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FAST INDICATORS OF LONG-TERM STABILITY*

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The problem of finding effective fast indicators of long-term stability for single particle betatronic motion in hadron accelerators is considered. Two indicators are analysed: the maximal Lyapunov exponent and the variation of the instantaneous tune. An automatic procedure to select stable from unstable particles is proposed. The guess of the dynamic aperture provided by these fast indicators is checked against long-term tracking for the 4D Hénon map and for a simplified version of the SPS lattice used for experiments. Both indicators prove to be very effective to determine long-term stability with a limited number of turns.

Keywords: Dynamic aperture.

1 INTRODUCTION

During the past years, several approaches have been made to define fast indicators of long-term stability in single-particle betatron motion. The aim is to speed up the numerical simulations of the dynamic aperture, that for a large machine such as the planned LHC should be carried out for approximately 10^7 turns. One approach to long-term stability is based on the Lyapunov exponent: this tool has been applied both to celestial mechanics[1–3] and to accelerator physics[4,5] to select chaotic from regular motion with a limited number of turns. Under the assumption that all the chaotic particles are unstable, one obtains a stability criterion. Another powerful indicator is the variation of the instantaneous tune[6–8] that can provide an analogous criterion.

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On the other hand, following the spirit of the Nekhoroshev theorem and its generalization to symplectic mappings,[9–11] several methods for giving stability estimates over long but finite times have been defined.[12–17] The basic idea[12] is to compute an invariant of motion with the highest achievable precision; then, one numerically evaluates the drift in the invariant space for a limited number of turns, and uses this value to extrapolate a bound for a large but finite number of turns. In order to evaluate the reliability of these approaches, the check of the stability estimates against long-term tracking is crucial.

In this paper we follow the approach to long-term stability based on two early indicators, namely the Lyapunov exponent and the tune variation. The main aim is to propose an automatic procedure to determine long-term particle loss. The procedure relies on the definition of thresholds that depend on the number of turns. Moreover, we carry out an accurate check of the early indicators predictions against long-term particle loss, making a statistical analysis of a wide sampling of initial conditions.

The main result is that one can establish thresholds for defining automatic procedures for long-term estimates. We considered a 4D Hénon map and a simplified version of the SPS lattice used for experiments,[18] taking into account of nonlinearities, of low order resonances, and of linear coupling. For these models, it turns out that both indicators are very predictive: the tune variation provides already at $10^2 - 10^3$ turns rather good dynamic aperture estimates (within 5%), and the Lyapunov gives more precise estimates at $10^3 - 10^4$ turns. Moreover, the Lyapunov evaluated for a large number of turns (greater than 10^4) leads to a systematic underestimate of the dynamic aperture since there exists a set of initial conditions that are chaotic but stable. This effect can be relevant and for the analysed lattices is of the order of 10%. Further developments of this work will concern the analysis of the case where tune modulation is relevant, and the effect of off-momentum dynamics.

2 EARLY INDICATORS

2.1 Lyapunov Exponent

The Lyapunov exponent is a dynamical indicator that is related to the rate of divergence of two infinitesimally neighbour particles. Let x be an initial condition at a given section of the machine, and let $x^{(n)}$ be its phase space position after n turns. If we consider a neighbour initial condition $\hat{x} = x + \delta$, with $|\delta| \ll 1$, then the estimate of the maximal Lyapunov exponent at the n-th turn reads

$$\lambda(n) = \frac{1}{n} \log \frac{|\hat{x}^{(n)} - x^{(n)}|}{|\delta|}. \tag{1}$$

One can have the following behaviours:

- The orbit is regular, i.e., it lies on a 2D deformed invariant torus. Then, the rate of divergence of two close particles is linear with the number of iterates, and therefore $\lambda(n)$ tends to zero for $n \to \infty$.
- The orbit is chaotic. Then, the rate of divergence is exponential with the number of iterates, and therefore $\lambda(n)$ tends to a positive limit.

There exist several methods to estimate the Lyapunov exponent (see Refs. 1–4); in the followings we will use the direct implementation of the above definition (method of neighbour particles), plus the renormalization procedure as outlined in Ref. 3, which is crucial to avoid underestimates of the Lyapunov for orbits that are chaotic but stable for long times.

2.2 Tune Variation

Another method to select regular from chaotic trajectories, proposed by J. Laskar during the last decade,[6–8] is based on the variation of the instantaneous tune. Let $\nu_x(m : n)$ and $\nu_y(m : n)$ be the nonlinear tunes in the x and y plane respectively, computed over the $m, m+1, m+2, ..., n$ turns; then we define the variation of the tune $\tau(n)$ over two successive samples of turns $1, ..., n/2$ and $n/2 + 1, ..., n$ as

$$\tau(n) = \sqrt{\frac{1}{2} \sum_{i=x,y} [\nu_i(1 : n/2) - \nu_i(n/2 + 1 : n)]^2}. \tag{2}$$

Also in this case one can distinguish between two different behaviours:

- For regular trajectories, the tunes are well defined and therefore $\tau(n)$ converges to zero in the limit $n \to \infty$.
- For chaotic trajectories, the tune is not well defined and therefore $\tau(n)$ is bounded away from zero.

In order to use this method, it is crucial to have very precise tools to evaluate the tune also with a limited number of turns.[7,19]

2.3 Thresholds to Select Regular from Chaotic Motion

We propose the following automatic procedure to forecast long-term particle loss.

- The orbit is evaluated through tracking for n turns; if the particle is stable, the early indicator $\lambda(n)$ or $\tau(n)$ is computed.
- If the early indicator evaluated at n turns is greater than a threshold $\sigma(n)$, then we assume that the particle is unstable. Otherwise, we consider it as stable.

No tentative of estimating the diffusion time is made: particles are considered to be either regular (and therefore stable) chaotic (and therefore one assumes that sooner or later they will be lost).

One can prove that all the regular particles have a Lyapunov exponent $\lambda(n)$ that converges to zero according to the law $1/n\log(An)$, and the constant A is related to the derivative of the nonlinear tune with respect to the amplitude.[20] Therefore we define the threshold for the Lyapunov as

$$\sigma_\lambda(n) = \frac{1}{n}\log(A_\lambda n). \tag{3}$$

Fortunately, the dependence of the threshold on the constant A_λ is quite weak. In the followings, we will optimize the choice of A_λ through the check with long-term tracking for the Hénon map. Then, we will show that the same value of A_λ gives good long-term estimates also for the SPS model.

The threshold for the tune variation has been fixed to

$$\sigma_\tau(n) = \frac{A_\tau}{n}; \tag{4}$$

this can be justified by heuristic arguments: in fact, the dependence on the inverse of the number of turns is an upper bound to the precision associated to the tune estimate with n turns for generic signals. Also in this case, the constant will be optimized through long-term tracking for the Hénon map, and will then be used for the SPS.

2.4 Dynamic Aperture Definition

In order to obtain a precise estimate of the dynamic aperture, it is crucial to perform a wide sampling of initial conditions in the space x, y. In particular, we use a polar grid $x = r\cos\alpha$ and $y = r\sin\alpha$, scanning $\alpha \in [0, \pi/2]$ and $r \in [0, R]$. The dynamic aperture is then defined as the average distance of the border of stability $r(\alpha)$

$$D = \int_0^{\pi/2} r^4(\alpha)\sin(2\alpha)d\alpha; \tag{5}$$

where the factor $\sin(2\alpha)$ comes from the integration in a 4D phase space.[21,22] We neglect the dependence on the angles, that can be taken into account through the information folded in the dynamics (see Refs. 21,22).

3 ANALYSIS OF THE HÉNON MAP

The Hénon map represents the transfer map of a linear lattice with a single sextupole in the thin lens approximation. We fix to one the ratio between the beta functions evaluated in the sextupole, and the linear tunes to 0.168 and 0.201 respectively, i.e., very close to low order resonances. This model features a phase space where long-term losses are relevant.

We carried out an extensive sampling of initial conditions in the plane x, y; we considered the polar coordinates r, α defined in the previous section, using 20 steps in the angle α and 40 steps in $r \in [0, 0.6]$. We checked that for $r > 0.6$ all the particles are lost in a very short number of turns (less than 64). For each initial condition we computed the orbit for 10^7 turns.

Both early indicators were evaluated for four different number of iterates: $n_1 = 128$, $n_2 = 512$, $n_3 = 2048$ and $n_4 = 8192$. In Figure 1 we plot the histogram of the distribution of the Lyapunov exponents $\lambda(n/2)$, computed over the grid of initial conditions, for $n = n_1, n_2, n_3, n_4$. We marked in black the initial conditions that are lost before 10^7 turns. The distribution features a high peak with a sharp fall on the right part. The abscissa of the fall separates rather well the stable from the unstable particles, and therefore it appears to be the natural threshold of the Lyapunov. In fact, it turns out that the threshold evaluated numerically through the four histograms is very well interpolated by Equation (3), with $A_\lambda = 0.15$. For low number of turns ($n = 128$ and $n = 512$), most of the particles

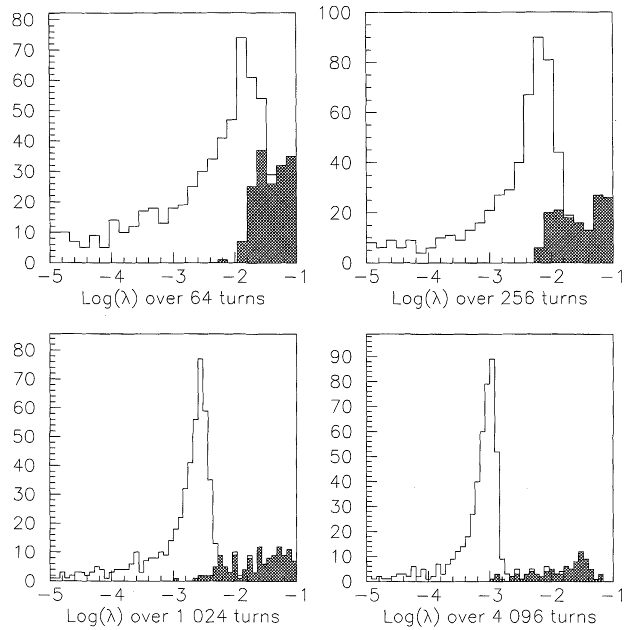

FIGURE 1 Distribution of the Lyapunov exponent (computed for four different increasing number of iterates) for the set of initial conditions of the 4D Hénon map at $\nu_1 = 0.168$ $\nu_2 = 0.201$ shown in Figure 1. Black parts of the histograms represents initial conditions that are lost before 10^7 turns.

whose early indicator prediction fails are unstable with Lyapunov lower than the threshold (intermittency). On the other hand, for higher number of turns ($n = 2048$ and $n = 8192$), most of the particles whose early indicator prediction is wrong are stable with large Lyapunov (stable chaos). This shows that for very large n the Lyapunov leads to a systematic underestimate of the dynamic aperture, since it assumes that all the chaotic particles will be lost. The relevance of this effect will be quantitatively evaluated in the following.

In Figure 2 the same histograms shown in Figure 1 are plotted for the tune variation. The distributions are wider: this is a good feature since it implies that the long-term estimate is less sensitive on the threshold. On the other hand, there is not a specific pattern that allows one to define a threshold without carrying out the long-term analysis. Using the long-term data, we empirically fixed the threshold using Equation (4), with $A_\sigma = 0.1$. For $n = 2048$ and $n = 8192$ a very large fraction of the particles

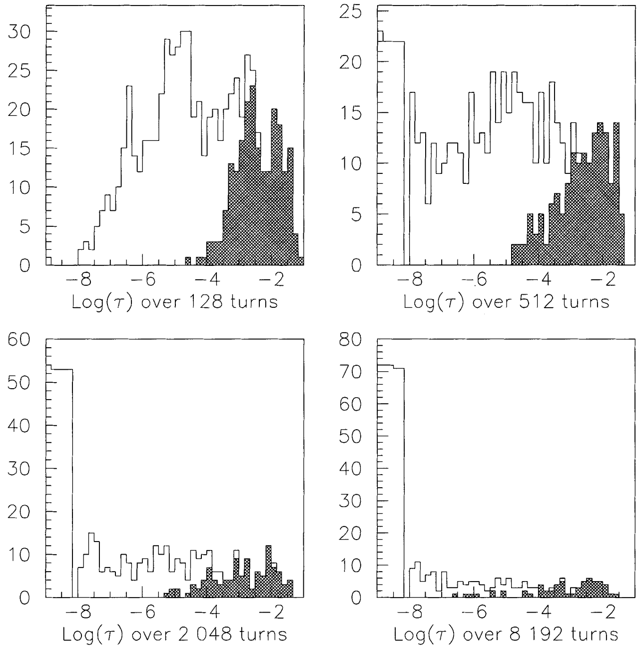

FIGURE 2 Distribution of the variation of the tune (computed for four different increasing number of iterates) for the set of initial conditions of the 4D Hénon map at $\nu_1 = 0.168$ $\nu_2 = 0.201$ shown in Figure 1. Black parts of the histograms represents initial conditions which are lost before 10^7 turns.

has a tune variation that is very small (less than 10^{-7}); no long-term loss is observed for these particles. This is another interesting feature that could allow one to define a conservative lower bound to long-term stability.

Having fixed the thresholds, we computed the dynamic aperture using the average defined in (5) through three methods: we define $D(n)$ as the dynamic aperture evaluated through particle loss at n turns, $D_\lambda(n)$ as the dynamic aperture evaluated through the Lyapunov exponent computed at $n/2$ turns, and $D_\tau(n)$ as the dynamic aperture evaluated through the tune variation over $n/2 + n/2$ turns.

In Table I we give the relative errors of these estimates compared to the dynamic aperture computed through particle loss at 10^7 turns, that has been taken as the reference value. One observes that the effect of long-term is rather relevant for this model: the dynamic aperture estimate

TABLE I Relative errors of the dynamic aperture estimates for the Hénon map model

n	*Particle loss*	*Lyapunov*	*Tune variation*
128	38%	11%	2%
512	30%	7%	–4%
2048	17%	0%	–5%
8192	9%	–5%	–5%
10^5	3%	–8%	

using the standard particle loss criterion through tracking at 2048 turns leads to an overestimate of about 20%. Moreover, the procedure based on the Lyapunov exponent allows one to have a more realistic estimate, especially for 1000–4000 turns where the errors due to stable chaos are compensated by intermittency. For 50 000 turns the Lyapunov is already considerably pessimistic, and this trend continues for larger number of turns. Finally, the procedure based on tune variation is rather precise with an extremely low number of turns ($n = 128$). Unfortunately, there is no improvement of the precision with the number of turns. It must be pointed out that, contrary to the Lyapunov analysis, the threshold parameter has been optimized through long-term tracking; for this reason the early indicator cannot be considered predictive for this model. We will show in the next section that using the same threshold one obtains good estimates for the considered SPS model.

4 ANALYSIS OF THE SPS

The used lattice is a simplification of the setup for nonlinear dynamics experiments.[18] The nonlinear part of the magnetic lattice is made up by 8 strong sextupoles normally used to extract the beam, and by 108 chromatic sextupoles placed near the main quadrupoles of the regular cells. The tune modulation due either to coupling with the synchrotron frequency or to ripples is neglected. Also the close orbit effects are not taken into account.

We considered the following working points:

- WP1: 26.6378 26.5319 – it is the working point of the SPS during operations.

TABLE II Relative errors of the dynamic aperture estimates for the SPS models

n	*Particle loss*	*Lyapunov*	*Tune variation*
128	14%	7%	5%
512	7%	5%	2%
2048	5%	2%	2%
8192	4%	1%	2%
10^5	1%	–7%	

- WP2: 26.6059 26.5373 and WP3: 26.8320 26.7990 – these are working points close to resonances of order five, that are used for the machine development experiments.
- WP4: 26.7200 26.6900 – it is the planned LHC working point.
- WP5: the same as WP4, plus some linear coupling that was included in order to check whether the tune variation is still effective in this case.

For each of the working points, we evaluated the histograms of the early indicators; one obtains very similar patterns to the case of the Hénon map. The comparison with long-term particle loss (10^6 turns) shows that the same thresholds worked out for the Hénon map hold. The dynamic aperture results are summarized in Table II, where an average over the five working points has been carried out. The long-term effect is less relevant compared to the analysed Hénon map: particle loss at 2048 turns gives the dynamic aperture within 5%. Nevertheless, both the Lyapunov and the tune variation give very similar results to the Hénon case.

References

[1] M. Hénon and C. Heiles, Astr. J., **69–1**, 73–79 (1964).
[2] R. Gonczi and C. Froeschle, *Cel. Mech.*, **25**, 271–80 (1981).
[3] G. Benettin, L. Galgani, A. Giorgilli and J. M. Strelcyn, *Meccanica*, **15–1**, 21–30 (1980).
[4] F. Schmidt, F. Willeke and F. Zimmermann, *Part. Accel.*, **35**, 249–256 (1991).
[5] F. Zimmermann, *Fourth European Particle Accelerator Conference*, edited by V. Sueller, *et al.* (World Scientific, Singapore, 1995) 327–331.
[6] J. Laskar, Icarus, **88**, 266–291 (1990).
[7] J. Laskar, C. Froeschlé and A. Celletti, *Physica D*, **56**, 253–269 (1992).
[8] J. Laskar, *Physica D*, **67**, 257–281 (1993).
[9] A. Giorgilli and L. Galgani, *Cel. Mech.*, **37**, 95–112 (1992).

[10] G. Turchetti, in *Number theory and physics*, J.M. Luck and P. Moussa, Springer Verlag, Berlin–Heidelberg, 1990) pp. 223.
[11] A. Bazzani, S. Marmi and G. Turchetti, *Cel. Mech.*, **47**, 333–359 (1990).
[12] R.L. Warnock and R.D. Ruth, *Physica D*, **56**, 88–215 (1992).
[13] H.S. Dumas and J. Laskar, *Phys. Rev. Lett.*, **70**, 2975–2979 (1993).
[14] J. Laskar, these proceedings.
[15] D. Robin and J. Laskar, these proceedings.
[16] J.S. Berg, these proceedings.
[17] G.H. Hoffstatter, these proceedings.
[18] W. Fischer, M. Giovannozzi and F. Schmidt, CERN SL (AP) **95–96** (1995).
[19] R. Bartolini, A. Bazzani, M. Giovannozzi, W. Scandale and E. Todesco, CERN SL (AP), **95–84** (1995).
[20] E. Todesco, M. Giovannozzi and W. Scandale, in preparation.
[21] E. Todesco and M. Giovannozzi, CERN SL (AP) **95–85** (1995 and *Phys. Rev. E*, in press).
[22] M. Giovannozzi and E. Todesco, these proceedings.