{Re}U - t(\mathrm{Re}V)^2 > 0 \ . \tag{20}$$

Let us consider states with $n_i = m_j = 0$. For $t = 1$, there are two states with $p_L^2 = 2$, $p_R^2 = 0$, carrying $b = \pm 2$. These two states are the additional vector multiplets which enhance the $U(1)_V$ to $SU(2)$ at $V = 0$. On the other hand, for $t = 2$, there are no such states with $p_L^2 = 2$, $p_R^2 = 0$, reflecting the fact that in this case there is no gauge symmetry enhancement of the $U(1)_V$ at $V = 0$.

Note that the Narain lattice L of signature $(3, 2)$ associated with (19) is given by $L = \Lambda \oplus U(-1)$, where $U(-1)$ denotes the hyperbolic plane $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and where $\Lambda = U(-1) \oplus$ $< 2t > = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2t \end{pmatrix}$ in a basis which we will denote by (f_2, f_{-2}, f_3) [19]. In the basis $j_1 = f_{-2} + f_2, j_3 = f_{-2}, j_4 = 2f_2 + 2f_{-2} - f_3$, L is equivalent to the matrix

$$I = \begin{pmatrix} -2 & -1 & -4 \\ -1 & 0 & -2 \\ -4 & -2 & -8 + 2t \end{pmatrix} \ . \tag{21}$$

Inspection of (13) and (15) shows that the matrix $-I$ is nothing but the intersection matrix of the $K3$ fibre of the Calabi–Yau threefolds (13) and (15) for $t = 1$ and $t = 2$, respectively.

Of special relevance to the computation of perturbative corrections to F_1 are those BPS states, whose tree-level mass vanishes at certain surfaces in the classical moduli space. These surfaces are surfaces in the Siegel modular threefold $\mathcal{A}_t = \Gamma_t \backslash \mathcal{H}_2$. Here, Γ_t denotes the target-space duality group, which is a paramodular group, $\Gamma_t \subset Sp_4(\mathbf{Q})$ [19]. For $t = 1$, $\Gamma_1 = Sp_4(\mathbf{Z})$. The condition $\mathcal{M} = 0$ is the condition for a rational quadratic divisor H_l [19] of discriminant

$$\mathrm{D}(l) = 2t(p_L^2 - p_R^2) = 4tm_1 n_1 + 4tn_2 m_2 + b^2. \tag{22}$$

The divisors H_l with discriminant $\mathrm{D}(l)$ determine the Humbert surface H_D in the Siegel modular threefold $\mathcal{A}_t$ [19]. Each Humbert surface H_D can be represented by a linear relation in T, U and V [19]. The Humbert surface H_1 corresponds to the surface $V = 0$. States becoming massless at $V = 0$ lay on the orbit $\mathrm{D}(l) = 1$, that is, on the orbit $n^T m = 0, b^2 = 1$. The Humbert surface H_{4t}, on the other hand, corresponds to the surface $T = U$. The states becoming massless on H_{4t} carry quantum numbers $n^T m = 1, b = 0$. They carry $p_L^2 = 2, p_R^2 = 0$, and hence, they correspond to the additional vector multiplets describing the enhancement of $U(1)_T \times U(1)_U$ [24].

5. The heterotic perturbative Wilsonian gravitational coupling F_1

The heterotic perturbative Wilsonian gravitational coupling F_1 for an S-T-U-V model is, up to ambiguities linear in T, U and V, given by [9,10]

$$F_1 = 24S \tag{23}$$
$$-\frac{2}{\pi} \sum_{\substack{k,l,b \in \mathbf{Z} \\ (k,l,b) > 0}} d(4tkl - b^2)Li_1(e^{-2\pi(kT + lU + bV)}),$$

where the condition $(k, l, b) > 0$ means again that: either $k > 0, l, b \in \mathbf{Z}$ or $k = 0, l > 0, b \in \mathbf{Z}$ or $k = l = 0, b > 0$. The expansion coefficients $d(4tN)$ $(N = kl - \frac{b^2}{4t})$ are determined in terms of

[10]

$$F(\tau) = \sum_{N \in \mathbf{Z}, \mathbf{Z} + \frac{4t-1}{4t}} c(4tN) q^N \ ,$$

$$E_2 F(\tau) = \sum_{N \in \mathbf{Z}, \mathbf{Z} + \frac{4t-1}{4t}} d(4tN) q^N \ . \qquad (24)$$

Here, the $c(4tN)$ denote the expansion coefficients occuring in the prepotential (1). The quantity $E_2 F(\tau)$ should be related to the gravitational threshold corrections, as follows [27,10],

$$\tilde{I}_{3,2} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Big[Z_{3,2} F(\tau) [E_2 - \frac{3}{\pi \tau_2}] - d(0) \Big] \ . (25)$$

The Wilsonian coupling F_1 should, on the other hand, also be expressible in terms of target-space duality modular forms, namely Siegel modular forms on $\mathcal{A}_t = \Gamma_t \backslash \mathcal{H}_2$. Below, we will rewrite F_1 in terms of Siegel modular forms.

5.1. The S-T-U-V models with $(N_V, N_H) = (5, 215 - 12n)$

For this class of models, the target space duality group is $\Gamma_1 = Sp_4(\mathbf{Z})$. The relevant Humbert surfaces are H_1 and H_4. The Siegel modular forms which vanish on H_1 and H_4 are given as follows [28]. The Siegel modular form which vanishes on the $T = U$ locus and has modular weight 0 is given by $\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5}$. It can be shown that, as $V \to 0$,

$$\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} \to (j(T) - j(U))^2 \ , \qquad (26)$$

up to a normalization constant. On the other hand, the Siegel modular form which vanishes on the $V = 0$ locus and has modular weight 0 is given by $\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}}$. It can be shown that, as $V \to 0$,

$$\mathcal{C}_5 \to V \left(\eta^{24}(T) \eta^{24}(U) \right)^{\frac{1}{2}} \ , \qquad (27)$$

up to a proportionality constant. Here, the Siegel form $\mathcal{C}_{12}$ is a modular form of weight 12 which generalises $\eta^{24}(T) \eta^{24}(U)$, that is

$$\mathcal{C}_{12} \to \eta^{24}(T) \eta^{24}(U) \qquad (28)$$

as $V \to 0$.

The Siegel modular forms $\mathcal{C}_5$ and $\mathcal{C}_{30}$ have infinite product expansions, as follows [28],

$$\mathcal{C}_5 = (qr^{-1}s)^{1/2} \prod_{\substack{k,l,b \in \mathbf{Z} \\ (k,l,b) > 0}} (1 - q^k r^b s^l)^{f(4kl - b^2)} \ ,$$

$$\mathcal{C}_{30} = (q^3 r^{-1} s^3)^{1/2}(q - s) \times \qquad (29)$$
$$\prod_{\substack{k,l,b \in \mathbf{Z} \\ (k,l,b) > 0}} (1 - q^k r^b s^l)^{f_2'(4kl - b^2)} \ ,$$

where $q = \exp(-2\pi T)$, $s = \exp(-2\pi U)$ and $r = \exp(-2\pi V)$. The coefficients $f(4kl - b^2)$ and $f_2'(4kl - b^2)$ are defined as follows [28]. Consider the expansion of the weak Jacobi form of weight 0 and index 1

$$\phi_{0,1} = \frac{\phi_{12,1}}{\eta^{24}} = \sum_{m \geq 0} \sum_{l \in \mathbf{Z}} f(4m - l^2) q^m r^l$$
$$= (r + 10 + r^{-1})$$
$$+ q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2)$$
$$+ q^2(\ldots) \ , \qquad (30)$$

where the sum over l is restricted to $4n - l^2 \geq -1$. Then, $f(N) = f(4m - l^2)$ if $N = 4m - l^2 \geq -1$, and $f(N) = 0$ otherwise. The coefficients $f_2'(N)$ are then given by $f_2'(N) = 8f(4N) + (2\left(\frac{-N}{2}\right) - 3)f(N) + f(\frac{N}{4})$. Here, $\left(\frac{D}{2}\right) = 1, -1, 0$ depending on whether $D \equiv 1 \bmod 8, 5 \bmod 8, 0 \bmod 2$.

Then, in analogy to the perturbative Wilsonian gravitational coupling F_1 for the S-T-U model [5,7]

$$F_1 = 24 S_{\text{inv}} - \frac{b_{\text{grav}}}{24\pi} \log \eta^{24}(T) \eta^{24}(U)$$
$$+ \frac{2}{\pi} \log(j(T) - j(U)) \ , \qquad (31)$$

the perturbative Wilsonian gravitational coupling for an S-T-U-V model should now be given by (in the chamber $T > U > 2V$) [10]

$$F_1 = 24 S_{\text{inv}} - \frac{b_{\text{grav}}}{24\pi} \log \mathcal{C}_{12} + \frac{1}{\pi} \log \frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5}$$
$$- \frac{1}{2\pi}(N_H' - N_V') \log \left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}} \right)^2 \ . \qquad (32)$$

Here, $N_V' = 2$ and $N_H' = 12n + 32$ denote the vector and the hyper multiplets which become massless at the $V = 0$ locus.

The invariant dilaton S_{inv} is given by [29]

$$S_{\text{inv}} = \tilde{S} + \frac{1}{10}L \ ,$$
$$\tilde{S} = S - \frac{4}{10}(\partial_T\partial_U - \frac{1}{4}\partial_V^2)h \ , \qquad (33)$$

where the role of the quantity L is to render S_{inv} free of singularities. It can be shown [10] that

$$\tilde{S} = S + \frac{1}{5\pi}\log\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} \qquad (34)$$
$$- \frac{3}{10\pi}(2+n)\log\left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}}\right)^2 + regular$$

and, hence,

$$L = -\frac{2}{\pi}\log\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} + \frac{3}{\pi}(2+n)\log\left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}}\right)^2 \ . \quad (35)$$

It follows that the Wilsonian gravitational coupling (32) can be rewritten as

$$F_1 = 24\tilde{S}$$
$$- \frac{1}{\pi}\left[\frac{19}{5}\log\mathcal{C}_{30}^2 + \frac{3}{5}(1-2n)\log\mathcal{C}_5^2\right] \ . \quad (36)$$

Note that the $\log\mathcal{C}_{12}$ terms have completely canceled out.

Comparison of (23) and (36)

$$F_1 = 24S - \frac{2}{\pi}\sum_{(k,l,b)>0}d_n(4kl-b^2)Li_1 \qquad (37)$$
$$= 24\tilde{S} - \frac{1}{\pi}\left[\frac{19}{5}\log\mathcal{C}_{30}^2 + \frac{3}{5}(1-2n)\log\mathcal{C}_5^2\right]$$

gives a highly non-trivial consistency check on (24) (here, we have ignored the issue of ambiguities linear in T, U and V). It yields, using the product expansions for $\mathcal{C}_5$ and $\mathcal{C}_{30}$ given in (29), that

$$d_n(N) = -\frac{6}{5}Nc_n(N) - \frac{19}{5}f_2'(N)$$
$$- \frac{3}{5}(1-2n)f(N) \ , \qquad (38)$$

where $N = 4kl - b^2 \in 4\mathbf{Z}, 4\mathbf{Z} + 3$. In order to show that (38) really holds, consider introducing the hatted quantities [9,10]

$$\hat{Z} = \frac{1}{72}\frac{(E_4^2\widehat{E_{4,1}} - E_6\widehat{E_{6,1}})}{\Delta} \ ,$$
$$J_C = \frac{2E_6\widehat{E_{6,1}}}{\Delta} + 81\hat{Z} \ , \qquad (39)$$

as well as

$$\tilde{Z}(\tau) = \hat{Z}(4\tau) = 2\sum_{N\in 4\mathbf{Z}\,or\,4\mathbf{Z}+3}f(N)q^N, \quad (40)$$
$$\tilde{J}_C(\tau) = J_C(4\tau) = \sum_{N\in 4\mathbf{Z},4\mathbf{Z}+3}c_J(N)q^N. \quad (41)$$

Then, it can be verified that

$$f_2'(N) = \frac{1}{2}c_J(N) + 6f(N) \ . \qquad (42)$$

It can also be shown that, for $m = 4, 6$ [9]

$$\Theta_q E_m = \frac{m}{12}(E_2 E_m - E_{m+2}) \ ,$$
$$\Theta_q\hat{E}_{m,1} = \frac{2m-1}{24}\left(E_2\hat{E}_{m,1} - \hat{E}_{m+2,1}\right) \ ,$$
$$\Theta_q\tilde{E}_{m,1} = \frac{2m-1}{6}\left(\tilde{E}_2\tilde{E}_{m,1} - \tilde{E}_{m+2,1}\right) \ , (43)$$

where

$$\tilde{E}_2(\tau) = E_2(4\tau) \ , \tilde{E}_{m,1}(\tau) = \hat{E}_{m,1}(4\tau) \ , \qquad (44)$$

and where $\Theta_q = q\frac{d}{dq}$. Then, by using (42) as well as (43), it can be shown that (38) indeed holds [10].

5.2. The S-T-U-V models with $(N_V, N_H) = (5, 149)$

For this class of models, the target-space duality group is Γ_2 [19]. Thus, the appropriate Siegel modular forms for this class of models will be different from the ones considered in the previous subsection. Of relevance for the following are going to be the Siegel modular forms with divisors H_1 and H_8, denoted by Δ_2 and $\Psi_{12}^{(2)}$ in [19], with modular weight 2 and 12, respectively. Both Δ_2 and $\Psi_{12}^{(2)}$ enjoy infinite product expansions, as follows [19],

$$\Delta_2 = q^{1/4}r^{-1/2}s^{1/2} \times$$
$$\prod_{\substack{k,l,b\in\mathbf{Z}\\(k,l,b)>0}}(1 - q^k r^b s^l)^{f_2(8kl-b^2)} \ ,$$
$$\Psi_{12}^{(2)} = q\prod_{\substack{k,l,b\in\mathbf{Z}\\(k,l,b)>0}}(1 - q^k r^b s^l)^{c_2(8kl-b^2)} \ , \quad (45)$$

where $q = \exp(-2\pi T)$, $s = \exp(-2\pi U)$ and $r = \exp(-2\pi V)$. The expansion coefficients $f_2(8kl-$

b^2) are the expansion coefficients of the following weak Jacobi form of weight 0 and index 2 [19]

$$\phi_{0,2} = \frac{\phi_{2,2}}{\eta^4} = \sum_{m \geq 0} \sum_{l \in \mathbf{Z}} f_2(8m - l^2) q^m r^l$$
$$= (r + 4 + r^{-1}) \qquad (46)$$
$$+ q(r^{\pm 3} - 8r^{\pm 2} - r^{\pm 1} + 16) + q^2(\ldots),$$

whereas the expansion coefficients $c_2(8kl - b^2)$ are the expansion coefficients of the nearly holomorphic Jacobi form of weight 0 and index 2 [19]

$$\Psi_{0,2} = \frac{E_{12,2}}{\eta^{24}} = \sum_{m \geq -1} \sum_{l \in \mathbf{Z}} c_2(8m - l^2) q^m r^l$$
$$= q^{-1} + 24 + q(\ldots) . \qquad (47)$$

Finally, we will denote the analogue of $\mathcal{C}_{12}$ by X_{12}. The Siegel form X_{12} should be such that it doesn't vanish in the interior of moduli space, and that it reduces to $\eta^{24}(T)\eta^{24}(U)$ in the limit $V \to 0$. Below, we will see that X_{12} will in the end drop out of the relevant expressions.

Thus, the Siegel modular form which vanishes on the $T = U$ locus and has modular weight 0 is given by $\frac{\Psi_{12}^{(2)}}{X_{12}}$. As $V \to 0$,

$$\frac{\Psi_{12}^{(2)}}{X_{12}} \to T - U , \qquad (48)$$

up to a normalization constant. On the other hand, the Siegel modular form which vanishes on the $V = 0$ locus and has modular weight 0 is given by $\frac{\Delta_2^6}{X_{12}}$. As $V \to 0$,

$$\frac{\Delta_2^6}{X_{12}} \to V^6 , \qquad (49)$$

up to a proportionality constant.

The perturbative Wilsonian gravitational coupling should then, in analogy to (32), be given by (in the chamber $T > U > 2V$)

$$F_1 = 24 S_{\text{inv}} - \frac{b_{\text{grav}}}{24\pi} \log X_{12} + \frac{2}{\pi} \log \frac{\Psi_{12}^{(2)}}{X_{12}}$$
$$- \frac{1}{6\pi}(N_H' - N_V') \log \frac{\Delta_2^6}{X_{12}} . \qquad (50)$$

Using that $b_{\text{grav}} = 336$, $N_V' = 0$ and $N_H' = 96$, it follows that

$$F_1 = 24 S_{\text{inv}} + \frac{2}{\pi} \log \Psi_{12}^{(2)} - \frac{96}{\pi} \log \Delta_2 . \qquad (51)$$

Note that, in contrast to (32), the dependence on X_{12} already drops out when using S_{inv}.

The invariant dilaton S_{inv} is given by [29]

$$S_{\text{inv}} = \tilde{S} + \frac{1}{10} L ,$$
$$\tilde{S} = S - \frac{4}{10}(\partial_T \partial_U - \frac{1}{8}\partial_V^2)h , \qquad (52)$$

where the role of the quantity L is to render S_{inv} free of singularities. Using (1), it follows that, as $T \to U$,

$$\partial_T \partial_U h = -\frac{1}{\pi} \log(T - U)$$
$$= -\frac{1}{\pi} \log \frac{\Psi_{12}^{(2)}}{X_{12}} + regular \qquad (53)$$

and that, as $V \to 0$,

$$-\partial_V^2 h = -\frac{1}{\pi} c(-1) \log V$$
$$= \frac{1}{12\pi} N_H' \log \frac{\Delta_2^6}{X_{12}} + regular . \qquad (54)$$

Hence,

$$\tilde{S} = S - \frac{4}{10\pi} \left(-\log \Psi_{12}^{(2)} + 6 \log \Delta_2 \right)$$
$$+ regular \qquad (55)$$

and

$$L = -\frac{4}{\pi} \left(\log \Psi_{12}^{(2)} - 6 \log \Delta_2 \right) . \qquad (56)$$

It follows that

$$F_1 = 24 \tilde{S} - \frac{1}{\pi} \left[\frac{38}{5} \log \Psi_{12}^{(2)} + \frac{192}{5} \log \Delta_2 \right] . \qquad (57)$$

Note that X_{12} has again cancelled out.

Comparison of (23) and (57)

$$F_1 = 24 S - \frac{2}{\pi} \sum_{\substack{k,l,b \in \mathcal{Z} \\ (k,l,b) > 0}} d(8kl - b^2) Li_1$$
$$= 24 \tilde{S} - \frac{1}{\pi} \left[\frac{38}{5} \log \Psi_{12}^{(2)} + \frac{192}{5} \log \Delta_2 \right] \qquad (58)$$

gives again a highly non-trivial consistency check on (24) (here, we have again ignored the issue of ambiguities linear in T, U and V). It yields, using the product expansions for Δ_2 and $\Psi_{12}^{(2)}$ given in (45), that

$$-2d(N) = \frac{6}{5} Nc(N) + \frac{38}{5} c_2(N) + \frac{192}{5} f_2(N), \quad (59)$$

where $N = 8kl - b^2 \in 8\mathbf{Z}, 8\mathbf{Z} + 7$. This relation can be explicitly checked for some of the coefficients. Consider, for instance, the cases where $N = -8, -1, 0$. Then, using that

$$
\begin{aligned}
d(-8) &= c(-8) = 1 \ , \ \ d(-1) = c(-1) = -48 \ , \\
d(0) &= c(0) - 24c(-8) = -168
\end{aligned}
\tag{60}
$$

as well as

$$
\begin{aligned}
f_2(-1) &= 1 \ , \ \ f_2(0) = 4 \ , \\
c_2(-8) &= 1 \ , \ \ c_2(0) = 24 \ ,
\end{aligned}
\tag{61}
$$

it is straightforward to check that (59) indeed holds.

Acknowledgement

REFERENCES

1. S. Kachru and C. Vafa, Nucl. Phys. **B 450** (1995) 69, hep-th/9505105.
2. V. Kaplunovsky, J. Louis and S. Theisen, Phys. Lett. **B 357** (1995) 71, hep-th/9506110.
3. A. Klemm, W. Lerche and P. Mayr, Phys. Lett. **B 357** (1995) 313, hep-th/9506112.
4. I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. **B 455** (1995) 109, hep-th/9507115.
5. G. Curio, Phys. Lett. **B 366** (1996) 131, hep-th/9509042; Phys. Lett. **B 368** (1996) 78, hep-th/9509146.
6. J. A. Harvey and G. Moore, Nucl. Phys. **B 463** (1996) 315, hep-th/9510182.
7. G. L. Cardoso, G. Curio, D. Lüst, T. Mohaupt and S.–J. Rey, Nucl. Phys. **B 464** (1996) 18, hep-th/9512129.
8. G. L. Cardoso, G. Curio, D. Lüst and T. Mohaupt, Phys. Lett **B 382** (1996) 241, hep-th/9603108.
9. T. Kawai, hep-th/9607078.
10. G. L. Cardoso, G. Curio and D. Lüst, hep-th/9608154.
11. J. A. Harvey and G. Moore, hep-th/9611176.
12. P. Aspinwall and J. Louis, Phys. Lett. **B 369** (1996) 233, hep-th/9510234.
13. P. Candelas and A. Font, hep-th/9603170.
14. G. Aldazabal, A. Font, L. E. Ibáñez and F. Quevedo, Nucl. Phys. **B 461** (1996) 85, hep-th/9510093.
15. P. Berglund, S. Katz, A. Klemm and P. Mayr, hep-th/9605154.
16. G. Aldazabal, A. Font, L. E. Ibáñez and A. M. Uranga, hep-th/9607121.
17. A. Klemm, P. Mayr and C. Vafa, hep-th/9607139.
18. M. Green, J. Schwarz and P. West, Nucl. Phys. **B 254** (1985) 327.
19. V. A. Gritsenko and V. V. Nikulin, alg-geom/9611028.
20. J. Louis, J. Sonnenschein, S. Theisen and S. Yankielowicz, Nucl. Phys. **B 480** (1996) 185, hep-th/9606049.
21. S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Commun. Math. Phys. **167** (1995) 301.
22. S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, Nucl. Phys. **B 365** (1991) 431.
23. C. Vafa, Nucl. Phys. **B 447** (1995) 261, hep-th/9505023.
24. G. L. Cardoso, D. Lüst and T. Mohaupt, Nucl. Phys. **B 450** (1995) 115, hep-th/9412209.
25. T. Kawai, Phys. Lett. **B 371** (1996) 59, hep-th/9512046.
26. C. D. D. Neumann, hep-th/9607029.
27. M. Henningson and G. Moore, hep-th/9608145.
28. V. A. Gritsenko and V. V. Nikulin, alg-geom/9603010.
29. B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. **B 451** (1995) 53, hep-th/9504006.