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F^4 TERMS IN $N = 4$ STRING VACUA †

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ABSTRACT

We discuss $F^4_{\mu\nu}$ terms in torroidal compactifications of type-I and heterotic $SO(32)$ string theory. We give a simple argument why only short BPS multiplets contribute to these terms at one loop, and verify heterotic-type-I duality to this order. Assuming exact duality, we exhibit in the heterotic calculation non-zero terms that are two-loop, three-loop and non-perturbative on the type-I side.

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Introduction. BPS states play a special role in theories with extended ($N \geq 2$) supersymmetry. The fact that they form multiplets which are shorter than the generic representation of the supersymmetry algebra implies relations between their mass, charges and values of moduli which are valid in the exact quantum theory. For $N \geq 4$ these relations are furthermore purely classical, and they ensure that BPS states are either stable or, at worse, marginally-unstable. Stable BPS states can thus be traced all the way to strong coupling, and their existence with appropriate multiplicities has constituted the main test of the various duality conjectures.

Another remarkable feature of BPS states is that they saturate certain one-loop terms in the effective low-energy action. This fact has been articulated clearly by Harvey and Moore [1] in the context of heterotic $N = 2$ thresholds, though it was implicit in much of the earlier work, such as for instance refs. [2]. BPS-saturated terms are furthermore typically related, by supersymmetry, to anomalies, and are thus expected to obey non-renormalization theorems. This makes them a precious tool for checking duality conjectures. Tseytlin [3] has in particular used such $F^4_{\mu\nu}$ terms, in order to test the conjectured duality between the type-I and heterotic-string theories in ten dimensions [4, 5]. In this paper we will extend Tseytlin's analysis to torroidal compactifications.

The effective gauge-field action of open-string theory is closely related to the phase-shift or velocity-dependent forces between D-branes [6, 7, 8, 9, 10]. BPS saturation and a non-renormalization assumption of the leading $o(v^4)$ interaction are, furthermore, a crucial ingredient in the recent interesting conjecture by Banks et al [11] concerning M-theory in the infinite-momentum frame. Despite their close relation the two calculations differ however in some significant ways. For instance in the effective-action calculation one subtracts diagrams with massless closed strings in the intermediate channels. These diagrams must be kept in the D-brane calculation, where they are regulated effectively by the world-volume dimensional reduction. Our analysis does not therefore translate into the D-brane context immediately, but it raises by analogy some interesting questions.

Supertrace formulae. BPS saturation at one loop follows from supertrace formulae [12] involving powers of helicity and R-symmetry charges. These are easier to discuss in terms of generating functionals. Define for instance

$$Z_{rep}(y) = str\ y^{2\lambda} \tag{1}$$

where the supertrace stands for a sum over bosonic minus fermionic states of the representation, and λ is the eigenvalue of a generator of the little group: $SO(3)$ or $SO(2)$ in the massive, respectively massless case in four dimensions. For a particle of spin j we have

$$Z_{[j]} = \begin{cases} (-)^{2j}\left(\frac{y^{2j+1}-y^{-2j-1}}{y-1/y}\right) & \text{massive} \\ \\ (-)^{2j}(y^{2j}+y^{-2j}) & \text{massless} \end{cases} \tag{2}$$

When tensoring representations the generating functionals get multiplied,

$$Z_{r\otimes\tilde{r}} = Z_r Z_{\tilde{r}} \ . \tag{3}$$

The supertrace of the nth power of helicity can be extracted from the generating functional through

$$str\ \lambda^n = (y^2\frac{d}{dy^2})^n\ Z(y)|_{y=1} \ . \tag{4}$$

Consider now $N=2$ multiplets. The supersymmetry algebra contains four fermionic charges that may act independently: two of them raise the helicity by one half unit, while the other two lower it by the same amount. For the generic massive (long) multiplet all charges act non-trivially on some "ground state" of spin j and one finds

$$Z^{N=2}_{long} = Z_{[j]}\ (1-y)^2(1-1/y)^2 \ . \tag{5a}$$

For a massless or a short massive multiplet half of the supercharges have a trivial action so that one finds instead

$$Z^{N=2}_{short} = (2)\times Z_{[j]}\ (1-y)(1-1/y) \ , \tag{5b}$$

where the factor 2 is required in the massive case, since short massive multiplets carry charge and are thus necessarily complex. Familiar examples of short massive multiplets include the monopoles ($j=0$) and charged gauge bosons ($j=\frac{1}{2}$) of pure N=2 Yang-Mills theories. An immediate consequence of eqs. (5) is that *only for short (BPS) multiplets is str* $\lambda^2 \neq 0$.

Let us turn next to the $N=4$ algebra. This has four raising and four lowering fermionic charges, all of which can act independently in a generic massive (long) representation,

$$Z^{N=4}_{long} = Z_{[j]}\ (1-y)^4(1-1/y)^4 \ . \tag{6a}$$

Short representations, which include all the massless as well as some massive multiplets, annihilate half the supercharges so that

$$Z^{N=4}_{short} = (2)\times Z_{[j]}\ (1-y)^2(1-1/y)^2 \ . \tag{6b}$$

This is precisely the content of a long $N = 2$ representation. $N = 4$ has also intermediate (or semi-long) multiplets, which annihilate one-quarter of supercharges, and for which

$$Z^{N=4}_{semi-long} = 2 \times Z_{[j]}\ (1-y)^3(1-1/y)^3 \quad . \tag{6c}$$

The factors of two take again into account that massive short and intermediate multiplets have charge and are thus necessarily complex. It follows trivially from the above expressions that $str\lambda^2 = 0$ always, $str\lambda^4 \neq 0$ only for short multiplets, and $str\lambda^6 \neq 0$ in both the short and the intermediate case.

The discussion can be extended easily to take into account R-symmetry charges. These are simply helicities in some (implicit) internal dimensions: there is a single R-charge for $N = 2$, and three independent charges, corresponding to the Cartan generators of $SO(6)$, in the $N = 4$ case. To get a non-zero result for short, intermediate or long multiplets in the latter case, one must insert in the supertrace at least four, six, respectively eight powers of helicity and/or R-charges.

Type-I effective action. Let us turn now to the one-loop calculation of the effective gauge-field action in type-I theory. In the background-field method the one-loop free energy in d non-compact dimensions reads [13]

$$\begin{aligned} \mathcal{F}^{(1)}_I(B) = -\frac{V^{(d)}}{8\pi}\int_0^\infty \frac{dt}{t}(2\pi^2 t)^{1-\frac{d}{2}}\times \\ \times Str\ \frac{q\mathcal{B}}{\sinh(\pi t\epsilon/2)}\ e^{-\frac{\pi t}{2}(M^2+2\lambda\epsilon)} \end{aligned} \tag{7}$$

where $B = \mathcal{B}Q$ is a background magnetic field pointing in some direction Q in group space, $q = q_L + q_R$ is the corresponding charge distributed between the two string endpoints, and ϵ is a non-linear function of the charges and the field that vanishes linearly with the latter

$$\epsilon(\mathcal{B}, q_L, q_R) \simeq q\mathcal{B} + o(\mathcal{B}^3)\ . \tag{8}$$

In the weak-field limit and for low spins this is a familiar field-theory expression: it follows directly from the fact that elementary charged particles have gyromagnetic ratio 2 and a spectrum given by equally-spaced Landau levels. The effects of non-minimal coupling for an open string are captured essentially by the replacement $q\mathcal{B} \to \epsilon$.

The supertrace in eq. (7) runs over all charged string states. For any given supermultiplet the mass and charges are however common, so that its contribution is proportional to

$$str\ e^{-\pi t\epsilon\lambda} = \sum_{n=0}^{\infty}\frac{(\pi t\epsilon)^{2n}}{(2n)!}\ str\lambda^{2n}\ , \tag{9}$$

where we have used the fact that odd powers of helicity trace out automatically to zero. Since the ϵ-expansion is an expansion in weak-field, the various non-renormalization statements at one loop follow directly from the properties of helicity supertraces and eqs. (7,9). Thus in $N = 2$ theories the first non-zero term, proportional to $str\lambda^2$, is the one-loop gauge kinetic function: it only receives contributions from short (BPS) multiplets, as has been noted previously by using identities of θ-functions [15, 14]. In $N = 4$ theories the gauge coupling constant is not corrected at one loop. The first non-zero term, proportional to $str\lambda^4$ is quartic in the background field, and only receives contributions from short $N = 4$ multiplets. This was noted again through θ-function identities in the D-brane context in refs. [10, 8]. The following term of order $o(F^6)$ is also determined, incidentally, by short BPS states. This is because long multiplets do not contribute to $str\lambda^6$, and there are no intermediate multiplets in the perturbative type-I spectrum.

Let us take now a closer look at the quartic term arising in N=4 (torroidal) compactifications. The only perturbative charged BPS states are the multiplets of the $SO(32)$ gauge bosons, together with all their Kaluza-Klein descendants. For these states the mass is equal to the internal momentum, so that after some straightforward algebra one finds

$$\begin{aligned} \mathcal{F}_I^{(1)}/V^{(d)} = -\frac{\mathcal{B}^4}{2^9\pi^4}\int_0^\infty \frac{dt}{t}(2\pi^2 t)^{4-\frac{d}{2}}\times \\ \times \sum_{\substack{\text{Chan}\\ \text{Patton}}} (q_L+q_R)^4 \sum_{p\in a_L+a_R+{}^*\Gamma} e^{-\pi t p^2/2} + o(\mathcal{B}^6) \end{aligned} \tag{10}$$

where ${}^*\Gamma$ stands for the $(10-d)$-dimensional lattice of Kaluza-Klein momenta, which must be shifted from the origin in the presence of non-vanishing Wilson lines. Each end-point charge takes 32 values, but the sum runs only over antisymmetric states. For ease of notation we will from now on suppress the $o(\mathcal{B}^6)$ terms when writing effective actions.

The above expression is strictly-speaking formal, since it diverges at the $t \to 0$ limit of integration. This is an open-string ultraviolet divergence, but can be also interpreted as coming from an on-shell dilaton or graviton that propagates between two non-vanishing tadpoles. We are interested in the effective (Wilsonian) action, so this divergence due to exchange of massless particles must be subtracted away. The right procedure is to change variables to the closed-string proper time l, which is related to t differently for the annulus and Möbius-strip contributions,

$$l = \begin{cases} 1/\mathrm{t} & \text{annulus} \\ 1/4\mathrm{t} & \text{Möbius strip} \end{cases} \tag{11}$$

Separating the two topologies amounts to writing the sum over Chan-Patton states as an unconstrained sum over all left- and all right- endpoints, minus the diagonal. After performing also a Poisson resummation the result reads

$$\mathcal{F}_I^{(1)} = -\frac{\mathcal{B}^4 V^{(10)}}{2^{10}\pi^6}\int_0^\infty dl\ \frac{1}{2}\sum_{w\in\Gamma}\times\Big\{\sum_{|L>,|R>}(q_L+q_R)^4 e^{-w^2 l/2\pi + iw\cdot(a_L+a_R)} - \\ -4\times\sum_{|L>=|R>}(2q_L)^4 e^{-2w^2 l/\pi + 2iw\cdot a_L}\Big\} \tag{12}$$

where Γ is now the compactification lattice. Our conventions are such that $w = 2\pi Rm$ for a circle of radius R.

The divergence in the above expression comes from the $w = 0$ piece, as all other terms are exponentially-small in the $l \to \infty$ region. Thanks to the factor 4 that multiplies the Möbius-strip contribution, this divergence is proportional to $(trB^2)^2$. It corresponds *precisely* to the tadpole $\to$ massless-propagator$\to$ tadpole diagram, that must be removed in the effective action [14]. Switching-off the Wilson-lines for simplicity, and changing integration variable once again for the Möbius contribution, we thus obtain our final expression

$$\int \mathcal{L}_I^{(1)} = -\frac{V^{(10)}}{2^{10}\pi^6}\ \{24\ trB^4 + 3(trB^2)^2\}\times\int_0^\infty dl \sum_{w\in\Gamma-\{0\}} e^{-w^2 l/2\pi} \tag{13}$$

Since in the decompactification limit all $w \neq 0$ terms disappear, we have just shown in particular *that the 10d effective type-I Lagrangian has no one-loop F^4 corrections.* This is in agreement with 10d heterotic-type-I duality [3] as we will discuss in detail in the following section. The fact that only open BPS states contribute to the amplitude is in this respect crucial: it ensures that the string-scale does not enter in the expression for $\mathcal{F}_I^{(1)}$, which must thus cancel entirely when passing to the effective action in ten dimensions. More generally, after compactification, the fact that $\mathcal{F}_I^{(1)}$ does not depend on α' implies that all corrections to the effective Lagrangian at one loop come from integrating out the Kaluza-Klein modes of massless 10d string states.

Heterotic-type-I duality. The predictions of this string-string duality [4, 5] for the effective action in 10d, have been worked out and checked against earlier calculations by Tseytlin [3]. In summary, there exist two superinvariants quartic in the gauge-field strength [16], which are only distinguished by the group-index contractions:

$$I_1 = t_8\ trF^2 trF^2 - \frac{1}{4}\epsilon_{10}\ C\ trF^2 trF^2\ , \tag{14b}$$

and

$$I_2 = t_8 \; trF^4 - \frac{1}{4}\epsilon_{10} \; CtrF^4 \; . \tag{14a}$$

Here C is the antisymmetric 2-form, ϵ_{10} the Levi-Civita tensor, and t_8 the covariant extension of the well-known light-cone-gauge zero-mode tensor. The parity-even part of I_1 appears however also independently, in the supersymmetric completion of the Chern-Simmons-modified two-derivative action. Since all these superinvariants have anomaly-cancelling pieces one may expect that they appear at one given order in the loop expansion. In heterotic $SO(32)$ theory in ten dimensions the two-derivative action comes from the sphere, I_1 does not appear at all, while I_2 appears at one loop only. Duality maps the σ-model metrics and string couplings as follows [4]:

$$\lambda^h = 1/\lambda^I \;, \quad G^h_{\mu\nu} = G^I_{\mu\nu}/\lambda^I \tag{15}$$

Simple power counting then shows that the parity-even parts of I_2 and I_1 should arise in type-I theory from surfaces of Euler number, respectively, minus one (disk, projective plane) and one (disk with two holes etc). This is compatible with the absence of all quartic terms at Euler number zero, as we have concluded.

Consider now torroidal compactifications. The gauge-field-dependent one-loop free energy in heterotic $SO(32)$ theory reads [2]

$$\mathcal{F}^{(1)}_h = -\frac{V^{(d)}(\lambda^I)^{4-d/2}}{2^{10}\pi^6} \times \int_{Fun} \frac{d^2\tau}{\tau_2^2} \; \frac{\Gamma^{10-d,10-d}}{(2\pi^2\tau_2)^{d/2-5}} A(F,\bar\tau) \tag{16}$$

Here $\Gamma^{10-d,10-d}$ stands for the usual sum over the Lorentzian Narain lattice, which factorizes in the integrand because we assumed zero Wilson lines, and

$$\begin{aligned} A(F,\bar\tau) = t_8 \; trF^4 + \frac{1}{2^9\cdot 3^2}\Big[\frac{E_4^3}{\eta^{24}} + \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} \\ - 2\frac{\hat{E}_2 E_4 E_6}{\eta^{24}} - 2^7\cdot 3^2\Big] \; t_8(trF^2)^2 \end{aligned} \tag{17}$$

with E_{2n} the nth Eisenstein series:

$$E_2 = \frac{12}{i\pi}\partial_\tau \log\eta = 1 - 24\sum_{n=1}^{\infty}\frac{n\,q^n}{1-q^n} \tag{18a}$$

$$E_4 = \frac{1}{2}\left(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8\right) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3 q^n}{1-q^n} \tag{18b}$$

$$E_6 = \frac{1}{2}(\vartheta_2^4+\vartheta_3^4)(\vartheta_3^4+\vartheta_4^4)(\vartheta_4^4-\vartheta_2^4) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5 q^n}{1-q^n} \tag{18c}$$

The E_{2n}'s are modular forms of weight $4n$ except for E_2 which must be modified to

$$\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2} \tag{18d}$$

The powers of λ^I in front of expression (16) come from the fact that we used type-I normalizations for the metric: $d/2$ of these powers are due to the space-time volume, and the other four to the tensor t_8. As for the fact that all holomorphic dependence in the integral appears through the sum over the Lorentzian lattice, this is a result of BPS saturation [1]. It can be derived by an argument similar to the one for the open string, except that the background field now only couples to the helicity coming from the left (supersymmetric) sector.

The Lorentzian lattice involves a sum over both momenta and windings on the $(10-d)$-dimensional torus. Setting to zero the antisymmetric tensor background, which is a Ramond-Ramond field in type-I theory, and using again type-I normalizations for the compactification torus, we have

$$\Gamma^{10-d,10-d} = \sum_{\substack{p\in {}^*\Gamma \\ w\in\Gamma}} e^{-\pi\tau_2 p^2\lambda^I/2 - \tau_2 w^2/2\pi\lambda^I + i\tau_1 p\cdot w} \ . \tag{19}$$

Now since inside the fundamental domain, τ_2 is bounded away from the origin, all terms with non-zero winding are non-perturbatively small at weak λ^I. This is consistent with the fact that winding heterotic strings are solitonic D-branes on the type-I side [5]. The remaining momentum lattice can be Poisson-resummed back and written as follows:

$$\frac{\Gamma^{10-d,10-d}}{(2\pi^2\tau_2)^{d/2-5}} = V_\Gamma(\lambda^I)^{d/2-5} \times \sum_{\tilde{w}\in\Gamma} e^{-\tilde{w}^2/2\pi\lambda^I\tau_2} + \ o(e^{-1/\lambda^I}) \ . \tag{20}$$

We will now plug the above expression into eq. (16), and perform the modular integration. The $\tilde{w}=0$ term can be integrated explicitly, using the formulae

$$\begin{aligned} I(0,0,0) = \pi/3 \quad &, \quad I(1,1,1) = -48\pi \\ I(0,3,0) = 240\pi \quad &, \quad I(2,2,0) = 48\pi \end{aligned} \tag{21a}$$

where we defined

$$I(m,n,k) = \int_{Fun} \frac{d^2\tau}{\tau_2^2} \, \frac{\hat{E}_2^m E_4^n E_6^k}{\eta^{24p}} \tag{21b}$$

subject to the modular-invariance condition $6p = m + 2n + 3k$. In what concerns the $\tilde{w} \neq 0$ terms, we may extend their integration regime to the entire strip $-\frac{1}{2} < \tau_1 < \frac{1}{2}$, modulo a non-perturbatively small error. This makes the τ_1 integration straightforward, since only terms without exponential $\bar\tau$-dependence in $A(F,\bar\tau)$ survive:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 A(F,\bar\tau) = trF^4 + \frac{1}{8}(trF^2)^2 \times \left[1 - \frac{15}{2\pi\tau_2} + \frac{63}{8\pi^2\tau_2^2}\right] \tag{22}$$

Putting all this together, redefining $\tau_2 \equiv 1/l\lambda^I$, and doing some tedious algebra leads to our final expression for the heterotic one-loop free energy at weak type-I coupling::

$$\mathcal{F}_h^{(1)} = -\frac{V^{(10)}}{2^{10}\pi^6}\Bigg\{ t_8\, trF^4\, \left(\frac{\pi}{3\lambda^I} + \int_0^\infty dl\ \mathcal{K}\right) + \\ + \frac{1}{8} t_8 \left(trF^2\right)^2 \int_0^\infty dl\ \mathcal{K}\ \left(1 - \frac{15l}{2\pi}\lambda^I + \frac{63l^2}{8\pi^2}(\lambda^I)^2\right) + o(e^{-1/\lambda^I})\Bigg\} \tag{23a}$$

where here

$$\mathcal{K} = \sum_{\tilde w \in \Gamma - \{0\}} e^{-\tilde w^2 l/2\pi}\ . \tag{23b}$$

The leading, $o(1/\lambda^I)$ term in this expression corresponds to the type-I disk-diagram [3]. The constant piece should be compared to the sum of the Möbius-strip and annulus, given by eq. (13). These are indeed identical, if one notes that for a simple magnetic field $t_8F^4 = 24B^4$. The remaining terms, as well as the moduli-independent contribution of the heterotic sphere-diagram [3]

$$\mathcal{F}_h^{(0)} = \frac{V^{(10)}}{2^{10}}\ \lambda^I\ t_8\ (trF^2)^2 \tag{24}$$

correspond to two- and three-loop diagrams on the type-I side. If we assume exact duality and no further corrections on the heterotic side, we conclude that beyond three loops there are only instanton corrections on the type-I side.

Afterword. The effective expansion parameter in eq. (23) is α'_h/R^2, where $\alpha'_h = \lambda^I\alpha'_I$ is the heterotic Regge slope, and R is a typical radius of the compactification torus. Stretched heterotic strings are (non-perturbative) charged BPS states on the type-I side, so it is not surprising that they should control at least part of the F^4 terms in $\mathcal{L}_{eff}$. The role of analogous degrees of freedom, as well as of the two-loop renormalization, eq. (24), in the D-brane context must be elucidated further. The study of fundamental-string scattering [17] may shed some different light on these issues.

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