EXERCISES IN EQUIVARIANT COHOMOLOGY AND TOPOLOGICAL THEORIES

R. STORA
Laboratoire de Physique Théorique ENSLAPP [a], B.P. 110,
F-74941 Annecy-le-Vieux Cedex, France
and
Theory Division, CERN, CH-1211, Geneva 23, Switzerland.

Equivariant cohomology is suggested as an alternative algebraic framework for the definition of topological field theories constructed by E. Witten circa 1988. It also enlightens the classical Faddeev Popov gauge fixing procedure.

1 Introduction

Before going into the subject of this talk, I would like to describe some concrete exercises done by Claude and I which represent a very small portion of the numerous discussions we had, mostly by exchange of letters. We happened to be both guests of the CERN theory division during the academic year 1972-1973.

The perturbative renormalization of gauge theories was still a hot subject, and, whereas most of our colleagues considered the problem as solved we were both still very innocent. I happened to be scheduled for a set of lectures for the "Troisième cycle de la Suisse Romande" in the spring 1973, on the subject "Models with renormalizable Lagrangians: Perturbative approach to symmetry breaking", and I decided to conclude those lectures with a summary of the known constructions related to gauge theories, mostly at the classical level, except for a heuristic derivation of the now called [1] Slavnov Taylor identities, taking seriously the Faddeev Popov ghost and antighost as local fields. What had to be done was indicated in A. Slavnov's preprint which I had remarked: perform a gauge transformation of parameter $m^{-1}\bar{\xi}$ where m is the Faddeev Popov operator and $\bar{\xi}$ the source of the antighost field. That strange trick was due to E.S. Fradkin and I.V. Tyutin as indicated in Slavnov's preprint. At the time, I was not aware of J.C. Taylor's paper which came to my attention much later. Anyway, Claude and I carried out that calculation whose result is reported in the notes, with details in an appendix for which the authors (A. Rouet and I) thank Claude Itzykson for generous help [2]. It is that form of the identity which, a few months later drew Carlo Becchi and Alain Rouet's atten-

[a]URA 1436 du CNRS, associée à l'Ecole Normale Supérieure de Lyon et à l'Université de Savoie.

tion, leading them to the remark that the gauge fixed Faddeed Popov action possesses a symmetry naturally called the Slavnov symmetry. A year later, when the paper by E.S. Fradkin and G.A. Vilkovisky on the quantization of canonical systems with constraints came out, Claude and I had a conversation on the telephone and we found we had both noticed that paper. I suggested that the action they proposed possessed a Slavnov symmetry. A couple of days later, Claude called me back and gave me the formula -at least in the case of gauge constraints- which I immediately forgot. When I met E.S. Fradkin in Moscow in the fall 1976, I told him about Claude's finding, and there followed the first article by I.A. Batalin and G.A. Vilkovisky who unfortunately thank me for suggesting the problem, and do not mention Claude at all.

These are only two examples of the innumerable discussions we had on physics and other things as well, mostly in writing, because life did not make our trajectories intersect so often. The last long series of discussions I had with him took place in Turku, Finland, at the meeting of the spring 1991. Almost every evening, we were ambulating around the big lawn in front of the dining room, trying to reconstruct, at his request, the arguments which produce the existence of 27 straight lines on an unruled third degree surface. That was a prelude to his later work on enumerative geometry.

Generous, he was; intelligent he was; cultivated he was; we remain deprived of patiently gathered wisdom, a rather rare item.

Returning to technicalities I will now try to describe a few facts about the Lagrangian formulation of topological -more precisely cohomological- field theories, constructed by E. Witten from 1988 on, in as much as they are relevant to our poor understanding of gauge theories. That is to say I will insist on the field theory aspects in particular, the distinction between fields and observables, even though a host of beautiful results and conjectures have been obtained otherwise.

Equivariant cohomology is roughly forty five years old, and yet, does not belong to most theoretical physicists' current mathematical equipment. The easy parts, namely, definitions, terminology, elementary properties are described in the appendix whose content is freely used throughout the text.

Section 2 is devoted to a reminder on dynamical gauge theories and a formal description of the Faddeev Popov gauge fixing procedure in terms of notions belonging to the theory of foliations [3].

Section 3 describes some aspects of "cohomological" topological theories with emphasis on some of the features which distinguish them from dynamical theories at the algebraic level provided by the Lagrangian descriptions.

2 Formal aspects of dynamical gauge theories

Here are a few considerations on formal aspects of the Faddeev Popov gauge fixing procedure which allowed to handle, thanks to the very strong consequences of locality, the ultraviolet difficulties found in the perturbative treatment of theories of the Yang Mills type. This can be found in most textbooks and usually proceeds via factoring out of the relevant functional integral the infinite volume of the gauge group produced by the gauge invariance of the functional measure. There is a more satisfactory strategy sketched in J. Zinn Justin's book [4] which avoids this unpleasant step, and fits more closely mathematical constructions now classical in the theory of foliations [3].

The set up is as follows:

M_4 is a smooth space time manifold, which one may choose compact without boundary, in euclidean field theory. $P(M,G)$ is a principal G bundle over M_4, $\bigcup_i (U_i \times G)$ modulo glueing maps above $U_i \cap U_j$, where $\{U_i\}$ is an open covering of M). G is a compact Lie group referred to as the structure group. $\mathcal{A}$ is the set of principal connections a on $P(M,G)$ (Yang Mills fields). On M_4

$$a_M = \sum_\alpha a^\alpha_\mu(x) dx^\mu e_\alpha \qquad e_\alpha : \text{basis of Lie } G \tag{1}$$

On $P(M,G)$, locally,

$$a = g^{-1} a_M \; g + g^{-1} dg \qquad (x,g) \text{ local coordinates in } U \times G \tag{2}$$

$$F(a) = da + \frac{1}{2}[a,a] \tag{3}$$

is the curvature of a (the field strength).

$\mathcal{A}$ is acted upon by $\mathcal{G}$, the gauge group, i.e. the group of vertical automorphisms of $P(M,G)$ ("gauge transformations"). Upon suitable restrictions, $\mathcal{A}$ is a principal $\mathcal{G}$ bundle over $\mathcal{A}/\mathcal{G}$, the set of gauge orbits.

Dynamical gauge theories are models in which the fields are the a's (and, possibly matter fields), and the observables are gauge invariant functions of the a's (or functions on $\mathcal{A}/\mathcal{G}$).

For historical as well as technical reasons related to locality, one chooses models specified by a local gauge invariant action

$$S_{YM}(a) = \frac{1}{4g^2} \int_{M_4} trF \wedge *F. \tag{4}$$

Heuristically, one considers the $\mathcal{G}$ invariant measure on $\mathcal{A}$

$$\Omega_{YM} = e^{-S_{YM}(a)} \underbrace{\wedge \, \delta a}_{\mathcal{D}a} \tag{5}$$

If $\{X_\alpha\}$ denotes a basis of fundamental vertical vector fields representing the action of Lie $\mathcal{G}$ on $\mathcal{A}$, one constructs the Ruelle Sullivan [5] current

$$\Omega_{RS} = i(\underset{\alpha}{\Lambda} X_\alpha)\Omega_{YM} \tag{6}$$

which is closed and horizontal, therefore basic: (cf. Appendix A)

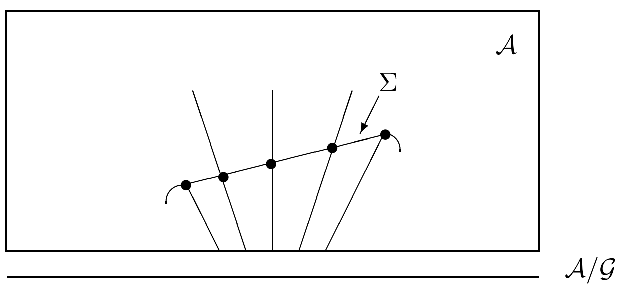
$$\begin{aligned} \delta\Omega_{RS} &= 0 \\ i(X_\alpha)\Omega_{RS} &= 0 \end{aligned} \tag{7}$$

hence

$$\ell(X_\alpha)\Omega_{RS} = 0 \tag{8}$$

It follows in particular that Ω_{RS} is invariant under field dependent gauge transformations.

Given a gauge invariant observable $\mathcal{O}(a)$, the question is to integrate it against Ω_{RS}, or rather to integrate its image as a function on $\mathcal{A}/\mathcal{G}$ against the image of Ω_{RS} as a top form on $\mathcal{A}/\mathcal{G}$.



Choose a local section Σ (transverse to the fibers) with local equations

$$g(a) = 0 \tag{9}$$

and corresponding local coordinates $\dot{a}$ so that a local parametrization of $\mathcal{A}$ is given by

$$a = \dot{a}^g \tag{10}$$

i.e. all a's are, locally gauge transforms of points on the chosen transversal manifold.

One can represent the transverse measure associated with the chosen section as follows:

$$< \mathcal{O} >_\Omega = \int_\Sigma \mathcal{O}(\dot{a})\Omega_{RS|\Sigma} \quad = \quad \int_\Sigma \mathcal{O}(\dot{a})\Omega_{RS|\Sigma} \underbrace{\int_{fiber} \delta(g) \wedge \delta g}_{=1 \qquad !}$$

$$= \int \mathcal{O}(a)\Omega_{RS}\ \delta(g) detm(\wedge g^{-1}\delta g) \tag{11}$$

where the volume $\wedge g^{-1}\delta g$ is chosen so that

$$i(\wedge_{\dot\alpha} X_{\dot\alpha})(\wedge g^{-1}\delta g) = 1 \tag{12}$$

and m is given by

$$m = \frac{\delta g}{\delta a} D_a \tag{13}$$

Thus

$$\Omega_{RS}(\wedge g^{-1}\delta g) = \Omega_{YM} \tag{14}$$

and the result follows:

$$< \mathcal{O} >_{\Omega} = \int \mathcal{O}(a)\ \delta(g)\ \det m\ \Omega_{YM} \tag{15}$$

This, of course only holds if $\mathcal{O}(a)$ has its support inside the chosen chart. By construction, the result is independent of the choice of a local section, two local sections differing by a field dependent gauge transformation.

The final outcome is to replace Ω_{YM} by

$$\Omega_{YM\Phi\Pi} = \Omega_{YM}\ \Omega_{\Phi\Pi} \tag{16}$$

where

$$\Omega_{\Phi\Pi} = \int \mathcal{D}\bar\omega \mathcal{D}\omega \mathcal{D}b\ e^{i<b,g(a)>+<\bar\omega,m\omega>} \tag{17}$$

where we have used the Stueckelberg Nakanishi Lautrup Lagrange multiplier b, the Faddeev Popov fermionic ghost ω, the Faddeev Popov fermionic Lagrange multiplier (antighost) $\bar\omega$. The modern reading of the exercise done with Claude is that not only $\Omega_{YM\Phi\Pi}$ is invariant under the operation s

$$\begin{aligned} sa &= -\mathcal{D}_a\omega \\ s\omega &= -\frac{1}{2}[\omega,\omega] \qquad s^2 = 0 \\ s\bar\omega &= -ib \\ sb &= 0 \end{aligned} \tag{18}$$

but, thanks to the introduction of the b-field,

$$i < b, g > + < \bar\omega, m\omega > = s\,(- < \bar\omega, g >) \tag{19}$$

This allows to discuss perturbative renormalization using all the power of locality. The useful part involves the local cohomology of Lie $\mathcal{G}$ in terms of which the observables can be defined and which also classifies obstructions to gauge invariance due to quantum deformations (i.e. anomalies).

We shall see in the next section that the cohomology involved in topological theories is different !

Of course the above discussion is local over orbit space, and a constructive procedure to glue the charts is missing. This is the Gribov problem.

3 Cohomological Theories

E. Witten's 1988 paper [6] contains several things. First, invoking "twisted $N = 2$ supersymmetry" E. Witten gets an action $S(a, \psi, \varphi; ...)$ where ψ resp φ is a 1 resp 0 form with values in Lie G and the dots represent a collection of Lagrange multiplier fields. Then it is observed that

$$QS = 0 \tag{20}$$

with

$$\begin{array}{cc} Qa = \psi & \quad \text{infinitesimal} \\ Q\psi = D_a\varphi & \quad Q^2 = \text{gauge transformation} \\ Q\varphi = 0 & \quad \text{of parameter } \varphi \end{array} \tag{21}$$

Furthermore there is an identity of the form

$$\int trF \wedge F = S - Q\chi(a, \psi, \varphi; ...) \tag{22}$$

where χ is gauge invariant.

The observables are classified according to the gauge invariant cohomology of Q, with the example

$$\begin{aligned} Q\ tr\ F \wedge F &= -d\ tr\ 2F\psi \\ Q\ tr\ 2F\psi &= -d\ tr\ (\psi \wedge \psi + 2F\varphi) \\ Q\ tr(\psi_\wedge \psi + 2F\psi) &= -d(2\psi\varphi) \\ Q\ tr\ 2\psi\varphi &= -d\ tr\ \varphi^2 \\ Q\ tr\ \varphi^2 &= 0 \end{aligned} \tag{23}$$

It follows that integrating the polynomials exhibited in these descent equations over cycles of the correct dimensions yields (non trivial !) elements of the cohomology of Q whose correlation functions are conjectured to reproduce Donaldson's polynomials.

Very soon after the appearance of E. Witten's article, L. Baulieu and I.M. Singer [7] remarked that Eq.(22) can be rewritten as

$$S = \int tr\ F \wedge F + Q\chi(a, \psi, \varphi; ...) \tag{24}$$

so that this action looks like the gauge fixing of a topological invariant. Furthermore, at the expense of introducing a Faddeev Popov ghost ω, Q can be replaced by s:

$$\begin{aligned} sa &= \psi - \mathcal{D}_a\omega \\ s\psi &= -D_a\Omega + [\psi, \omega] \qquad s^2 \equiv 0 \\ s\omega &= \Omega - \frac{1}{2}[\omega, \omega] \\ s\Omega &= -[\omega, \Omega] \end{aligned} \tag{25}$$

(For homogeneity in the notations, we have replaced φ by Ω).

This has however a defect, namely, s has no cohomology and therefore is not adequate to describe the physics of the model.

Inspired by an article by J. Horne [8], devoted to a supersymmetric formulation of this model, S. Ouvry, R. S. and P. van Baal [9] solved that difficulty by phrasing J. Horne's observation as follows: S and χ are not only gauge invariant but also are independent of ω !

In other words they are invariant under

$$\begin{array}{lll} I(\lambda), L(\lambda), & \lambda \in \text{Lie } \mathcal{G} & \\ I(\lambda)\omega = \lambda & I(\lambda) \text{ other} = 0 & \\ L(\lambda)\omega = [\lambda, \omega] & L(\lambda) \text{ other} = & \text{infinitesimal gauge} \\ & & \text{transformation of parameter } \lambda \end{array} \tag{26}$$

and, one can verify that

$$L(\lambda) = [I(\lambda), s]_+ \tag{27}$$

The cohomology that defines the physics of the model is the basic cohomology of s for the operation $\{I(\lambda), L(\lambda)\}$. This is not empty and coïncides with that of Q. Looking into that direction was suggested during a seminar by P.§ Braam at the CERN theory division in the spring 1988. There it was stated that the subject was the equivariant cohomology of $\mathcal{A}$ (restricted to $F = *F$). Further geometrical interpretations of $\psi\omega\Omega$ were given by L. Baulieu and I.M. Singer [7] and the general set up was precisely phrased in terms of equivariant cohomology by J. Kalkman [10] who developed the algebraic equipment further. Two general types of equivariant cohomology classes are involved in the present models:

- Mathaï Quillen [11] representatives of Thom class of vector bundles (Gaussian deformations of covariant δ functions). Those occur in the action.

- Equivariant characteristic classes of vector bundles. They are expressed in terms of an arbitrary invariant connection[12]. They provide the known topological observables. In the case where the manifold to be quotiented is a principal bundle, Cartan's "theorem 3"[13] transforms equivariant cohomology classes into basic cohomology classes, by the substitution $\omega \rightarrow \tilde{\omega}, \Omega \rightarrow \tilde{\Omega}$, where $\tilde{\omega}$ is a connection and $\tilde{\Omega}$ its curvature. It is expressible in terms of another identity in which integral representation of both bosonic and fermionic δ functions provides other terms in the action:

$$\int \mathcal{D}\omega \mathcal{D}\Omega \; \delta(\omega - \tilde{\omega}) \; \delta(\Omega - \tilde{\Omega}) = 1 \tag{28}$$

This can only be understood if ω is introduced, although it does not always appear in the action.

We shall now illustrate these general recipes in the case of topological Yang Mills theories (YM_4^{top}).

The observables are constructed as universal cohomology classes of $\mathcal{A}/\mathcal{G}$ as follows: consider the G bundle $P(M,G) \times \mathcal{A}$ and, on it, the $\mathcal{G}$ invariant G connection a (a zero form on $\mathcal{A}$, a one form on $P(M,G)$).

The equivariant curvature of a, in the intermediate scheme (see appendix A) is

$$R^{eq.}_{int} = F(a) + \psi + \Omega \tag{29}$$

with

$$\psi = \delta a. \tag{30}$$

In the Weil scheme, we are interested in

$$R^{eq.}_{w} = F(a) + \psi + \Omega \tag{31}$$

with

$$\psi = \delta a + \mathcal{D}_a \omega. \tag{32}$$

This is the object first considered by L. Baulieu, I.M. Singer [7].

The equivariant characteristic class $tr(R^{eq.}_{w})^2$ fulfills

$$(d + \delta) \; tr(R^{eq.}_{w})^2 = 0 \tag{33}$$

which provides the descent equations (Eq.23). Replacing ω by $\tilde{\omega}, \Omega$ by $\tilde{\Omega}$, where $\tilde{\omega}$ is a $\mathcal{G}$ connection on $\mathcal{A}$, provides a basic form on $P(M,G) \times \mathcal{A}$.

One may choose [7,11]

$$\tilde{\omega} = -D_a^* \frac{1}{D_a^* D_a} \delta a \tag{34}$$

provided reducible connections are excluded.

Let now $\mathcal{O}_i(a, \psi, \omega, \Omega)$ be equivariant classes of $\mathcal{A}$ obtained by integration over cycles in M with the proper dimension. We want to find an integral representation in terms of fields of the form on $\mathcal{A}/\mathcal{G}$ corresponding to a basic form $\mathcal{O} = \prod_i \mathcal{O}_i$ and, in the case of a form of maximal degree ("top form") of its integral.

Let $\tilde{a}$ be coordinates of a local section Σ

$$g(\tilde{a}) \equiv 0 \quad \frac{\delta g}{\delta a} \delta \tilde{a} \equiv \frac{\delta g}{\delta a} (\tilde{\psi} - D_{\tilde{a}} \tilde{\omega}) \equiv 0 \tag{35}$$

We have

$$\mathcal{O}(a, \psi, \tilde{\omega}, \tilde{\Omega})_{|\Sigma} = \mathcal{O}\left(\tilde{a}, \delta \tilde{a} + D_{\tilde{a}} \tilde{\omega}_{|\Sigma}, \tilde{\omega}_{|\Sigma}, \tilde{\Omega}_{|\Sigma}\right) \tag{36}$$

This defines a cohomology class on $\mathcal{A}/\mathcal{G}$, independently of the choice of Σ, because of the basicity of $\mathcal{O}$. The expression at hand can be expressed through the introduction of a collection of δ-functions.

First, in the case of YM_4^{top}, one has to restrict to $F = *F$, which goes through a δ function or a smeared gaussian thereof according to the Mathaï Quillen formula (cf. Ref.[11] and appendix A).

The replacement $\omega \to \tilde{\omega}$ $\Omega \to \tilde{\Omega}$ can be carried out using the δ functions of Eq.(28):

$$\begin{aligned} & \int \delta(\omega - \tilde{\omega}) \delta(\Omega - \tilde{\Omega}) \mathcal{D}\omega \mathcal{D}\Omega \\ & = \int \mathcal{D}\bar{\omega} \mathcal{D}\bar{\Omega} \mathcal{D}\omega \mathcal{D}\Omega \ e^{(s+\delta)(\bar{\Omega}(\omega - \tilde{\omega}))} \end{aligned} \tag{37}$$

where s is extended to

$$\begin{aligned} s\bar{\Omega} &= \bar{\omega} - [\omega, \bar{\Omega}] \\ s\bar{\omega} &= [\Omega, \bar{\Omega}] - [\omega, \bar{\omega}] \end{aligned} \tag{38}$$

If $\tilde{\omega}$ is the solution of a local equation e.g.

$$D_a^* \tilde{\Psi} = D_a^* (\delta a + D_a \tilde{\omega}) \tag{39}$$

this can be rewritten, thanks to the cancellation of determinants, as:

$$\int \mathcal{D}\omega \mathcal{D}\Omega \mathcal{D}\bar{\omega} \mathcal{D}\bar{\Omega} \ e^{s(\bar{\Omega} D^* \Psi)} \tag{40}$$

Other local choices can be made, e.g. the flat connection determined by the local section Σ [14], but, in this case, a change of local section produces a change of representative in the cohomology class under consideration due to the associated change of connection.

Finally, the restriction to Σ goes via the insertion of the δ function identity

$$\int \delta(a-\tilde{a})\delta(\psi-\tilde{\psi})\mathcal{D}a\mathcal{D}\psi = 1 \tag{41}$$

This can be rewritten as

$$\int \mathcal{D}a\mathcal{D}\psi \int \mathcal{D}\bar{\alpha}\mathcal{D}\bar{\psi}\ e^{(s+\delta)(\bar{\psi}(a-\tilde{a}))} = 1 \tag{42}$$

with

$$\begin{aligned} s\bar{\psi} &= \bar{\alpha} - [\omega, \bar{\psi}] \\ s\bar{\alpha} &= [\Omega, \bar{\psi}] - [\omega, \bar{\alpha}] \end{aligned} \tag{43}$$

Integrating over all a's and Ψ's yields a field theory representation of forms on orbit space, as advocated in ref.[14]. Integrating over the superfiber (the tangent bundle of a fiber with Grassmann variables on the vectorial part) yields a formal field theory representation of the integral over orbit space of a basic top form. In terms of the local equations Eq.(35), this can be rewritten as

$$\int \mathcal{D}a\mathcal{D}\psi \int \mathcal{D}\bar{\alpha}\mathcal{D}\bar{\psi}\ e^{s(\bar{\gamma}g(a))} = 1 \tag{44}$$

with

$$\begin{aligned} s\bar{\gamma} &= \beta + \omega \cdot \bar{\gamma} \\ s\beta &= -\Omega \cdot \bar{\gamma} + \omega.\beta \end{aligned} \tag{45}$$

where the dot denotes the action of $\mathcal{G}$ on the bundle over $\mathcal{A}$ of which g is a section.

If $\mathcal{O}$ is a top form, integration transforms the integration over the fiber, in Eqs (42, 43) into integration over $\mathcal{A}$, after localizing $\mathcal{O}$ inside the domain of Σ. The result is then a functional integral of the exponential of an action of the form $s\chi$. If this representation involves ultraviolet problems one may conjecture that, besides the necessity to include in χ all terms consistent with power counting the gauge fixing term in Eq.(44) has to be written in the form $sW\chi$ where W is another operation which anticommutes with s and involves a Faddeev Popov ghost field, its graded partner, and the corresponding antighosts. This however is still waiting for confirmation.

In support of the relevance of these constructions, one may give a few examples:

i) The equivariant curvature Eq.(31),(33) precisely yields the observables constructed by E. Witten via the interpretation given by L. Baulieu, I.M. Singer. The same method yields the observables constructed by C. Becchi, R. Collina, CÉ Imbimbo [14] in the case of 2-d topological gravity (see also L. Baulieu, I.M. Singer [7]).

ii) Recent work by M. Kato [15] and collaborators remarking the equivalence of some pairs of topological conformal models through similarity transformations of the form e^R is interpretable by $R = i_M(\omega)$, in J. Kalkman's language [10].

iii) The identification in topological actions of terms which fix a choice of connection is an additional piece of evidence [6], [14].

4 Conclusion

The formalism of equivariant cohomology provides an elegant algebraic set up for topological theories of the cohomological type. Its relationship with $N = 2$ supersymmetry via twisting is still mysterious and may still require some refinements before it provides some principle of analytic continuation. At the moment, it is still a question whether topological theories can be treated as field theories according to strict principles [14] or whether the formal integral representations they provide can at best suggest mathematical conjectures to be mathematically proved or disproved.

Acknowledgments

I wish to thank C. Becchi and C. Imbimbo for numerous discussions about their work on 2d topological gravity. I also wish to thank R. Zucchini for discussions about his recent work.

Appendix A

Equivariant Cohomology

Example 1.

M is a smooth manifold with a smooth action of a connected Lie group $\mathcal{G}; \Omega^*(M)$ is the exterior algebra of differential forms on M, d_M the exterior differential; $\lambda \in \text{Lie}\ \mathcal{G}$ is represented by a vector field $\underline{\lambda} \in \text{Vect}M. i_M(\lambda) = i(\underline{\lambda})$ operates on $\Omega^*(M)$ by contraction with $\underline{\lambda}$; the Lie derivative is defined by

$$\ell_M(\lambda) = \ell(\underline{\lambda}) = [i(\underline{\lambda}), d_M]_+ \tag{46}$$

One has

$$\begin{aligned}
[i_M(\lambda), i_M(\lambda')]_+ &= 0 \\
[\ell_M(\lambda), i_M(\lambda')]_- &= i_M([\lambda, \lambda']) \\
[\ell_M(\lambda), \ell_M(\lambda')]_- &= \ell_M([\lambda, \lambda'])
\end{aligned} \tag{47}$$

Forms $\omega \in \Omega^*(M)$ such that

$$i_M(\lambda)\omega = 0 \quad \forall \lambda \in \text{Lie } \mathcal{G} \tag{48}$$

are called horizontal.

Forms $\omega \in \Omega^*(M)$ such that

$$\ell_M(\lambda)\omega = 0 \quad \forall \lambda \in \text{Lie } \mathcal{G} \tag{49}$$

are called invariant.

Forms which are both horizontal and invariant are called basic.

The basic de Rham cohomology is the cohomology of d_M restricted to basic forms.

Generalization.

E is a graded commutative differential algebra with differential d_E and two sets of graded derivations $i_E(\lambda)$ (of grading -1) $\ell_E(\lambda)$ (of grading 0) fulfilling Eq.(47), with M replaced by E. The notions of horizontal and invariant elements similarly generalize as well as that of basic cohomology.

Example 2: The Weil algebra of $\mathcal{G} : W(\mathcal{G})$.

$$W(\mathcal{G}) = \wedge(\text{Lie } \mathcal{G})^* \otimes S\left((\text{Lie } \mathcal{G})^*\right) \tag{50}$$

whose factors are generated by ω, of grading $1, \Omega$ of grading 2, with values in Lie $\mathcal{G}$. We define the differential d_w by

$$\begin{aligned}
d_W \ \omega &= \Omega - \frac{1}{2}[\omega, \omega] \\
d_W \ \Omega &= [\omega, \Omega]
\end{aligned} \tag{51}$$

$i_W(\lambda), \ell_W(\lambda)$ by

$$\begin{aligned}
i_W(\lambda)\omega &= \lambda \quad i_W(\lambda)\Omega = 0 \\
\ell_W(\lambda) &= [i_W(\lambda), d_W]_+ : \\
\ell_W(\lambda)\omega &= [\lambda, \omega] \\
\ell_W(\lambda)\Omega &= [\lambda, \Omega]
\end{aligned} \tag{52}$$

Definition: The equivariant cohomolgy of M is the basic cohomology of $W(\mathcal{G})\otimes\Omega^*(M)$ for the differential d_W+d_M and the action $i_W(\lambda)+i_M(\lambda)$, $\ell_W(\lambda)+\ell_M(\lambda)$.

This is the Weil model of equivariant cohomology.

One can define the intermediate model according to J. Kalkman[10] by applying the algebra automorphism

$$x \to e^{-i_M(\omega)}x \tag{53}$$

which transforms the differential into

$$d_{int} = d_W + d_M + \ell_M(\omega) - i_M(\Omega) \tag{54}$$

and the operation into

$$\begin{aligned} i_{int}(\lambda) &= i_W(\lambda) \\ \ell_{int}(\lambda) &= \ell_W(\lambda) + \ell_M(\lambda) \end{aligned} \tag{55}$$

From this one easily sees that the equivariant cohomology is that of $[\Omega^*(M) \otimes S\left((\mathrm{Lie}\ \mathcal{G})^*\right)]^{\mathcal{G}}$ with the differential

$$d_C = d_M - i_M(\Omega) \tag{56}$$

where the superscript $\mathcal{G}$ denotes $\mathcal{G}$-invariant elements. This is the Cartan model [13], [10]. If M is a principal $\mathcal{G}$ bundle with a connection $\tilde{\omega}$, the mapping

$$\omega \to \tilde{\omega} \quad \Omega \to \tilde{\Omega} \tag{57}$$

where $\tilde{\Omega}$ is the curvature of $\tilde{\omega}$, maps isomorphically the equivariant cohomology of M into its basic cohomology, independently of the choice of $\tilde{\omega}$. This is Cartan's theorem 3 [13].

There are two standard ways to produce non trivial equivariant cohomology classes:

i) [12] If the action of $\mathcal{G}$ can be lifted to a principal bundle $P(M,K)$ with structure group K, and Γ is a $\mathcal{G}$ invariant connection on $P(M,K)$, the intermediate equivariant curvature is defined as

$$R^{eq}_{int}\ (\Gamma) = D_{int}\Gamma + \frac{1}{2}[\Gamma,\Gamma] = R(\Gamma) - i_P(\Omega)\Gamma \tag{58}$$

One has

$$\begin{aligned} i_{int}(\lambda)\ R^{eq}_{int}(\Gamma) &= 0 \\ \ell_{int}(\lambda)\ R^{eq}_{int} &= [\lambda, R^{eq}_{int}(\lambda)] \end{aligned} \tag{59}$$

It follows that any K invariant polynomial of Lie K, P_{inv} yields an equivariant "characteristic" cohomology class. This can be written in the Weil model using Kalkman's automorphism and is at the root of the construction of topological observables [6], [14].

ii) If $E(X,V)$ is a vector bundle over the manifold X, reducible to $\mathcal{G}$, one may write

$$E(X,V) = P(X,\mathcal{G}) \otimes_{\mathcal{G}} V \tag{60}$$

where P is the associated frame bundle.

There is a basic cohomology class, the universal Thom class obtained as follows [11]:

$$\tau_0 \equiv \delta(v) \wedge dv = N_0 \int db\ d\bar{\omega}\ e^{i<b,v>+<\bar{\omega},dv>} \tag{61}$$

for some normalization constant N_0 where b and $\bar{\omega} \in V^*$, the dual of V, $\int d\bar{\omega}$ means Berezin integration, and $<\ ,\ >$ denotes the duality pairing. Introducing s by

$$\begin{aligned}
s\ v &= dv + \omega v \equiv \psi + \omega v \\
s\ dv &= -\Omega v + \omega dv \\
s\ \omega &= \Omega - \frac{1}{2}[\omega,\omega] \\
s\ \Omega &= -[\omega,\Omega] \\
s\ \bar{\omega} &= -ib - \bar{\omega}\omega \\
s\ ib &= -ib\omega + \bar{\omega}\Omega
\end{aligned} \tag{62}$$

One may write

$$\tau_0 = \delta(v)(\wedge dv) = N_0 \int db\ d\bar{\omega}\ e^{s<\bar{\omega},V>} \tag{63}$$

It is easy to prove that

$$\tau = N_0 \int db\ d\bar{\omega}\ e^{s[<\bar{\omega},v>-i(\bar{\omega},b)]} \tag{64}$$

where $(\bar{\omega},b)$is a $\mathcal{G}$ invariant bilinear form on $\mathcal{G}^*$, is an equivariant class of V, with fast decrease. Replacing ω by $\tilde{\omega}$, a connection on $P(X,\mathcal{G})$, yields a basic class of $E(X,V)$, once written in the Weil scheme ($\psi_{Weil} = dv - \omega v$, whereas $\psi_{int} = dv$). The extension of the s-operation to the integration variables brings a substantial simplification to the original calculations.

The substitution of v by a section $v(x)$ transforms τ into the cohomology class associated with the submanifold of X defined by $v(x) = 0$.

Formula 64 gives the Mathaï Quillen representative of the Thom class of $E(X, V)$ and leads to a gaussianly spread Dirac current of the submanifold in question.

As a last example, used in the text, let us describe the Ruelle Sullivan [3,5] class associated with an invariant closed form ω on M:

$$\omega_{RS} = i(\wedge_\alpha e_\alpha)\omega \tag{65}$$

where e_α is a basis of Lie $\mathcal{G}$.

That ω_{RS} is both closed and invariant follows from the closedness and invariance of ω, and horizontality is trivial $(i(e_\alpha)i(e_\alpha) = 0)$.

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