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REMARKS ON FINITE $\mathcal{W}$ ALGEBRAS

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Abstract

The property of some finite $\mathcal{W}$ algebras to be the commutant of a particular subalgebra of a simple Lie algebra $\mathcal{G}$ is used to construct realizations of $\mathcal{G}$.

When $\mathcal{G} \simeq so(4,2)$, unitary representations of the conformal and Poincaré algebras are recognized in this approach, which can be compared to the usual induced representation technique. When $\mathcal{G} \simeq sp(2,\mathbb{R})$ or $sp(4,\mathbb{R})$, the anyonic parameter can be seen as the eigenvalue of a $\mathcal{W}$ generator in such $\mathcal{W}$ representations of $\mathcal{G}$.

The generalization of such properties to the affine case is also discussed in the conclusion, where an alternative of the Wakimoto construction for $\widehat{sl}(2)_k$ is briefly presented.

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1 Preliminaries

The subject of this talk concerns a special class of algebras, which have been called "finite $\mathcal{W}$ algebras" [1] and which we will denote FWAs. First constructed from the zero modes of the (known) $\mathcal{W}$ algebras, the FWAs present two appealing features.

First, they can be seen as a good laboratory for studying properties of the usual -or affine- $\mathcal{W}$ algebras, which depend of a complex variable: this is due to the relative simplicity of their commutation relations, compared to the affine case. The second point is that they constitute their own a rather interesting field of investigation, as well as applications, in mathematical physics. It is this second aspect of finite $\mathcal{W}$ algebras that we wish to raise and develop hereafter, just mentioning in the conclusion the generalization of our results to affine algebras.

The plan of this review will be the following:

-Section 2 contains a brief introduction on finite $\mathcal{W}$ algebras, with some definitions and notations.

-Section 3 deals with the construction of (a class of) finite $\mathcal{W}$ algebras starting from a simple Lie algebra $\mathcal{G}$. In our approach, the FWA appears as the commutant, in a generalization of the enveloping algebra of $\mathcal{G}$, of a $\mathcal{G}$-subalgebra. Such an approach can be seen as providing a definition of (a class of) FWAs, and also as an explicit method for determining the commutant of (a class of) subalgebras of a simple Lie algebra.

Then, it is this property of FWAs that will be exploited to construct realizations of a simple Lie algebra $\mathcal{G}$. More precisely, it can be shown that knowing a special realization of $\mathcal{G}$ in terms of differential operators of the $\mathcal{G}$ generators, new $\mathcal{G}$ realizations can be constructed owing to a suitably chosen FWA. This will be the subject of section 4, where the method is illustrated on the $sl(2,\mathbb{R})$ algebra. Then:

-Section 5 presents such a $\mathcal{W}$ realization for the four dimensional conformal algebra $so(4,2)$. Unitary representations of the conformal algebra, as well as its Poincaré subalgebra, can be recognized in this approach, which can be compared with the usual induced representation technique.

-Section 6 is devoted to $\mathcal{W}$ realizations of the $sp(2)$ and $sp(4)$ cases. These Lie algebras can be considered as the algebras of observables for a system of two identical particles in $d=1$ and in $d=2$ dimensions, in the Heisenberg quantization scheme. In each case, it will be possible to relate the anyonic parameter to the eigenvalues of a $\mathcal{W}$ generator.

Finally we conclude by a short discussion on the extension of these results to the affine case.

2 Definitions and notations

As a general definition of a finite $\mathcal{W}$ algebra, we can propose the following :

It is an algebra over a field k (we will limit to $\mathbb{R}$ *or* $\mathbb{C}$*) the two corresponding internal laws being the usual addition and multiplication with an extra internal law —the commutator— which is antisymmetric, k-bilinear, satisfies the Jacobi identity and closes polynomially.*

We will talk about a classical $\mathcal{W}$ algebra when the algebra is endowed with a Kirillov-Poisson structure, and the so-called commutator is the Poisson bracket (PB); we will speak about a quantum $\mathcal{W}$ algebra when the commutator is simply $[A,B]=AB-BA$ for any couple of elements A and B.

With this definition, the enveloping algebra of a finite dimensional Lie algebra appears as a particular case of $\mathcal{W}$ algebra. But let us propose an example:

Consider the algebra generated by the four elements E, F, H and C with the commutation relations (C.R.):

$$[H,E]=E \quad [H,F]=-F \quad [E,F]=H^2+C \tag{2.1}$$
$$\text{with } [C,E]=[C,F]=[C,H]=0$$

It seems natural to compare this algebra with the $sl(2)$ one generated by e, f, h:

$$[h,e]=e \quad [h,f]=-f \quad [e,f]=2h \tag{2.2}$$

The algebra (2.1) can be obtained from the zero modes of the (affine) $\mathcal{W}$ algebra made by four generators, of spin 2, 3/2, 3/2 and 1 under the spin 2 Virasoro generator, and sometimes called the Bershadsky algebra $\mathcal{W}_3^{(2)}$.The zero mode of the spin 2 element is C, the ones of spin 3/2 are E and F, while the one of spin 1 is H. One must note that commuting the zero modes of two different spin 3/2 generators does not uniquely provide zero modes in the other generators. The standard procedure to eliminate the non-zero modes consists in considering the action of such elements on any highest-weight representation of the $\mathcal{W}$ algebra; then projecting out both sides of the C.R. on each h.w. state will allow us to get the C.R. of (2.1), the positive modes annihilating the h.w. states.

The $\mathcal{W}_3^{(2)}$ algebra belongs to the large class of $\mathcal{W}$ algebras, symmetries of Toda theories, and which can be constructed from WZW models by using the Hamiltonian reduction technique[2]. The first step consists in imposing (first-class) constraints on the components of the conserved currents in the considered WZW model. Such constraints imply gauge transformations, and the associated $\mathcal{W}$ algebra will then be obtained by determining the corresponding gauge-invariant polynomial quantities.

In the following, $\mathcal{G}$ will stand for the Lie algebra of a simple, real, connected and non-compact Lie group G. Let $\{t_a\}, a=1...dim\mathcal{G}$, be a basis of $\mathcal{G}$ and $\{J^a\}$ the dual basis in $\mathcal{G}^*$

$$[t_a,t_b]=f_{ab}^c t_c \quad J^a(t_b)=\delta_b^a \tag{2.3}$$

We introduce the metric on $\mathcal{G}$ in a representation R:

$$\eta_{ab}=<t_a,t_b>=tr_R(t_a t_b) \text{ and } \eta_{ab}\eta^{bc}=\delta_a^c \tag{2.4}$$

We can then define on $\mathcal{G}^*$ a Poisson-Kirillov structure that mimicks the commutator (we have identified $\mathcal{G}^*$ and $\mathcal{G}$):

$$\{J^a,J^b\}=f_c^{ab}J^c \ \text{ with } \ f_c^{ab}=\eta^{ad}\eta^{be}\eta_{cg}f_{de}^g \tag{2.5}$$

Actually, from the simple Lie algebra $\mathcal{G}$, the different sets of constraints one wishes to impose, and therefore the different $\mathcal{W}$ algebras one can construct, are in one-to-one correspondence with the embeddings of the $sl(2)$ algebra in $\mathcal{G}$. To each such embedding can be associated a grading of $\mathcal{G}$, given by the eigenvalue of the $sl(2)$ Cartan generator h: $\mathcal{G} = \oplus_{p=-m}^{+m}\mathcal{G}$ with $[h, X] = pX$ for any $X \in \mathcal{G}, p \in \frac{1}{2}\mathbb{Z}$; we then have the property $[\mathcal{G}_p, \mathcal{G}_q] \subset \mathcal{G}_{p+q}$, with the convention that $\mathcal{G}_p = \{0\}$ when $|p| > m$. More generally, we write:

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_0 \oplus \mathcal{G}_- \tag{2.6}$$

the constraints being imposed, in the current matrix $J = J^a t_a$, on the components of $\mathcal{G}_-$.

Since any $sl(2)$ subalgebra in a classical simple $\mathcal{G}$ algebra is principal in a subalgebra $\mathcal{H}$ of $\mathcal{G}$ (we discard the exceptional algebras case), it is quite usual to denote the corresponding $\mathcal{W}$ algebra by $\mathcal{W}(\mathcal{G}, \mathcal{H})$. Taking as an example the $\mathcal{G} = sl(3)$ case, two $\mathcal{W}$ algebras can be constructed in this way: the $\mathcal{W}(sl(3), sl(3))$ one, which in the affine case is generated by the two fields W_2 and W_3 of respective spin 2 and 3, usually also written as $\mathcal{W}_3^{(1)}$, and the $\mathcal{W}(sl(3), sl(2))$ one, presented just above as the Bershadsky $\mathcal{W}$ algebra and also denoted as $\mathcal{W}_3^{(2)}$.

3 Realization of finite $\mathcal{W}$ algebras

In the usual Hamiltonian reduction approach [2], we start by imposing (first-class) constraints on the $\mathcal{G}_-^*$ part of the J-matrix. Following Dirac's prescription, these first-class constraints generate a gauge invariance on the J^a's, i.e. in the classical case:

$$J \to J^g = exp\left(c_\alpha \{J^\alpha, \cdot\}_{cons.}\right)(J) \tag{3.1}$$

where the $\{\ ,\ \}_{cons.}$ means that one has to impose the constraint conditions on the r.h.s. of the Poisson bracket. Developing J^g with the help of the gradation and the use of constraints, we can, using the relations (2.3-2.5), rewrite J^g as:

$$J^g = exp\left(c^{\bar\alpha}[t_{\bar\alpha}, \cdot]\right)(J) = g_+^{-1} J g_+ \tag{3.2}$$

where $g_+ = exp(c^{\bar\alpha} t_{\bar\alpha})$ with the parameters $c^{\bar\alpha} = \eta^{\bar\alpha\alpha} c_\alpha$ and $t_{\bar\alpha} \in \mathcal{G}_+$. Thus the gauge transformations can be seen as conjugation on $\mathcal{G}$ by elements of the subgroup G_+.

Finally, while fixing the gauge, one obtains, in the components of J^g, gauge-invariant quantities, i.e. quantities which Poisson-commute with the constraints.

The main idea of our construction [3, 4] is not to impose constraints anymore on $\mathcal{G}_-^*$ once the gradation is chosen, but however to use the G_+ conjugation as in (3.2):

$$J_{tot}^g = g_+^{-1} J_{tot}\ \ g_+ \ \text{ with } \ g_+ \in G_+ \tag{3.3}$$

(we denote the J-part by J_{tot} in order to emphasize that no restrictions on its components have been put).

Then, developing J_{tot}^g by using the same rules as before, we get:

$$J_{tot}^g = exp\left(c_\alpha \{J^\alpha, \cdot\}\right)(J_{tot}) \tag{3.4}$$

where now the PBs are computed without using what were the constraints. Thus if one finds quantities which are invariant under the coadjoint transformations, these objects will have strongly vanishing PBs with the elements J^α. Let us add that, although the transformations we are looking at have the same form as the gauge transformations described at the beginning of this section, they are not gauge transformations, not being associated with constraints. Thus the construction we present is strictly algebraic, but we will see that the technique can be applied to physical problems.

Then we are looking for quantities which Poisson-commute with the $\mathcal{G}^*_-$ part of $\mathcal{G}^*$. Let us translate this problem into the Lie algebra $\mathcal{G}$. Such a quantization can be easily performed thanks to the Lie isomorphism between $\mathcal{G}^*$ and $\mathcal{G}$, and a symmetrization procedure that maps polynomials in $\mathcal{G}^*$ onto elements of $\mathcal{U}(\mathcal{G})$. Indeed, the isomorphism i between $\mathcal{G}^*$ and $\mathcal{G}$ defined by:

$$i(J^a) = t^a = \eta^{ab} t_b \tag{3.5}$$

where η^{ab} is the inverse matrix of the metric η_{ab}, can be extended as a vector space homomorphism from $\mathcal{G}^*$ polynomials into $\mathcal{U}(\mathcal{G})$ with the rule:

$$i(J^{a_1} J^{a_2} \cdots J^{a_n}) = S(t^{a_1} \cdot t^{a_2} \cdots t^{a_n}) \quad \forall n \tag{3.6}$$

where $S(\cdot, \cdots, \cdot)$ stands for the symmetrized product of the generators t^a; S is normalized by $S(X, \cdots, X) = X^n$.

At this stage, one could realize that the finite $\mathcal{W}$ algebras that we wish to construct have some connection with the commutant of a subalgebra in $\mathcal{G}$. Actually, after developing a symmetry fixing procedure, and limiting to a class of finite $\mathcal{W}$ algebras that are of special interest for the rest of this talk, we can announce the following result [4]:

Theorem: *Any finite $\mathcal{W}(\mathcal{G}, \mathcal{S})$ algebra, with $\mathcal{S} = \mu sl(2)$ regular subalgebra of $\mathcal{G}$, can be seen as the commutant in a (localization of) the enveloping algebra $\mathcal{U}(\mathcal{G})$ of some $\mathcal{G}$-subalgebra $\tilde{\mathcal{G}}$.*

Moreover, let H be the Cartan generator of the diagonal $sl(2)$ in $\mathcal{S}$. If we call $\mathcal{G}_-, \mathcal{G}_0$ and $\mathcal{G}_+$ the eigenspaces of respectively negative, null and positive eigenvalues under H, then $\tilde{\mathcal{G}}$ decomposes as $\tilde{\mathcal{G}} = \mathcal{G}_- \oplus \tilde{\mathcal{G}}_0$, where $\tilde{\mathcal{G}}_0$ is a subalgebra of $\mathcal{G}_0$, which can be uniquely determined.

Let us briefly comment this property. First of all, the $\mathcal{G}$ grading obtained from regular subalgebras $\mathcal{S} = \mu sl(2)$ are always such that the $\mathcal{G}_-$ part is Abelian. This means in particular that, in the determination of quantities that commute with $\mathcal{G}_-$, the elements of $\mathcal{G}_-$ themselves appear. In order to get rid of these undesirable quantities, one can think about increasing the subalgebra $\mathcal{G}_-$ up to another one $\tilde{\mathcal{G}}$, in such a way that the commutant of $\tilde{\mathcal{G}}$ provides exactly the $\mathcal{W}$ algebra one wishes to obtain.

Note that more general finite $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebras than the ones mentioned above can be obtained as a commutant, but one will then have to extend $\tilde{\mathcal{G}}_0$ to a part of $\mathcal{G}_+$ in $\mathcal{G}$. The case of "affine" $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebra can also be treated within this framework.

Thus the $\mathcal{W}$ algebras that one can construct are written in terms of all the generators of the algebra $\mathcal{G}$. There is, however, a price to pay: the generators obtained this way show up as functions $P(t_a)/Q(t_\alpha)$ with P a polynomial in all the t_a's and Q a smooth function in the center of the $\tilde{\mathcal{G}}$ Lie derivative (i.e. $\mathcal{Z}([\tilde{\mathcal{G}}, \tilde{\mathcal{G}}])$). It can be shown that the $\mathcal{W}$ generators form a polynomial basis of the commutant of $\widetilde{\mathcal{G}}$ in a generalization of the enveloping algebra $\mathcal{U}(\mathcal{G})$. The one we consider is the localization $\mathcal{U}(\mathcal{G})_{\mathcal{S}}$, where $\mathcal{S} = \mathcal{Z}([\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}])$, which contains apart from $\mathcal{U}(\mathcal{G})$ itself, quotients $u^{-1}v, vu^{-1}$, where $u \in \mathcal{S}, u \neq 0$ and $v \in \mathcal{U}(\mathcal{G})$, or an extension of this latter allowing elements like $u^r, u \in \mathcal{S}, r \in \frac{1}{2}\mathbb{Z}$. Let us emphasize that the technique that is summarized and briefly commented on above leads to a purely algebraic construction of a class of finite, as well as affine, $\mathcal{W}$ algebras. Consequently, it can also be considered as a way of defining (a family of) $\mathcal{W}$ algebras.

4 $\mathcal{W}$ realizations of simple Lie algebras

The property of a $\mathcal{W}$ algebra to appear as the commutant of a $\mathcal{G}$-subalgebra can be used to build, from a special realization of $\mathcal{G}$, a large set of $\mathcal{G}$-representations [3][5]. Indeed one knows how to construct a realization of $\mathcal{G}$ with differential operators on the space of smooth functions $\varphi(x_1, \cdots, x_n)$ with $n = dim\mathcal{G}_-$. In this picture, when $\mathcal{G}_- \equiv \mathcal{G}_{-1}$, the abelianity of the $\mathcal{G}_{-1}$ part allows each $\mathcal{G}_{-1}$ generator to act by direct multiplication:

$$\varphi(x_1, \cdots, x_n) \to x_i \varphi(x_1, \cdots, x_n) \quad \text{with} \quad i = 1, \cdots, n \tag{4.1}$$

—cf. action of the translation group— while the generators of the $\mathcal{G}_0 \oplus \mathcal{G}_+$ part will be represented by polynomials in the x_i and ∂_{x_i}.

It is from a particular —canonical— differential realization of $\mathcal{G}$ that new realizations will be constructed with the use of the finite $\mathcal{W}$ algebra mentioned above. Realization of the $\widetilde{\mathcal{G}}$ generators will not be affected in this approach. On the contrary, to the differential form of each generator in a certain supplementary subspace of $\widetilde{\mathcal{G}}$ in $\mathcal{G}$ will be added a sum of $\mathcal{W}$ generators, the coefficients of which $f(x_i, \partial_{x_i})$ are polynomials in the ∂_{x_i}'s. To each irreducible d-dimensional representation of the $\mathcal{W}$ algebra one can associate a matrix differential realization of $\mathcal{G}$ acting on vector functions $\bar{\varphi} = (\varphi_1, \cdots, \varphi_d)$ with $\varphi_i = \varphi_i(x_1, \cdots, x_n)$.

It is time to illustrate our technique on the simplest non-trivial example, i.e. $\mathcal{G} = sl(2, \mathbb{R})$.

Let us define:

$$J = J^- t_- + J^0 t_0 + J^+ t_+ = \begin{pmatrix} J^0 & J^+ \\ J^- & -J^0 \end{pmatrix} \tag{4.2}$$

with $\mathcal{G}_{+,0,-}$ generated by $t_{+,0,-}$ respectively.

By the action of an adequate G_+ element, namely:

$$g_+ = \begin{pmatrix} 1 & -J^0/J^- \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

one obtains (symmetry fixing):

$$J^{g_+} = g_+ \, J g_+^{-1} = \begin{pmatrix} 0 & \frac{J^+J^- + (J^0)^2}{J^-} \\ J^- & 0 \end{pmatrix} \tag{4.4}$$

It follows, after quantization:

$$J^+ J^- + (J^0)^2 \to \frac{1}{2}(t_+ t_- + t_- t_+) + t_0^2 = C_2 \tag{4.5}$$

that is, exactly the C_2 Casimir operator of $sl(2,\mathbb{R})$ generating the finite $\mathcal{W}$ algebra that we wish to determine (do not forget that we are expecting the zero mode of the Virasoro generator !).

On this simple example, we can convince ourselves that the commutant of $\mathcal{G}_-$ in $\mathcal{U}(\mathcal{G})_{\mathcal{S}}$ ($\mathcal{S}$ is generated by t_-) denoted by $Com(\mathcal{G}_-)$ is a polynomial algebra generated by $\{C_2, t_-, \frac{1}{t_-}\}$.

In order to get the C_2 element only, we will look for the commutant of a Lie algebra larger than $\mathcal{G}_-$; more precisely, we will obtain:

$$Com\ (\mathcal{G}_- \oplus \mathcal{G}_0) = Polyn\ (\{C_2\}). \tag{4.6}$$

Now, let us show how this construction can be applied to realizations of $sl(2,\mathbb{R})$. For such a purpose, consider the $sl(2)$ (differential) realization:

$$E_- = \frac{1}{2}x^2 \quad E_+ = -\frac{1}{2}\partial_x^2 \quad H = -(x^2\partial_{x^2} + \frac{1}{4}) \tag{4.7}$$

acting on smooth functions φ of the real variable x. We are in the conditions of eq. (4.1) for the $\mathcal{G}_{-1}$ part, which is one-dimensional. We also note that the eigenvalue of C_2 for this representation is $= -3/16$.

On this example, it is an easy calculation to write down from (4.7) new realizations of the algebra under consideration. Leaving the $\mathcal{G}_-$ generator unchanged, as well as the $\mathcal{G}_0$ one, a realization corresponding to the eigenvalue γ of C_2 is given by:

$$E_- = \frac{1}{2}x^2 \qquad E_+ = -\frac{1}{2}\partial_x^2 + \frac{\gamma + \frac{3}{16}}{x^2} \qquad H = -(x^2\partial_{x^2} + \frac{1}{4}) \tag{4.8}$$

The above expression of E_+ can also be obtained systematically in the following way (which, therefore, generalizes to the other cases). Coming back to eq. (4.4), let us formally act on J^{g_+} by g_+^{-1}:

$$J^{g_+} = \begin{pmatrix} 0 & \frac{"C_2"}{J^-} \\ J^- & 0 \end{pmatrix} \to g_+^{-1} J^{g_+} g_+ = J = \begin{pmatrix} J^0 & \frac{-(J^0)^2 + "C_2"}{J^-} \\ J^- & -J^0 \end{pmatrix} \tag{4.9}$$

where "C_2" denotes the Casimir element before quantization. By identification, one gets:

$$J^+ = -\frac{(J^0)^2}{J^-} + \frac{"C_2"}{J^-} \tag{4.10}$$

which shows a direct correspondence between J^+ and "C_2". J^+ is linear in "C_2" and vice-versa; this linearity property will survive at the quantum level:

$$E_+ = \frac{1}{E_-}(C_2 - H^2 - H) \tag{4.11}$$

which leads to the E_+ expression in (4.8).

Thus, from the special ("canonical") differential realization (4.7), our technique on the commutant has allowed to get a large class of $sl(2,\mathbb{R})$ representations.

5 Unitary irreducible representations of the conformal and Poincaré algebras

We now apply the results summarized in sections 3 and 4 to get $\mathcal{W}$ realizations of the $so(4,2)$ algebra and of its Poincaré subalgebra [5]. The $so(4,2)$ algebra is known as the conformal algebra in four dimensions in the Minkowski space, i.e. with the metric $g_{\mu\nu} = diag(1,-1,-1,-1)$. Its fifteen generators can be chosen and realized in the momentum representation as follows:

- four translations: $P_\mu = p_\mu$ $(\mu = 0,1,2,3)$ forming the $\mathcal{G}_-$ part.
- six Lorentz generators: $M_{\mu\nu} = i(p_\mu \partial_\nu - p_\nu \partial_\mu)$ forming with the dilatation $D = -i(p \cdot \partial + 4)$ the $\mathcal{G}_0$ part.
- four special conformal transformations

$$K_\mu = p_\mu \Box - 2p - \partial \cdot \partial_\mu - 8\partial_\mu \tag{5.1}$$

constituting the $\mathcal{G}_+$ part.

The corresponding grading operator is D, and in the above expressions ∂_μ stands for $\partial/\partial p^\mu, p$ is the quadrivector $(p_\mu), \partial$ the quadrivector (∂_μ) and $\Box = \partial \cdot \partial = g^{\mu\nu}\partial_\mu\partial_\nu = \partial^\mu\partial_\mu$.

We wish to construct the commutant of the P_μ in order to build a $\mathcal{W}$ realization of $\mathcal{G} = so(4,2)$. Note that $so(4,2)$ and $sl(4,\mathbb{R})$ are two different non compact real forms of the algebra $so(6) \sim su(4)$. If we had considered $sl(4,\mathbb{R})$, i.e. the maximally non-compact form of $\mathcal{G}$, the chosen gradation would correspond to the model $\mathcal{W}(sl(4), 2sl(2))$. Referring to section 3, we can take $\tilde{\mathcal{G}} = \mathcal{G}_- \oplus \tilde{\mathcal{G}}_0$, where:

$$\tilde{\mathcal{G}} = \{M_{13} - M_{01}, M_{23} - M_{02}, M_{03}, D\} \tag{5.2}$$

The commutant of $\tilde{\mathcal{G}}$ can therefore be seen as a compactified form of $\mathcal{W}(sl(4),\, 2sl(2))$. It contains seven generators: three generators $J_k, k = 1,2,3$, forming an $so(3)$ algebra; three other generators $S_l, l = 1,2,3$ forming a vector under this $so(3)$, their C.R.'s closing under a polynomial in the J_k and the seventh generator C_2, which is the second-order Casimir of $so(4,2)$. In summary the W C.R.'s read:

$$\begin{aligned}
[J_j, J_k] &= i\,\varepsilon_{jkl} J_l \\
[J_j, S_k] &= i\,\varepsilon_{jkl} S_l \\
[S_j, S_k] &= -i\,\varepsilon_{jkl}(2(J_1^2 + J_2^2 + J_3^2) - C_2 - 4)J_l \\
[C_2, J_j] &= [C_2, S_j] = 0 \qquad \{j,k,l\} = \{1,2,3\}.
\end{aligned} \tag{5.3}$$

It can directly be checked that this algebra satisfies, for each (real) value of the C_2 scalar, the defining C.R.'s of the Yangian [6] $Y(sl(2))$, with generators J_i and $S_i (i = 1,2,3)$. In other words, the $\mathcal{W}$ algebra defined by (5.3) provides a realization of[2] $Y(sl(2))$.

The $so(4,2)$ realization obtained by our $\mathcal{W}$ approach stands as follows, for $p^2 > 0$ (note that sign $(p^2) = +, 0, -$ is conserved in $so(4,2)$):

$$P_\mu = p_\mu \mathbb{1} \qquad \mu = 0,1,2,3 \tag{5.4}$$

[2]We thank Mo-Lin Ge for bringing our attention to this point.

$$
\begin{aligned}
M_{12} &= i(p_1\partial_2 - p_2\partial_1)\mathbb{1} + J_3 && (5.5)\\
M_{13} &= i(p_1\partial_3 - p_3\partial_1)\mathbb{1} - \frac{\sqrt{p^2}}{p_0+p_3}J_2 - \frac{p_2}{p_0+p_3}J_3 && (5.6)\\
M_{23} &= i(p_2\partial_3 - p_3\partial_2)\mathbb{1} + \frac{\sqrt{p^2}}{p_0+p_3}J_1 + \frac{p_1}{p_0+p_3}J_3 && (5.7)\\
M_{01} &= i(p_0\partial_1 - p_1\partial_0)\mathbb{1} - \frac{\sqrt{p^2}}{p_0+p_3}J_2 - \frac{p_2}{p_0+p_3}J_3 && (5.8)\\
M_{02} &= i(p_0\partial_2 - p_2\partial_0)\mathbb{1} + \frac{\sqrt{p^2}}{p_0+p_3}J_1 + \frac{p_1}{p_0+p_3}J_3 && (5.9)\\
M_{03} &= i(p_0\partial_3 - p_3\partial_0)\mathbb{1} && (5.10)\\
D &= -i(p\cdot\partial + 4)\mathbb{1} && (5.11)\\
K_0 &= (p_0\Box - 2p\cdot\partial\partial_0 - 8\partial_0)\mathbb{1} - \frac{2}{p_0+p_3}Z_3 + \frac{p_0}{p^2}Z_0\\
&\quad + \frac{1}{(p_0+p_3)\sqrt{p^2}}(p_1Z_1 + p_2Z_2) +\\
&\quad - \frac{2i\sqrt{p^2}}{p_0+p_3}\left(-(\frac{5}{2}\frac{p_2}{p^2} + \partial_2)J_1 + (\frac{5}{2}\frac{p_1}{p^2} + \partial_1)J_2\right)\\
&\quad - \frac{2i}{p_0+p_3}(p_2\partial_1 - p_1\partial_2)J_3 && (5.12)\\
K_1 &= (p_1\Box - 2p\cdot\partial\partial_1 - 8\partial_1)\mathbb{1} + \frac{1}{\sqrt{p^2}}Z_1 + \frac{p_1}{p^2}Z_0\\
&\quad - \frac{2i\sqrt{p^2}}{p_0+p_3}\left(\frac{5}{2}\frac{p_0+p_3}{p^2} + \partial_0 + \partial_3\right)J_2\\
&\quad - 2i\left(\frac{p_2}{p_0+p_3}(\partial_0+\partial_3) - \partial_2\right)J_3 && (5.13)\\
K_2 &= (p_2\Box - 2p\cdot\partial\partial_2 - 8\partial_2)\mathbb{1} + \frac{1}{\sqrt{p^2}}Z_2 + \frac{p_2}{p^2}Z_0\\
&\quad + \frac{2i\sqrt{p^2}}{p_0+p_3}\left(\frac{5}{2}\frac{p_0+p_3}{p^2} + \partial_0 + \partial_3\right)J_1\\
&\quad + 2i\left(\frac{p_1}{p_0+p_3}(\partial_0+\partial_3) - \partial_1\right)J_3 && (5.14)\\
K_3 &= (p_3\Box - 2p\cdot\partial\partial_3 - 8\partial_3)\mathbb{1} + \frac{2}{p_0+p_3}Z_3 + \frac{p_3}{p^2}Z_0\\
&\quad - \frac{1}{(p_0+p_3)\sqrt{p^2}}(p_1Z_1 + p_2Z_2)\\
&\quad + \frac{2i\sqrt{p^2}}{p_0+p_3}\left(-(\frac{5}{2}\frac{p_2}{p^2} + \partial_2)J_1 + (\frac{5}{2}\frac{p_1}{p^2} + \partial_1)J_2\right)\\
&\quad + \frac{2i}{p_0+p_3}(p_2\partial_1 - p_1\partial_2)J_3 && (5.15)
\end{aligned}
$$

where

$$Z_1 = 2S_1 + J_3J_1 + J_1J_3 \qquad Z_2 = 2S_2 + J_3J_2 + J_2J_3$$
$$Z_3 = S_3 - (J_1^2 + J_2^2) \qquad Z_0 = 2S_3 + C_2 - J_3^2 - 2(J_1^2 + J_2^2) \tag{5.16}$$

Let us focus for the moment on the expressions of the Poincaré generators and remark that only the $\vec{J}$-part of the $\mathcal{W}$ algebra shows up there. We recall the expressions of the Pauli-Lubanski-Wigner quadrivector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$, which satisfies:

$$[W_\mu, P_\nu] = 0 \qquad [W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \qquad W \cdot P = 0 \tag{5.17}$$

It is well-known that the irreducible representations of the Poincaré algebra are labelled by the eigenvalues of $P^2 = p^2$ and $W^2 = -s(s+1)p^2$, where s is the spin of the particle. Because of the relation $W \cdot P = 0$, the quadrivector W^μ possesses only 3 independent components. These generate the spin algebra $so(3)$ when p^2 is positive. This is recovered in a very natural way in our $\mathcal{W}$ algebra framework. Indeed, the generators J_k really play the role of the spin generators, since they can be rewritten as:

$$J_k = -\frac{1}{m}n_k \cdot W = -\frac{1}{m}(n_k)^\mu W_\mu \quad \text{and} \quad W^\mu = -m\sum_{k=1}^{k=3}(n_k)^\mu J_k \tag{5.18}$$

(since $P \cdot W = 0$)

$$\text{with} \qquad m = \sqrt{p^2} \qquad \text{and} \qquad W \cdot W = -P^2\vec{J}^2 \tag{5.19}$$

and where we have introduced the frame [7] of the "particle" of momentum p:

$$\begin{aligned}
n_0 &= (n_0)_\mu = \frac{1}{m}p = (\frac{p_0}{m}, \frac{p_1}{m}, \frac{p_2}{m}, \frac{p_3}{m}) \\
n_1 &= (n_1)_\mu = \frac{1}{p_0+p_3}(p_1, p_0+p_3, 0, -p_1) \\
n_2 &= (n_2)_\mu = \frac{1}{p_0+p_3}(p_2, 0, p_0+p_3, -p_2) \\
n_3 &= (n_3)_\mu = \frac{-m}{p_0+p_3}(1,0,0,-1) + \frac{1}{m}p
\end{aligned} \tag{5.20}$$

which obeys $n_\mu \cdot n_\nu = (n_\mu)^\rho(n_\nu)^\sigma g_{\rho\sigma} = g_{\mu\nu}$ and also $(n_\mu)^\rho(n_\nu)^\sigma g^{\mu\nu} = g^{\rho\sigma}$.

The Lorentz transformation $L(p)$, which moves the rigid referential frame (e_0, e_1, e_2, e_3) with $e_0 = (1, \vec{0}), e_1 = (0,1,0,0), e_2 = (0,0,1,0)$ and $e_3 = (0,0,0,1)$ to the p-frame (n_0, n_1, n_2, n_3), also relates the three-vector (J_i) to the four-vector $(\mathcal{W}_\mu)$

$$mJ = (0, m\vec{J}) \xrightarrow{L(p)} W = (W_\mu) \tag{5.21}$$

It is through $L(p)$ that the representations of the Poincaré group can be constructed from representations of the rotation subgroup. Indeed, the Lorentz transformation Λ acting on functions $\tilde{\varphi}$ of the p-variable in the U representation:

$${}^\Lambda\tilde{\varphi}(p) = U(\Lambda)\tilde{\varphi}(\Lambda^{-1}p) \tag{5.22}$$

is written more conveniently on the Wigner functions ψ defined by

$$\psi(p) = U(L(p)^{-1})\tilde{\varphi}(p) \tag{5.23}$$

as

$$\left({}^{\Lambda}\psi\right)(p) = U\left(L(p)^{-1}\Lambda L(\Lambda^{-1}p)\right)\psi(\Lambda^{-1}p). \tag{5.24}$$

We recognize in the product $L(p)^{-1}\Lambda L(\Lambda^{-1}p)$ a Wigner rotation, element of the $(m, 0, 0, 0)$-vector stabilizer, itself isomorphic to the $SO(3)$ group when $p^2 > 0$. It is exactly the infinitesimal part of (5.24) that we have in the expressions of the Poincaré generators displayed above.

We now look at the other generators of the conformal algebra. In the same way as we have introduced the Pauli-Lubanski-Wigner vector W^μ, let us define:

$$\begin{aligned} \Sigma_\mu &= -W^2\,P_\mu + P^2[\;P^\alpha M_{\alpha\mu}(D+i) - \\ &\quad -\frac{1}{2}(P_\mu\,P\!\cdot\!K - P^2\,K_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}W^\nu M^{\rho\sigma}] \end{aligned} \tag{5.25}$$

It satisfies in particular

$$[\Sigma_\mu, P_\nu] = 0 \quad \text{and} \quad \Sigma\cdot P = 0 \tag{5.26}$$

and we can prove that the generators S_i are connected to the quadrivector (Σ^μ) through

$$S_k = -\frac{1}{m^3}n_k\cdot\Sigma = -\frac{1}{m^3}(n_k)^\mu\Sigma_\mu \quad \text{and} \quad \Sigma^\mu = -m^3\sum_{k=1}^{k=3}(n_k)^\mu S_k \tag{5.27}$$

(since $P\cdot\Sigma = 0$) and also:

$$m^3 S = (0, m^3\vec{S}) \xrightarrow{L(p)} \Sigma = (\Sigma_\mu) \tag{5.28}$$

It would be interesting to add to these geometrical properties a physical meaning for (Σ_μ), which appears as a sort of conformal analogous of (W_μ).

Finally, as for the Poincaré subcase, we expect that the obtained W-realizations of the $so(4,2)$ algebra can be compared with the ones constructed via the induced representation method. This later approach can be found in ref. [8], where the classification of all the unitary ray representations of the $su(2,2)$ group with positive energy is achieved. A direct comparison with the construction of ref. [8] can be performed [5], which leads to a selection of finite-dimensional representations of the $\mathcal{W}$ algebra leading to the unitary conformal representations. In ref. [8] the induced representations are labelled by two non-negative (half) integers (j_1, j_2) associated with spinor representations of the Lorentz group D^{j_1,j_2}, and by d a real number associated to the dilatation. In particular the representations of positive masses satisfy the conditions:

$$d \geq j_1 + j_2 + 2 \quad \text{with} \quad j_1, j_2 \neq 0 \quad \text{and} \quad d > j_1 + j_2 + 1 \quad \text{with} \quad j_1, j_2 = 0 \tag{5.29}$$

Our task is greatly facilitated by the Miura transformation, which allows the $\mathcal{W}$ generators to be expressed in terms of generators of the $\mathcal{G}_0$ part, that is the Lorentz algebra generated by the rotations $\vec{R}$ and the boosts $\vec{B}$ to which has to be added the dilatation D. The result is quite simple:

$$\vec{J} = \vec{R} \qquad \vec{S} = \vec{R} \times \vec{B} - i(D-1)\vec{B} \qquad C_2 = \vec{R}^2 - \vec{B}^2 + D(D-4) \tag{5.30}$$

Owing to (5.30) we know how to associate, and explicitly construct, the $\mathcal{W}$ representation relative to the $so(4,2)$ unitary representation labelled by $(j_1, j_2; d)$ the condition (5.29) becoming:

$$\begin{array}{lll} c_2 \geq 2j_1(j_1+1) + 2j_2(j_2+1) + (j_2+j_2)^2 - 4 & \text{if} \;\; j_1 j_2 \neq 0 & \text{and} \\ c_2 > 3(j_1+j_2+1)(j_1+j_2-1) & \text{if} \;\; j_1 j_2 = 0 & \end{array} \tag{5.31}$$

where c_2 is the C_2 eigenvalue.

6 Anyons and $\mathcal{W}$ algebras

We now turn to the Heisenberg quantization for a system of two identical particles in $d=1$ and $d=2$ dimensions[9]. In each case a finite $\mathcal{W}$ algebra will be recognized [3] from the algebra of observables, and used for an algebraic treatment of intermediate statistics.

6.1 Two particles in $d = 1$

We must remark that such a one dimensional system of two identical particles has been proposed as anyon candidate. Indeed it can be formally related to a system of two identical vortices in a thin, incompressible superfluid film, the two spatial coordinates of the vortex center acting as canonically conjugate quantities [10].

The relative coordinate and momentum of the two particle system are denoted by:

$$x = x_{(1)} - x_{(2)} \qquad p = \frac{1}{2}(p_{(1)} - p_{(2)}) \tag{6.1}$$

and satisfy the C.R. $[x,p] = i$ in the quantum case.

Then the chosen observables, i.e. the quadratic polynomials homogeneous in x and p $(x^2, p^2, xp+px)$ close in the quantum case under the C.R.'s of $\mathcal{G} = sp(2,\mathbb{R}) \simeq sl(2,\mathbb{R})$ and we recognize the expressions already written in (4.7). The $\mathcal{W}$ treatment [3] leads directly to (4.8), in particular to the expression:

$$E_+ = \, -\frac{1}{2}\partial_x^2 \, + \, \frac{\gamma + 3/16}{x^2} \tag{6.2}$$

where we can recognize ($x = x_{(1)} - x_{(2)}$) the Calogero Hamiltonian. As discussed in [9] the parameter $\lambda = \gamma + 3/16$ can be directly related to the anyonic continuous parameter, with end point $\lambda = 0$ or $\gamma = -3/16$ corresponding to the boson and fermion cases.

6.2 Two particles in $d = 2$

Then, the algebra of observables is generated by the quadratic homogeneous polynomials in the relative coordinates x_j and p_j $(j = 1, 2)$. One gets a realization of the $\mathcal{G} = sp(4, \mathbb{R})$ Lie algebra, the generators of which can be conveniently separated into three subsets:

- the $\mathcal{G}_{-1}$ part with the three (commuting) coordinate operators:

$$u = (x_1)^2 + (x_2)^2 \qquad v = (x_1)^2 - (x_2)^2 \qquad w = 2x_1x_2 \tag{6.3}$$

- the $\mathcal{G}_{+1}$ part with the three (commuting) second order differential operators:

$$U = (p_1)^2 + (p_2)^2 \qquad V = (p_1)^2 - (p_2)^2 \qquad W = 2p_1p_2 \tag{6.4}$$

- and the $\mathcal{G}_0$ part isomorphic to $s\ell(2, \mathbb{R}) \oplus g\ell(1)$ with the four first order differential operators:

$$\begin{array}{ll} C_s = \frac{1}{4}\sum_{i=1}^{2}(x_ip_i + p_ix_i) & C_d = \frac{1}{4}(x_1p_1 + p_1x_1 - x_2p_2 - p_2x_2) \\ & \\ L = x_1p_2 - x_2p_1 & M = x_1p_2 + x_2p_1 \end{array} \tag{6.5}$$

C_s being the Abelian factor.

The finite $\mathcal{W}$ algebra associated with this $\mathcal{G}$-gradation is four-dimensional, and can be seen as a "deformed" $gl(2)$ algebra, i.e.:

$$\begin{array}{rcl} [S, Q] &=& -2iR \\ {[S, R]} &=& -2iQ \\ {[Q, R]} &=& -8iS(\mu - 2S^2) \\ {[\mu, Q]} &=& [\mu, R] = [\mu, S] = 0 \end{array} \tag{6.6}$$

We will not give explicit $\mathcal{W}$ realizations of the $sp(4, \mathbb{R})$ algebra in order not to overload the text, but rather concentrate our attention on the possible determination of an operator which carries the intermediate statistics in this framework.

By the following change of variables

$$u = r^2 \qquad v = r^2 \sin\theta\ \cos 2\phi \qquad w = r^2 \sin\theta\ \sin 2\phi \tag{6.7}$$

with $0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le M,$ the generator L in (6.5) becomes:

$$L = -i\frac{\partial}{\partial\phi} \tag{6.8}$$

We note that when working with univalued functions, this operator is not well defined on the set $L_2([0, \pi])$. More precisely, for L to be self-adjoint, we have to:

- either restrict it on functions satisfying

$$\psi(0) = \lambda\psi(\pi) \tag{6.9}$$

where λ plays the rôle of an anyonic parameter ($\lambda = 1$ characterizes the bosons, while $\lambda = -1$ characterizes fermions)

- or modify the explicit form of L

$$L = -i\frac{\partial}{\partial\phi} + \alpha \tag{6.10}$$

and apply it on functions such that

$$\psi(0) = \psi(\pi) \tag{6.11}$$

the anyonic parameters being here α ($\alpha = 0$ being the bosons, and $\alpha = 1$ being the fermions).

Choosing the second alternative, which is compatible with the bosonic case provided by the x_i, p_j representation, leads to adding a $\mathcal{W}$ contribution to the L defined in (6.8). Actually, it appears possible to propose different $\mathcal{W}$ realizations of the $sp(4,\mathbb{R})$ algebra, i.e. different expressions including $\mathcal{W}$ contributions to the $sp(4,\mathbb{R})$ generators; of course these different realizations will be equivalent once a finite dimensional $\mathcal{W}$ representation has been chosen. Among the different possibilities (see ref. [3]), the simplest one is the following:

$$L' = -i\frac{\partial}{\partial\phi} - S \tag{6.12}$$

For the one-dimensional representations of the $\mathcal{W}$ algebra, S becomes a number that can be non-zero. For higher dimensional $\mathcal{W}$ representations, S is a diagonal matrix, and this framework could lead to a generalization of anyons directly related to the enlarged $\mathcal{W}$ algebra (when compared with paragraph 6.1). A more complete discussion of the validity of this interpretation can be found in ref. [3].

As a conclusion, owing to a $\mathcal{W}$ algebra treatment, the one-parameter self-adjoint extension family of the angular momentum does allow the anyonic statistics to be incorporated.

7 Conclusion and perspectives

The characterization of the class of finite $\mathcal{W}$ algebras that we proposed, based on completely algebraic grounds, is conceptually simple. Such an algebra is defined in terms of the commutant, in a particular localization of the enveloping algebra $\mathcal{U}(\mathcal{G})$, of a subalgebra $\tilde{\mathcal{G}}$ of a simple Lie algebra $\mathcal{G}$. This approach is specially adapted to obtain new realizations of the Lie algebra $\mathcal{G}$ from a particular (differential) one. In other words, a class of $\mathcal{G}$ representations can be explicitly built with the help of a $\mathcal{G}$ differential realization and the knowledge of a particular W algebra. A direct comparison of this construction with the technique of induced representations has been given in section 5 for the cases of the four-dimensional conformal and Poincaré algebras. It was also shown in section 6 that this framework fits with the Heisenberg quantization for a system of two identical particles in two dimensions, the $\mathcal{W}$ algebra under consideration being interpreted as carrying the anyonic information.

Therefore, it appears reasonable to put some more effort in the physical applications, as well as in the mathematical developments of these objects. The first question that could be raised concerns the Heisenberg quantization for a system of more than $N = 2$ identical particles. For $N \geq 3$, the structure of the algebra of observables becomes much more complicated (see for example ref.[11]). The anyonic problem deserves more work, in particular if we keep in mind the relevance of the $\mathcal{W}_{1+\infty}$ algebra in the algebraic treatment of the quantum Hall effect (ref. [12]). As a second question, one could wonder about the occurrence of a finite $\mathcal{W}$ algebra as the symmetry algebra for a particular Hamiltonian; several examples corresponding to different types of potential have already been detected (see for example ref. [1]). It might be of some interest to bypass the mere observation and try to understand how the non-linearity of the symmetry algebra arises.

On the mathematical side, it looks promising to connect our approach with the general study of primitive ideals considered in ref. [13], in which the commutants of nilpotent algebras are directly involved. But the most natural field of investigation is of course the generalization of our constructions to the affine —or not finite— $\mathcal{W}$ algebra case.

Actually the realization of $\mathcal{W}$ algebras with the generators expressed in terms of all the generators of an affine Kac-Moody algebra has been performed in ref. [4]. As in the finite case, the (primary) $\mathcal{W}$ fields are written as quotients of polynomials. However, we have to stress that the denominators, in these quantities, simply commute with all the numerators, allowing in particular a computation of the operator product expansions (OPE) without special difficulties in the quantum framework. Such a construction might be seen as a sort of generalized Sugawara one, but without the restrictions to special values of the Kac Moody central extension, and without need of coset technique, as developed in ref. [14].

Finally, affine $\mathcal{W}$ algebras could also be used to obtain realizations of affine Kac-Moody ones[15][16]. Let us close this section by presenting such an approach in the $\widehat{sl}(2)_k$ case.

For such a purpose, we need first to define the $\widehat{sl}(2)_k$ currents $J_-(z)$, $J_0(z)$ and $J_+(z)$ which satisfy the OPE's

$$\begin{aligned} J_0(z)J_\pm(w) &\sim \pm\frac{1}{z-w}J_\pm(w) \qquad J_0(z)J_0(w) \sim \frac{k/2}{(z-w)^2} \\ J_+(z)J_-(w) &\sim \frac{k}{(z-w)^2} + 2\frac{J_0(w)}{z-w} \end{aligned} \tag{7.1}$$

where the $\sim$ symbol indicates that we restrict the OPE to its singular part.

Then, we introduce the $T(z)$ operator, which commutes with $J_0(z)$ as well as $J_-(z)$:

$$T = T_{sug} - \partial J_0 + :\frac{\partial J_-}{J_-}J_0: + k\left[\frac{3}{4}\left(\frac{\partial J_-}{J_-}\right)^2 - \frac{1}{2}\frac{\partial^2 J_-}{J_-}\right] \tag{7.2}$$

with $T_{sug} = \frac{1}{k+2} : J_0J_0 + \frac{1}{2}(J_+J_- + J_-J_+) :$, and $:\ :$ being the normal ordered product.

$T(z)$ generates the Virasoro algebra with central charge

$$c = 1 - 6\frac{(k+1)^2}{k+2} \tag{7.3}$$

In the following, we will use: $W(z) = (k+2)T(z)$, which for $k = -2$ reduces to: $J_0J_0 + \frac{1}{2}(J_+J_- + J_-J_+) : (z)$, and commutes also with $J_+(z)$.

Now, as in the Wakimoto construction [17], we start with a (β,γ) system satisfying:

$$\beta(z)\gamma(w) \sim \frac{1}{z-w} \tag{7.4}$$

which can be considered as the "affine" analogue of the (x,∂_x) pair. The realization of $\widehat{sl}(2)_k$ we obtain reads:

$$J_- = \gamma \qquad J_0 = -:\gamma\beta: -\frac{k+2}{4} : \frac{\partial\gamma}{\gamma} : \tag{7.5}$$

$$\begin{aligned} J_+ = -:\gamma\beta^2: -k\partial\beta \quad &+ \quad :\frac{1}{\gamma}W: +(k+2)\left[-\frac{1}{2} :: \frac{\partial\gamma}{\gamma} : \beta : \right. \\ &+ \left. \frac{1}{4}\left((k+1)\frac{\partial^2\gamma}{\gamma^2} - \frac{1}{4}(5k+6)\frac{(\partial\gamma)^2}{\gamma^3}\right)\right] \end{aligned} \tag{7.6}$$

We could say that in our construction, the Wakimoto $\phi(z)$ free field has been replaced by the Virasoro $W(z)$ operator. Fractional calculus technique [18] are well adapted to this framework, which, we hope, might have its interest in the computation of correlation functions.

By the Sugawara construction, we knew how to obtain, from an affine Lie algebra, a Virasoro realization. The alternative to the Wakimoto construction given just above allows us, starting from a Virasoro representation, to obtain new $\widehat{sl}(2)_k$ ones: shall we conclude that we have looped the loop?

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