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Waves, boosted branes and BPS states in M-theory

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Abstract

Certain type II string non-threshold BPS bound states are shown to be related to non-static backgrounds in 11-dimensional theory. The 11-d counterpart of the bound state of NS-NS and R-R type IIB strings wound around a circle is a pure gravitational wave propagating along a generic cycle of 2-torus. The extremal (q_1, q_2) string with non-vanishing momentum along the circle (or infinitely boosted black string) corresponds in $D = 11$ to a 2-brane wrapped around 2-torus with momentum flow along the (q_1, q_2) cycle. Applying duality transformations to the string-string solution we find the background representing the type IIA bound state of 2-brane and 0-brane. Its lift to 11 dimensions is simply a 2-brane finitely boosted in transverse direction. This 11-d solution interpolates between a static 2-brane (zero boost) and a gravitational wave in 11-th dimension (infinite boost). Similar interpretations are given for various bound states involving 5-branes. Relations between transversely boosted M-branes and 1/2 supersymmetric non-threshold bound states 2+0 and 5+0 complement those between M-branes with momentum in longitudinal direction and 1/4 supersymmetric threshold bound states 1+0 and 4+0. In the second part of the paper we establish the correspondence between the BPS states of type IIB strings on a circle and oscillating states of a fundamental supermembrane wrapped around a 2-torus. We show that the (q_1, q_2) string spectrum is reproduced

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by the membrane BPS spectrum, determined using a certain limit. This supports the picture suggested by Schwarz.

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1. Introduction

In view of recent suggestions that $D = 11$ supergravity may be a low-energy effective field theory of a more fundamental 'M-theory', it is important to clarify further how different p-branes and their BPS bound state configurations in $D = 10$ string theories can be understood from eleven-dimensional perspective (for recent reviews and refs. see [1,2,3]). At the level of effective field theory classical configurations, the threshold (i.e. zero binding energy) BPS bound states of p-branes (corresponding, in particular, to superpositions of D-branes with $p - p' = 4, 8$ [4,5,6,7]) originate from combinations of intersecting 2-branes and 5-branes in 11 dimensions [8,9,10,11,12,13].

As for non-threshold BPS bound states with non-zero binding energy (typically, with $M_{1+2} = \sqrt{M_1^2 + M_2^2}$) which cannot be viewed as n weakly coupled p-branes at equilibrium (and are not described by n independent harmonic functions), their 11-dimensional origin was not much discussed in the past. The basic example of such $D = 10$ configuration is the (q_1, q_2) bound state of the NS-NS and R-R strings in type IIB theory [14,15,1,16], from which various other similar bound states (corresponding, in particular, to superpositions of D-branes with $p - p' = 2, 6$ [4,6,17]) may be constructed by applying T and $SL(2, Z)$ dualities.

It was suggested in [14,1] from consideration of the nine-dimensional 0-brane spectra that the (q_1, q_2) string-string type IIB bound states should be related to the states of 2-brane wrapped around a 2-torus in $D = 11$. One of the aims of the present paper is to extend and complete what was done in [14,1] at two different levels. At the 'macroscopic' classical-solution level, we shall explicitly identify the $D = 11$ background which has the (q_1, q_2) string solution of [14] as T-dual of its dimensional reduction. We shall also determine the $D = 11$ counterparts of several type IIA non-threshold bound state backgrounds related to the string one by T and $SL(2, Z)$ dualities. At the microscopic quantum state level, we shall demonstrate that the mass spectrum of BPS states of (q_1, q_2) string indeed matches the spectrum of the corresponding oscillating states of wrapped membrane not only for the zero-mode parts of the masses, as was already shown in [14,1], but also including the oscillator parts.

Section 2 will be devoted to a discussion of classical type II $D = 10$ solutions corresponding to various non-threshold BPS bound states of branes and their $D = 11$ counterparts. We shall first determine (in Section 2.1) which $D = 11$ solution is related, by dimensional reduction and T-duality, to the (q_1, q_2) string type IIB background constructed in [14] by applying $SL(2, R)$ duality transformation to the fundamental $(1, 0)$ string solution of [18]. In the general case of the (q_1, q_2) string with momentum (which is a straightforward generalization of the solution of [14] and preserves 1/4 of maximal supersymmetry) the corresponding $D = 11$ solution is the extremal limit of a 2-brane [19] wrapped around a 2-torus and infinitely 'boosted' [9,20] along *generic* (q_1, q_2) cycle of the torus. In agreement with the interpretation in [14], the momentum (or the charge of the 'boost' harmonic function) of the (q_1, q_2) string is the winding number (charge) of the 2-brane, while the winding number of the (q_1, q_2) string is the momentum of the 2-brane boosted along the (q_1, q_2) cycle. The counterpart of the zero-momentum string is thus a zero-charge limit of boosted 2-brane, or simply a gravitational wave along the cycle of 2-torus.

For generic values of the charges, this 1/4 supersymmetric $D = 11$ solution can be thought of as a bound state of a 2-brane and a gravitational wave (or $2+\nearrow$, where the 'diagonal' direction of the arrow indicates a generic direction of the momentum flow on 2-torus). In analogy with the fundamental string case [18,21,22], it should represent, at macroscopic level, an excited BPS state of the membrane with the momentum being carried by oscillations propagating along the (q_1, q_2) cycle of the torus. Its dimensional reduction is type IIA solution representing a bound state of a fundamental string, 0-brane and a wave, carrying momentum along the string, $1 + 0 + \uparrow$. The T-duality along the string direction relates this to our starting point, type IIB bound state of the wave, R-R string and fundamental string, $\uparrow + 1_R + 1_{NS}$.

In general, bound states of type IIB branes, constructed by applying $SL(2, Z)$ duality transformations, can be also obtained from the 11-dimensional theory by starting with appropriate M-brane configurations and boosting them along a non-trivial (q_1, q_2) cycle of the 2-torus. This is in agreement with the observation [14,23] that $SL(2, Z)$ duality of type IIB theory has its origin in the toroidal $SL(2, Z)$ in $D = 11$ (in the case of $D = 11$ backgrounds with at least two isometries which we shall be considering here this was originally understood in [24]). We shall illustrate this further in Section 2.2 on several examples, starting with the (q_1, q_2) bound state of NS-NS and R-R 5-branes [14,16]. The 1/4 supersymmetric $1_{NS} + 1_R + \uparrow$ and $5_{NS} + 5_R + \uparrow$ configurations are the special cases of the non-threshold 1/8 supersymmetric configuration $(1 + 5)_{NS} + (1 + 5)_R + \uparrow$ which is an $SL(2, Z)$ rotation of the threshold bound state of the NS-NS string and solitonic 5-brane [25] or of its $SL(2, Z)$-dual R-R version [26]. Its $D = 11$ counterpart is shown to be the intersection of the 2-brane and 5-brane over a string [9] with a momentum flow along generic 'diagonal' direction of 2-brane torus, i.e. $5\perp 2 + \nearrow$.

In Section 2.3 we shall address the question of $D = 11$ interpretation of other non-threshold type II BPS bound states related to the (q_1, q_2) string by the duality transformations. We shall demonstrate that the type IIA bound state of 2-brane and 0-brane is just a dimensional reduction of the standard extremal $D = 11$ 2-brane solution *finitely* (i.e. $v < 1$) boosted along the isometric 11-th dimension which is *transverse* to the brane, or $2 \mapsto$. Analogous surprisingly simple interpretation (which is by no means obvious at the level of the type IIA backgrounds) applies also to $5 + 0$ (and $6 + 0$) bound states. The relations between boosted branes in $D = 11$ and $D = 10$ 'p-brane + 0-brane' bound states can be summarized as follows (cf. [13,7,27]): (a) 1/4 supersymmetric threshold bound states: $2 + \uparrow \longrightarrow 1 + 0$, $5 + \uparrow \longrightarrow 4 + 0$; (b) 1/2 supersymmetric non-threshold bound states: $2 \mapsto\longrightarrow 2+0$, $5 \mapsto\longrightarrow 5+0$. We shall discuss two natural generalizations, $2\perp 2 \mapsto$ and $5\perp 2 \mapsto$. In particular, $2\perp 1+0$ solution (T-dual to $1_{NS}+1_R+\uparrow$) is the reduction of the $D=11$ configuration $2\perp 2 \mapsto$ in which one of the 2-branes intersecting over a point [8] is finitely boosted in a direction of the other 2-brane orthogonal to it. We shall also comment on the existence of several different 11-dimensional solutions which reduce to $D=10$ solutions related by T and $SL(2,Z)$ dualities.

In Section 3 we shall address the relation between the type IIB strings and $D=11$ membrane at the 'microscopic' level, by explicitly identifying the quantum supermembrane states which correspond to the BPS spectrum of the bound states of type IIB strings. These excited membrane states have oscillations only along one (momentum) direction, in agreement with what is also implied by the macroscopic effective field theory picture. As was shown in [14,1], the zero-mode part of the mass of BPS states of type IIB (q_1,q_2) string on a circle is in perfect agreement with the zero-mode part of the mass of the fundamental supermembrane states wrapped around a 2-torus with a momentum along the (q_1,q_2) cycle of the torus. Our aim will be to check that the oscillator parts in the masses also agree. We shall argue that in order to determine the relevant BPS spectrum of the supermembrane it appears to be sufficient to omit the interaction term in the light-cone Hamiltonian of the wrapped membrane. Solving the resulting gaussian theory, we shall find that its oscillating membrane BPS states do, indeed, have the same masses as the BPS states of type IIB strings.

2. Boosted branes and non-threshold BPS bound states: classical solutions

Gravitational waves seem to play an important role in D=11 theory as 2-brane and 5-brane solutions. This may not be unexpected since pp-waves are known to preserve supersymmetry [28,29]; they are related, in $D=10$ theory, to the fundamental string background by T-duality [30,31]; and a single pp-wave (Schwarzschild background boosted to the speed of light) is the $D=11$ image of the 0-brane of $D=10$ type IIA theory [32]. Nevertheless, it is a bit surprising (though, in fact, implicit in [14]) that the $D=11$ solution that corresponds to a rather complicated-looking (q_1,q_2) string background of type IIB theory is just a dimensional reduction of the pure gravitational wave propagating along the (q_1,q_2) cycle of 2-torus in $M^{11}=T^2\times M^9$. Wrapping a 2-brane around T^2 corresponds to adding a wave (momentum) to the type IIB (q_1,q_2) string [14]. As in the case of the $D=10$ fundamental string [22,21], here the wave may be represented by transverse oscillations of the 2-brane propagating along the cycle (i.e. in *one* circular direction) and carrying the corresponding momentum. This suggests that, as in the string-theory case, there should exists the corresponding BPS states of the quantum supermembrane. This issue will be addressed in Section 3, while below we will discuss only the classical effective field theory solutions.

2.1. Wave along generic cycle of 2-torus in $D=11$ and $SL(2,Z)$ family of type IIB strings

In what follows we shall consider the 11-dimensional space $M^{11}=T^2\times M^9$ with the isometric rectangular 2-torus coordinates (y_1,y_2) having periods $(2\pi R_1, 2\pi R_2)$ (the case of more general T^2 is treated similarly). The gravitational wave propagating in y_1 direction $(i=1,...,8)$

$$ds^2_{11} = dudv + W(u,x)du^2 + dy_2^2 + dx_i dx_i \ , \quad \partial_x^2 W = 0 \ , \quad u,v = y_1 \pm t \ , \tag{2.1}$$

solves the $D=11$ Einstein equations for any harmonic function W. The case of $W=\frac{\mathcal{Q}}{|x|^6}$ is special as this pp-wave can be represented [28] as the extremal (infinite boost, zero mass) limit of the boosted Schwarzschild solution, or, more precisely, since y_1,y_2 are isometries, of a black 2-brane of zero charge,

$$\begin{aligned} ds^2_{11} &= -f(r)dt'^2 + dy_1'^2 + dy_2^2 + f^{-1}(r)dr^2 + r^2 d\Omega_7^2 \\ &= -dt^2 + dy_1^2 + dy_2^2 + \frac{\mu}{r^6}(\cosh\beta\ dt - \sinh\beta\ dy_1)^2 + f^{-1}(r)dr^2 + r^2 d\Omega_7^2 \\ &= -\hat{K}^{-1}(r)f(r)dt^2 + \hat{K}(r)[dy_1 + A(r)dt]^2 \ + dy_2^2 + f^{-1}(r)dr^2 + r^2 d\Omega_7^2 \ , \end{aligned} \tag{2.2}$$

where

$$\begin{gathered} t' = \cosh\beta\ t - \sinh\beta\ y_1 \ , \qquad y_1' = -\sinh\beta\ t + \cosh\beta\ y_1 \ , \\ f = 1 - \frac{\mu}{r^6} \ , \quad \hat{K} = 1 + \frac{\hat{\mathcal{Q}}}{r^6} \ , \quad A = -\frac{\mathcal{Q}}{r^6}\hat{K}^{-1} \ , \\ \hat{\mathcal{Q}} = \mu \sinh^2\beta \ , \qquad \mathcal{Q} = \mu\sinh\beta\cosh\beta \ . \end{gathered} \tag{2.3}$$

The boost parameter β is related to the Kaluza-Klein electric charge $\mathcal{Q}$, i.e. the momentum along the boost direction y_1. In the extremal limit $\mu \longrightarrow 0, \ \beta \longrightarrow \infty, \ \mathcal{Q}$ =fixed, the metric (2.2) takes the form (2.1) ($\hat{\mathcal{Q}} = \mathcal{Q}, \ K = \hat{K}$) with $W = K - 1 = \frac{\mathcal{Q}}{r^6}$.

The 1/2 supersymmetric extremal wave can be superposed with 1/2 supersymmetric extremal 2-brane (this can be shown, e.g., by boosting the black 2-brane [33] and taking the extremal limit). The resulting 1/4 supersymmetric background is [9,20]

$$ds^2_{11} = H_2^{1/3}(r)\Big(H_2^{-1}(r)\left[-dt^2 + dy_1^2 + dy_2^2 + W(r)(dt-dy_1)^2\right] + dx_i dx_i \Big) \ , \tag{2.4}$$

$$C_3 = H_2^{-1} dt\wedge dy_1 \wedge dy_2 \ , \qquad H_2 = 1 + \frac{Q}{r^6} \ , \quad W = \frac{\mathcal{Q}}{r^6} \ .$$

More generally, $W(r)$ in (2.4) can be replaced by any solution $W(u,x)$ of the Laplace equation $\partial_x^2 W = 0$. In particular, as in the string case [22,21], choosing $W = f_i(u)x^i$ corresponds to adding a wave of membrane oscillations propagating along y_1 (f_i is related to the profile of oscillations). The macroscopic properties of such solution are the same as of the solution with $W = \frac{\mathcal{Q}}{r^6}$ with $\mathcal{Q}$ being proportional to the asymptotic value of momentum carried by the wave ($\sim \langle[\int du f_i(u)]^2\rangle$).

It is important to stress that one is able to construct a supersymmetric BPS background by making an infinite boost in *one* isometric direction only (boosting in y_1 and y_2 directions with two independent parameters β_a and taking the limit $\mu \longrightarrow 0, \beta_a \longrightarrow \infty$ does not give a new solution). Similar observation applies in the general p-brane case and suggests that the BPS states of p-branes are always 'string-like', i.e. have non-vanishing momentum flow in *one* direction only. This should also follow directly from the BPS requirement and supersymmetry algebra (cf. [34]).

Dimensionally reducing the '2-brane plus wave' (or $2+\uparrow$) solution (2.4) to $D = 10$ along the 'spectator' direction y_2 we get the 'fundamental string plus wave' (or $1+\uparrow$) background. Under the T-duality in y_1, it transforms into the same type of solution of type IIB theory $1_{(1,0)}+\uparrow$ (with $H_2 \leftrightarrow K = 1 + W$, i.e. $Q \leftrightarrow \mathcal{Q}$). If instead we reduce along the boost direction y_1, we get the type IIA solution, which can be interpreted as a bound state of the fundamental string and the 0-brane, $1 + 0$. The T-dual solution of type IIB theory is the boosted R-R string, or $1_{(0,1)}+\uparrow$. This suggests that reducing to $D = 10$ along some 'mixed' direction we should get a more general class of solutions which will interpolate between the two special cases $1+\uparrow$ and $1 + 0$, or, when T-duality transformed into type IIB theory, between $1_{(1,0)}+\uparrow$ and $1_{(0,1)}+\uparrow$. This is equivalent to putting the wave (i.e. momentum) along some generic cycle of the 2-torus while reducing to $D = 10$ along the orthogonal cycle.

For the simple rectangular torus $ds^2 = R^2|d\sigma_1+\tau d\sigma_2|^2$, $\sigma_a \in (0, 2\pi)$, $y_a = R\sigma_a$, $\tau = i$, the direction of the (q_1, q_2) cycle[1] is specified by

$$\cos\theta = \frac{q_1}{\sqrt{q_1^2 + q_2^2}} \ , \qquad \sin\theta = \frac{q_2}{\sqrt{q_1^2 + q_2^2}} \ . \tag{2.5}$$

The background corresponding to the 2-brane boosted along the (q_1, q_2) cycle is then found from (2.4) by a rotation

$$y_1' = \cos\theta \ y_1 + \sin\theta \ y_2 \ , \qquad y_2' = -\sin\theta \ y_1 + \cos\theta \ y_2 \ , \tag{2.6}$$

i.e.[2]

$$ds^2_{11} = H_2^{1/3}\left(H_2^{-1}[-dt^2 + dy_1^2 + dy_2^2 + W(dt - \cos\theta \ dy_1 - \sin\theta \ dy_2)^2] + dx_i dx_i\right) \ ,$$

[1] The integers q_1, q_2 can be taken to be relatively prime since rescaling them by common factor does not change the direction of the cycle.

[2] This background has, of course, a straightforward non-extremal generalization, cf. [20].

$$C_3 = H_2^{-1} dt \wedge dy_1 \wedge dy_2 \ . \tag{2.7}$$

The quantized momentum has components (nq_1, nq_2) and the modulus $n\sqrt{q_1^2+q_2^2}$, implying the change in the quantization condition for the Kaluza-Klein charge $\mathcal{Q}$

$$\mathcal{Q} = c_0 \frac{n}{R} \quad \longrightarrow \quad \mathcal{Q}_q = c_0 \frac{n}{R_q} = \mathcal{Q}\sqrt{q_1^2+q_2^2} \ , \tag{2.8}$$

where $c_0 = \frac{1}{3}\kappa_9^2/\omega_7$ and n is an integer.

Putting the boost along the (q_1, q_2) cycle of the torus restores the symmetry between y_1 and y_2 directions: the background (2.7) is invariant under $y_1 \leftrightarrow y_2$, $q_1 \leftrightarrow q_2$, so that reductions along y_1 or y_2 give, for generic (q_1, q_2), similar $D = 10$ type IIA backgrounds related simply by $q_1 \leftrightarrow q_2$.

Let us now show that the $D = 11$ counterpart of the original zero-momentum (q_1, q_2) string solution of type IIB theory [14] is the limit of (2.7) in which the 2-brane charge is set equal to zero,[3] $Q = 0$, $H_2 = 1$, i.e. a boosted 'membrane without a membrane', or simply a gravitational wave travelling along a cycle of T^2.

Rewriting the resulting metric as

$$\begin{aligned} ds^2_{11} &= -dt^2 + dy_1^2 + dy_2^2 + W(dt - \cos\theta \ dy_1 - \sin\theta \ dy_2)^2] + dx_i dx_i \\ &= \ -K^{-1}dt^2 + K\tilde{K}^{-1}[dy_1 - \cos\theta \ WK^{-1}dt]^2 \\ &+ \tilde{K}[dy_2 - \sin\theta \ W\tilde{K}^{-1}dt + \sin\theta\cos\theta \ W\tilde{K}^{-1}dy_1]^2 \ , \end{aligned} \tag{2.9}$$

where

$$K \equiv 1 + W \ , \qquad \tilde{K} \equiv 1 + \sin^2\theta \ W \ , \qquad W = \frac{\mathcal{Q}_q}{r^6} \ , \tag{2.10}$$

one can easily read off the type IIA solution which follows upon the dimensional reduction along $y_2 \equiv y_{11}$ direction.

The resulting 1/2 supersymmetric type IIA background can be interpreted as a non-threshold BPS bound state of a gravitational wave (carrying momentum in a compact y_1-direction) and a 0-brane. When $q_2 = 0$ there is no 0-brane and when $q_1 = 0$ there is no momentum, i.e. we just get a 0-brane in the space with one compact isometric direction or 0_1. The R-R vector field originating from off-diagonal $(11, \mu)$ components of the metric has both time-like and internal spatial parts, which transform under T-duality into the ty_1 component of the R-R 2-tensor $B^{(2)}_{\mu\nu}$ and the R-R scalar χ of type IIB theory. Using the T-duality transformation rules or directly the relation between a $D = 11$ background with two isometries and the corresponding type IIB background given in [24] we find exactly the

[3] The non-extremal limit of this solution is a finitely boosted black (Schwarzschild) 2-brane of zero charge.

type IIB (q_1,q_2) string solution obtained in [14] by applying the $SL(2,Z)$ transformation to the fundamental string of [18][4]

$$ds^2_{10B} = \tilde{K}^{1/2}[K^{-1}(-dt^2 + dy_1^2) + dx_i dx_i] \ , \tag{2.11}$$

$$e^{2\phi} = K^{-1}\tilde{K}^2 \ , \qquad \chi = \sin\theta\cos\theta \ W\tilde{K}^{-1} \ ,$$

$$B^{(1)}_{ty_1} = -\cos\theta \ WK^{-1} \ , \qquad B^{(2)}_{ty_1} = -\sin\theta \ WK^{-1} \ .$$

Note that $dB^{(1)} + idB^{(2)} = e^{i\theta} dK^{-1}\wedge dt \wedge dy_1$. The special cases are the NS-NS string ($q_1 = 1, q_2 = 0$, i.e. $\theta = 0,\ \tilde{K} = 1$), and the R-R string ($q_1 = 0, q_2 = 1$, i.e. $\theta = \frac{\pi}{2},\ \tilde{K} = K$). This is the background corresponding to the the simplest 2-torus $\tau = i$ or to the trivial vacuum $\rho_0 = \chi_0 + ie^{-\phi_0} = i$ [14]. Generalization of the correspondence between (2.9) and (2.11) to the case of an arbitrary 2-torus is straightforward. In particular, in the case of the rectangular torus with different radii of y_1 and y_2, i.e. $\tau_2 = R_1/R_2$, one can change $q_2 \longrightarrow \frac{R_1}{R_2} q_2$ in the above expressions and thus replace the (q_1,q_2) string tension factor (in the string frame) $\sqrt{q_1^2+q_2^2}$ by $\sqrt{q_1^2 + e^{-2\phi_0} q_2^2} = \sqrt{q_1^2 + (\frac{R_1}{R_2})^2 q_2^2}$ [14].

All works in a similar way if we add back the 2-brane, i.e. start with (2.7) with $H_2 \neq 1$ and reduce down to $D = 10$ along y_2. The very simple $D = 11$ solution (2.7) then leads to a rather complicated $D = 10$ type IIA background which can be interpreted as an effective field theory representation of a 1/4 supersymmetric non-threshold bound state of a fundamental string, 0-brane and a wave, i.e. $1 + 0 + \uparrow$. T-duality along y_1 transforms it into the boosted version of the type IIB (q_1,q_2) string which differs from (2.11) only by an extra wave term in the metric

$$ds^2_{10B} = \tilde{K}^{1/2}\left(K^{-1}\left[-dt^2 + dy_1^2 + (H_2 - 1)(dt - dy_1)^2\right] + dx_i dx_i\right) \ . \tag{2.12}$$

The 2-brane charge Q $(= c_0 \frac{w_0 R}{\alpha'})$ thus becomes the momentum of the (q_1,q_2) string. The total momentum of the wave is just the tension of the (q_1,q_2) string.

This provides an explicit classical-solution realization of the observations in [14] based on the expected form of the $D = 9$ 0-brane spectrum. As in the case of the fundamental NS-NS string [22,21], the momentum flow has also a microscopic representation in terms of right-moving string oscillations, i.e. there exists a similar solution which is supported by an oscillating string source and has the same asymptotic value of momentum.

[4] We shall always use string-frame metric in $D = 10$. Here y_1 is the coordinate dual to y_1 in (2.9).

2.2. Type IIB bound states of 1-branes and 5-branes from $2\perp 5$ in $D=11$

Analogous considerations apply in general to type IIB solutions obtained by making an $SL(2,Z)$ transformation on a 'generating' background with at least one or more compact isometries. Since according to [24,14] the action of $SL(2,Z)$ should correspond to a modular transformation of the compactification 2-torus in $D=11$, such $SL(2,Z)$ families of solutions should originate from the $D=11$ counterpart of the generating solution which is boosted and reduced down to $D=10$ along generic cycle of the 2-torus.

Let us consider the $SL(2,Z)$ bound states of NS-NS and R-R 5-branes in type IIB theory [14,16] and explain their 11-dimensional origin. To get an idea of what kind of $D=11$ backgrounds we should expect to find, it is useful to consider the special cases, describing a type IIB solution, its possible T-dual images in type IIA theory and $D=11$ solutions from which the latter follow upon dimensional reduction:

(1) solitonic NS-NS 5-brane of IIB theory with a momentum flow along one of its directions $5_{NS}+\uparrow$ is T-dual to: (a) bound state of 5-brane and a wave $5+\uparrow$ in type IIA theory (if duality is performed along one of 4 'unboosted' directions), which is the type IIA image of boosted 11-dimensional 5-brane reduced along an isometric coordinate orthogonal to it,[5] $(5+\uparrow)_1$; (b) bound state of 5-brane and fundamental string, $5+1$ (if duality is performed along the boosted direction), which is a reduction of the $D=11$ $5\perp 2$ solution describing an orthogonal intersection of 5-brane and 2-brane over a line [9].

(2) R-R 5-brane of IIB theory with a momentum flow along one of its directions, $5_R+\uparrow$, is T-dual to: (a) $(4+\uparrow)_1$ which is the reduction of $(5+\uparrow)_1$ in $D=11$; (b) $4\perp 1$ which is another reduction of $5\perp 2$ in $D=11$.

Let us first ignore the momentum flow and consider the (q_1,q_2) bound state $5_{NS}+5_R$ obtained by applying the $SL(2,R)$ rotation to 5_{NS} of [2,35]. Its T-dual can be interpreted as a 1/2 supersymmetric non-threshold bound state of the solitonic 5-brane and R-R 4-brane in type IIA theory, i.e. $5+4$. The image of the latter in $D=11$ is 5_1, i.e. the 5-brane in $M^{11}=S^1_{y_2}\times M^{10}$, reduced down to $D=10$ along the mixture of the isometric direction y_2 transverse to the 5-brane, and a 5-brane direction y_1 (which will be orthogonal to the 4-brane within the type IIA 5-brane). These two directions form the 2-torus related to type IIB $SL(2,Z)$ duality which is the analogue of T^2 in the previous section.

The $D=11$ 5_1 background can be viewed as the limiting case of the $5\perp 2$ solution when the charge of 2-brane is set equal to zero (with y_2 being the 2-brane direction orthogonal to 5-brane). The reduction is done along the (q_1,q_2) cycle of the 2-torus around which the 2-brane is wrapped. When $q_1=0, q_2=1$ we get simply 4_1 (the reduction goes along a 5-brane direction y_1 common to 5 and 2). When $q_1=1, q_2=0$ we get the type IIA 5-brane (the reduction goes along the 2-brane direction y_2 transverse to the 5-brane).

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[5] We shall use subscript to indicate the existence of a number of periodic isometric directions orthogonal to p-brane, on which harmonic functions in the solution do not depend.

Next, we can add momentum along some special direction on 5-branes, i.e. start with type IIB background $5_{NS}+5_R+\uparrow$. It is natural to add also a string lying along the same direction, i.e. to consider the $SL(2,Z)$ rotation of the 1/8 supersymmetric background $(1+5)_{NS}+\uparrow$ [25], representing the threshold bound state of a boosted fundamental string and solitonic 5-brane. The resulting (q_1,q_2) configuration $(1+5)_{NS}+(1+5)_R+\uparrow$ ($n=3,4,5,6$; y_2 is reserved for eleventh coordinate)

$$ds^2_{10B} = H_5^{1/2}H_{15}^{1/2}\Big(H_1^{-1}H_5^{-1}\big[-dt^2+dy_1^2+V(dt-dy_1)^2\big]+H_5^{-1}dy_ndy_n+dx_idx_i\Big)\ , \quad (2.13)$$

$$e^{2\phi}=H_1^{-1}H_5^{-1}H_{15}^2\ , \qquad \chi=\sin\theta\cos\theta\ (H_1-H_5)H_{15}^{-1}\ , \qquad V=\frac{\mathcal{Q}}{r^2}\ ,$$

$$dB^{(1)}+idB^{(2)}=(\cos\theta+i\sin\theta)(dH_1^{-1}\wedge dt\wedge dy_1+*dH_5)\ ,$$

$$H_{15}\equiv\cos^2\theta\ H_5+\sin^2\theta\ H_1\ , \quad H_{1,5}=1+\frac{Q_{q(1,5)}}{r^2}\ , \quad Q_{q(1,5)}=Q_{1,5}\sqrt{q_1^2+q_2^2}\ ,$$

is parametrized by 5 charges and includes as special cases $1_{NS}+1_R+\uparrow$ (i.e., in particular, reduces to (2.11) when $H_5=1$, $K=H_1$, $\tilde{K}=H_{15}$, $V=0$) and $5_{NS}+5_R+\uparrow$ ($H_1=1$). Its mass is $M=\frac{2\pi^2}{\kappa_5^2}\sqrt{q_1^2+q_2^2}(Q_1+Q_5)$. T-duality along y_1 converts this into a complicated type IIA solution which can be interpreted as a 1/8 supersymmetric non-threshold bound state of a 5-brane, 4-brane, 0-brane and boosted string, i.e. $5+4+0+1+\uparrow$.

Lifting the latter to $D=11$ along $y_{11}\equiv y_2$ and setting first the momentum to zero (i.e. $V=0$) we find the following metric

$$ds^2_{11}=H_5^{3/2}\Big(-H_1^{-1}H_5^{-1}dt^2+H_1H_{15}^{-1}[dy_1-\cos\theta\ (H_1-1)H_1^{-1}dt]^2 \quad (2.14)$$

$$+H_5H_{15}[dy_{11}-\sin\theta\ (H_1-1)H_{15}^{-1}dt+\sin\theta\cos\theta\ (H_1-H_5)H_{15}^{-1}dy_1]^2$$

$$+H_5^{-1}dy_ndy_n+dx_idx_i\Big)\ ,$$

which reduces to (2.9) when there are no 5-branes $H_5=1, H_1=K$. This can be re-written simply as the 'rotated' 5_1-brane

$$ds^2_{11}=H_5^{2/3}\left[H_5^{-1}(-dt^2+dy_1'^2+dy_ndy_n)+dy_2'^2+dx_idx_i\right]\ , \quad (2.15)$$

where y_1',y_2' are the rotated coordinates as in (2.6). As expected, the same turns out to be true for non-vanishing momentum: the background one finds can be put into the form (cf. (2.7); $y_2\equiv y_{11}$)

$$ds^2_{11}=H_5^{2/3}H_2^{1/3}\Big(H_5^{-1}H_2^{-1}[-dt^2+dy_1'^2+W(dt-dy_1')^2] \quad (2.16)$$

$$+ H_5^{-1} dy_n dy_n + H_2^{-1} dy_2'^2 + dx_i dx_i \Big) ,$$

$$dC_3 = dH_2^{-1} \wedge dt \wedge dy_1 \wedge dy_2 + *dH_5 \wedge dy_2' , \quad H_2 \equiv 1 + V , \quad W \equiv H_1 - 1 , \qquad (2.17)$$

i.e. is just the 'rotated' version of $5\bot 2+ \uparrow$, i.e. of the intersection of the 2-brane and 5-brane boosted along the common string [9,10] where the momentum now flows along the (q_1, q_2) cycle of the (y_1, y_2)-torus around which the 2-brane is wrapped.

Special cases of the corresponding type IIA and type IIB solutions include (depending on direction of reduction): $2\bot 5 \longrightarrow 2\bot 4$ in IIA, related by T-duality to $(1+5)_R$ in IIB; $2\bot 5 \longrightarrow 1\bot 4$ in IIA, related by T-duality to $5_R + \uparrow$ in IIB; $2\bot 5 \longrightarrow 1+5$ in IIA , related by T-duality to $5_{NS} + \uparrow$ or $(1+5)_{NS}$ in IIB. Reduction down to $D=5$ gives a family of regular extremal black hole solutions related to the simplest NS-NS [25] and R-R [26] ones by U-duality. Reduction along the four 5-brane directions gives a family of solitonic strings in $D=6$.

We conclude that the $5\bot 2+ \nearrow$ background (2.16) provides a unified $D=11$ description of various 1-brane and 5-brane bound state configurations of type IIB theory.[6]

2.3. $2+0$ and $5+0$ bound states as M-branes boosted in transverse direction

The type IIA 1/4 supersymmetric threshold bound states of a string and 0-brane $1+0$ and of 4-brane and 0-brane 4+0 have a simple origin in $D=11$: the corresponding solutions are readily obtained by dimensional reduction from extremal M-brane configurations with momentum along the brane $2+\uparrow$ and $5+\uparrow$, or black M-branes infinitely boosted in one *parallel* direction [9,36] (for a discussion of alternative $D=11$ embeddings of $1+0$ and $4+0$ and also the embedding of $8+0$ see [13]).

The existence of $1_{NS} + 1_R$ type IIB bound state suggests by T and $SL(2,Z)$ duality the existence of other 1/2 supersymmetric *non-threshold* type IIA bound states that can be interpreted as $2+0$, $5+0$, etc.[7] In view of the above identification of the $D=11$ counterpart of the string-string bound state background it is of interest (in particular, in connection with [27]) to enquire about the $D=11$ origin of such bound states. As we shall demonstrate below, the corresponding classical solutions are indeed very simple: they are just a coordinate (Lorentz) transformation of the extremal M-brane background

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[6] In addition to the special cases of $(1+5)_{NS} + (1+5)_R + \uparrow$ there are also 1/2 supersymmetric non-threshold bound state of the type $1_{NS} + 5_R$, i.e. the fundamental string lying on R-R 5-brane (and its S-dual $1_R + 5_{NS}$). Such solution can be constructed by T-duality in 4 transverse directions from $(1_{NS} + 1_R)_4$. Its T-dual (along string direction) counterpart in IIA theory is 4-brane finitely boosted in transverse dimension, $4 \mapsto$ (see below).

[7] The existence of related D-brane bound states was noted in [4,6]. For a discussion of $2+0$ bound state in D-brane representation see [17].

and describe an M-brane *finitely* boosted in the direction *transverse* to its internal space. Dimensional reduction along this direction gives $2+0$ and $5+0$ type IIA solutions. These solutions interpolate between the standard static M-brane (zero boost) and plane wave (infinite boost). As will follow from the relation to the $1_{NS}+1_R$ solution, the coefficient in the harmonic function of the transversely boosted brane depends on the boost parameter and goes to zero in the limit of the infinite boost when the $D=11$ solution becomes just a plane wave (and the corresponding type IIA solution becomes a 0-brane).

We shall start with the explicit construction of $2+0$ solution by duality transformations[8] from $1_{NS}+1_R$ background of [14] and then lift the result to $D=11$. Assuming that the harmonic function W in (2.11) does not depend on two of the transverse coordinates (to be denoted as y_2, y_3), i.e. starting with [14] with two extra isometries, $(1_{NS}+1_R)_2$, we may apply the T-duality in these two isometric directions to obtain the type IIB solution which has the interpretation of a bound state of a fundamental string and 3-brane, $1_{NS}+3$,

$$ds^2_{10B} = \tilde{K}^{1/2}[K^{-1}(-dt^2+dy_1^2)+\tilde{K}^{-1}(dy_2^2+dy_3^2)+dx_s dx_s] \ , \qquad (2.18)$$

$$e^{2\phi}=K^{-1}\tilde{K} \ , \qquad \chi=0 \ , \qquad D_{ty_1y_2y_3}=\sin\theta\ W\tilde{K}^{-1} \ ,$$

$$B^{(1)}_{ty_1}=-\cos\theta\ WK^{-1} \ , \quad B^{(2)}_{y_2y_3}=\sin\theta\cos\theta\ W\tilde{K}^{-1} \ ,$$

where K and $\tilde{K}$ have the same form as in (2.10), and we do not list other nonvanishing components of the 4-tensor determined by self-duality of its field strength. This background interpolates between the fundamental string 1_2 ($q_1=1, q_2=0$, i.e. $\theta=0,\ \tilde{K}=1$) and the 3-brane ($q_1=0, q_2=1$, i.e. $\theta=\frac{\pi}{2},\ K=\tilde{K}$). The simple $SL(2,Z)$ transformation which interchanges the NS-NS and R-R strings and leaves 3-brane invariant then gives similar 1_R+3 background

$$ds^2_{10B}=K^{1/2}[K^{-1}(-dt^2+dy_1^2)+\tilde{K}^{-1}(dy_2^2+dy_3^2)+dx_s dx_s] \ , \quad e^{2\phi}=K\tilde{K}^{-1} \ , \qquad (2.19)$$

with the same values of D_4,χ and interchanged values of $B^{(1)}$ and $B^{(2)}$. The final step is to apply the T-duality in y_1 direction

$$ds^2_{10A}=K^{1/2}[-K^{-1}dt^2+\tilde{K}^{-1}(dy_2^2+dy_3^2)+dy_1^2+dx_s dx_s] \ , \qquad (2.20)$$

$$e^{2\phi}=K^{3/2}\tilde{K}^{-1} \ , \qquad C_{ty_2y_3}=-\sin\theta\ W\tilde{K}^{-1} \ ,$$

$$A_t=-\cos\theta\ WK^{-1} \ , \qquad B_{y_2y_3}=\sin\theta\ \cos\theta\ W\tilde{K}^{-1} \ .$$

This background can be interpreted as $(2+0)_1$ since it interpolates between the 0-brane ($q_2=0,\ \tilde{K}=1$) and the 2-brane wrapped around y_2,y_3 ($q_1=0,\ K=\tilde{K}$) in the space

[8] When reduced to $D=7$ these are U-duality transformations between the corresponding 0-brane charges.

with one extra isometry (y_1). As y_1 does not play any special role, the restriction of the extra isometry can be relaxed (i.e. y_1 can be added to x_s) obtaining the $2+0$ background which is spherically-symmetric in all 7 transverse coordinates.

This type IIA solution can now be lifted to $D=11$ (cf. (2.7),(2.9))

$$ds^2_{11} = \tilde{K}^{1/3}[-K^{-1}dt^2 + K\tilde{K}^{-1}(dy_{11} - \cos\theta\, W K^{-1}dt)^2 + \tilde{K}^{-1}(dy_2^2 + dy_3^2) + dx_i dx_i] \ , \quad (2.21)$$

$$C_3 = -\sin\theta\; W\tilde{K}^{-1} dt\wedge dy_2 \wedge dy_3 + \sin\theta\cos\theta\; W\tilde{K}^{-1} dy_{11}\wedge dy_2\wedge dy_3 \ .$$

This can be re-written simply as

$$ds^2_{11} = \tilde{H}_2^{1/3}\Big[\tilde{H}_2^{-1}(-d\tilde{t}^2 + dy_2^2 + dy_3^2) + d\tilde{y}^2_{11} + dx_i dx_i\Big] \ , \quad (2.22)$$

$$dC_3 = d\tilde{H}_2^{-1}\wedge d\tilde{t}\wedge dy_2\wedge dy_3 \ , \qquad \tilde{H}_2 \equiv \tilde{K} = 1 + \sin^2\theta\ \frac{\mathcal{Q}_q}{r^6} \ ,$$

$$\tilde{t} \equiv \frac{1}{\sin\theta}(t - \cos\theta\ y_{11}) \ , \qquad \tilde{y}_{11} \equiv \frac{1}{\sin\theta}(y_{11} - \cos\theta\ t) \ , \qquad -\tilde{t}^2 + \tilde{y}^2_{11} = -t^2 + y^2_{11} \ .$$

This is just the 2-brane (2.4) with harmonic function $H_2 = \tilde{H}_2$ boosted to a velocity $v = \cos\theta \le 1$ in the isometric transverse direction y_{11} (this background will be denoted as $2 \mapsto$). This matches perfectly with the $2+0$ interpretation of the corresponding type IIA solution!

The background (2.22) interpolates between the standard extremal 2-brane and a plane wave: for $q_1 = 1, q_2 = 0$ (i.e. infinite boost $v = 1$, $\sin\theta = 0$, $\tilde{H}_2 = \tilde{K} = 1$) we get the gravitational wave along y_{11} direction which reduces to 0-brane in $D = 10$, while for $q_1 = 0, q_2 = 1$ (i.e. zero boost $\sin\theta = 1$, $H_2 = \tilde{K} = 1 + \frac{Q}{r^6}$, $Q \equiv \mathcal{Q}$) we find just the static 2_1 which reduces to type IIA 2-brane. In that sense (2.22) can be interpreted also as a 1/2 supersymmetric non-threshold bound state of a 2-brane and a wave in transverse direction. The energy is given by the usual relativistic expression $E^2 = M^2 + p^2 \sim \sin^2\theta\ \mathcal{Q}_q^2 + \cos^2\theta\ \mathcal{Q}_q^2 = \mathcal{Q}_q^2$, with the first (second) term being zero in the infinite (zero) boost limit. This gives a 'kinematic' interpretation to the non-threshold BPS bound state mass formula. This is not surprising, given the explanation in terms of a D-brane (Born-Infeld) action for the tension of the bound state of IIB strings [37].

The above discussion admits the following generalization. If we start with the *boosted* version (2.12) of the type IIB (q_1, q_2) string we find the following sequence of 1/4 supersymmetric non-threshold bound states: $(1_{NS} + 1_R + \uparrow)_2 \longrightarrow 1_{NS} + \uparrow + 3 \longrightarrow 1_R + \uparrow + 3 \longrightarrow 2\bot 1 + 0$. The last one is a generalization of the 1/4 supersymmetric threshold configuration $2\bot 1$ of 2-brane orthogonally intersected by a fundamental string over a point where an extra 0-brane is now placed. Recalling that $2\bot 1$ is the dimensional reduction of $2\bot 2$ in $D = 11$ [8,9] it is clear that $2\bot 1 + 0$ should be a dimensional reduction of $2\bot 2 \mapsto$

where one 2-brane is finitely boosted in the transverse direction which is the longitudinal direction of the other 2-brane, i.e.

$$ds_{11}^2 = \tilde{H}_{2(1)}^{1/3} H_{2(2)}^{1/3} \big[- H_{2(1)}^{-1} H_{2(2)}^{-1} d\tilde{t}^2 + \tilde{H}_{2(1)}^{-1} (dy_2^2 + dy_3^2) + H_{2(2)}^{-1} (d\tilde{y}_{11}^2 + dy_1^2) + dx_s dx_s \big],$$

$$C_3 = \tilde{H}_{2(1)}^{-1} d\tilde{t} \wedge dy_2 \wedge dy_3 + H_{2(2)}^{-1} d\tilde{t} \wedge d\tilde{y}_{11} \wedge dy_1 \ , \qquad (2.23)$$

where $\tilde{t}, \tilde{y}_{11}$ and $\tilde{H}_{2(1)}$ are defined as above. y_2, y_3 and y_1, y_{11} are the coordinates of the two 2-branes, and the first brane is boosted along y_{11} direction of the second.

Another chain of duality transformations $(1_{NS} + 1_R)_4 \longrightarrow 1_{NS} + 5_R \longrightarrow 1_R + 5_{NS} \longrightarrow (0+5)_1 \longrightarrow 5+0$ leads to the type IIA bound state of a solitonic 5-brane and a 0-brane. Its image in $D = 11$ is simply the 5-brane finitely boosted in the transverse isometric direction, i.e. $5 \mapsto$. The corresponding background is the obvious analog of (2.22). Similarly, the $6+0$ bound state is dimensional reduction of the $D = 11$ KK monopole [32] boosted in 11-th direction to a velocity $v = \cos\theta$.

Starting with $(1_{NS} + 1_R + \uparrow)_4$ we arrive at 1/4 supersymmetric type IIA solution $5+1+0$ describing the bound state of a 5-brane with a fundamental string and a 0-brane. This is a non-threshold bound state, except for the special cases $1+0$ and $5+1$ which are at threshold. Its $D = 11$ image is $2 \perp 5 \mapsto$, i.e. the orthogonal intersection of 5-brane and 2-brane over a string with 5-brane finitely boosted along the direction of 2-brane orthogonal to it. In the limit of the infinite boost the 5-brane charge goes to zero and this configuration becomes just the longitudinally boosted 2-brane, $2+\uparrow$ (which reduces to $1+0$). The corresponding solution is the direct analog of the above expressions (cf. also (2.16))

$$ds_{11}^2 = \tilde{H}_5^{2/3} H_2^{1/3} \Big[\tilde{H}_5^{-1} H_2^{-1} (-d\tilde{t}^2 + dy_1^2) + H_5^{-1} dy_n dy_n + H_2^{-1} d\tilde{y}_{11}^2 + dx_i dx_i \Big] \ , \qquad (2.24)$$

$$dC_3 = dH_2^{-1} \wedge dt \wedge dy_1 \wedge d\tilde{y}_{11} + *d\tilde{H}_5 \wedge d\tilde{y}_{11} \ , \qquad \tilde{H}_5 \equiv 1 + \sin^2\theta \ \frac{Q_{q5}}{r^6} \ . \qquad (2.25)$$

where the 2-brane coordinates are y_1 and y_{11}.

Although the $D = 10$ solutions $1_{NS} + 1_R$, $2+0$ and $5+0$ (or their generalizations $1_{NS} + 1_R + \uparrow$, $2 \perp 1 + 0$ and $5+1+0$) are related by T and $SL(2,Z)$ dualities, their $D = 11$ counterparts – the wave along generic cycle of 2-torus, $2 \mapsto$ and $5 \mapsto$ (or 2-brane boosted in (q_1, q_2) longitudinal direction $2+\nearrow$, $2 \perp 2 \mapsto$ and $2 \perp 5 \mapsto$) do not seem to be related in any obvious way.

There is one more example of this dichotomy: $2+0$ is T-dual to $4+2$ which in turn results upon dimensional reduction [38] from another $D = 11$ solution which interpolates between the 2-brane and 5-brane backgrounds. It can be interpreted as a 2-brane lying within a 5-brane [39] (see also [40,38]). In the notation used here it is given by, cf. (2.16),(2.21)[9]

$$ds^2_{11} = K^{1/3}\tilde K^{1/3}\Big[K^{-1}(-dt^2 + dy_1^2 + dy_2^2) + \tilde K^{-1}(dy_3^2 + dy_4^2 + dy_5^2) + dx_i dx_i\Big] \ , \quad (2.26)$$

$$K = 1 + W \ , \qquad \tilde K = 1 + \sin^2\theta \ W \ , \qquad W = \frac{Q_q}{r^3} \ ,$$

$$dC_3 = \cos\theta \ dK^{-1}\wedge dt\wedge dy_1\wedge dy_2 + \sin\theta \ * dK - 6\cot\theta \ d\tilde K^{-1}\wedge dy_3\wedge dy_4\wedge dy_5 \ .$$

Since for $\theta = 0$ this background reduces to the 2-brane ($\tilde K = 1, K = H_2$) and for $\theta = \pi/2$ to the 5-brane ($K = \tilde K = H_5$), and in view of the close analogy with similar solutions discussed above, it can be interpreted as describing their 1/2 supersymmetric non-threshold bound state 2+5. Reducing to $D = 10$ along a 5-brane coordinate one finds the +2 solution, while the reduction along 2-brane coordinate gives a similar bound state of a 4-brane and a fundamental string $4+1$. The latter ($(4+1)_1$) is T-dual to a non-threshold bound state of a R-R 5-brane and a fundamental string, $5_R + 1_{NS}$, in type IIB theory, which thus has $5+2$ bound state as its $D = 11$ counterpart.

It remains an interesting question whether the above $D = 11$ solutions with boosts are somehow connected to this static solution. This may give a hint about a relation between branes and waves, similar to the T-duality relation between fundamental strings and waves.

The $D = 11$ solutions discussed above have many generalizations. One may combine the threshold intersecting solutions with longitudinal momentum along common string direction [8,9,10,11,12] with boosts of branes in transverse directions. One may also combine intersection solutions with static non-threshold $2+5$ bound states of [39] (some of such solutions were discussed in [41]).[10]

3. Correspondence between microscopic BPS states of type IIB strings and quantum supermembrane

It was suggested in [14,1] that the spectrum of BPS states of type IIB (q_1, q_2) string on a circle should be in correspondence with the BPS spectrum of the fundamental supermembrane wrapped around a 2-torus with a momentum along the (q_1, q_2) cycle of the torus. What was shown in [14] is that the zero-mode parts of the spectra do match. Since

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[9] This form of the $2+5$ solution is found if one first obtains $4+2$ by dualities from our basic starting point $1_{NS} + 1_R$ (2.11) and then lifts $4+2$ to $D = 11$.

[10] We expect that one of such solutions reduces to $4+2\perp 2+0$ bound state discussed in D-brane approach in [17].

the states in question are, in general, oscillating states it is important to check that the oscillator parts in the masses (and the constraints) also agree.

As was argued in [14], the BPS spectrum of (q_1, q_2) string should be the same as that of perturbative $(1,0)$ string provided the tension is rescaled by $\sqrt{q_1^2 + e^{-2\phi_0} q_2^2}$. The problem which we address below is how to determine the *oscillator* part of the quantum supermembrane BPS spectrum, given that this is a complicated interacting theory (assuming it is well-defined as a quantum theory in the first place).

For a membrane wrapped around the torus, the light-cone gauge Hamiltonian H turns out to contain a 'free' (quadratic) H_0 part and an interaction part H_{int}. The theory can be solved in a special limit of large torus area in which the interacting term H_{int} drops out. Our main assumption will be that the masses of the BPS states of the wrapped supermembrane do not receive corrections in decreasing the torus area while keeping fixed the torus modular parameters and the $D=10$ string tension $T_2 = 1/2\pi\alpha'$ (in this supersymmetric problem one expects that the masses of the BPS states with given charges should not receive quantum corrections depending on continuous parameters like radii of the 2-torus, i.e. masses should remain the same as for $H_{\text{int}} = 0$). Solving the gaussian theory with $H = H_0$ we find that the resulting oscillating membrane BPS states do, indeed, have the same masses as the BPS states of type IIB strings. This provides strong support for the suggestion of [14].

It should be emphasized that this correspondence between the BPS states of (q_1, q_2) string and wrapped membrane with 'quadratic' Hamiltonian is quite non-trivial, as the 'quadratic' approximation reproduces not only the perturbative NS-NS $(1,0)$ part of string spectrum but also the R-R $(0,1)$ and the general (q_1, q_2) parts as well. It is remarkable that these (non-perturbative) bound states of type IIB superstring theory admit an interpretation in terms of excitations of a fundamental supermembrane with certain values of Kaluza-Klein momentum and winding number.

This complements the relation between the (q_1, q_2) string and $D=11$ membrane with momentum flow along the (q_1, q_2) cycle of 2-torus established at the macroscopic effective field theory level in Section 2. In the case of the fundamental type II strings the classical 'string+wave' solutions are in correspondence with microscopic BPS states [18,21,22], with momentum being carried by chiral (right-moving) string oscillations. This suggests that the 'membrane+wave' solution (2.7) (more precisely, its oscillating generalization mentioned in Section 2) should also correspond to microscopic membrane BPS states with longitudinal momentum carried by membrane oscillations propagating along *one* direction – the (q_1, q_2) cycle of the torus. Indeed, as shown below, these are the membrane states whose masses coincide with the masses of the BPS states of (q_1, q_2) string.

3.1. Supermembrane theory on $\mathbf{R}^9 \times S^1 \times S^1$

The membrane states investigated here are excitations of a membrane with non-trivial winding number around the target-space torus. The spectrum of the light-cone membrane Hamiltonian in this case is discrete (an earlier study of the spectrum on $\mathbf{R}^9 \times S^1 \times S^1$ is in ref. [42]).

Let X^1 and X^2 represent the compact coordinates, with periods $2\pi R_1$ and $2\pi R_2$. X^2 will be viewed as an extra 'eleventh' coordinate absent in superstring theory. Let $\sigma, \rho \in [0, 2\pi)$ be the world-volume spatial dimensions. The transverse, single-valued coordinates X^i, $i = 3, ..., 10$, and the canonical momenta P^i can be expanded in a complete set of functions on the torus[11]

$$X^i(\sigma,\rho) = \sqrt{\alpha'} \sum_{k,m} X^i_{(k,m)} e^{ik\sigma + im\rho} \ , \quad P^i(\sigma,\rho) = \frac{1}{(2\pi)^2\sqrt{\alpha'}} \sum_{k,m} P^i_{(k,m)} e^{ik\sigma+im\rho} \ , \tag{3.1}$$

where

$$\alpha' \equiv (4\pi^2 R_2 T_3)^{-1} \ , \qquad [T_3] = cm^{-3} \ . \tag{3.2}$$

The canonical commutation relations imply

$$[X^i_{(k,m)}, P^j_{(k',m')}] = i\delta_{k+k'}\delta_{m+m'}\delta^{ij} \ . \tag{3.3}$$

For a membrane wrapped around the rectangular target-space torus so that

$$X^1(\sigma + 2\pi, \rho) = X^1(\sigma,\rho) + 2\pi w_1 R_1 \ , \quad X^2(\sigma, \rho + 2\pi) = X^2(\sigma,\rho) + 2\pi w_2 R_2 \ , \tag{3.4}$$

one has

$$X^1(\sigma,\rho) = w_1 R_1 \sigma + \tilde{X}^1(\sigma,\rho) \ , \quad \tilde{X}^1(\sigma,\rho) = \sqrt{\alpha'}\sum_{k,m} X^1_{(k,m)} e^{ik\sigma+im\rho} \ , \tag{3.5}$$

$$X^2(\sigma,\rho) = w_2 R_2 \rho + \tilde{X}^2(\sigma,\rho) \ , \quad \tilde{X}^2(\sigma,\rho) = \sqrt{\alpha'}\sum_{k,m} X^2_{(k,m)} e^{ik\sigma+im\rho} \ . \tag{3.6}$$

The winding number that counts how many times the toroidal membrane is wrapped around the target-space torus is

$$w_0 = \frac{1}{4\pi^2 R_1 R_2} \int d\sigma d\rho \ \{X^1, X^2\} = w_1 w_2 \ , \qquad \{X, Y\} \equiv \partial_\sigma X \partial_\rho Y - \partial_\rho X \partial_\sigma Y \ . \tag{3.7}$$

A membrane with $w_0 \neq 0$ is topologically protected against usual supermembrane instabilities [43]. It is convenient to choose the light-cone gauge to remove the single-valued

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[11] We shall use the indices k, l for the Fourier components in σ, and indices m, n, p for Fourier components in ρ.

part $\tilde{X}^1$ of X^1, i.e. $X^+ = x^+ + \alpha' p^+$, $X^\pm \equiv (X^0 \pm \tilde{X}^1)/\sqrt{2}$. The standard light-cone Hamiltonian of the supermembrane is given by [44,45]

$$H = H_{\rm B} + H_{\rm F} \ ,$$

$$H_{\rm B} = 2\pi^2 \int d\sigma d\rho \left[P_a^2 + \tfrac{1}{2} T_3^2 (\{X^a, X^b\})^2 \right] \ , \tag{3.8}$$

$$H_{\rm F} = -T_3 p^+ \int d\sigma d\rho \ \bar{\theta} \Gamma_a \{X^a, \theta\} \ , \tag{3.9}$$

where $a = 1, 2, ..., 10$ and $\tilde{X}^1 = 0$. Here θ_α $(\alpha = 1, ..., 16)$ is a real $SO(9)$ spinor. The (mass)^2 operator $M^2 = 2p^+p^- - p_a^2$ is given by $2H - p_a^2$, where p_a is the center-of-mass momentum of the membrane, $p_a = \int d\sigma d\rho P_a$. For simplicity of presentation, in what follows we will omit the fermionic terms in the formulas (their inclusion is straightforward, see, e.g., [42,46]).

The Hamiltonian has a residual symmetry associated with area-preserving diffeomorphims, which can be fixed by setting

$$\tilde{X}_2(\sigma, \rho) = X_2^C(\rho) \equiv \sqrt{\alpha'} \sum_m X_{2(0,m)} e^{im\rho} \ . \tag{3.10}$$

One can split $P_2(\sigma, \rho) = P_2^C + \hat{P}_2$, where $P_2^C = P_2^C(\rho)$ belongs to the (Cartan) subspace generated by $e^{im\rho}$, and $\hat{P}_2$ to the complement. The local constraints can be solved for $\hat{P}_2$ in terms of $\tilde{X}_2$ and the transverse coordinates and momenta X_i, P_i. $\hat{P}_2$ can be ignored in the problem of BPS masses we are interested here (in the large radius limit considered below $\hat{P}_2$ gives subleading contributions of order $O(\frac{1}{R_2})$, see ref. [46]).

Separating the winding contributions, we can put the Hamiltonian in the form

$$H_{\rm B} = H_0 + H_{\rm int} \ , \tag{3.11}$$

$$H_0 = 2\pi^2 T_3^2 A w_0^2 + 2\pi^2 \int d\sigma d\rho \big[P_a^2 + T_3^2 R_2^2 w_2^2 \big(\partial_\sigma X_a\big)^2 + T_3^2 R_1^2 w_1^2 \big(\partial_\rho X_a\big)^2 \big] \ , \tag{3.12}$$

$$H_{\rm int} = \pi^2 T_3^2 \int d\sigma d\rho \big(\{X^a, X^b\}\big)^2 \ , \qquad A = 4\pi^2 R_1^2 R_2^2 \ , \tag{3.13}$$

where $i = 3, ..., 10$ and $a, b = 2, ..., 10$, and $X_2 = X_2^C$. As discussed in [46], the spectrum of this Hamiltonian is discrete provided $w_0 = w_1 w_2 \neq 0$ (for $w_0 = 0$ one or both terms in the quadratic part of the potential vanish, flat directions remain, leading to a continuous spectrum). Inserting the expansions (3.1), (3.5) and (3.6), we obtain

$$\alpha' H_0 = \frac{1}{2} \sum_{k,m} \left[P^a_{(k,m)} P^a_{(-k,-m)} + \omega_{km}^2 X^a_{(k,m)} X^a_{(-k,-m)} \right] + \frac{R_1^2 w_0^2}{2\alpha'} \ , \tag{3.14}$$

$$\alpha' H_{\rm int} = \frac{1}{4g^2} \sum_{m,n,p} \sum_{k,l,l'} (pk' - m'k)(nl - ml')\ X^a_{(l',n)} X^a_{(k,p)}\ X^b_{(k',m')} X^b_{(l,m)}\ , \tag{3.15}$$

where $m' = -m - n - p$, $k' = -k - l - l'$ and

$$g^2 \equiv \frac{R_2^2}{\alpha'} = 4\pi^2 R_2^3 T_3\ , \qquad \omega_{km} = \sqrt{w_2^2 k^2 + w_1^2 m^2 \tau_2^2}\ , \qquad \tau_2 = \frac{R_1}{R_2}\ . \tag{3.16}$$

g has the interpretation of type IIA string coupling. The operator $\alpha' H_0$ is the generator of translations in the world-volume time τ. For the center-of-mass coordinate $X^i_{(0,0)}$ one obtains the simple time dependence $X^a_{(0,0)}(\tau) = x^a + \alpha' p^a \tau$.

The system described by (3.14),(3.15) can be represented as a direct product of the classical zero-mode system (which is known explicitly for all values of R_1, R_2) and a complicated interacting system of oscillation modes, which determines quantum corrections to the masses of states. The latter can be solved exactly in the special limit $R_1, R_2 \to \infty, T_3 \to 0$ with α' and τ_2 fixed since then the interaction term drops out.[12] In this limit $g^2 \to \infty$, and the Hamiltonian becomes that of an infinite set of harmonic oscillators labelled by (k, m). In this supersymmetric model, one may argue that since the *quantum* corrections to the masses of BPS states should not depend on continuous parameters, their values should thus be the same as in this limiting case, i.e. in the case when the interaction term drops out. This is what we shall assume below.

Let us introduce the standard creation and annihilation operators

$$X^i_{(k,m)} = \frac{1}{2}(\omega_{km})^{-1/2}\big(a^i_{(k,m)} + i b^i_{(k,m)} + a^{i\dagger}_{(k,m)} + i b^{i\dagger}_{(k,m)}\big)\ , \quad X^i_{(-k,-m)} = \big(X^i_{(k,m)}\big)^\dagger\ , \tag{3.17}$$

$$P^i_{(k,m)} = -\frac{i}{2}(\omega_{km})^{1/2}\big(a^i_{(k,m)} + i b^i_{(k,m)} - a^{i\dagger}_{(k,m)} - i b^{i\dagger}_{(k,m)}\big)\ , \quad P^i_{(-k,-m)} = \big(P^i_{(k,m)}\big)^\dagger \tag{3.18}$$

for $k = 0, 1, 2, ...$, $m = 1, 2, ...$, and,

$$X^i_{(k,-m)} = \tfrac{1}{2}(\omega_{km})^{-1/2}\big(c^i_{(k,m)} + i d^i_{(k,m)} + c^{i\dagger}_{(k,m)} + i d^{i\dagger}_{(k,m)}\big)\ , \tag{3.19}$$

$$P^i_{(k,-m)} = -\frac{i}{2}(\omega_{km})^{1/2}\big(c^i_{(k,m)} + i d^i_{(k,m)} - c^{i\dagger}_{(k,m)} - i d^{i\dagger}_{(k,m)}\big)\ , \tag{3.20}$$

for $k = 1, 2, ...$, $m = 0, 1, 2, ...$ Similar operators are introduced for $X_{2(0,m)}$, $P_{2(0,m)}$, understanding that $a_{2(k,m)} = b_{2(k,m)} = c_{2(k,m)} = d_{2(k,m)} = 0$ if $k \neq 0$. Then the quadratic part of the Hamiltonian becomes[13]

$$\alpha' H_0 = \tfrac{1}{2}\alpha'(p_i^2 + p_1^2 + p_2^2) + \frac{R_1^2 w_0^2}{2\alpha'} + \mathcal{H}\ , \tag{3.21}$$

[12] Note that if, instead, we take the limit $R_2 \to \infty$ with R_1 fixed (or vice versa), then flat directions remain for the constant modes in σ (or for the constant modes in ρ). The system is then equivalent to the one discussed in ref. [46], with an infinite number of zero modes.

[13] We are ignoring normal ordering constants since they will cancel out once fermionic contributions are incorporated; see below.

$$\mathcal{H} = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\omega_{km}\left[a^{a\dagger}_{(k,m)}a^{a}_{(k,m)} + b^{a\dagger}_{(k,m)}b^{a}_{(k,m)}\right] + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\omega_{km}\left[c^{i\dagger}_{(k,m)}c^{i}_{(k,m)} + d^{i\dagger}_{(k,m)}d^{i}_{(k,m)}\right].$$

Using eq. (3.3) one can check the standard commutation rules of the form $[a, a^\dagger] = 1$. The time dependence of the mode operators is $a^i_{(k,m)}(\tau) = e^{i\omega_{km}\tau}a^i_{(k,m)}(0)$, etc.

It is convenient to define also another set $(\alpha, \tilde{\alpha})$ of mode operators

$$a^a_{(k,m)} = i\frac{\alpha^a_{(k,m)} - \tilde{\alpha}^a_{(k,m)}}{\sqrt{2\omega_{km}}} \ , \quad b^a_{(k,m)} = \frac{\alpha^a_{(k,m)} + \tilde{\alpha}^a_{(k,m)}}{\sqrt{2\omega_{km}}} \ , \tag{3.22}$$

$$c^a_{(k,m)} = i\frac{\alpha^a_{(k,-m)} - \tilde{\alpha}^a_{(k,-m)}}{\sqrt{2\omega_{km}}} \ , \quad d^a_{(k,m)} = \frac{\alpha^a_{(k,-m)} + \tilde{\alpha}^a_{(k,-m)}}{\sqrt{2\omega_{km}}} \ , \tag{3.23}$$

satisfying $(k, m = 1, 2, ...)$

$$[\alpha^i_{(k,m)}, \alpha^j_{(k',m')}] = \omega_{(k,m)}\delta_{k+k'}\delta_{m+m'}\delta^{ij} \ , \qquad \omega_{(k,m)} = \epsilon(k)\ \omega_{km} \ , \tag{3.24}$$

where $\epsilon(k)$ is the sign function, and similar relations for the $\tilde{\alpha}^i_{(k,m)}$. Then the oscillator part of the Hamiltonian becomes

$$\mathcal{H} = \tfrac{1}{2}\sum_{m,k}^{\infty}\left[\alpha^a_{(-k,-m)}\alpha^a_{(k,m)} + \tilde{\alpha}^a_{(-k,-m)}\tilde{\alpha}^a_{(k,m)}\right] \ . \tag{3.25}$$

Explicitly, the time dependence of X^i is as follows

$$X^a = x^a + \alpha' p^a\tau + i\sqrt{\tfrac{\alpha'}{2}}\sum_{(k,m)\neq(0,0)}\omega^{-1}_{(k,m)}\left[\alpha^a_{(k,m)}e^{ik\sigma+im\rho} + \tilde{\alpha}^a_{(k,m)}e^{-ik\sigma-im\rho}\right]\ e^{i\omega_{(k,m)}\tau}.$$

The requirement that X^- is single-valued leads to the following global constraints for the physical states (see e.g. [45,42,46])

$$\mathbf{P}^{(\sigma)} = \frac{1}{2\pi\alpha'}\int_0^{2\pi}d\sigma\ \partial_\sigma X^a\dot{X}^a \equiv 0 \ , \quad \mathbf{P}^{(\rho)} = \frac{1}{2\pi\alpha'}\int_0^{2\pi}d\rho\ \partial_\rho X^a\dot{X}^a \equiv 0 \ . \tag{3.26}$$

The operators $\mathbf{P}^{(\sigma)}$, $\mathbf{P}^{(\rho)}$ generate translations in σ and ρ. These conditions were discussed in ref. [46] for the space $\mathbf{R}^{10}\times S^1$. It is easy to generalize the expressions for the present case. Let the momenta in the directions X^1 and X^2 be given by

$$p_1 = \frac{l_1}{R_1} \ , \quad p_2 = \frac{l_2}{R_2} \ , \qquad l_1, l_2 \in \mathbf{Z} \ .$$

Inserting the expansions for X^a in terms of the mode operators we find[14]

$$N^+_\sigma - N^-_\sigma = w_1 l_1 \ , \qquad N^+_\rho - N^-_\rho = w_2 l_2 \ , \tag{3.27}$$

[14] There are changes in the notation relative to ref. [46]. Now $N^+_\sigma, N^-_\sigma, N^+_\rho, N^-_\rho$ stand for $\mathbf{N}, \tilde{\mathbf{N}}, \mathbf{N}^+, \mathbf{N}^-$, and $\tilde{\alpha}_{(k,-m)} \to \tilde{\alpha}_{(k,m)}$. Also, in [46], $w_1 = 0$, $w_2 = 1$, $Q = l_2$.

where

$$N_\sigma^+ = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{km}} \alpha^i_{(-k,-m)} \alpha^i_{(k,m)} \ , \qquad N_\sigma^- = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{km}} \tilde{\alpha}^i_{(-k,-m)} \tilde{\alpha}^i_{(k,m)} \ , \tag{3.28}$$

$$N_\rho^+ = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{m}{\omega_{km}} \left[\alpha^a_{(-k,-m)} \alpha^a_{(k,m)} + \tilde{\alpha}^a_{(-k,m)} \tilde{\alpha}^a_{(k,-m)} \right] \ , \tag{3.29}$$

$$N_\rho^- = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{m}{\omega_{km}} \left[\alpha^a_{(-k,m)} \alpha^a_{(k,-m)} + \tilde{\alpha}^a_{(-k,-m)} \tilde{\alpha}^a_{(k,m)} \right] \ . \tag{3.30}$$

As usual, the Fock vacuum $|0\rangle$ is defined as the state annihilated by the $a^i_{(k,m)}$, $b^i_{(k,m)}$, $c^i_{(k,m)}$, $d^i_{(k,m)}$, and $p_a|0\rangle = 0$, and the Fock space is generated by the states constructed by successive applications of the creation operators to the vacuum. The physical Hilbert space thus consists of all states in the Fock space obeying the conditions (3.27).

3.2. Matching the membrane and type IIB string BPS spectra

The 'nine-dimensional' membrane mass operator is given by

$$M^2 = 2p^+p^- - p_i^2 = 2H_0 - p_i^2 = \frac{l_1^2}{R_1^2} + \frac{l_2^2}{R_2^2} + \frac{w_0^2 R_1^2}{\alpha'^2} + \frac{2}{\alpha'} \mathcal{H} \ , \tag{3.31}$$

where $\mathcal{H}$ is given by (3.25). Our aim is now to compare the BPS part of this membrane spectrum with the type IIB (q_1, q_2) string BPS spectrum given in [14], where the correspondence between the spectra was established at the zero-mode level. We shall show that it extends to the oscillator level as well.

From the ten-dimensional string-theory point of view, the Kaluza-Klein momentum $p_2 = l_2/R_2$ corresponds to the Ramond-Ramond charge. Let us first consider the perturbative or $(1,0)$ string states with $l_2 = 0$, the winding number l_1, the Kaluza-Klein momentum w_0/R'_1 and the mass (note that under T-duality relating IIA and IIB spectra $R_1 \to R'_1 = \alpha'/R_1$)

$$M_{\mathrm{IIB}}^2 = \frac{l_1^2 {R'_1}^2}{\alpha'^2} + \frac{w_0^2}{{R'_1}^2} + \frac{2}{\alpha'}(N_R + N_L) \ , \qquad N_R - N_L = l_1 w_0 \ , \tag{3.32}$$

and compare general BPS states (having $N_L = 0$ or $N_R = 0$) with the corresponding $(l_2 = 0)$ oscillating states in the membrane spectrum (3.31). For a BPS saturated state, the mass should take the minimal possible value compatible with the charges. For the membrane, these are the states with $N_\sigma^- = N_\rho^- = 0$. The constraint equations (3.27) become

$$N_\sigma^+ = l_1 w_1 \ , \qquad N_\rho^+ = 0 \ . \tag{3.33}$$

The condition $N^+_\rho = 0$ implies that such states are constructed by applications of $\alpha^i_{(-k,0)}$ to the vacuum. For these states, $\omega_{k0} = w_2 k$, and (cf. (3.25), (3.28)) $\mathcal{H} = w_2 N^+_\sigma = w_1 w_2 l_1$, and so that

$$M^2 = \frac{l_1^2}{R_1^2} + \frac{w_0^2 R_1^2}{\alpha'^2} + \frac{2}{\alpha'} l_1 w_1 w_2 = \left(\frac{l_1}{R_1} + \frac{w_0 R_1}{\alpha'} \right)^2 . \tag{3.34}$$

This coincides with the masses of perturbative string BPS states obtained upon setting $N_L = 0$ in (3.32).

The mass formula for perturbative states of a type IIB string with charges (q_1, q_2) given in [14] is

$$M^2_{\rm IIB} = \frac{l_1^2 R_1'^2}{\alpha'^2} + \frac{w_0^2}{R_1'^2} + \frac{l_2^2}{R_2^2} + 4\pi T_{(q_1,q_2)} (N_L + N_R) , \tag{3.35}$$

$$N_R - N_L = n w_0 , \tag{3.36}$$

where $l_1 = n q_1, \ l_2 = n q_2$, with q_1, q_2 being co-prime, and

$$T_{(q_1,q_2)} = \frac{T_2}{n} \sqrt{l_1^2 + \tau_2^2 l_2^2} , \qquad T_2 = (2\pi\alpha')^{-1} , \qquad \tau_2 = \frac{R_1}{R_2} = e^{-\phi_0} . \tag{3.37}$$

Eq. (3.35) should be exact for BPS states [14]. The case of NS-NS string (3.35) corresponds to $l_2 = 0, q_1 = 1, q_2 = 0$. In the case of the R-R string $l_1 = 0, \ q_1 = 0, q_2 = 1$ one finds that for BPS states with $N_L = 0$

$$M^2_{\rm IIB} = \frac{l_2^2}{R_2^2} + \frac{w_0^2 R_1^2}{\alpha'^2} + 4\pi T_{(0,1)} \big(N_L + N_R\big) = \left(\frac{l_2}{R_2} + \frac{w_0 R_1}{\alpha'} \right)^2 . \tag{3.38}$$

In the supermembrane spectrum (3.31) the minimal mass for given charges is obtained for states with $N^-_\rho = N^-_\sigma = 0$. For the counterpart of the R-R states with $l_1 = 0$, the constraints (3.27) are similar to (3.33) $N^+_\sigma = 0, \ N^+_\rho = l_2 w_2$. N^+_σ is a sum of positive definite terms, so it is equal to zero only if each term in it vanishes. The most general physical state satisfying $N^+_\sigma = N^-_\sigma = 0$ is thus obtained by acting by $\alpha^i_{(0,-m)}$ on the vacuum. On these states, N^+_ρ and $\mathcal{H}$ take the form (see (3.25))

$$N^+_\rho = \frac{1}{w_1 \tau_2} \sum_{m=1}^{\infty} \alpha^a_{(0,-m)} \alpha^a_{(0,m)} , \qquad \mathcal{H} = w_1 \tau_2 N^+_\rho = l_2 w_0 \tau_2 , \tag{3.39}$$

so that

$$M^2 = \frac{l_2^2}{R_2^2} + \frac{w_0^2 R_1^2}{\alpha'^2} + \frac{2}{\alpha'} \tau_2 l_2 w_0 = \left(\frac{l_2}{R_2} + \frac{w_0 R_1}{\alpha'} \right)^2 , \tag{3.40}$$

in agreement with the type IIB string expression (3.38).

The previous two examples of comparison with the $(1,0)$ and $(0,1)$ string spectra illuminate the fact that the BPS states in the supermembrane spectrum have excitations only in one direction – along the momentum vector. This is also what is suggested by correspondence between the (q_1,q_2) string and wrapped membrane with momentum along (q_1,q_2) cycle at the level of the classical solutions discussed in Section 2. Let us now identify the relevant oscillation states in the supermembrane spectrum for generic (q_1,q_2). Performing the rotation as in (2.6) $y_1=\cos\theta\ X_1+\sin\theta\ X_2$, $\ y_2=-\sin\theta\ X_1+\cos\theta\ X_2$ (with $q_2\to\frac{R_1}{R_2}q_2$ in (2.5), i.e. $\tan\theta=q_2R_1/q_1R_2$, for generic R_1,R_2) we may align the momentum with the direction y_1. The map between the target-space torus and the toroidal membrane surface is given by the zero-mode part

$$y_1^0=w_1R_1\cos\theta\ \sigma+w_2R_2\sin\theta\ \rho\ =\frac{R_1}{\sqrt{q_1^2+q_2^2\tau_2^2}}(w_1q_1\sigma+w_2q_1\rho)\ , \tag{3.41}$$

$$y_2^0=-w_1R_1\sin\theta\ \sigma+w_2R_2\cos\theta\ \rho\ =\frac{R_1}{\sqrt{q_1^2+q_2^2\tau_2^2}}(-w_1q_2\sigma+w_2q_1\rho)\ .$$

For an oscillation mode $\alpha^i_{(k,m)}e^{i(k\sigma+m\rho)}$ the factor in the exponent can be written as

$$k\sigma+m\rho=\Big(\frac{k}{w_1R_1}\cos\theta+\frac{m}{w_2R_2}\sin\theta\Big)y_1^0+\Big(-\frac{k}{w_1R_1}\sin\theta+\frac{m}{w_2R_2}\cos\theta\Big)y_2^0\ ,$$

so that the the condition that there are no oscillations along y_2 becomes $\alpha^i_{(k,m)}=0$ if $w_2R_2k\sin\theta\neq w_1R_1m\cos\theta$, i.e. if $w_2kq_2\neq w_1mq_1$. Thus the relevant states are constructed using $\alpha^i_{(-k,-m_0)}$ with

$$m_0=\frac{w_2q_2}{w_1q_1}k\ . \tag{3.42}$$

For such states,

$$\mathcal{H}=\sum_{k=1}^{\infty}\alpha^i_{(-k,-m_0)}\alpha^i_{(k,m_0)}\ ,\qquad \omega_{km_0}=\frac{kw_2}{q_1}\sqrt{q_1^2+q_2^2\tau_2^2}\ ,$$

and the constraints become (understanding that $\alpha^i_{(k,m_0)}=0$ if $m_0\notin\mathbf{Z}$)

$$N_\sigma^+=\frac{q_1}{w_2\sqrt{q_1^2+q_2^2\tau_2^2}}\sum_{k=1}^{\infty}\alpha^i_{(-k,-m_0)}\alpha^i_{(k,m_0)}=l_1w_1\ , \tag{3.43}$$

$$N_\rho^+=\frac{q_2}{w_1\sqrt{q_1^2+q_2^2\tau_2^2}}\sum_{k=1}^{\infty}\alpha^i_{(-k,-m_0)}\alpha^i_{(k,m_0)}=l_2w_2\ . \tag{3.44}$$

The membrane BPS mass formula is then

$$M^2=\frac{l_1^2}{R_1^2}+\frac{l_2^2}{R_2^2}+\frac{w_0^2R_1^2}{\alpha'^2}+\frac{2\mathcal{H}}{\alpha'}\ , \tag{3.45}$$

with

$$\mathcal{H} = \frac{w_2}{q_1}\sqrt{q_1^2 + q_2^2\tau_2^2}\; N_\sigma^+ \;\; = \;\; w_0\sqrt{l_1^2 + l_2^2\tau_2^2}\; . \tag{3.46}$$

Using (3.43), (3.45) and (3.46), we thus obtain

$$M^2 = \left(\sqrt{\frac{l_1^2}{R_1^2} + \frac{l_2^2}{R_2^2}} + \frac{w_0 R_1}{\alpha'}\right)^2 . \tag{3.47}$$

Remarkably, this agrees with the Schwarz string mass formula (3.35), (3.36) for BPS states with $N_L = 0$.

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