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A NEW SUPERSYMMETRY

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ABSTRACT

We propose a new supersymmetry in field theory that generalizes standard supersymmetry and we construct field theoretic models that provide some of its representations. This symmetry combines a finite number of ordinary four dimensional supersymmetry multiplets into a single multiplet with a new type of Kaluza-Klein embedding in higher dimensions. We suggest that this mechanism may have phenomenological applications in understanding family unification. The algebraic structure, which has a flavor of W-algebras, is directly motivated by S-theory and its application in black holes. We show connections to previous proposals in the literature for 12 dimensional supergravity, Yang-Mills, (2,1) heterotic superstrings and Matrix models that attempt to capture part of the secret theory behind string theory.

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1 New supersymmetry

The usual $N = 1$ superalgebra in four dimensions is

$$\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} = \sigma^\mu_{\alpha\dot{\beta}}\, p_\mu \ , \tag{1}$$

where $\alpha, \dot{\beta}$ are Weyl spinor indices corresponding to $(1/2, 0)$ and $(0, 1/2)$ representations of the Lorentz group $SO(3,1)$. The simplest example of the new superalgebras that we will discuss in this paper is the $N = 1$ case

$$\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} = \sigma^\mu_{\alpha\dot{\beta}}\, p_\mu v_+ \ . \tag{2}$$

Here $v_+ = v_{0'} + v_{1'}$ is the light-cone component of a new operator $v_m = i\tilde{\partial}_m$ acting as a momentum in an additional 1+1 dimensional space $y^m = (y^{0'}, y^{1'})$ beyond the usual four dimensions x^μ. We will see in section-3 that it is possible to interpret these as the 12th and 11th dimensions respectively. Note that there are two time coordinates x^0 and $y^{0'}$; we will show that in our approach no problems arise due to this fact and that four dimensional physics looks quite conventional. This superalgebra has an isometry group $SO(3,1) \otimes SO(1,1)$ which is the direct product of the Lorentz groups in the x^μ and y^m spaces. The group $SO(1,1)$ consists of a single parameter corresponding to a boost that mixes the y^m . Its action on the Weyl spinors is an overall scale transformation with weight 1/2, while its action on the vector v_+ is an overall scale transformation with weight 1.

In this paper we construct field theory models that provide representations of this $N = 1$ superalgebra. The fields $\Phi(x, y)$ depend on the 4D x^μ and on the 2D $y^m = y^\pm = y^{0'} \pm y^{1'}$. We take derivatives with respect to them ∂_μ and $\tilde{\partial}_\pm$ such that $p_\mu v_+ = -\partial_\mu \tilde{\partial}_+$. On this space the form of the superalgebra is

$$\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} = -\sigma^\mu_{\alpha\dot{\beta}}\, \partial_\mu \tilde{\partial}_+ \ . \tag{3}$$

If one considers an expansion of the fields in 2D momentum modes

$$\Phi(x, y) = \sum_{k_\pm} \Phi_k(x)\ e^{-i\left(k_+ y^+ + k_- y^-\right)} \tag{4}$$

one sees that for each mode the new superalgebra reduces to the standard $N = 1$ superalgebra, except for a rescaling of the momentum $p^\mu k_+$ by a different amount for each mode. Hence the effect of this type of "Kaluza-Klein expansion" on the 4D mass spectrum is very different than the usual one. We will suggest that the extra space y^m may be related to family structure (of quarks and leptons) through this new type of expansion.

As explained in section-3 our original motivation for considering this superalgebra comes from recent developments in S-theory. Some sectors of S-theory may be described as sectors in which the 12D superalgebra simplifies to

$$\{Q_\alpha, Q_\beta\} = \gamma^{MN}_{\alpha\beta}\ p_M\, v_N \tag{5}$$

There is a related version that applies to 13D as well as to compactifications to lower dimensions (see [1][2] and section-3). The compactified form was recently used to explain the presence of up to 12 (or 13) hidden dimensions in supersymmetric black holes [2]. This development provided a strong motivation since the relevance of the hidden dimensions and of the superalgebra was demonstrated in a potentially physical system. There were additional hints that such new superalgebras provide a framework for understanding some deeper structures. In particular, the 12D version in (5) with all possible eigenvalues of the operators $p_M = i\partial/\partial x^M$, $v_N = i\partial/\partial y^N$ that satisfy certain BPS constraints was first suggested in [1] as a basis for extending 11D supergravity to 12D supergravity. It was later used with a fixed eigenvalue of v_N (that breaks $SO(10,2)$) in a super Yang-Mills theory [3], and in a related matrix model [4] that recasts a matrix version [5] of 11D M-theory [6] to a 12D version. The same algebra was also understood to be present in a (2,1) heterotic string [7]. From the point of view of the superalgebra and S-theory the models in [3][4] [7] are incomplete because of the fixed v_N. As explained in section-3, fully $SO(10,2)$ covariant generalizations of these proposals must exist by allowing all eigenvalues of v_N (as in section-2). Furthermore such models may be regarded as intermediate steps toward the construction of the secret theory behind string theory. These provide additional motivations for studying this type of superalgebra.

In section-3 we will show how the superalgebra (2) is embedded in S-theory and how it is generalized in all possible ways up to 13 higher dimensions. But at a simpler level one may also consider the $N=1$ superalgebra (2) on its own merit as a symmetry structure in field theory. Thus, a study of its representations in field theory is undertaken in section-2. In this context we have discovered a new Kaluza-Klein mechanism for embedding families in higher dimensions. We find that several families belong together in the same supermultiplet. In section-2 we provide first examples of representations and family generation mechanisms that may be generalized in several ways. The fact that this approach may have phenomenological implications is both surprizing and welcomed. Connections to M-,F-,S-theories are also given in section-3. In Section-4 we present the first steps of a superfield formalism and conclude with some observations.

2 Field theoretic representations

The simplest multiplet of standard $N = 1$ supersymmetry is the scalar multiplet that contains the fields $(\phi, \psi_\alpha, F)(x)$ that appear in the Wess-Zumino (WZ) model. For the new $N = 1$ superalgebra we will present four different representations that connect to the WZ representation upon Kaluza-Klein reduction (there may also be others). We will refer to them as the scalar-vector multiplets with the fields

$$(\phi, \psi_\alpha, V_+)(x, y) \quad or \quad (\phi', \psi'_\alpha, V'_-)(x, y) \tag{6}$$

and the scalar-scalar multiplets with the fields

$$(\varphi, \chi_\alpha, f)(x, y) \quad or \quad (\varphi', \chi'_\alpha, f')(x, y). \tag{7}$$

The new $N = 1$ superalgebra has the isometry $SO(3, 1) \times SO(1, 1)$, therefore every field must correspond to a representation of this group. Since we wish to connect to the WZ fields we take a triplet of fields that are (scalar, chiral spinor, scalar) under $SO(3, 1)$, and next we choose their $SO(1, 1)$ properties as follows. The group $SO(1, 1)$ contains a single parameter corresponding to boosts that mix the $(y^{0'}, y^{1'})$, but rescales the light-cone components $y_\pm$ with opposite factors $y_\pm \to \Lambda^{\pm 1} y_\pm$. Under the boosts the fields (ϕ, ψ_α, V_+) undergo scale transformations with the scale factor raised to the powers $(0, 1/2, 1)$ respectively. The complex conjugates of these fields $(\bar{\phi}, \bar{\psi}_{\dot{\alpha}}, \bar{V}_+)$ have the same $SO(1, 1)$ scales. Similarly $(\phi', \psi'_\alpha, V'_-)$ and their complex conjugates are assigned $SO(1, 1)$ weights $(0, -1/2, -1)$, also $(\varphi, \chi_\alpha, f)$ have weights $(0, 1/2, 0)$ and $(\varphi', \chi'_\alpha, f')$ have weights $(0, -1/2, 0)$. The name of the multiplets "scalar-vector" refers to the properties of V_+, V'_- that are scalars under $SO(3, 1)$ and vectors under $SO(1, 1)$. Similarly "scalar-scalar" refers to f, f' that are scalars under both $SO(3, 1)$ and $SO(1, 1)$.

2.1 Scalar-vector multiplets

We guess the following supersymmetry transformation rules for (ϕ, ψ_α, V_+) by imitating the old WZ transformation rules and by requiring consistency with the isometries $SO(3,1) \times SO(1,1)$

$$\begin{aligned}
&\delta\phi = -\varepsilon'^{\alpha}\psi_\alpha,\ \delta\psi_\alpha = \partial_\mu\tilde{\partial}_+\phi\,\sigma^\mu_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}} + \tilde{\partial}_+\tilde{\partial}_- V_+\varepsilon_\alpha,\ \delta V_+ = \bar{\varepsilon}_{\dot{\beta}}\bar{\sigma}^{\dot{\beta}\alpha}_\mu\partial^\mu\tilde{\partial}_+\psi_\alpha \qquad (8)\\
&\delta\bar{\phi} = -\bar{\psi}_{\dot{\alpha}}\bar{\varepsilon}'^{\dot{\alpha}},\ \delta\bar{\psi}_{\dot{\alpha}} = \varepsilon'^{\beta}\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\tilde{\partial}_+\bar{\phi} + \tilde{\partial}_+\tilde{\partial}_-\bar{V}_+\bar{\varepsilon}_{\dot{\alpha}},\ \delta\bar{V}_+ = \partial^\mu\tilde{\partial}_+\bar{\psi}_{\dot{\alpha}}\,\bar{\sigma}^{\dot{\alpha}\beta}_\mu\varepsilon_\beta
\end{aligned}$$

The fermionic parameters $\varepsilon', \varepsilon$ mix the pairs (ϕ, ψ) and (ψ, V_+) respectively. We will build Lagrangians that are invariant under this transformation for arbitrary global fermionic parameters $\varepsilon', \varepsilon$. However for the closure of the algebra we find that $\varepsilon', \varepsilon$ must be related as given below.

Taking hermitian conjugation (bar) turns the index α into a dotted index $\dot{\alpha}$, and also interchanges the order of anticommuting variables. Applying $C = i\sigma_2$ raises or lowers the index. We have used it in the following definitions

$$\begin{aligned}
\sigma^\mu_{\alpha\dot{\beta}} &\equiv (1, \vec{\sigma}), \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \equiv C(1, \vec{\sigma}^*)C^T = (1, -\vec{\sigma})\,, &(9)\\
(\sigma^\mu)^\dagger &= \sigma^\mu, \quad (\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu, \quad \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu}. &(10)
\end{aligned}$$

Since we have anticommuting variables, the following rules apply

$$\varepsilon'^{\alpha}\psi_\alpha = \varepsilon'_\beta C^{\beta\alpha}\psi_\alpha = \psi_\alpha C^{\alpha\beta}\varepsilon'_\beta = \psi^\beta\varepsilon'_\beta = -\psi_\alpha\varepsilon'^{\alpha}. \qquad (11)$$

Then the transformation rules are consistent with hermitian conjugation.

By applying two infinitesimal transformations (8) and antisymmetrizing $[\delta_1, \delta_2]\,\phi$, $[\delta_1, \delta_2]\,\psi_\alpha$, $[\delta_1, \delta_2]\,V_+$ we find the closure of the algebra by demanding consistency with eq.(2)

$$\begin{aligned}
[\delta_1, \delta_2]\,\phi &= -\varepsilon'^{\alpha}_2\,(\delta_1\psi_\alpha) - (1 \longleftrightarrow 2)\\
&= -\varepsilon'^{\alpha}_2\left(\partial_\mu\tilde{\partial}_+\phi\,\sigma^\mu_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}}_1 + (\tilde{\partial}_+\tilde{\partial}_- V_+)\varepsilon_{1\alpha}\right) - (1 \longleftrightarrow 2)\\
&= \left(\varepsilon'^{\alpha}_1\sigma^\mu_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}}_2 - \varepsilon'^{\alpha}_2\sigma^\mu_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}}_1\right)\partial_\mu\tilde{\partial}_+\phi \qquad (12)
\end{aligned}$$

The V_+ term drops out provided we take

$$\varepsilon'_\alpha = A\varepsilon_\alpha \qquad (13)$$

where A is any complex number. Similarly

$$\begin{aligned}
[\delta_1,\delta_2]\, V_+ &= \bar\varepsilon_{2\dot\beta}\bar\sigma_\mu^{\dot\beta\alpha}\partial^\mu\tilde\partial_+\left(\partial_\nu\tilde\partial_+\phi\,\sigma^\nu_{\alpha\dot\gamma}\bar\varepsilon_1^{\prime\dot\gamma}+\tilde\partial_+\tilde\partial_- V_+\varepsilon_{1\alpha}\right)-(1\longleftrightarrow 2)\\
&= \bar\varepsilon_{2\dot\beta}\bar\varepsilon_1^{\prime\dot\beta}\partial^\mu\partial_\mu\tilde\partial_+^2\phi+\bar\varepsilon_{2\dot\beta}\bar\sigma_\mu^{\dot\beta\alpha}\varepsilon_{1\alpha}\,\tilde\partial_-\tilde\partial_+^2\partial^\mu V_+-(1\longleftrightarrow 2)\\
&= \left(\bar\varepsilon_{2\dot\beta}\bar\sigma_\mu^{\dot\beta\alpha}\varepsilon_{1\alpha}-\bar\varepsilon_{1\dot\beta}\bar\sigma_\mu^{\dot\beta\alpha}\varepsilon_{2\alpha}\right)\tilde\partial_-\tilde\partial_+^2\partial^\mu V_+
\end{aligned}\tag{14}$$

Notice that

$$\begin{aligned}
\varepsilon_2^{\prime\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon_1^{\prime\dot\beta} &= -\bar\varepsilon_1^{\prime\dot\beta}\sigma^{T\mu}_{\dot\beta\alpha}\varepsilon_2^{\prime\alpha}=-\bar\varepsilon'_{1\dot\gamma}C^{\dot\gamma\dot\beta}\sigma^{T\mu}_{\dot\beta\alpha}\varepsilon'_{2\delta}C^{\delta\alpha}\\
&= -\bar\varepsilon'_{1\dot\gamma}\left(C\sigma^{T\mu}C^T\right)^{\dot\gamma\delta}\varepsilon'_{2\delta}=-\bar\varepsilon'_{1\dot\gamma}\left(\bar\sigma^\mu\right)^{\dot\gamma\delta}\varepsilon'_{2\delta}
\end{aligned}\tag{15}$$

Therefore, if we use (13) we obtain

$$\varepsilon_2^{\prime\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon_1^{\prime\dot\beta}=-\bar\varepsilon_{1\dot\gamma}\left(\bar\sigma^\mu\right)^{\dot\gamma\delta}\varepsilon_{2\delta}\;\;|A|^2\tag{16}$$

which gives the form

$$[\delta_1,\delta_2]\,V_+=\left(\varepsilon_1^{\prime\beta}\sigma^\mu_{\beta\dot\alpha}\bar\varepsilon_2^{\prime\dot\alpha}-\varepsilon_2^{\prime\beta}\sigma^\mu_{\beta\dot\alpha}\bar\varepsilon_1^{\prime\dot\alpha}\right)\left(-\frac{1}{|A|^2}\tilde\partial_+\tilde\partial_-\right)\partial^\mu\tilde\partial_+V_+\tag{17}$$

In order to have the same closure property as in (12) we must require

$$\left(\tilde\partial_+\tilde\partial_-+|A|^2\right)V_+=0.\tag{18}$$

The field is on shell in the extra dimensions but is *not* on shell from the point of view of 4D (for the scalar-scalar representation presented below the field is fully off-shell). We will see that this restriction follows from a Lagrangian that is fully invariant without using any mass shell conditions. Next consider the spinor

$$\begin{aligned}
[\delta_1,\delta_2]\,\psi_\alpha &= \partial_\mu\tilde\partial_+\left(\delta_1\phi\right)\,\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon_2^{\prime\dot\beta}+\tilde\partial_+\tilde\partial_-\left(\delta_1V_+\right)\varepsilon_{2\alpha}-(1\longleftrightarrow 2)\\
&= -\partial_\mu\tilde\partial_+\left(\varepsilon_1^{\prime\gamma}\psi_\gamma\right)\,\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon_2^{\prime\dot\beta}+\tilde\partial_+\tilde\partial_-\left(\bar\varepsilon_{1\dot\beta}\bar\sigma_\mu^{\dot\beta\gamma}\partial^\mu\tilde\partial_+\psi_\gamma\right)\varepsilon_{2\alpha}-(1\longleftrightarrow 2)\\
&= \frac{1}{2}\varepsilon'_1\sigma_\nu\bar\varepsilon'_2\;\partial_\mu\tilde\partial_+\left(\sigma^\mu\bar\sigma^\nu\right)\psi-\frac{1}{2}\left(\bar\varepsilon_1\bar\sigma_\nu\varepsilon_2\right)\,\tilde\partial_+^2\tilde\partial_-\partial_\mu\left(\sigma^\nu\bar\sigma^\mu\right)\psi-(1\longleftrightarrow 2)\\
&= \frac{1}{2}\left(\varepsilon'_1\sigma_\nu\bar\varepsilon'_2-\varepsilon'_2\sigma_\nu\bar\varepsilon'_1\right)\left(\sigma^\mu\bar\sigma^\nu+\sigma^\nu\bar\sigma^\mu\left(-\frac{1}{|A|^2}\tilde\partial_+\tilde\partial_-\right)\right)\partial_\mu\tilde\partial_+\psi
\end{aligned}\tag{19}$$

Note that we got an extra minus sign from the interchange of fermions from line 2 to line 3. The two terms combine to give the desired closure

$$[\delta_1, \delta_2]\,\psi_\alpha = (\varepsilon_2' \sigma_\nu \bar{\varepsilon}_1' \, - \varepsilon_1' \sigma_\nu \bar{\varepsilon}_2')\,\partial^\nu \tilde{\partial}_+ \psi_{\;\alpha} \;\; , \tag{20}$$

provided the spinor satisfies

$$\left(\tilde{\partial}_+ \tilde{\partial}_- + |A|^2\right)\psi_\alpha = 0. \tag{21}$$

Since both ψ_α and V_+ are restricted, ϕ must also be restricted by

$$\left(\tilde{\partial}_+ \tilde{\partial}_- + |A|^2\right)\phi = 0, \tag{22}$$

in order to have consistent transformation properties. As we will see in section-3, solutions of a generalized BPS condition involves a time-like condition on v^2. The condition $v^2 = -\tilde{\partial}_+ \tilde{\partial}_- = |A|^2$ is consistent with this case, and it is interesting that it followed from the closure of the superalgebra. As we will see, this is just what we need in order to connect correctly to standard physics.

So far there is no mass shell condition or equation of motion required in 4D. Therefore these are the properties of the scalar-vector representation (ϕ, ψ_α, V_+) independent of any dynamics. Note that $|A|$ plays the role of a label for the representation (like a Casimir eigenvalue). Next we consider a Lagrangian.

2.1.1 Free supersymmetric Lagrangian

The *free* Lagrangian we propose is $\mathcal{L}_0 + \mathcal{L}_0'$

$$\begin{aligned}
\mathcal{L}_0 &= \partial_\mu \tilde{\partial}_+ \bar{\phi}\; \partial^\mu \tilde{\partial}_- \phi + \bar{\psi}_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\beta} \partial_\mu \tilde{\partial}_- \psi_\beta + \tilde{\partial}_- \bar{V}_+ \,\tilde{\partial}_- V_+ , \\
\mathcal{L}_0' &= \phi' \left(\tilde{\partial}_+ \tilde{\partial}_- + |A|^2\right)\phi + \psi'^\alpha \left(\tilde{\partial}_+ \tilde{\partial}_- + |A|^2\right)\psi_\alpha \\
&\quad + V_-' \left(\tilde{\partial}_+ \tilde{\partial}_- + |A|^2\right) V_+ + h.c.
\end{aligned} \tag{23}$$

$\mathcal{L}_0$ contains the original fields (ϕ, ψ_α, V_+) and is hermitian up to total derivatives. $\mathcal{L}_0'$ is included to impose the constraints through the equations of motion of the Lagrange multipliers $(\phi', \psi'^\alpha, V_-')$. Their $SO(1,1)$ weights have to be the opposite of the original fields because of invariance under $SO(1,1)$.

This Lagrangian is unconventional because of the number of derivatives applied on the fields, and because of the two time coordinates. According to old wisdom one should expect problems with ghosts. However one should note that in either the x-space or the y -space there are at the most two derivatives. Also the y-space will be compactified in a Kaluza-Klein approach. We will see below that this structure leads to conventional physics in four dimensions without any problems, i.e. there are no ghosts in the spectrum of this model.

Each term of the Lagrangian is separately invariant under the transformations of (ϕ, ψ_α, V_+) and of $(\phi', \psi'^\alpha, V'_-)$ (given below) for arbitrary global parameters $\varepsilon', \varepsilon$, *without using any constraints.* The constraints that are needed to close the algebra follow as an equation of motion of the auxiliary fields. Thus, before using the constraints there is an even larger supersymmetry. We demonstrate the larger symmetry by applying the supersymmetry transformations on $\pounds_0$

$$\begin{aligned} \delta\pounds_0 \;=\; & \left(\partial_\mu \tilde{\partial}_+ \delta\bar{\phi}\right) \partial^\mu \tilde{\partial}_- \phi + \bar{\psi}_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\beta} \partial^\mu \tilde{\partial}_- \left(\delta\psi_\beta\right) + \left(\tilde{\partial}_- \delta\bar{V}_+\right) \tilde{\partial}_- V_+ \\ & + \partial_\mu \tilde{\partial}_+ \bar{\phi} \, \left(\partial^\mu \tilde{\partial}_- \delta\phi\right) + \delta\bar{\psi}_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\beta} \partial^\mu \tilde{\partial}_- \psi_\beta + \tilde{\partial}_- \bar{V}_+ \tilde{\partial}_- \left(\delta V_+\right) \end{aligned} \tag{24}$$

Substituting from (8) we have for the first line

$$\begin{aligned} & -\partial_\mu \tilde{\partial}_+ \bar{\psi}_{\dot{\alpha}} \bar{\varepsilon}'^{\dot{\alpha}} \, \partial^\mu \tilde{\partial}_- \phi + \bar{\psi}_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\beta} \partial^\mu \tilde{\partial}_- \left(\partial_\nu \tilde{\partial}_+ \phi \, \sigma^\nu_{\beta\dot{\gamma}} \bar{\varepsilon}'^{\dot{\gamma}} + \tilde{\partial}_+ \tilde{\partial}_- V_+ \varepsilon_\beta\right) \\ & + \tilde{\partial}_- \left(\partial^\mu \tilde{\partial}_+ \bar{\psi}_{\dot{\alpha}} \, \bar{\sigma}_\mu^{\dot{\alpha}\beta} \varepsilon_\beta\right) \tilde{\partial}_- V_+ \ , \end{aligned} \tag{25}$$

which is a total derivative (without using the constraints)

$$\begin{aligned} & \partial^\mu \left(\bar{\psi}_{\dot{\alpha}} \bar{\varepsilon}'^{\dot{\alpha}} \, \tilde{\partial}_+ \tilde{\partial}_- \partial_\mu \phi \ + \bar{\psi}_{\dot{\alpha}} \, \bar{\sigma}_\mu^{\dot{\alpha}\beta} \varepsilon_\beta \, \tilde{\partial}_+ \tilde{\partial}_-^2 V_+ \right) \\ & - \tilde{\partial}_+ \left(\partial_\mu \bar{\psi}_{\dot{\alpha}} \bar{\varepsilon}'^{\dot{\alpha}} \, \partial^\mu \tilde{\partial}_- \phi + \partial^\mu \bar{\psi}_{\dot{\alpha}} \, \bar{\sigma}_\mu^{\dot{\alpha}\beta} \varepsilon_\beta \, \tilde{\partial}_-^2 V_+\right) \\ & + \tilde{\partial}_- \left(\tilde{\partial}_- V_+ \, \tilde{\partial}_+ \partial^\mu \bar{\psi}_{\dot{\alpha}} \, \bar{\sigma}_\mu^{\dot{\alpha}\beta} \varepsilon_\beta\right) . \end{aligned} \tag{26}$$

Similarly, the second line gives

$$\begin{aligned} & \partial_\mu \tilde{\partial}_+ \bar{\phi} \, \partial^\mu \tilde{\partial}_- \left(-\varepsilon'^\alpha \psi_\alpha\right) + \tilde{\partial}_- \bar{V}_+ \, \left(\bar{\varepsilon}_{\dot{\beta}} \bar{\sigma}_\mu^{\dot{\beta}\alpha} \partial^\mu \tilde{\partial}_+ \tilde{\partial}_- \psi_\alpha\right) \\ & + \left(\varepsilon'^\gamma \sigma^\nu_{\gamma\dot{\alpha}} \partial_\nu \tilde{\partial}_+ \bar{\phi} + \tilde{\partial}_+ \tilde{\partial}_- \bar{V}_+ \, \bar{\varepsilon}_{\dot{\alpha}}\right) \, \bar{\sigma}_\mu^{\dot{\alpha}\beta} \partial^\mu \tilde{\partial}_- \psi_\beta \end{aligned} \tag{27}$$

which is also a total derivative

$$\partial^\mu \left(\partial^\nu \tilde{\partial}_+ \bar{\phi} \, \varepsilon' \sigma_{\nu\mu} \tilde{\partial}_- \psi \ \right) + \tilde{\partial}_+ \left(\tilde{\partial}_- \bar{V}_+ \, \bar{\varepsilon}_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\beta} \partial^\mu \tilde{\partial}_- \psi_\beta\right) . \tag{28}$$

Now, we turn to $\mathcal{L}'_0$. Its variation under the supertransformation gives

$$\begin{aligned}\delta\mathcal{L}'_0 &= \delta\phi'\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\phi + \delta\psi'^\alpha\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\psi_\alpha + \delta V'_-\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)V_+ \\ &\quad + \phi'\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)(-\varepsilon'^\alpha\psi_\alpha) + V'_-\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\left(\bar{\varepsilon}_{\dot{\beta}}\bar{\sigma}^{\dot{\beta}\alpha}_\mu\partial^\mu\tilde{\partial}_+\psi_\alpha\right) \\ &\quad + \psi'^\alpha\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\left(\partial_\mu\tilde{\partial}_+\phi\,\sigma^\mu_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}} + \tilde{\partial}_+\tilde{\partial}_- V_+\varepsilon_\alpha\right) + h.c. \end{aligned} \tag{29}$$

We get $\delta\mathcal{L}'_0$ =total derivative, without using the constraints, provided the Lagrange multipliers transform under supersymmetry as follows

$$\delta\phi' = -\partial_\mu\tilde{\partial}_+\psi'^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}},\ \ \delta\psi'^\alpha = \phi'\varepsilon'^\alpha + \partial^\mu\tilde{\partial}_+V'_-\bar{\varepsilon}_{\dot{\beta}}\bar{\sigma}^{\dot{\beta}\alpha}_\mu,\ \ \delta V'_- = \tilde{\partial}_+\tilde{\partial}_-\psi'^\alpha\varepsilon_\alpha \tag{30}$$

So, the total free Lagrangian is supersymmetric for arbitrary $\varepsilon, \varepsilon'$ without using any constraints. This larger symmetry closes into a larger set of bosonic operators included in S-theory. We will not discuss the larger symmetry in any detail but one can see its general structure in section-3. The smaller superalgebra (2) is represented correctly only after we use the constraints

$$\varepsilon' = A\varepsilon, \quad \tilde{\partial}_+\tilde{\partial}_- + |A|^2 = 0. \tag{31}$$

It is possible to write an additional piece in the free supersymmetric Lagrangian involving only the primed fields. In that case the primed fields become propagating fields instead of being Lagrange multipliers. This Lagrangian is invariant without using the constraints provided the number of derivatives $\tilde{\partial}_m$ on the fermion ψ' is cubic. Because of the high derivatives, and because we are interested in interpreting the primed fields as non-propagating fields, we refrain from adding this additional term in the present model.

2.1.2 On mass shell degrees of freedom

We can now analyze the equations of motion and determine the content of the degrees of freedom on mass shell

$$\bar{\sigma}^{\dot{\alpha}\beta}_\mu\partial_\mu\tilde{\partial}_-\psi_\beta = -\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\psi'^{\dot{\alpha}}, \quad \tilde{\partial}^2_- V_+ = -\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)V'_- \tag{32}$$

$$\tilde{\partial}_+\tilde{\partial}_-\partial^\mu\partial_\mu\phi = -\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\phi', \quad \left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)[\phi, \psi_\alpha, V_+] = 0, \tag{33}$$

The only solutions of the last equation are expressed as a linear combination of the following complete basis

$$e^{-i\left(k_+y^+ + k_-y^-\right)}\left[\phi_k(x), \psi_{k\alpha}(x), V_{+k}(x)\right], \quad with\ \ k_+k_- = |A|^2. \tag{34}$$

In the other equations the primed fields must be in the same basis (to match the $y^{\pm}$ dependence). However, the operator $\tilde{\partial}_+\tilde{\partial}_- + |A|^2$ applied on the primed fields vanishes on this basis. Therefore, the only solutions are of the form

$$\left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\left[\phi', \psi'_\alpha, V'_-\right] = 0, \quad \left(\tilde{\partial}_+\tilde{\partial}_- + |A|^2\right)\left[\phi, \psi_\alpha, V_+\right] = 0 \tag{35}$$

which reduce the original equations of motion to massless field equations in 4D since $\tilde{\partial}_+\tilde{\partial}_-$ or $\tilde{\partial}_-$ cannot vanish on these fields,

$$\partial^\mu\partial_\mu\phi = 0, \quad \bar{\sigma}_\mu^{\dot{\alpha}\beta}\partial_\mu\psi_\beta = 0, \quad V_+ = 0. \tag{36}$$

So the modes $\phi_k(x), \psi_{k\alpha}(x)$ are ordinary massless bosonic and fermionic fields in 4D, while $V_{+k}(x) = 0$. For each k^m one has the degrees of freedom of a scalar multiplet of ordinary $N = 1$ supersymmetry. The Hilbert space constructed with these degrees of freedom has no ghosts.

2.1.3 Families and Kaluza-Klein compactification

The massless modes are labeled by the Kaluza-Klein momenta k^m in 11th and 12th dimensions. Let us assume that these dimensions are compactified so that the momenta are quantized as follows

$$k_\pm = \frac{n_\pm}{R_\pm} \tag{37}$$

where $n_\pm$ are integers. Then we must choose

$$|A|^2 = \frac{n}{R_+R_-} \tag{38}$$

where n is a positive fixed integer, and the integers $n_\pm$ must take all possible values such that

$$n_+n_- = n. \tag{39}$$

n is a label of the representation. For fixed n the solutions of (39) are given as follows

$$\begin{aligned} n &= 1: \quad (n_+, n_-) = (1,1) \\ n &= 2: \quad (n_+, n_-) = (2,1), (1,2) \\ n &= 3: \quad (n_+, n_-) = (3,1), (1,3) \\ n &= 4: \quad (n_+, n_-) = (4,1), (1,4), (2,2) \\ &\quad\ \ etc. \end{aligned} \tag{40}$$

where we have listed only the positive values, assuming a positivity condition for both $k_\pm$.

We see that the on mass shell physical modes correspond to free fields that satisfy the operator conditions

$$p^2 = 0, \qquad v^2 = \frac{n}{R_+ R_-}. \tag{41}$$

where n/R_+R_- characterizes the fixed geometry in the compactified 11th and 12th dimensions (see section-3 for an interpretation of these conditions as generalized BPS constraints). For fixed geometry there are only a finite number of solutions as determined by n. This could be interpreted as a "family quantum number". We have obtained a finite number of families for fixed n because $k_\pm$ are *both* quantized. Without such a quantization the number of solutions of $k_+k_- = |A|^2$ is infinite. Furthermore if $n_\pm$ are integers but n is not an integer there are no solutions at all. With the quantization of all three integers $n, n_\pm$ number theory comes to the rescue to give a finite number of solutions.

The Kaluza-Klein momenta in the new SUSY multiply the usual momenta $k_+ p_\mu$, not add. Therefore their effect is similar to the slope parameter α' of strings (but unlike α', the factor k_+ is not necessarily a constant on all fields). For massless particles their presence does not change the mass, therefore we just get repetitions of massless particles, i.e. families.

So far the model is non-interacting. We expect that in the presence of interactions, for example gauge or gravitational interactions, the operators p_μ, v_m would be replaced by covariant derivatives in the closure of the superalgebra. The analog of the BPS conditions (41) would then become Laplacians and Dirac operators in the presence of interactions. Then one may consider their solutions in the presence of non-trivial geometries in y^m space (analogs of Calabi-Yau, etc., but now in a space with Minkowski signature). Also, as in sections 3 and 4, one may add the other c compactified dimensions and consider $SO(c+1,1)$ instead of $SO(1,1)$. Evidently the geometry is bound to modify the number of permitted solutions interpreted as families. While there are some similarities between this embedding of families in the geometries of higher dimensions, the equations and the mechanisms are different as compared to the more familiar Kaluza-Klein mechanism. We have seen already from (40) that there are new possibilities that had not emerged before. This unexpected wealth of possibilities may have fruitful phenomenogical applications.

2.2 Scalar-scalar representations

Instead of the $SO(1,1)$ vector V_+ of the previous subsection we now take an $SO(1,1)$ scalar f and consider the supersymmetry transformation rules of the fields $(\varphi, \chi_\alpha, f)$ that are consistent with the isometries $SO(3,1)\times SO(1,1)$

$$\begin{aligned} \delta\varphi &= -\varepsilon'^{\alpha}\chi_\alpha, \;\; \delta\chi_\alpha = \partial_\mu\tilde{\partial}_+\varphi\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon'^{\dot\beta} + s\tilde{\partial}_+ f\varepsilon_\alpha, \;\; \delta f = \bar\varepsilon_{\dot\beta}\bar\sigma^{\dot\beta\alpha}_\mu\partial^\mu\chi_\alpha \\ \delta\bar\varphi &= -\bar\chi_{\dot\alpha}\bar\varepsilon'^{\dot\alpha}, \;\; \delta\bar\chi_{\dot\alpha} = \varepsilon'^{\beta}\sigma^\mu_{\beta\dot\alpha}\partial_\mu\tilde{\partial}_+\bar\varphi + s^*\tilde{\partial}_+\bar f\bar\varepsilon_{\dot\alpha}, \;\; \delta\bar f = \partial^\mu\bar\chi_{\dot\alpha}\,\bar\sigma^{\dot\alpha\beta}_\mu\varepsilon_\beta \end{aligned} \tag{42}$$

where s is some complex number to be determined. In this case the $SO(1,1)$ weight of f is 0, which is to be contrasted to the previous case. Note again that we have two independent parameters $\varepsilon', \varepsilon$. The Lagrangian presented below is invariant under arbitrary $\varepsilon', \varepsilon, s$. Closure of the algebra will require a relation between these parameters, but it will not require any mass shell conditions as shown below.

By applying two infinitesimal transformations (8) and antisymmetrizing $[\delta_1,\delta_2]\,\varphi$, $[\delta_1,\delta_2]\,\chi_\alpha$, $[\delta_1,\delta_2]\,f$ we find the closure of the algebra consistent with eq.(2)

$$\begin{aligned} [\delta_1,\delta_2]\,\varphi &= -\varepsilon_2'^{\alpha}\,(\delta_1\chi_\alpha) - (1\longleftrightarrow 2) \\ &= -\varepsilon_2'^{\alpha}\left(\partial_\mu\tilde{\partial}_+\varphi\,\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon_1'^{\dot\beta} + s\tilde{\partial}_+ f\varepsilon_{1\alpha}\right) - (1\longleftrightarrow 2) \\ &= \left(\varepsilon_1'^{\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon_2'^{\dot\beta} - \varepsilon_2'^{\alpha}\sigma^\mu_{\alpha\dot\beta}\bar\varepsilon_1'^{\dot\beta}\right)\partial_\mu\tilde{\partial}_+\varphi \end{aligned} \tag{43}$$

The f term drops out provided we take

$$\varepsilon'_\alpha = A\varepsilon_\alpha \tag{44}$$

where A is any complex number. Similarly

$$\begin{aligned} [\delta_1,\delta_2]\,f &= \bar\varepsilon_{2\dot\beta}\bar\sigma^{\dot\beta\alpha}_\mu\partial^\mu\left(\partial_\nu\tilde{\partial}_+\varphi\,\sigma^\nu_{\alpha\dot\gamma}\bar\varepsilon_1'^{\dot\gamma} + s\tilde{\partial}_+ f\varepsilon_{1\alpha}\right) - (1\longleftrightarrow 2) \\ &= \bar\varepsilon_{2\dot\beta}\bar\varepsilon_1'^{\dot\beta}\partial^\mu\partial_\mu\tilde{\partial}_+\varphi + s\bar\varepsilon_{2\dot\beta}\bar\sigma^{\dot\beta\alpha}_\mu\varepsilon_{1\alpha}\,\tilde{\partial}_+\partial^\mu f - (1\longleftrightarrow 2) \\ &= s\left(\bar\varepsilon_{2\dot\beta}\bar\sigma^{\dot\beta\alpha}_\mu\varepsilon_{1\alpha} - \bar\varepsilon_{1\dot\beta}\bar\sigma^{\dot\beta\alpha}_\mu\varepsilon_{2\alpha}\right)\tilde{\partial}_+\partial^\mu f \\ &= -\frac{s}{|A|^2}\left(\varepsilon_1'^{\beta}\sigma^\mu_{\beta\dot\alpha}\bar\varepsilon_2'^{\dot\alpha} - \varepsilon_2'^{\beta}\sigma^\mu_{\beta\dot\alpha}\bar\varepsilon_1'^{\dot\alpha}\right)\partial^\mu\tilde{\partial}_+ f \end{aligned} \tag{45}$$

In order to have the same closure property as in (12) we must require

$$s = -|A|^2 . \tag{46}$$

Next consider the spinor

$$\begin{aligned}
[\delta_1, \delta_2]\, \chi_\alpha &= \partial_\mu \tilde{\partial}_+ (\delta_1 \varphi)\, \sigma^\mu_{\alpha\dot{\beta}} \bar{\varepsilon}_2'^{\dot{\beta}} + s\tilde{\partial}_+ (\delta_1 f)\, \varepsilon_{2\alpha} - (1 \longleftrightarrow 2) \\
&= -\partial_\mu \tilde{\partial}_+ \left(\varepsilon_1'^\gamma \chi_\gamma\right) \sigma^\mu_{\alpha\dot{\beta}} \bar{\varepsilon}_2'^{\dot{\beta}} + s \left(\bar{\varepsilon}_{1\dot{\beta}} \bar{\sigma}^{\dot{\beta}\gamma}_\mu \partial^\mu \tilde{\partial}_+ \chi_\gamma\right) \varepsilon_{2\alpha} - (1 \longleftrightarrow 2) \\
&= \frac{1}{2} \varepsilon_1' \sigma_\nu \bar{\varepsilon}_2'\, \partial_\mu \tilde{\partial}_+ (\sigma^\mu \bar{\sigma}^\nu)\, \chi - \frac{s}{2} (\bar{\varepsilon}_1 \bar{\sigma}_\nu \varepsilon_2)\, \tilde{\partial}_+ \partial_\mu (\sigma^\nu \bar{\sigma}^\mu)\, \chi - (1 \longleftrightarrow 2) \\
&= \frac{1}{2} \left(\varepsilon_1' \sigma_\nu \bar{\varepsilon}_2' - \varepsilon_2' \sigma_\nu \bar{\varepsilon}_1'\right) \left(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu \left(-\frac{s}{|A|^2}\right)\right) \partial_\mu \tilde{\partial}_+ \chi
\end{aligned} \tag{47}$$

The two terms combine to give the desired closure provided we use again $s = -|A|^2$.

$$[\delta_1, \delta_2]\, \chi_\alpha = \left(\varepsilon_2' \sigma_\nu \bar{\varepsilon}_1' - \varepsilon_1' \sigma_\nu \bar{\varepsilon}_2'\right) \partial^\nu \tilde{\partial}_+ \chi_{\ \alpha} \ . \tag{48}$$

In this version we did not need to impose any mass shell constraints in order to close the algebra.

2.2.1 Free supersymmetric Lagrangian

The *free* Lagrangian we start with is $\mathcal{L}_1$

$$\mathcal{L}_1 = \partial_\mu \tilde{\partial}_+ \bar{\varphi}\, \partial^\mu \tilde{\partial}_- \varphi + \bar{\chi}_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\beta}_\mu \partial_\mu \tilde{\partial}_- \chi_\beta - s \tilde{\partial}_+ \bar{f}\, \tilde{\partial}_- f \tag{49}$$

Applying the supersymmetry transformations on $\mathcal{L}_0$ we have

$$\delta \mathcal{L}_1 = total \ \ derivative, \tag{50}$$

for any $\varepsilon', \varepsilon, s$. The equations of motion that follow from this free Lagrangian require $v^2 f \equiv -\tilde{\partial}_+ \tilde{\partial}_- f = 0$ and $p^2 v^2 = 0$ on both φ, χ. There are two classes of solutions

$$\begin{aligned}
(i) \ &: \ p^2 = 0, \qquad v^2 \neq 0, \\
(ii) \ &: \ p^2 \neq 0, \qquad v^2 = 0.
\end{aligned} \tag{51}$$

Neither class is physically satisfactory. In class (i) the solution for f is zero while the other fields are massless. This seems fine, but since v^2 is not determined there are an infinite number of massless families from the point of view of 4D. In class (ii) p^2 is not required to be on shell by the equations of motion. To avoid these problems we add an additional part to the free Lagrangian to enforce the constraint $v^2 = M^2 \geq 0$ via Lagrange multipliers

$$\mathcal{L}_{12} = \varphi' \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) \varphi + \chi'^{\alpha} \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) \chi_\alpha + f' \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) f + h.c \tag{52}$$

Now, we turn to check the invariance of $\mathcal{L}'_{12}$. Its variation under supertransformation gives

$$\begin{aligned} \delta\mathcal{L}_{12} \;\; = \;\; & \delta\varphi' \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) \varphi + \delta\chi'^{\alpha} \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) \chi_\alpha + \delta f' \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) f \\ & +\varphi' \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) \left(-\varepsilon'^{\alpha}\chi_\alpha\right) + f' \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) \left(\bar{\varepsilon}_{\dot{\beta}}\bar{\sigma}^{\dot{\beta}\alpha}_{\mu}\partial^\mu\chi_\alpha\right) \\ & +\chi'^{\alpha} \left(\tilde{\partial}_+\tilde{\partial}_- + M^2\right) \left(\partial_\mu\tilde{\partial}_+\varphi\, \sigma^{\mu}_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}} + s\tilde{\partial}_+ f\varepsilon_\alpha\right) + h.c. \end{aligned} \tag{53}$$

We get $\delta\mathcal{L}'_{12}$ =total derivative, without using the constraints, provided the primed fields transform under supersymmetry as follows

$$\delta\varphi' = -\partial_\mu\tilde{\partial}_+\chi'^{\alpha}\, \sigma^{\mu}_{\alpha\dot{\beta}}\bar{\varepsilon}'^{\dot{\beta}}, \quad \delta\chi'^{\alpha} = \varepsilon'^{\alpha}\varphi' + \bar{\varepsilon}_{\dot{\beta}}\bar{\sigma}^{\dot{\beta}\alpha}_{\mu}\partial^\mu f', \quad \delta f' = s\tilde{\partial}_+\chi'^{\alpha}\varepsilon_\alpha \tag{54}$$

Remarkably, this transformation closes $[\delta_1, \delta_2]\,(\varphi', \chi', f')$ as desired without requiring any mass shell constraints. So, the total free Lagrangian is supersymmetric, and the supersymmetry algebra closes as desired, provided $\varepsilon' = A\varepsilon,$ and $s = -M^2$ as before. The free model with

$$\mathcal{L}_0^{(1)} = \mathcal{L}_1 + \mathcal{L}_{12} \tag{55}$$

has a physically satisfactory 4D mass spectrum. Class (ii) is completely eliminated while in class (i) there are a finite number of families as in the scalar-vector model of the previous subsection provided

$$M^2 = \frac{n}{R_+R_-}, \tag{56}$$

and n is a positive integer.

2.2.2 scalar-scalar hyper-multiplet

We could have stopped here, but we also wish to build more models by exploring the possibility of adding a supersymmetric free Lagrangian involving only the primed fields. There is one that satisfies $\delta\pounds_2$ =total derivative:

$$\pounds_2 = \varphi'\bar{\varphi}' + \chi'^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\partial_\mu\tilde{\partial}_+\bar{\chi}'^{\dot{\beta}} - \frac{1}{s}\partial_\mu f'\partial^\mu\bar{f}'. \tag{57}$$

By itself the spectrum of this Lagrangian has some of the problems of $\pounds_1$. However, when added to the previous terms it becomes interesting. Each term in the following total Lagrangian is supersymmetric separately

$$\pounds_0^{(2)} = \pounds_1 + \gamma\pounds_{12} + \pounds_2. \tag{58}$$

Note that there is an additional parameter γ that cannot be absorbed into normalizations of the fields. In this model there are two scalar-scalar representations that are coupled to each other in a supersymmetric invariant way. The significance of this coupling is that now there are two propagating fermions $\chi^\alpha, \chi'^\alpha$ both of which are left handed $SO(3,1)$ spinors $(1/2,0)$ but they have opposite $SO(1,1)$ chiralities (or weights $\pm 1/2$). Together they are equivalent to a full Dirac spinor of $SO(1,1)$ as well as of $SO(3,1)$, and their mixing term in $\gamma\pounds_{12}$ is the analog of a fermion mass term with mass $m \sim \gamma(\tilde{\partial}_+\tilde{\partial}_- + M^2)$ from the point of view of 4D. This interpretation is better understood by analyzing the coupled equations. One now finds that φ', f are completely solved in terms of the other fields and the remaining two complex bosons and two chiral fermions form a massive hyper-multiplet of ordinary supersymmetry in 4D. These remaining fields can be expanded in Kaluza-Klein modes, where the modes have quantized momenta labeled by (n_+, n_-). Each mode satisfies the following mass shell condition

$$p^2 = \frac{\gamma^2}{k^2}\left(k^2 - M^2\right)^2 \tag{59}$$

$$= \frac{\gamma^2}{R_+R_-}\frac{(n_+n_- - n)^2}{n_+n_-} \tag{60}$$

where we have used $k^2 = \frac{n_+n_-}{R_+R_-}$ and $M^2 = \frac{n}{R_+R_-}$. Unlike our previous examples, here k^2 or the product n_+n_- is not fixed. A plot of p^2 versus n_+n_- shows the following effects. For $n_+n_- = n$ there are massless modes, $p^2 = 0$, provided n is an integer, and this fixed integer (which is a label of the representation) determines the number of massless families as in (40). In addition, there are an infinite number of massive Kaluza-Klein modes for $n_+n_- \neq n$

such that their 4D mass gets bigger for n_+n_- increasing toward +infinity as well as for n_+n_- decreasing toward zero. There is a mass gap away from from zero mass, $n_+n_- = n$, since p^2 has quantized values in units of $\frac{\gamma^2}{R_+R_-}$. These features are compatible with a physical interpretation of the spectrum in 4D. However this model is not entirely satisfactory because the spectrum of p^2 contains tachyons when n_+n_- is negative ($k^2 \sim$ space-like). Additional input is needed to prevent n_+n_- from being negative. Perhaps interactions, or an appropriate interpretation of the extra y-space in terms of p-branes, will suggest how to impose $k^2 \geq 0$. This issue does not arise in the other models presented in this paper because for them k^2 is fixed and positive.

3 S-theory origins and generalizations

In this section we describe the algebra (2) in the context of a more general framework in order to display its connections to a secret theory behind string theory, and to provide a basis for generalizations.

The goal in S-theory [1] is to extract information about the secret theory behind string theory by combining the representation structure of a generalized superalgebra with other information that may be available about the secret theory through some of its limits such as string theory, p-brane theory, D-branes and the likes. This strategy is similar to the one used in the 1960's, with symmetries and current algebras on the one hand and experimental input on the other, which eventually led to the discovery of the Standard Model.

S-theory has two types of superalgebras with 32 real supergenerators and 528 real bosonic generators: the $SO(10,2)$ covariant type-A in 12 dimensions and the $SO(9,1) \times SO(2,1)$ covariant type-B in 13 dimensions. By a change of basis the same superalgebras may be rewritten in bases that display other symmetry structures. From the point of view of 10 dimensions the $32_{A,B}$ spinors correspond to two 16-component spinors, such that for the type-A the 10D-chiralities are opposite while for type-B the 10D-chiralities are the same, as in type-A and type-B string theories. The two types may be embedded in a 13D superalgebra by considering the 64-component spinor space of $SO(11,2)$. Then two different A, B projections reduce the 64-component spinor into distinguishable $32_{A,B}$ fermions, and those pick out the sets $528_{A,B}$ out of the $\frac{1}{2}64 \times 65 = 78 + 286 + 1716$ bosons classified as antisymmetric tensors with

2,3,6 indices under $SO(11,2)$. The A and B types are T-dual to each other such that T-duality mixes the 13th dimension with the others[1]. Therefore, even though there is no $SO(11,2)$ covariant formalism, thanks to T-duality of string theory we already know that there is a sense in which all 13 dimensions are connected to each other in the complete secret theory.

We remind the reader that one cannot consider more than 32 real supercharges *in the flat limit* of the secret theory. If there were more than 32, they would show up in 4D as more than $N = 8$ supersymmetries, and this is not permitted by the absence of massless interacting particles with helicities higher than 2, in the flat limit. Similarly, with $32_{A,B}$ supercharges there cannot be more than $528_{A,B}$ bosonic generators since 528 is the number of independent components of a 32×32 symmetric matrix. Special forms of these superalgebras are obtained in representations in which some of the $528_{A,B}$ bosons or some of the $32_{A,B}$ fermions vanish. The basic hypothesis of S-theory is that in the complete secret theory all of these operators are realized non-trivially when all of its sectors are taken into account. In the curved version of the secret theory more supercharges may exist, but they should vanish as the curvature vanishes. In considering curved spaces one is interested in what happens to the superalgebra of the $32_{A,B}$ supercharges that survive in the flat limit. Those can close only on the same set of $528_{A,B}$ bosons, but some of the latter, as well as some of the $32_{A,B}$ fermions, could satisfy non Abelian commutation relations depending on the nature of the curved space. As suggested in [1] various curved space models may be described as contractions of supergroups such as $OSp(1/32)$, $OSp(1/64)$ and other non-Abelian supergroups.

All sectors of the secret theory would fall into some representation of S-theory. Such sectors include well known theories such as super Yang-Mills, supergravity and superstring theory. Furthermore M- and F- theories can be viewed in the same light. This is because the type-A superalgebra contains the superalgebra of 11D M-theory [6], while the type-B superalgebra contains the superalgebra of 12D F-theory [8], so these theories could be embedded in a larger theory in 12D and 13D respectively. Various new compactifications [9] of the secret theory also seem to be consistent with the overall 12D or 13D algebraic structure of S-theory (Abelian and non-Abelian). Construct-

[1] In the complete theory probably only the self T-dual subset are actually the same operators while the remainder are T-dual without being identical operators.

ing simple explicit models that provide representations of the generalized superalgebra of S-theory is likely to shed more light on the dynamical structure of the secret theory behind string theory. Section-2 is a small step in this direction and it should provide an example of the idea expressed in this paragraph.

3.1 Type-A

Starting with the type-A superalgebra that contains a 2-brane and a self-dual 6-brane in 12D [2]

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = (S_A)_{\alpha\beta} \\ &S_A = \frac{1+\gamma_{13}}{2} C\left(\gamma^{M_1M_2}\ Z_{M_1M_2} + \gamma^{M_1\cdots M_6}\ Z^+_{M_1\cdots M_6}\right) \end{aligned} \tag{61}$$

and then reducing to 4 dimensions, one obtains the generalized $N=8$ superalgebra in 4D in a particular basis, extended with all possible 528 bosonic generators [2]

$$\begin{aligned} \{Q_{\alpha a}, Q_{\beta b}\} &= (i\sigma_2)_{\alpha\beta}\ z_{ab} + (i\sigma_2\vec{\sigma})_{\alpha\beta} \cdot \vec{F}_{ab} \\ \left\{\bar{Q}_{\dot{\alpha}\dot{a}}, \bar{Q}_{\dot{\beta}\dot{b}}\right\} &= (i\sigma_2)_{\dot{\alpha}\dot{\beta}}\ z^*_{\dot{a}\dot{b}} + (i\sigma_2\vec{\sigma})_{\dot{\alpha}\dot{\beta}} \cdot \vec{F}^*_{\dot{a}\dot{b}} \\ \left\{Q_{\alpha a}, \bar{Q}_{\dot{\beta}\dot{b}}\right\} &= \sigma^\mu_{\alpha\dot{\beta}}\ \left(\gamma^m_{a\dot{b}}\, P_{\mu m} + \gamma^{m_1m_2m_3}_{a\dot{b}} A_{\mu m_1 m_2 m_3}\right) \end{aligned} \tag{62}$$

where μ, m are Lorentz indices for $SO(3,1)$, $SO(c+1,1)$ respectively, $c=6$ is the number of compactified string dimensions, and the extra $(1,1)$ correspond to the 11th and 12th dimensions. The pair of spinor indices $\alpha a, \dot{\alpha}\dot{a}$ are Weyl spinor indices for the spacetime $SO(3,1)$ and internal $SO(c+1,1)$ groups, such that the Weyl projection is simultaneously left-handed or simultaneously

[2] The 12D momentum operator $P_M\gamma^M_{\alpha\beta}$ cannot appear, and $Z_{M_1M_2}$ is not the 12D Lorentz generator. So, this algebra is not the extension of the conformal superalgebra in 12D. The Z's have to do with p-brane open boundaries in flat and curved dimensions, or with wrappings of p-branes in dimensions with non-trivial topologies. The embedding of 11D in 12D with this interpretation was presented in 1995 in a conference [11] as the first suggestion of 12 dimensions as a step beyond the 11D M-theory. Since this superalgebra is type-A, not type-B, this 12D is distinguishable than the one suggested later in F-theory [8]. Also, in the type-B superalgebra, one must distinguish the explicit isometry $SO(2,1)$ that acts on 3D (including the 13th dimension) from the $SL(2)$ of U-duality that is not an explicit isometry of the superalgebra, but is used in describing a 12D F-theory.

right handed for both indices (because of the 12D Weyl projection $1+\gamma_{13}$). The ordinary $N=8$ supersymmetry with all of its Lorentz scalar central extensions is obtained for $c=6$ by keeping only $z_{ab}, z^*_{\dot{a}\dot{b}}, P_{\mu 0'}$, and setting the remaining Lorentz non-scalar operators to zero. In that sector one may transform to a basis with an $SU(8)$ symmetry, such that the momentum $P_{\mu 0'}$ is a singlet under the $SU(8)$. This $SU(8)$ is the maximal compact group of $E_{7,7}$ of U-duality [10]. The isometry group $SO(c+1,1)=SO(7,1)$ is not in this $SU(8)$ or even in the $E_{7,7}$ because $P_{\mu 0'}$ is not a singlet under $SO(c+1,1)$. The web of these symmetries is described further in [11][1] and it has been used to explain how the black hole entropy in 4D and 5D contains information up to 12 (or 13) hidden dimensions [2].

The form of the superalgebra given above may be taken with other values of c, as we will do below in order to study simpler systems with fewer supersymmetries. In particular $c=0,1$ contains $N=1,2$ supersymmetry.

3.2 some sectors

S-theory suggests that the other operators beyond $z_{ab}, z^*_{\dot{a}\dot{b}}, P_{\mu 0}$ (i.e. the Lorentz non-scalars) also play a role in the secret theory. Hence we are interested in exploring models that provide representations of the more general algebra even if they correspond to a simplified sector of the algebra in which some of the operators vanish, as long as some of the novel features that relate to the Lorentz non-scalars are included. With this in mind, a greatly truncated version of the 12D type-A superalgebra (61) was first suggested in [1] by taking $Z_{M_1 M_2}=\frac{1}{2}(p_{M_1}v_{M_2}-p_{M_2}v_{M_1})$ and $Z^{+}_{M_1\cdots M_6}=0$

$$\{Q_\alpha, Q_\beta\} = \gamma^{MN}_{\alpha\beta}\ p_M v_N\ . \tag{63}$$

The generic representations of this superalgebra are long supermultiplets of minimum dimension $2^{32/2}$ with 2^{15} bosons and 2^{15} fermions. Shorter multiplets also exist provided one imposes the generalized BPS constraint

$$\det\left(\gamma^{MN}_{\alpha\beta}\ p_M v_N\right) = \left[p^2 v^2 - (p\cdot v)^2\right]^{16} = 0. \tag{64}$$

There are three classes of 12D *covariant* solutions of the BPS constraint: (i) neither p^2, v^2 is zero, (ii) one of them is zero, (iii) both of them are zero. When p^2 or v^2 are non-zero their signs classify different representations. The physical signs and solutions must be imposed through the details of a physical theory. Some such input is the interpretation of the $Z_{M_1M_2},\ Z^+_{M_1\cdots M_6}$ in terms of p-brane boundaries. Each solution is distinct in the sense that $SO(10,2)$ transformations cannot relate them, but obviously, solution (i) contains (ii) and (iii), and solution (ii) contains (iii) as special cases. Examples of physical representations and the issue of signs were discussed in section-2 (for the $N=1$ case).

For cases (i) and (ii) there are 16 zero and 16 non-zero supercharges and the minimal supermultiplet of dimension $2^{16/2}$ contains 128 bosons plus 128 fermions. This is the same set of massless states of 11D membrane theory [12], 10D string theory, or 11D supergravity, but in the present case they are part of the spectrum of a 12D secret theory that contains these theories. Considering the low energy limit in a field theory context, this supergravity multiplet would be realized on bi-local fields $\Phi(x^M, y^M)$ on which p_M, v_M act as derivatives $i\partial/\partial x^M, i\partial/\partial y^M$ respectively. When the BPS constraints are satisfied with

$$v^2 = time - like, \quad v \cdot p = 0 \tag{65}$$

and $p^2 = 0$ on shell, these fields are directly connected to 11D supergravity fields by a Kaluza-Klein reduction in the y-space and keeping only one eigenvalue of v_M. Hence the unreduced theory must be the long sought $SO(10,2)$ supergravity, as suggested in [1]. The non-locality is a remnant of the extended objects that are needed to realize this type of superalgebra.

Similarly, for case (iii) there are 24 zero and 8 non-zero supercharges and the minimal supermultiplet has dimension $2^{8/2}$. So the Yang-Mills super multiplet provides a basis for realizing the superalgebra as the $(10,2)$ extension of 10D super Yang-Mills theory. However, such a field theory may be realized co-covariantlyvariantly provided one uses bi-local fields that satisfy the constraints

$$v^2 = 0, \quad p \cdot v = 0 \tag{66}$$

and take $p^2 = 0$ on shell.

The superalgebra (63) suggested in [1] later found realizations as the supersymmetry algebra for a series of intriguing models: a 12D Yang-Mills theory [3], a 12D heterotic $(2,1)$ -string [7], a covariant matrix model [4] for a possible large-N matrix description of M-theory [5]. These models are incomplete from the point of view of representations of the superalgebra (63) and S-theory. As suggested in [2] to complete the representation space one must take all eigenvalues of p_M, v_M that are consistent with a given solution of the constraints (i,ii,iii), rather than taking one of them as a *constant* light-like vector. This requires bi-local fields, as in section-2. Bi-local fields, that include all the Kaluza-Klein modes consistent with a solution of the BPS constraints, contain a finite or an infinite number of eigenvalues of v_M (as in section-2). Only if all such Kaluza-Klein modes are included can one maintain the 12D covariance. As currently known, the models in [3][7][4] correspond to keeping one Kaluza-Klein mode (the constant vector) in an expansion of another complete theory.

Another sector of the superalgebra (62) was shown to be relevant for supersymmetric extremal black holes [2]. In this sector one sets to zero all bosonic operators except for $z_{ab}, z^*_{\dot{a}\dot{b}}$ and take the special factorized form $P_{\mu m} = p_\mu v_m$. The resulting superalgebra is covariant under $SO(3,1)\times SO(c+1,1)$ which keeps track of all 12 dimensions. It was shown that the black hole entropy is invariant under this isometry and that it contains information about the hidden 12th or 13th dimension. To do so all eigenvalues of v_m had to be allowed. This is the first instance in which all eigenvalues of v_m showed up in a physical system, thus providing encouragement for pursuing this approach further.

In sections-1,2 of this paper we have considered a sector along the lines of the last paragraph. In this sector the discussion is simpler, and perhaps more relevant for possible physical applications. We have also specialized to the sector of zero central charges since they may be included in later investigations. Then one has

$$\left\{Q_{\alpha a}, \bar{Q}_{\dot{\beta}\dot{b}}\right\} = \sigma^{\mu}_{\alpha\dot{\beta}}\, \gamma^{m}_{a\dot{b}}\, p_\mu v_m \tag{67}$$

where μ, m are Lorentz indices for $SO(3,1)$, $SO(c+1,1)$ respectively as in (62). In the Weyl sector the gamma matrices may be represented by hermitian matrices as follows

$$\sigma^{\mu}\ p_\mu = p_0 + \vec{\sigma}\cdot\vec{p}, \quad \gamma^m v_m = v_{0'} + \vec{\gamma}\cdot\vec{v}\ , \tag{68}$$

where in the time directions $\mu = 0, m = 0'$ the Weyl projected gamma matrices are proportional to unity. The resulting superalgebra may also be viewed as a reduction of the12D superalgebra of (63) in which the BPS constraints are satisfied in a sector [3] . In this way with a minimal set of

[3]More generally (63) allows also the components p_m and v_μ. In their presence we must

operators one can still probe some novel sector of S-theory that respects the isometry $SO(3,1)\times\ SO(c+1,1)$.

3.3 Type-B sectors

Similarly, one may start from the type-B superalgebra that is covariant under $SO(9,1)\times SO(2,1)$ [1] and rewrite it in a 4D basis by using an explicit covariance $SO(3,1)\times SO(c)\times SO(2,1)$, with $c=6$. Then the indices on the 32 spinors are $Q_{\alpha Aa}$, with $\alpha=1,2$, denoting an $SO(3,1)$ spinor, $A=1,2,3,4$ denoting an $SO(6)$ spinor and $a=1,2$ denoting an $SO(2,1)$ spinor. $\bar{Q}_{\dot{\alpha}\dot{A}a}$ is the hermitian conjugate of the 16 complex $Q_{\alpha Aa}$. The 10D vector index is split into a 4D index μ and a 6D index j. Then the 4D, N=8 superalgebra, with 528 real bosonic generators, can be put into the following form that keeps track of all 13 dimensions labeled by $\mu=0,1,2,3;\ j=1,\cdots,6;\ m=0',1',2'$

$$\begin{aligned}
\{Q_{\alpha Aa},Q_{\beta Bb}\} &= (i\sigma_2)_{\alpha\beta}\left[\gamma^j_{AB}\,(i\tau_2\tau^m)_{ab}\,(P_{jm}+iX_{jm})+\gamma^{ijk}_{AB}\,(i\tau_2)_{ab}\,Y_{ijk}\right]\\
&\quad +(i\sigma_2\vec{\sigma})_{\alpha\beta}\left[\gamma^i_{AB}\,(i\tau_2)_{ab}\,\vec{Y}_i+\gamma^{ijk}_{AB}\,(i\tau_2\tau_m)_{ab}\,\vec{X}^m_{ijk}\right]\\
\left\{\bar{Q}_{\dot{\alpha}\dot{A}\dot{a}},\bar{Q}_{\dot{\beta}\dot{B}\dot{b}}\right\} &= (i\sigma_2)_{\dot{\alpha}\dot{\beta}}\left[\gamma^j_{\dot{A}\dot{B}}\,(i\tau_2\tau^m)_{ab}\,(P_{jm}-iX_{jm})+\gamma^{ijk}_{\dot{A}\dot{B}}\,(i\tau_2)_{ab}\,Y^*_{ijk}\right]\\
&\quad +(i\sigma_2\vec{\sigma})_{\alpha\beta}\left[\gamma^i_{AB}\,(i\tau_2)_{ab}\,\vec{Y}_i+\gamma^{ijk}_{AB}\,(i\tau_2\tau_m)_{ab}\,\vec{X}^m_{ijk}\right] \qquad (69)\\
\left\{Q_{\alpha Aa},\bar{Q}_{\dot{\beta}\dot{B}\dot{b}}\right\} &= \sigma^\mu_{\alpha\dot{\beta}}\begin{pmatrix}\delta_{A\dot{B}}\ (i\tau_2\tau^m)_{ab}\,P_{\mu m}+\delta_{A\dot{B}}\ (i\tau_2)_{ab}\ y_\mu\\ +\gamma^{ij}_{A\dot{B}}\ (i\tau_2)_{ab}\,Y_{\mu ij}+\gamma^{ij}_{A\dot{B}}\ (i\tau_2\tau_m)_{ab}\ X^m_{\mu ij}\end{pmatrix}
\end{aligned}$$

The 528_B bosons labeled by the letters P,Y,X come from a relabeling of the 528_B bosons that were denoted by the same letters in the $SO(9,1)\times SO(3,1)$ covariant basis of [1]. Following the route of reasoning that led to eq.(67), a truncation of this superalgebra gives the form

$$\left\{Q_{\alpha Aa},\bar{Q}_{\dot{\beta}\dot{B}b}\right\}=\sigma^\mu_{\alpha\dot{\beta}}\,\delta_{A\dot{B}}\,(i\tau_2\tau^m)_{ab}\ p_\mu v_m \qquad (70)$$

where we may use the $SO(2,1)$ gamma matrices $\tau^m=(-i\tau_2,\tau_3,-\tau_1)$ to get

$$i\tau_2\tau^m v_m=v_{0'}+\tau_1 v_{1'}+\tau_3 v_{2'} \qquad (71)$$

which is consistent with the notation in (68).

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have bi-local fields, but we have specialized to ordinary local fields in 12D by working in the sector in which p_m and v_μ are zero. In this way we have lost the full $SO(10,2)$ but still have an isometry that keeps track of all 12 dimensions. Case (i) cannot be realized in this sector, but cases (ii) and (iii) remain.

3.4 Reduction to N=1

Both forms (67,70) become the standard $N=8$ supersymmetry if one keeps only one eigenvalue of a time-like v^m, since then it is possible to use the isometry to rotate v^m to the form $\gamma^m v_m = 1$. Similarly, they reduce to standard $N=4$ supersymmetry if v^m is light-like and fixed. However, if one allows all possible eigenvalues of a time-like or light-like v^m, just as all possible eigenvalues of p^μ are allowed, then the representation space is much richer and includes novel sectors of S-theory. The presence of 12 or 13 dimensions manifests itself through the two distinct forms (67,70) in the corresponding representation spaces. Our purpose is to construct some simple models that provide explicit representations of this new type of supersymmetry, with all possible eigenvalues of v_m, as in section-2, with the hope that such models will shed some light on the dynamics of S-theory.

To begin with, one may start the analysis by neglecting the c compactified dimensions altogether, and keep only the 11th and 12th dimensions in (67) or the 11th, 12th and 13th dimensions in (70). This corresponds to setting $c=0$ in eq.(67,70) and neglecting the spinor indices that correspond to the $c=6$ dimensions. In doing so we are really studying a much simpler system with fewer supersymmetries (as if the supersymmetry has been broken down from $N=8$ to $N=1$). With $N=8$ supersymmetry we must have supergravity. To avoid such complicated systems in the initial stages of this program, we prefer to start with $N=1$ supersymmetry and slowly work our way to higher N. Thus, in the present paper our goal is to provide some examples of representations for $N=1$. For higher N one will need to deal with a more complicated set of auxiliary fields in building up the representations.

In the simplified $c=0$ case, $\gamma^m_{a\dot{b}} v_m$ in (67) becomes a 1×1 matrix

$$\gamma^m_{a\dot{b}} v_m = v_{1'} + v_{0'} \equiv v_+ \tag{72}$$

where $1', 0'$ represent the 11th and 12th dimensions respectively. The indices $a, \dot{b} = 1$ will be suppressed from now on, and the new superalgebra will be written in the form given in (2). This connects to the standard $N=1$ supersymmetry if only one Kaluza-Klein mode of v_m is kept. Similarly in the type-B superalgebra (70) setting $c=0$ corresponds to neglecting the indices $A, \dot{B}$ and writing

$$\left\{ Q_{\alpha a}, \bar{Q}_{\dot{\beta} b} \right\} = \sigma^\mu_{\alpha\dot{\beta}}\ p_\mu\ \left(v_{0'} + \sigma_1 v_{1'} + \sigma_3 v_{2'} \right)_{ab} \tag{73}$$

This connects to standard $N=2$ supersymmetry if only one Kaluza-Klein mode of v_m is kept. The same form follows from the type-A superalgebra (67) if one takes $c=1$. Then $v_{2'}$ represents one of compactified string dimensions c rather than the 13th dimension. The map between these two equivalent cases corresponds to T-duality that mixes the 13th dimension with the compactified string dimensions.

4 superspace

One may wonder whether the results in section-2 can be recast in a superfield formalism. The hope is that this would provide the calculus of representation theory for our superalgebra and help in constructing and analyzing interacting theories. Here we give a brief summary of such an attempt, which is an imitation of the ordinary superfield formulation with some new twists. However our formulation is only partially successful because the calculus of the representations turns out to be more tricky than just the superfield formulation. It also requires a non-trivial product rule for superfields which remains to be constructed.

Consider the superalgebra (67) with $SO(3,1)\times SO(c+1,1)$ isometry. Introduce fermionic coordinates $\theta^{\alpha a}$ and their hermitian conjugates $\bar{\theta}^{\dot{\alpha}\dot{a}}$classified by the isometry in one to one correspondence to the supercharges. Then the following representation of supercharges satisfy the algebra

$$Q_{\alpha a}=\frac{\partial}{\partial\theta^{\alpha a}}-\frac{1}{2}\sigma^{\mu}_{\alpha\dot{\beta}}\gamma^{m}_{a\dot{b}}\bar{\theta}^{\dot{\beta}\dot{b}}\partial_{\mu}\tilde{\partial}_{m},\quad \bar{Q}_{\dot{\beta}\dot{b}}=\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}\dot{b}}}-\frac{1}{2}\theta^{\alpha a}\sigma^{\mu}_{\alpha\dot{\beta}}\gamma^{m}_{a\dot{b}}\partial_{\mu}\tilde{\partial}_{m} \tag{74}$$

Furthermore we may introduce covariant derivatives that antianticommute with both of these charges

$$D_{\alpha a}=\frac{\partial}{\partial\theta^{\alpha a}}+\frac{1}{2}\sigma^{\mu}_{\alpha\dot{\beta}}\gamma^{m}_{a\dot{b}}\bar{\theta}^{\dot{\beta}\dot{b}}\partial_{\mu}\tilde{\partial}_{m},\quad \bar{D}_{\dot{\beta}\dot{b}}=\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}\dot{b}}}\text{-}+\frac{1}{2}\theta^{\alpha a}\sigma^{\mu}_{\alpha\dot{\beta}}\gamma^{m}_{a\dot{b}}\partial_{\mu}\tilde{\partial}_{m} \tag{75}$$

A general superfield $\Phi(x,y,\theta,\bar{\theta})$ is a double polynomial in powers of $\theta,\bar{\theta}$ with coefficients that are ordinary fields that have consistent $SO(3,1)\times SO(c+1,1)$ assignments. A supersymmetry transformation of all the fields is given as

$$\delta\Phi(x,y,\theta,\bar{\theta})=\left(\varepsilon^{\alpha a}Q_{\alpha a}+\bar{\varepsilon}^{\alpha a}\bar{Q}_{\alpha a}\right)\Phi(x,y,\theta,\bar{\theta}) \tag{76}$$

The supersymmetry transformation of the components is read off by comparing the powers of $\theta, \bar{\theta}$ on both sides.

As in usual supersymmetry, we introduce the concept of a chiral superfield defined by

$$\bar{D}_{\dot{\beta}\dot{b}}\Phi^{(chiral))}(x, y, \theta, \bar{\theta}) = 0 \tag{77}$$

The solution of this equation is

$$\Phi^{(chiral))}(x, y, \theta, \bar{\theta}) = \exp\left(\frac{1}{2}\theta^{\alpha a}\sigma^{\mu}_{\alpha\dot{\beta}}\gamma^{m}_{a\dot{b}}\bar{\theta}^{\dot{\beta}\dot{b}}\partial_{\mu}\tilde{\partial}_{m}\right)F\left(x, y, \theta\right) \tag{78}$$

where $F\left(x, y, \theta\right)$ is the general polynomial involving only θ. Note that, because of the double derivative, the exponential factor is not the translation operator on the x, y coordinates.

Now, let's specialize to the $N = 1$ case, for which $a, \dot{b} = 1$ and therefore this index is suppressed. The chiral superfield can have at the most two powers of θ. If one respects the $SO(1,1)$ assignments one finds, for example, the scalar-scalar chiral supermultiplet

$$\Phi^{(chiral))} = \exp\left(\frac{1}{2}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}\tilde{\partial}_{+}\right)\left(\phi + \theta^{\alpha}\chi_{\alpha} + \theta^{\alpha}\theta_{\alpha}\tilde{\partial}_{+}f\right)(x, y). \tag{79}$$

The supersymmetry transformation applied as a differential operator in the form of eq.(76) gives the transformation rules displayed in section-2 for the components (ϕ, χ_{α}, f). Closure of the superalgebra is guaranteed by the construction of eq.(74). Evidently this property is automatically generalized to higher dimensions by the superfield formalism. For the purpose of defining (at least some) representations, as above, the superfield formalism given here is clearly useful.

One may think that this formulation supplies the technique for writing interactions. Unfortunately this does not work in a straightforward manner. If one takes a function of the superfield $W(F)$, e.g. a polynomial, and constructs a new chiral superfield

$$\exp\left(\frac{1}{2}\theta^{\alpha a}\sigma^{\mu}_{\alpha\dot{\beta}}\gamma^{m}_{a\dot{b}}\bar{\theta}^{\dot{\beta}\dot{b}}\partial_{\mu}\tilde{\partial}_{m}\right)W(F) \tag{80}$$

then the transformation law of the original superfields F are not compatible with the transformation applied on the superfield $W(F)$, if super transformations are applied naively by $\delta = \varepsilon^{\alpha a}Q_{\alpha a} + \bar{\varepsilon}^{\alpha a}\bar{Q}_{\alpha a}$ as a differential operator.

This is because $Q_{\alpha a}$ contains the double derivative structure $\partial_\mu \tilde{\partial}_m$ which is not distributive as a single derivative structure. That is, it does not satisfy the Leibnitz rule on naive products of the superfield. Hence representations are not combined into new irreducible ones by naive superfield manipulations. The correct combination rules remain to be discovered. This probably requires the construction of a "star product" of superfields that is compatible with the Leibnitz rule. At this stage it is not clear whether component methods or superfield methods will be more efficient in providing the techniques for constructing interactions.

In this paper we concentrated on the simplest $N = 1$ superalgebra (67) with only two new dimensions y^m and constructed some of its representations in the context of field theory. Our purpose was to provide some concrete examples of representations and to show that they can be connected to familiar 4D physics. Surprizingly, a new mechanism for embedding a few families in higher compactified dimensions emerged. The possibility of phenomenological applications is intriguing and encouraging for pursuing further the ideas in this paper. The construction of representations of the higher dimensional cases is aided by the superfield formalism suggested here.

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