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PRECISE MEASUREMENT OF THE BETATRON TUNE

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In circular accelerators, sophisticated algorithms for frequency analysis can provide very precise estimates of the betatron tune from a relatively short sequence of turn-by-turn measurements of the beam position. They rely either on analytical interpolations of the Fast Fourier Transform, or on the numerical estimate of the Continuous Fourier Transform.

In this paper we review these methods, and the analytical estimates of the frequency error as a function of the sample size. Furthermore, we evaluate numerically the influence of noise in the determination of the tune. Finally, we present experimental applications to the LEP machine: they provide an accurate estimate of the detuning with amplitude due to the chromaticity sextupoles.

Keywords: Tune measurement.

1 INTRODUCTION

In circular accelerators, the tune can be measured by sampling the transverse position of the beam for N turns and by performing the Fast Fourier Transform (FFT) of the stored data. This approach has an intrinsic error proportional to $1/N$. In routine situations, N is chosen of the order of a thousand, as to reduce the tune error below 10^{-3}. However, there are special cases where either a better resolution or a faster measurement is desired. For instance, it is important to estimate the precise dependence of the tune with the oscillation amplitude or the precise distance of the working point from some harmful resonance, whenever strong non-linear magnetic fields perturb the regular

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motion of the particles. Furthermore one could try to measure the tune using a very limited number of turns, whenever the initial beam deflection is smeared out too quickly, either by filamentation or by head-tail or radiation damping.

In these and other situations, it is of paramount importance to make use of more efficient algorithms, as those suggested by E. Asseo[1–3] and by J. Laskar.[4–6] The first one is based on the analytical interpolation of the FFT, while the second one relies on the search of the maximum of the Continuous Fourier Transform. Both of them provide tune estimates more accurate than those of a plain FFT, as discussed in Ref. 7.

Another issue is the effect of the finite resolution of the instrumentation used to measure the beam position. In computer simulations, this is obviously negligible. In real measurements, instead, the finite resolution of the beam position monitor introduces a noise, that modifies the frequency response of the beam.[8,9]

In the absence of noise, the outlined algorithms for tune determination have an intrinsic error proportional to $1/N^2$. The error becomes proportional to $1/N^4$, when the raw data are treated with a Hanning filter. In presence of noise, the determination of the tune is less precise. We checked numerically that the error is still smaller than that of a plain FFT, i.e. smaller than $1/N$, at least for a signal to noise ratio varying in the range between 1 and 10^{-3}.

As an experimental application, we discuss the measurement of the detuning with the amplitude in LEP, under condition of strong damping (the initial oscillations of the beam disappear in about 150 turns).

The plan of the paper is the following. In Section 2 we recall the methods to compute the tune and the estimate of the algorithmic error as a function of N. In Section 3 we discuss the effect of noise. In Section 4 we discuss the measurement of the detuning with the amplitude in LEP at 20 GeV.

2 TUNE DETERMINATION AND ERROR ESTIMATE

A standard way to measure the tune consists in displacing transversally the particles by a fast kicker, in observing the beam position over N turns, and in Fourier analyzing the stored data. The tune is in general deduced by inspecting the power spectrum computed by an FFT and by choosing the value corresponding to the maximum frequency response. There are two basic methods to improve the resolution of the spectral analysis. Both of them rely on the assumption that the spectrum of the transverse oscillations of the beam contains a limited number of peaks. They correspond to the eigenfrequencies of the motion or to combinations of them driven by either linear or non-linear coupling, or by the interplay of synchrotron and betatron motion. In general, the harmonics of these peaks decrease very rapidly, and therefore can be neglected.

2.1 Fourier Series (FS)

The Fourier Series (FS) algorithm allows to compute the tune, using N consecutive values of the orbit. The time series $\{z(1), z(2), ..., z(N)\}$ of one of the orbit coordinates, can be expanded as a linear combination of N orthonormal functions:

$$z(n) = \sum_{j=1}^{N} \phi(\nu_j) \exp(2\pi i n \nu_j) \qquad \nu_j = \frac{j}{N}; \tag{1}$$

the coefficients $\phi(\nu_j)$, representing the amplitude spectrum, are given by the inverse formula

$$\phi(\nu_j) = \frac{1}{N} \sum_{n=1}^{N} z(n) \exp(-2\pi i n \nu_j). \tag{2}$$

One assumes that the N samples $z(n)$ are in fact extended in a periodic sampled signal of period N.

2.1.1 Fast Fourier Transform (FFT) In principle, the computation of the FS for a signal of N samples requires N^2 operations; indeed, if N is a power of two, one can define an algorithm that computes the FS by using only $N \log N$ operations: this method is called Fast Fourier Transform (FFT).

The error associated with the FFT is due to the discreteness of the frequencies ν_j, and is given by

$$|\epsilon_{\rm FFT}| \le \frac{C_{\rm FFT}}{N} \qquad \text{where} \qquad N = 2^M \qquad C_{\rm FFT} = \frac{1}{2}. \tag{3}$$

The FFT provides a very fast estimate of the complete Fourier spectrum, however, the evaluation of the main frequency is made with a poor precision.

2.1.2 Interpolated FFT Since the error in the FFT estimate is due to the discreteness of the spectrum, one can try to obtain better results by interpolating it around the main peak. The tune is then the abscissa of the maximum of the interpolating function. Following the approach outlined by Asseo,[1–3] we use as interpolating function the spectrum of a pure sinusoidal signal with unknown frequency ν_{Fint}:

$$|\phi(\nu_j)| = \left| \frac{\sin N\pi(\nu_{\text{Fint}} - \nu_j)}{N \sin \pi(\nu_{\text{Fint}} - \nu_j)} \right|. \tag{4}$$

For large N, the accuracy is given by

$$|\epsilon_{\text{Fint}}| \leq \frac{C_{\text{Fint}}}{N^2}. \tag{5}$$

This estimate holds, provided that the distance in frequency between the main peak and the closest one is larger than $1/N$ (for more details see Ref. 7).

2.1.3 Interpolated FFT with Data Windowing A standard approach used in signal processing theory to improve the Fourier analysis is based on filtering the data $z(n)$ using weight functions $\chi(n)$ (see Ref. 10). In this case, the FS of the orbit reads

$$\phi(\nu_j) = \frac{1}{N} \sum_{n=1}^{n} z(n)\chi(n) \exp(-2\pi i n \nu_j). \tag{6}$$

If we consider a Hanning filter

$$\chi(n) = 2 \sin^2\left(\frac{\pi n}{N}\right); \tag{7}$$

then, the spectrum of a pure sinusoidal signal of frequency ν_0 is given by

$$|\phi(\nu_j)| = \left| \frac{\sin^2(\pi/N)\ \sin \pi N(\nu_0 - \nu_j)\ \cos \pi(\nu_0 - \nu_j)}{N \sin \pi(\nu_0 - \nu_j)\ \sin \pi \left(\nu_0 - \nu_j + \frac{1}{N}\right)\ \sin \pi \left(\nu_0 - \nu_j - \frac{1}{N}\right)} \right|. \tag{8}$$

The effect of the filter is to increase the width of the main peak centered at ν_0 and to decrease the amplitude of the sidelobes. In fact their height as a function of N is reduced to $1/N^3$ while without filter it is $1/N$.

By applying the same reasoning used for the case of the interpolated FFT, it is possible to show that the expression (8) can be used as interpolating function: in this case ν_0 represents the unknown interpolating frequency. In this case the error scales as:

$$|\epsilon_{\text{Fint}}| \leq \frac{C_{\text{FHan}}}{N^4}. \tag{9}$$

The details can be found in Ref. 7.

2.2 Fourier Transform Methods

2.2.1 Fourier Transform (FT) Another very effective approach for spectral analysis which has been extensively used in the literature[4–6,11] is based on the Fourier Transform (FT). Let us consider a continuous function $f(t)$, where $t \in \mathbf{R}$. Then, it can be expanded as a linear combination of an infinite number of orthonormal functions:

$$f(t) = \int_{-\infty}^{+\infty} \phi(\nu) \exp(2\pi i \nu t) d\nu. \tag{10}$$

The function $\phi(\nu)$ is the FT of $f(t)$, and is given by the inverse formula

$$\phi(\nu) = \frac{1}{N} \int_{0}^{N} f(t) \exp(-2\pi i \nu t) dt. \tag{11}$$

In our case we have a discrete system, whose orbit $z(n)$ is defined only for integer times. Therefore one has to replace the integral (11) with a discrete sum:

$$\phi(\nu) = \frac{1}{N} \sum_{n=0}^{N} z(n) \exp(-2\pi i \nu n). \tag{12}$$

Contrary to the case of the interpolated FFT, no analytical formulas are available; indeed, the maximum can be computed using ν_{FFT} as a first guess, an by applying standard numerical tools such as the bisection method or the Newton method.

The scaling laws are the same as in the case of the interpolated FFT. For large N, provided that the distance $\Delta\nu$ between the main frequency and the closest one is greater than $1/N$, one has

$$|\epsilon_{\mathrm{FT}}| \leq \frac{C_{\mathrm{FT}}}{N^2}. \tag{13}$$

Otherwise the error is independent of N.

2.2.2 FT and Data Windowing As already mentioned, data windowing considerably improves the precision of the method: one defines an FT as

$$\phi(\nu) = \frac{1}{N}\sum_{n=0}^{N} z(n)\chi(n)\exp(-2\pi i\nu n) \tag{14}$$

where $\chi(n)$ is a window function. In the case of a Hanning window [see Equation (7)] one obtains an error estimate which scales like N^{-4}:

$$|\epsilon_{\mathrm{FTHan}}| \leq \frac{C_{\mathrm{FTHan}}}{N^4} \tag{15}$$

This result is proven in Ref. 7.

3 EFFECT OF NOISE

There are several perturbing effects that reduce the precision associated to the measure of the transverse beam position. In general, one can assume that the main source of uncertainty is related to the granularity of the Analog-to-Digital-Converter (ADC), used to digitize the position signal. This means that all the other perturbations such as the non-linear response of the pick-up, its finite resolution, the electronic noise, the distortion due to the cable transmission, and so on, are negligible compared to the step of the ADC. The least significant bit (LSB) of the ADC is equal to zero or one, randomly. In the frequency domain, the random variation of the LSB introduces a white noise whose r.m.s. amplitude is $1/2$ LSB. The effect on the tune precision can be investigated using a simple numerical model that contains a main sinusoidal wave of frequency ν_0 (with its harmonics), together with a secondary sinusoidal wave of frequency ν_1 (with its harmonics). The amplitude of the harmonics is assumed to decrease rapidly. We therefore consider

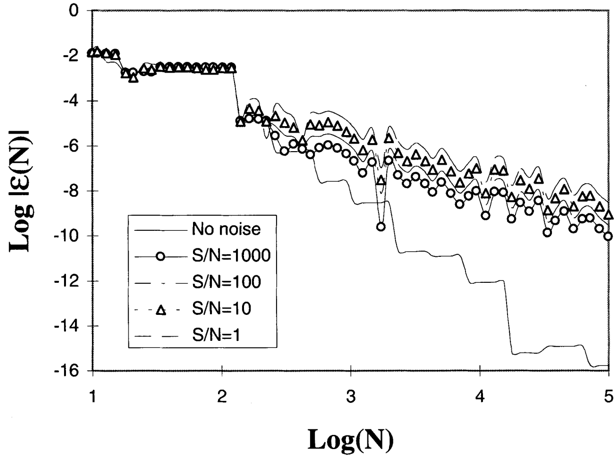

FIGURE 1 Tune error $\epsilon(N)$ versus N for different values of the signal over noise power ratio s/n. The signal is generated by two sine waves plus harmonics. The tune is computed using the interpolation of the FFT plus Hanning filtering. Five cases, referring to different signal to noise ratio, are shown.

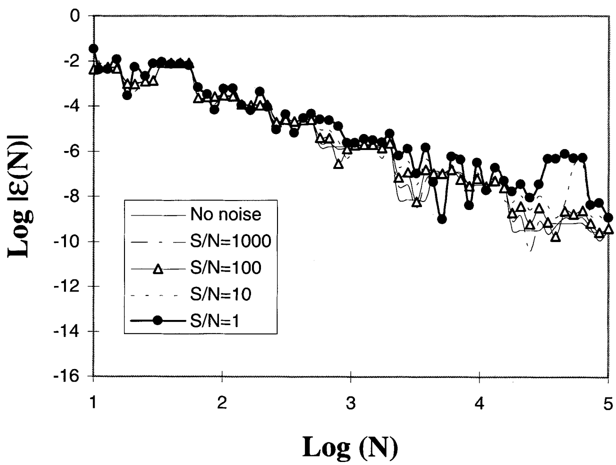

FIGURE 2 Tune error $\epsilon(N)$ versus N for different values of the signal over noise power ratio s/n. The signal is generated by two sine waves plus harmonics. The tune is computed using the interpolation of the FFT without filtering. Five cases, referring to different signal to noise ratio, are shown.

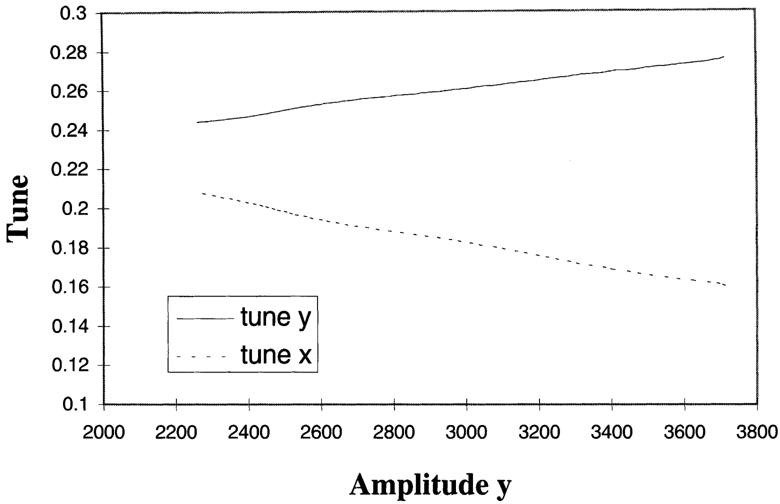

FIGURE 3 Horizontal and vertical tune as a function of vertical amplitude. The tune is computed using a moving window of 64 turns. The vertical amplitude, averaged over the 64 turn window, is in μm. The method used is the interpolation of the FFT without filtering.

$$z(n) = e^{2\pi i \nu_0 n} + \sum_k a_k \, e^{2\pi i \nu_0 k n} + b_0 e^{2\pi i \nu_1 n} + \sum_k b_k \, e^{2\pi i \nu_1 k n} + r(n) \text{ LSB} , \tag{16}$$

where $|a_k|$, $|b_k| < 1$, $n \in N$, and $r(n)$ is a random variable equal to 0 or 1.

In our simulations we chose $\nu_0 = 0.28$ and $\nu_1 = 0.31$ to simulate the working point of the LHC. We consider five harmonics of ν_0 and ν_1, and we assume that their amplitudes decrease exponentially with their order. The main tune ν_0 is found using the interpolated FFT algorithm. The reference case, without noise, is compared to several cases, where the signal to noise power ratio (s/n) vary over a large range. Due to the noise, the precision associated to the tune is reduced, as shown in Figures 1 and 2. In particular, the beneficial effect of the Hanning filter is completely lost, even if the noise is small $(s/n = 1000)$. On the other hand, the effect of the noise has a weak dependence on the s/n ratio. In addition, the tune error scales always better than $1/N$, even in the extreme case when the noise and the signal powers are equal.

4 TUNE MEASUREMENTS IN LEP

Archived data of LEP have been used to illustrate how one can measure the tune as a function of the amplitude even in the presence of a strong damping.

The horizontal and the vertical beam oscillations following a vertical deflection of about 5 mm are collected and stored for 1024 consecutive turns. The horizontal motion, induced by the residual linear coupling, has an initial amplitude of about 1.5 mm. The oscillations persist for 150 turns only, since the too high circulating current induces head-tail damping. The horizontal and the vertical tunes are both computed with the interpolated FFT algorithm. The computation is made using the first 64 values of the horizontal and the vertical positions respectively. The vertical amplitude of oscillation is evaluated by averaging over the same set of 64 turns. The calculation is iterated over several successive sets of 64 turns. In each iteration the window of 64 turns moves by one turn until the whole range of data is scanned. The results are shown in Figure 3. The vertical tune increases with the vertical amplitude, whereas the horizontal tune decreases with it. This is in a qualitative agreement with the expected behavior of the LEP machine with injection optics. Quantitative comparisons are not yet available. The ADC step of the LEP pick-ups is of the order of 60 μm, therefore $s/n \geq 10$ even in the last set of 64 data. Using the results of the simulations presented in the previous section, we can conclude that the tune accuracy is likely to be of the order of 10^{-3}, or better. A deeper investigation is needed to confirm that.

5 CONCLUSIONS

We presented some of the advanced methods to determine the betatron tune in circular particle accelerators. We discussed their precision, and pointed out the harmful effect of the electronic noise due to the analog-to-digital conversion of the beam position signal. Finally, we presented an experimental application to evaluate the detuning with amplitude in LEP, in the presence of a strong damping.

Our experimental analysis was on purpose concise and mostly oriented to apply the theoretical results to real measurements. The applications of the modern methods of Fourier analysis discussed here are extremely appealing both in experimental and numerical studies of beam dynamics. These approaches are certainly propaedeutic to a deeper investigation of existing or planned accelerators.

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