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Operator solutions of two-dimensional chiral gauge theories

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Abstract

The exact operator solutions of two-dimensional anomaly-free chiral abelian gauge theories are obtained. We show that anomaly-cancellation conditions arise as consistency requirements of these solutions. For a certain class of flavour symmetries, fermion condensates are constructed. These are shown to violate the fermion-number conservation rule. The models are extended to include massive fermions. We propose a bosonised lagrangian for the massive theory and verify that it complies with the Gupta-Bleuler condition.

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1 Introduction

Two-dimensional gauge theories have provided us with valuable informations since the pioneering work of Schwinger on two-dimensional quantum electrodynamics [1]. The subject evolved rapidly with the discovery of the operator solution of this model [2]. It was subsequently realised that this solution is of a key importance for the understanding of fundamental physical problems [2, 3] such as confinement and the structure of the θ -vacuum.

Any operator solution of this kind can only be of physical relevance if it respects gauge symmetries. In chiral gauge theories, the gauge field couples to a non-conserved current and the gauge symmetry is broken. In Salam-Weinberg model, different multiplets of right and left chiralities are introduced in order to restore the gauge symmetry. The gauge anomalies only cancel for a certain combination of multiplets. This anomaly-cancellation condition lead to the discovery of the top quark.

In this paper, we obtain the operator solution of a general two-dimensional quantum theory with $SU_R(n) \times SU_L(q)$ symmetry. As in Salam-Weinberg model, the anomalies cancel between different multiplets. We start by deriving the anomaly-cancellation condition in Section 2. In Section 3, we write the gauge and the fermion fields in terms of bosonic variables. The full operator solutions are then found by requiring that these fields obey Dirac and Maxwell equations. These solutions are similar to the operator solutions of the flavoured Schwinger model [4]. Massive excitations are obtained from the massive mesons. These are, in turn, constructed from the gauge field and their masses are generated by a dynamical Higgs mechanism. The massless operators build the θ vacua which are labelled by two angles θ_1 and θ_2 . Out of these solutions, condensates with non-vanishing fermion numbers are also constructed. We calculate the two-point functions of these condensates. The non-vanishing result shows that the fermion-number conservation rule is violated. In Section 4, we study the massive theory. We propose an ansatz for the bosonised massive lagrangian, using the principle of form invariance. This lagrangian depends on free parameters. In order to fix them, we identify the combinations of the fields that are required to remain massless by the BRST condition. This then insures the validity of our lagrangian. The last section is devoted to conclusions and comments.

2 The massless model

The interaction of massless right and left chiral fermions (ψ and λ) of charges e and e' with a U(1) gauge field (A_μ) is described by the Lagrangian density ²

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{j=1}^n \bar{\psi}_j \gamma^\mu (i\partial_\mu + eA_\mu \frac{1+\gamma_5}{2}) \psi_j \\ & + \sum_{l=1}^q \bar{\lambda}_l \gamma^\mu (i\partial_\mu + e'A_\mu \frac{1-\gamma_5}{2}) \lambda_l, \end{aligned} \quad (1)$$

where the space-time indices are $\mu, \nu = \{0, 1\}$. The coupling of the gauge field to the non-conserved currents,

$$J^+ = \frac{e}{2} \sum_j \bar{\psi}_j \gamma^+ \psi_j = e\psi_-^+ \psi_- \quad (2)$$

and

$$J^- = \frac{e'}{2} \sum_l \bar{\lambda}_l \gamma^- \lambda_l = e'\lambda_+^+ \lambda_+ \quad (3)$$

renders this model anomalous. The breakdown of the gauge symmetry is made manifest in the anomaly equation which expresses the non-vanishing of the total divergence of the chiral currents, *i.e.*,

$$\partial^\mu J_\mu = \frac{1}{4\pi} (ne^2 - qe'^2) \epsilon^{\mu\nu} F_{\mu\nu}. \quad (4)$$

A well-known method of dealing with an anomalous model is to introduce extra degrees of freedom [5] into the theory. In this procedure, the chiral theory is replaced by a gauge-invariant theory which incorporates the additional modes. The original anomalous theory can be recovered by a suitable choice of gauge [4]. However, it is believed that a reliable physical theory cannot contain gauge anomalies. The gauge symmetries can be restored by imposing the anomaly cancellation condition on the theory. Specifically, such a cancellation is achieved by requiring that the coefficient of the anomaly equation vanishes, *i.e.*,

$$ne^2 = qe'^2 \Rightarrow \left(\frac{e}{e'}\right)^2 = \frac{q}{n}. \quad (5)$$

²In our notation [4], $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^\mu \gamma^5 = \epsilon^{\mu\nu} \gamma_\nu$, $\epsilon_{01} = +1$ and $\tilde{p}_\mu = \epsilon_{\mu\nu} p^\nu$.

We shall study this anomaly-free quantum model and solve it in terms of elementary quantum scalar fields.

3 Quantum solution of the massless theory

In order to obtain the complete operator solution to the model (1), we need to solve the equations of motion and to insure that the physical Hilbert space is consistent with Maxwell equations.

The Dirac equations of motion, obtained from the Lagrangian (1) is ³

$$i\partial\eta + \tilde{e}A\frac{1 + \epsilon\gamma_5}{2}\eta = 0, \quad (6)$$

where η is ψ_i or λ_i , $\epsilon = \pm 1$ and $\tilde{e} = e, e'$. The most general expression for the gauge field is

$$A_\mu = -(\gamma\tilde{\partial}_\mu\Sigma + \delta\partial_\mu\eta), \quad (7)$$

where Σ and η are any scalar fields. Consequently, the field strength takes the form,

$$F_{\mu\nu} = \gamma\epsilon_{\mu\nu}\square\Sigma. \quad (8)$$

Similarly, the fermions can be expressed in terms of bosonic fields φ_f , ζ_f , η and Σ as follows ⁴:

$$\psi_{+,f} = \left(\frac{\mu}{2\pi}\right)^{1/2}K_f \exp\left[\frac{i}{4}\pi + i\sqrt{\pi}\left(-\varphi_f + \int_{x^1} dy^1 \dot{\varphi}_f(x^0, y^1)\right)\right], \quad (9)$$

$$\begin{aligned} \psi_{-,f} &= \left(\frac{\mu}{2\pi}\right)^{1/2}K_f \exp\left[-\frac{i}{4}\pi + i\sqrt{\pi}\left(\varphi_f + \int_{x^1} dy^1 \dot{\varphi}_f(x^0, y^1)\right)\right] \\ &\times \exp\left[i(e\gamma\Sigma + e\delta\eta)\right], \end{aligned} \quad (10)$$

$$\lambda_{+,f'} = \left(\frac{-\mu}{2\pi}\right)^{1/2}K_f \exp\left[\frac{i}{4}\pi + i\sqrt{\pi}\left(-\zeta_f + \int_{x^1} dy^1 \dot{\zeta}_{f'}(x^0, y^1)\right)\right]$$

³Note that the Dirac equations are similar to those obtained for flavoured Schwinger model.

⁴The fields Σ and η do not appear in the expressions for ψ_+ or λ_- which are the free chiral components.

$$\times \exp \left[-i(e'\gamma'\Sigma + e'\delta'\eta') \right], \quad (11)$$

$$\lambda_{-,f'} = \left(\frac{\mu}{2\pi}\right)^{1/2} K_f \exp \left[\frac{-i}{4}\pi + i\sqrt{\pi}(-\zeta_{f'} + \int_{x^1} dy^1 \dot{\zeta}_{f'}(x^0, y^1)) \right], \quad (12)$$

where the Klein-factors K_f insure the correctness of the commutation relations [4]. The gauge and the fermion fields satisfy the Dirac equations provided $\gamma = \gamma'$, $\delta = \delta'$ and the fields ζ_f , φ_f and η are massless.

In addition to the Dirac equations, Maxwell equations,

$$\partial_\mu F^{\mu\nu} + e \sum_j \bar{\psi} \gamma^\nu \frac{1 + \gamma_5}{2} \psi + e' \sum_l \bar{\lambda} \gamma^\nu \frac{1 - \gamma_5}{2} \lambda = 0, \quad (13)$$

should also be satisfied⁵. Using eqs. (9-12), and after some manipulations [4], we obtain the currents,

$$J_-^\psi = \bar{\psi} \gamma_- \psi = -\frac{1}{\sqrt{\pi}} \partial_- \varphi_f - \frac{e}{\pi} \delta \partial_- \eta - \frac{e}{\pi} \gamma \partial_- \Sigma, \quad (14)$$

$$J_+^\psi = \bar{\psi}_f \gamma_+ \psi_f = \frac{1}{\sqrt{\pi}} \partial_+ \varphi_f, \quad (15)$$

$$J_-^\lambda = -\frac{1}{\sqrt{\pi}} \partial_- \zeta_f, \quad (16)$$

$$J_+^\lambda = \frac{1}{\sqrt{\pi}} \partial_+ \zeta_f + \frac{e'}{\pi} \delta \partial_+ \eta + \frac{e'}{\pi} \gamma \partial_+ \Sigma. \quad (17)$$

On inserting these currents into the Maxwell equations (13), we obtain for $\nu = -$,

$$\gamma \partial_+ \square \Sigma + \frac{e'}{\sqrt{\pi}} \partial_+ \sum_f \zeta_f + \frac{e'}{\pi} q^2 \delta \partial_+ \eta + e q'^2 \frac{\gamma}{\pi} \partial_+ \Sigma = 0, \quad (18)$$

and for $\nu = +$,

$$-\gamma \partial_- \square \Sigma - \frac{e}{\sqrt{\pi}} \partial_- \sum_f \varphi_f - \frac{1}{\pi} \delta e^2 n \partial_- \eta - \frac{\gamma}{\pi} n e^2 \partial_- \Sigma = 0. \quad (19)$$

The massless terms (*i.e.* the terms containing massless fields) in the above two equations have to vanish separately. This is only possible

⁵In fact, Maxwell equations can only be satisfied on the physical Hilbert space.

if the space of states is chosen in such a way that the longitudinal currents

$$L_+ = \partial_+ \left[\frac{e'}{\sqrt{\pi}} \sum_f \zeta_f + \frac{qe'^2}{\pi} \delta\eta \right] \quad (20)$$

and

$$L_- = \partial_- \left[\frac{e}{\sqrt{\pi}} \sum_f \varphi_f + \frac{qe^2}{\pi} \delta\eta \right] \quad (21)$$

vanish weakly, *i.e.*, the physical subspace is defined to be that generated by states which obey

$$\langle \text{phys} | L_\mu | \text{phys} \rangle = 0. \quad (22)$$

In other words, the Gupta-Bleuler (or BRST) condition must be satisfied.

In contrast to the fields η , ζ_f and φ_f , which are massless, the bosonic field Σ obeys a massive equation. For this field, the contradictory equations of motion,

$$(\gamma \square + \frac{\gamma}{\pi} qe'^2) \Sigma = 0 \quad (23)$$

and

$$(\gamma \square + \frac{\gamma}{\pi} ne^2) \Sigma = 0, \quad (24)$$

only coincide if the anomaly-cancellation condition (5) is imposed on the charges e and e' . Consequently, the mass of the Σ field is ⁶

$$m_\Sigma^2 = \frac{ne^2}{\pi} = \frac{qe'^2}{\pi}. \quad (25)$$

The parameters γ and δ can now be computed. We require the vanishing of the norm of L_μ and impose canonical quantisation condition on η , taking care of the negative metric character of the commutator ⁷. We then obtain

$$\frac{ne^2}{\pi} \left[4 - \frac{ne^2\delta^2}{\pi} \right] = 0 \Rightarrow \delta = \frac{1}{e} \sqrt{\frac{\pi}{n}} = \frac{1}{e'} \sqrt{\frac{\pi}{q}} \quad (26)$$

⁶This corresponds to the dynamical Higgs mechanism.

⁷The fields ζ_f and φ_f are also canonically quantised, but with positive metric.

Similarly, by imposing the canonical commutation relation on the gauge field A_1 ⁸ and on the scalar field Σ , we obtain

$$\gamma = \frac{1}{e} \sqrt{\frac{\pi}{n}} = \frac{1}{e'} \sqrt{\frac{\pi}{q}}. \quad (27)$$

Next, we construct the physical operators. We define $\varphi = \frac{1}{\sqrt{n}} \sum_f \varphi_f$ and $\zeta = \frac{1}{\sqrt{q}} \sum_{f'} \zeta_{f'}$. The longitudinal currents (20, 21) simplify to

$$L_+ = e' \sqrt{\frac{q}{\pi}} \partial_+(\zeta + \eta) \quad (28)$$

and

$$L_- = e \sqrt{\frac{n}{\pi}} \partial_-(\varphi + \eta). \quad (29)$$

The negative metric part of the theory can now be isolated by re-expressing the physical operators in terms of the combined fields $\varphi + \eta$ and $\zeta + \eta$. The field ψ_- depends on φ and η and the field λ_+ on ζ and η . However, physical quantities can only depend on the combinations $\varphi + \eta$ or $\zeta + \eta$, due to the physical state condition (22). Also, the fields with opposite chiralities ψ_+ and λ_- should remain invariant. Thus we need a chiral gauge transformation. Taking these considerations into account, we define the following transformations⁹:

$$\begin{aligned} \psi_- &\rightarrow e^{i\sqrt{\frac{\pi}{n}} \int_{x^1} \dot{\eta}} \psi_- \quad , \quad \psi_+ \rightarrow \psi_+, \\ \lambda_+ &\rightarrow e^{-i\sqrt{\frac{\pi}{q}} \int_{x^1} \dot{\eta}} \lambda_+ \quad , \quad \lambda_- \rightarrow \lambda_-, \\ A_- &\rightarrow A_- - \frac{1}{e} \sqrt{\frac{\pi}{n}} \partial_- \eta, \\ A_+ &\rightarrow A_+ + \frac{1}{e} \sqrt{\frac{\pi}{n}} \partial_+ \eta. \end{aligned} \quad (30)$$

The physical fields are now given by,

$$\begin{aligned} \psi_{-f} &= \left(\frac{\mu}{2\pi}\right)^{1/2} K_f e^{i\sqrt{\frac{\pi}{n}} \Sigma} \\ &\times \exp i\sqrt{\pi} \left[\varphi_f - \frac{1}{\sqrt{n}} \varphi + \int (\dot{\varphi}_f - \frac{1}{\sqrt{n}} \dot{\varphi}) \right] \\ &\times \exp i\sqrt{\frac{\pi}{n}} \left[\eta + \varphi + \int (\dot{\eta} + \dot{\varphi}) \right] \end{aligned} \quad (31)$$

⁸The gauge field is $A_1 = \gamma \partial_0 \Sigma$ and the momentum conjugate to it is $\pi_1 = F^{01} = \gamma \square \Sigma = -\gamma \frac{n e}{\pi} \Sigma$.

⁹The partition function remains invariant as a result of the fine tuning of the charges e and e' .

and

$$\begin{aligned}
\lambda_{+f'} &= \left(\frac{\mu}{2\pi}\right)^{1/2} K_{f'} e^{-i\sqrt{\frac{\pi}{q}}\Sigma} \\
&\times \exp i\sqrt{\pi}\left[-\zeta_{f'} + \frac{1}{\sqrt{q}}\zeta + \int(\dot{\zeta}_{f'} - \frac{1}{\sqrt{q}}\dot{\zeta})\right] \\
&\times \exp -i\sqrt{\frac{\pi}{q}}\left[\eta + \zeta - \int(\dot{\eta} + \dot{\zeta})\right], \tag{32}
\end{aligned}$$

which depend on the required combination of fields. The last exponentials appearing in the expressions for the physical fields, *i.e.*,

$$\sigma(x) = \exp i\sqrt{\frac{\pi}{n}}\left[\eta + \varphi + \int(\dot{\eta} + \dot{\varphi})\right] = e^{i\theta_1} \tag{33}$$

and

$$\rho(x) = \exp -i\sqrt{\frac{\pi}{q}}\left[\eta + \zeta - \int(\dot{\eta} + \dot{\zeta})\right] = e^{i\theta_2}, \tag{34}$$

act as constant operators on the physical Hilbert space.¹⁰ This gives rise to the angles θ_1 and θ_2 . Thus, the vacuum has a structure similar to that of the Schwinger model—it is degenerate and is labelled by the angles θ_1 and θ_2 .

For general values of n and q , the solutions are similar to those of the flavoured Schwinger model. However, for rational values of $\sqrt{n/q}$ a special situation arises. One can construct condensates in the following way:

$$\eta_{f_1, \dots, f_{\sqrt{n'}}, f'_1, \dots, f'_{\sqrt{q'}}} = \psi_{f_1} \cdots \psi_{f_{\sqrt{n'}}} \bar{\lambda}_{f'_1} \cdots \bar{\lambda}_{f'_{\sqrt{q'}}}, \tag{35}$$

where n' and q' are such that $\sqrt{n'}$ and $\sqrt{q'}$ are integers and obey $\sqrt{n'/q'} = \sqrt{n/q}$. It is also possible to define the duals

$$\tilde{\eta}_{f_1, \dots, f_{n'-\sqrt{n'}}, f'_{q'-\sqrt{q'}}} = \psi_{f_1} \cdots \psi_{f_{n'-\sqrt{n'}}} \bar{\lambda}_{f'_1} \cdots \bar{\lambda}_{f'_{q'-\sqrt{q'}}}, \tag{36}$$

The η - $\tilde{\eta}$ -operators have non-vanishing two-point functions $\langle \eta_{\{f\}\{f'\}}(x) \tilde{\eta}_{\{\tilde{f}\}\{\tilde{f}'\}}(y) \rangle$ which break the fermion-number conservation

¹⁰This can be shown by using the two-point-functions relations,

$$\langle \varphi(x) \varphi(y) \rangle = \langle \zeta(x) \zeta(y) \rangle = -\langle \eta(x) \eta(y) \rangle.$$

law[6]¹¹. Let us take the example of the $SU(4)_R \times U(1)_L$ theory, which contains particles of the type

$$\eta(x)_{ij} = \psi_i \psi_j \bar{\lambda}(x) \quad (37)$$

and where the dual coincides with the original condensate, after a permutation of indices. We are interested in computing the correlators for long distances. In that case, we can drop the contribution of the massive field, and we are left with

$$\begin{aligned} \langle \eta_{12}(x) \eta_{34}(0) \rangle &\approx \\ &\approx \langle e^{-\frac{i}{2}\sqrt{\pi}(\varphi_1^{(+)} + \varphi_2^{(+)} - \varphi_3^{(+)} - \varphi_4^{(+)})(x)} e^{-\frac{i}{2}\sqrt{\pi}(\varphi_1^{(+)} + \varphi_2^{(+)} - \varphi_3^{(+)} - \varphi_4^{(+)})(0)} \rangle, \\ &= e^{-\frac{1}{4}4\ln(-x)} = [-x^-]^{-1} \quad . \end{aligned} \quad (38)$$

where the label-+ means the combination $\varphi^{(+)} = \varphi + \int_{x^1} dy^1 \dot{\varphi}(x^0, y^1)$. Notice the use of the infrared regularization of the exponential of the massless field in two dimensions, crucial in the present computation. In general, such two-point correlators are expected to vanish. Here they do not, in view of instanton effect as already announced in [6]. These excitations can be created in a world where $\sqrt{n/q}$ takes on rational values. Moreover, these condensates only have a -1 power decay (that is as a fermion) for very particular models, such as the example $SU(4) \times U(1)$ above, discussed in [6]. If we take the example of a model $SU(9) \times U(1)$, we have for the corresponding correlator the expression

$$\langle \eta_{123}(x) \tilde{\eta}_{456789}(0) \rangle \approx [-x^-]^{-1} \quad . \quad (39)$$

For a general symmetry $SU(n^2) \times SU(q^2)$, with both n and q not unit, we obtain

$$\langle \eta_{ff'}(x) \tilde{\eta}_{\tilde{f}\tilde{f}'}(0) \rangle \approx [-x^-]^{-1} [-x^+]^{-1} \quad . \quad (40)$$

Analogously, for a symmetry $SU(n^2) \times U(1)$ a correlator

$$\langle \eta(x_1) \cdots \eta(x_n) \rangle \approx \prod_{i < j} [x_i^- - x_j^-]^{-1} \quad (41)$$

where all indices are different does not vanish either.

¹¹Such two-point functions can be computed by known techniques of two-dimensional quantum field theory [4].

4 The massive theory

Unlike the massless theory, the massive theory is not exactly solvable. However, to obtain a bosonised massive lagrangian, the mass term of the original lagrangian can be treated perturbatively. Mass terms of the bosonised lagrangian are constructed by using the fields appearing in the massless bosonised theory. In addition, one should allow for possible renormalisation constants. This procedure is known as the principle of form invariance [8].

The massive theory is described by the lagrangian

$$\mathcal{L}_m = \mathcal{L} + \mathcal{L}_{\text{mass}}, \quad (42)$$

where \mathcal{L} is given by (1) and

$$\mathcal{L}_{\text{mass}} = m \sum_f \psi_{+f}^\dagger \psi_{-f} + m' \sum_{f'} \lambda_{+f'}^\dagger \lambda_{-f'} + c.c.. \quad (43)$$

The combinations $\psi_+^\dagger \psi_-$ or $\lambda_+^\dagger \lambda_-$ in the mass terms are determined by the charge superselection rules. The massive lagrangian can be bosonised by using the formulae (9-12), obtained in the massless case. We find,

$$\begin{aligned} \mathcal{L}_{\text{mass}}^{(0)} &= m \sum_f \cos[2\sqrt{\pi}\varphi_f + \sqrt{\frac{\pi}{n}}(\eta + \Sigma)] \\ &+ m' \sum_{f'} \cos[2\sqrt{\pi}\zeta_{f'} + \sqrt{\frac{\pi}{q}}(\eta + \Sigma)]. \end{aligned} \quad (44)$$

The bosonised massive lagrangian obtained from a simple extension of the massless lagrangian is, however, incomplete. The equations of motion are inconsistent and the physical operators fail to satisfy the BRST conditions.

Nevertheless, expression (44) gives us an insight into the possible form of the correct lagrangian. It shows that all the fields in the massive theory are interacting. We also use the principle of form invariance, discussed above, to propose the ansatz

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= m \sum_f \cos[\alpha\varphi_f + \beta\eta + \gamma\Sigma + \sigma\Sigma + \sigma\varphi + \epsilon\zeta] \\ &+ m' \sum_{f'} \cos[\alpha'\varphi_{f'} + \beta'\eta + \gamma'\Sigma + \sigma'\Sigma + \sigma'\varphi + \epsilon'\zeta]. \end{aligned} \quad (45)$$

The equations of motion obtained from the total lagrangian which includes the above mass terms are,

$$\begin{aligned}
(\square + \mu^2)\Sigma &+ m\gamma \sum_f \sin[\alpha\varphi_f + \beta\eta + \gamma\Sigma + \sigma\varphi + \epsilon\zeta] \\
&+ m'\gamma' \sum_{f'} \sin[\alpha'\zeta_{f'} + \beta'\eta + \gamma'\Sigma + \sigma'\varphi + \epsilon'\zeta] = 0, \quad (46)
\end{aligned}$$

$$\begin{aligned}
\square\varphi_f &+ m\alpha \sum_f \sin[\alpha\varphi_f + \beta\eta + \gamma\Sigma + \sigma\varphi + \epsilon\zeta] \\
&+ m\frac{\sigma}{\sqrt{n}} \sum_{\bar{f}} \sin[\alpha\varphi_{\bar{f}} + \beta\eta + \gamma\Sigma + \sigma\varphi + \epsilon\zeta] \\
&+ m'\alpha' \sum_{\bar{f}'} \sin[\alpha'\zeta_{\bar{f}'} + \beta'\eta + \gamma'\Sigma + \sigma'\zeta + \epsilon'\varphi] = 0, \quad (47)
\end{aligned}$$

$$\begin{aligned}
\square\zeta_f &+ m'\alpha' \sum_f \sin[\alpha'\varphi_f + \beta'\eta + \gamma'\Sigma + \sigma'\zeta + \epsilon'\varphi] \\
&+ m'\frac{\sigma'}{\sqrt{q}} \sum_{\bar{f}'} \sin[\alpha'\zeta_{\bar{f}'} + \beta'\eta + \gamma'\Sigma + \sigma'\zeta + \epsilon'\varphi] \\
&+ m\alpha \sum_{\bar{f}} \sin[\alpha\varphi_{\bar{f}} + \beta\eta + \gamma\Sigma + \sigma\varphi + \epsilon\zeta] = 0, \quad (48)
\end{aligned}$$

$$\begin{aligned}
-\square\eta &+ m\beta \sum_f \sin[\alpha\varphi_f + \beta\eta + \gamma\Sigma + \sigma\varphi + \epsilon\zeta] \\
&+ m'\beta' \sum_{f'} \sin[\alpha'\zeta_{f'} + \beta'\eta + \gamma'\Sigma + \sigma'\varphi + \epsilon'\zeta] = 0, \quad (49)
\end{aligned}$$

The equations of motion of the fields which are relevant for the BRST conditions follow from these;

$$\begin{aligned}
\square(\varphi + \eta) &= -m\left(\frac{\alpha}{\sqrt{n}} + \sigma - \beta\right) \sum_f \sin[\alpha\varphi_f + \beta\eta + \gamma\Sigma + \sigma\varphi + \epsilon\zeta] \\
&- m'(\epsilon' - \beta') \sum_{f'} \sin[\alpha'\zeta_{f'} + \beta'\eta + \gamma'\Sigma + \sigma'\varphi + \epsilon'\zeta], \quad (50)
\end{aligned}$$

$$\begin{aligned}
\square(\zeta + \eta) &= -m'\left(\frac{\alpha'}{\sqrt{q}} + \sigma' - \beta'\right) \sum_{f'} \sin[\alpha'\zeta_{f'} + \beta'\eta + \gamma'\Sigma + \sigma'\varphi + \epsilon'\zeta] \\
&- m(\epsilon - \beta) \sum_f \sin[\alpha\varphi_f + \beta\eta + \gamma\Sigma + \sigma\varphi + \epsilon\zeta]. \quad (51)
\end{aligned}$$

The above combined operators are required, by the Gupta-Bleuler condition, to be free massless fields (see eqs. (28) and (29)). Therefore,

we obtain

$$\beta = \frac{\alpha}{\sqrt{n}} + \sigma = \epsilon \quad (52)$$

and

$$\beta' = \frac{\alpha'}{\sqrt{q}} + \sigma' = \epsilon'. \quad (53)$$

The dimensions of the mass operators,

$$\begin{aligned} \dim(\bar{\psi}\psi) &= \frac{\alpha^2 + 2\alpha\sigma/\sqrt{n} + \sigma^2 + \gamma^2 + \epsilon^2 - \beta^2}{4\pi}, \\ &= \frac{\alpha^2 + \sigma^2 + \gamma^2 + 2\alpha\sigma/\sqrt{n}}{4\pi}, \end{aligned} \quad (54)$$

$$\dim(\bar{\lambda}\lambda) = \frac{\alpha'^2 + \sigma'^2 + \gamma'^2 + 2\alpha'\sigma'/\sqrt{n}}{4\pi}, \quad (55)$$

determine the range of the allowed values of the parameters γ and γ' . If, as in the massive Schwinger model, we require the absence of Thirring interactions and fix the parameter α to be $2\sqrt{\pi}$, the mass operator will have dimension greater than one [4]. On evaluating the operator-product expansion of the canonical fermion fields, we also find that $\sigma' = \sigma = 0$. Under such conditions, we obtain

$$\dim(\bar{\psi}\psi) = \frac{\alpha^2 + \gamma^2}{4\pi} \rightarrow 1 + \frac{\gamma^2}{4\pi}, \quad (56)$$

$$\dim(\bar{\lambda}\lambda) \rightarrow 1 + \frac{\gamma'^2}{4\pi}. \quad (57)$$

The mass perturbation expansion is only well-defined if the dimensions of these mass operators do not exceed 2. This is required for renormalizability. Therefore, we conclude that, $\gamma \leq 2\sqrt{\pi}$ and $\gamma' \leq 2\sqrt{\pi}$.

5 Conclusion and Comments

We have constructed the full operator solutions of chiral gauge theories and obtained the anomaly-cancellation conditions by requiring the consistency of these solutions. The solutions together with the structure of the θ -vacuum are similar to their counterparts in the Schwinger model. It would be interesting to see if these theories also have phase structures similar to the screening and confining phases of the Schwinger model.

For symmetry groups $SU(n)_R \times SU(q)_L$ with $\sqrt{n/q}$ a rational number, we have constructed condensates. For long distances these condensates behave like fermions. We have shown that they violate fermion-number conservation law. It remains to be seen if this phenomenon can be generally and satisfactorily explained in terms of instantons, as discussed in the case of symmetry $SU(4) \times U(1)$. In the bosonised theory, the fermion number is a topological number and the instantons can interplay between different vacua. Thus, the non-conservation of the fermion number can be traced back to instanton effects. The generalization of these results to a nonabelian gauge group would be welcome,[7] but the nonabelian bosonisation formulas are more involved, and do not permit a straightforward solution.

We have formulated a bosonised massive lagrangian by using the principle of form invariance, the BRST condition and mass perturbation techniques. However, it remains a challenging open problem to find a bosonisation formula which represents massive fermions as operator functions of bosonic variables.

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