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R–R Scalars, U–Duality and Solvable Lie Algebras[1]

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Abstract

We consider the group theoretical properties of R–R scalars of string theories in the low–energy supergravity limit and relate them to the solvable Lie subalgebra $\mathbb{G}_s \subset \mathbf{U}$ of the U–duality algebra that generates the scalar manifold of the theory: $\exp[\mathbb{G}_s] = U/H$. Peccei–Quinn symmetries are naturally related with the maximal abelian ideal $\mathcal{A} \subset \mathbb{G}_s$ of the solvable Lie algebra. The solvable algebras of maximal rank occurring in maximal supergravities in diverse dimensions are described in some detail. A particular example of a solvable Lie algebra is a rank one, $2(h_{2,1}+2)$–dimensional algebra displayed by the classical quaternionic spaces that are obtained via c–map from the special Kählerian moduli spaces of Calabi–Yau threefolds.

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1 Introduction

One of the most intriguing features of string theory, in its perturbative formulation, is the existence of two kind of scalars, those coming from the Neveu–Schwarz sector (N–S) and those coming from the Ramond–Ramond sector (R–R). The former fields have sometimes the interpretation of moduli of a conformal field–theory, while the latter do not have such a property [1]. However, in an effective lagrangian formulation for the string light–states, the R–R scalars are linked to N–S scalars by supersymmetries exchanging left and right movers and more interestingly by U–dualities [2], which, for continuous transformations, are related to the non–compact symmetries present in extended supergravities [3], [4]. These symmetries, not present in the perturbative string spectrum, are conjectured to be symmetries at the non–perturbative level, at least under the restriction $U \rightarrow U(\mathbb{Z})$. Indeed supergravity theories in diverse dimensions [5] constitute a nested web filling a plane whose axes are the space–time dimensions D and the number of supersymmetry charges N. Our recently improved understanding of non perturbative string theory has taught us to regard all the lagrangians in the web as different effective actions describing the interaction of the light fields in different corners of a single *quantum theory*. The glue that keeps the various parts of the web together is provided by duality transformations [6]. Although the ideas are conceptually new their mathematical realization occurs by means of structures that have been known for many years. Indeed the relevant duality transformation groups are nothing else but the well known hidden symmetries of supergravity governing the structure of the scalar sector [7]. In every dimension D and for each value of N the n_s scalar fields φ^I can be interpreted, at least locally, as the coordinates of an appropriate Riemanian manifold $\mathcal{M}_{scalar}$ whose metric $g_{IJ}(\varphi)$ appears in the scalar kinetic term

$$\mathcal{L}^{scal}_{kin} = \frac{1}{2}\, g_{IJ}(\varphi)\, \partial^\mu \varphi^I\, \partial_\mu \varphi^J \tag{1}$$

Let U be the group of isometries (if any) of the scalar metric $g_{IJ}(\varphi)$. The elements of U correspond to global symmetries of the σ–model lagrangian $\mathcal{L}^{scal}_{kin}$. If the action of U were not extended from the scalar fields also to the other fields and in particular to the vector fields or higher rank p–form potentials, U could not be promoted to a symmetry of the full theory. This is clear from the fact that the scalar fields are related to the spinor and vector fields, and/or $(p+1)$–form potentials, by supersymmetry transformations. The implementation of U isometries in a supersymmetric consistent way is the basic issue of supergravity hidden symmetries. Indeed a generic feature of supergravity theories at $D=4$ is that they have a symmetry under "duality" which acts non linearly on the scalars and linearly on the field-strengths and their duals, that are fitted together into a single suitable symplectic representation of U [8].

In diverse dimensions D the U–duality group acts linearly on generic $(p+1)$–potentials unless $p+2=\frac{D}{2}$ in which case it acts linearly on the field strengths and their duals [5].

The low energy supergravities divide into two classes:

1. the first, containing the $D=4$, $N\leq 2$ and the $D=5$, $N=2$ cases, is the class where the scalar manifold $\mathcal{M}_{scalar}$ can admit isometries, but it is not necessarily a homogeneous space U/H

2. All the other theories have scalar manifolds which are necessarily homogeneus coset manifolds U/H and this class comprises, in particular, the $D=4$, $N>2$ theories and all the maximally extended supergravities in $D\leq 11$

In the first class of theories the local scalar geometries defined at string tree level acquire perturbative and non–perturbative quantum corrections. In the second class, which will be the main focus of this paper, the local scalar geometry given by the natural Riemannian metric defined on U/H is protected by supersymmetry against quantum corrections.

In this paper we investigate the solvable algebra $Solv(U/H)=\mathbb{G}_s\subset U$ that generates the Riemannian manifold U/H in such a way that $\exp[\mathbb{G}_s]=U/H$ and hence $\dim\mathbb{G}_s=\dim U-\dim H$.

Solvable Lie algebras appeared first in the supergravity literature to classify quaternionic manifolds with a transitive, solvable group of motions [9], [10][11] [12][13].

The construction of $\mathbb{G}_s$ is available in standard textbooks [14]. It suffices to say that $\mathbb{G}_s$ contains a non compact Cartan part $\mathcal{H}_K$ which is the abelian set of semisimple generators in $\mathbb{K}$ defined by the Cartan decomposition

$$\mathbf{U}=\mathbb{H}\oplus\mathbb{K} \tag{2}$$

and other nilpotent generators coming from the positive roots of $\mathbf{U}$. Among them, of particular relevance is the maximal abelian ideal $\mathcal{A}$ whose elements have the physical interpretation of being the translational (Peccei–Quinn) isometries of the theory.

The use of the solvable Lie algebra representation allows one to regard the coset manifold U/H as the group manifold of the corresponding solvable group with its own advantages. For instance all the geometric notions (metric, connection, curvature) are translated into an algebraic language and, in particular, one obtains an intrinsic privileged set of coordinates for the manifold where each scalar field is in one to one correspondence with a generator of the solvable algebra.

The natural question which arises is therefore that of finding an intrinsic algebraic characterization of the R–R scalars (i.e. R–R generators) with respect to the N–S scalars (i.e. N–S generators), which is otherwise obscure in the effective supergravity formalism.

In this paper we show how to obtain this characterization by decomposing the U–duality algebra, and hence its solvable subalgebra, with respect to its S–duality and T–duality subalgebras. Indeed the distinction between Neveu–Schwarz (N–S) and Ramond (R–R) scalars is T–duality invariant.

Mastering the structure of the solvable Lie algebra appears to be relevant in different respects. A particularly significant one is partial supersymmetry breaking. It appears from recent results [15], [16], obtained in the context of N=2 theories, that partial SUSY breaking $N = 2 \rightarrow N = 1$, with zero vacuum energy, can be obtained precisely by gauging generators in the maximal abelian ideal $\mathcal{A} \subset \mathbb{G}_s$ of the solvable algebra [16]. In this respect fields of the maximal abelian ideal contain the flat directions after gauging.

We expect the same to be true in other extended theories with the eigenvalues of the gravitino mass matrix parametrized by the charges of the fields with respect to $\mathcal{A}$. It goes without saying that, whenever these charges are of R–R type [17] they carry a non–perturbative significance so that knowledge of $\mathcal{A}$ and of its N–S, R–R splitting is a fundamental prerequisite. This is the information we present in this paper.

It is hoped that some of the properties outlined in the present investigation may also be useful to explore features of R–R scalars when the moduli space gets quantum corrected. This is known to occur, through D–two branes instanton effects in type IIA theory [18] compactified on Calabi–Yau manifolds, as a consequence of second quantized mirror symmetry [19]. In particular such corrections should resolve conifold singularities [20], [23], [24] in the construction of quaternionic manifolds by c–map [13] from the complex structure moduli space of Calabi–Yau threefolds.

2 Solvable Lie Algebras: the machinery.

In this section we will deal with a general property according to which any homogeneous non-compact coset manifold may be expressed as a group manifold generated by a suitable solvable Lie algebra. [9]

Let us start by giving few preliminar definitions. A *solvable* Lie algebra $\mathbb{G}_s$ is a Lie algebra whose n^{th} order (for some $n \geq 1$) derivative algebra vanishes:

$$\begin{aligned} \mathcal{D}^{(n)}\mathbb{G}_s &= 0 \\ \mathcal{D}\mathbb{G}_s = [\mathbb{G}_s, \mathbb{G}_s] \quad ; &\qquad \mathcal{D}^{(k+1)}\mathbb{G}_s = [\mathcal{D}^{(k)}\mathbb{G}_s, \mathcal{D}^{(k)}\mathbb{G}_s] \end{aligned}$$

A *metric* Lie algebra $(\mathbb{G}, h)$ is a Lie algebra endowed with an euclidean metric h. An important theorem states that if a Riemannian manifold $(\mathcal{M}, g)$ admits a transitive group of isometries $\mathcal{G}_s$ generated by a solvable Lie algebra $\mathbb{G}_s$ of the same dimension as $\mathcal{M}$, then:

$$\begin{aligned} \mathcal{M} \sim \mathcal{G}_s &= exp(\mathbb{G}_s) \\ g_{|e\in\mathcal{M}} &= h \end{aligned}$$

where h is an euclidean metric defined on $\mathbb{G}_s$. Therefore there is a one to one correspondence between Riemannian manifolds fulfilling the hypothesis stated above and solvable metric Lie algebras $(\mathbb{G}_s, h)$.
Consider now an homogeneous coset manifold $\mathcal{M} = \mathcal{G}/\mathcal{H}$, $\mathcal{G}$ being a non compact real form of a semisimple Lie group and $\mathcal{H}$ its maximal compact subgroup. If $\mathbb{G}$ is the Lie algebra generating $\mathcal{G}$, the so called Iwasawa decomposition ensures the existence of a solvable Lie subalgebra $\mathbb{G}_s \subset \mathbb{G}$, acting transitively on $\mathcal{M}$, such that [14]:

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{G}_s \qquad \dim\, \mathbb{G}_s = \dim\, \mathcal{M} \tag{3}$$

$\mathbb{H}$ being the maximal compact subalgebra of $\mathbb{G}$ generating $\mathcal{H}$.
In virtue of the previously stated theorem, $\mathcal{M}$ may be expressed as a solvable

group manifold generated by $\mathbb{G}_s$. The algebra $\mathbb{G}_s$ is constructed as follows [14]. Consider the Cartan decomposition

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \tag{4}$$

Let us denote by $\mathcal{H}_K$ the maximal abelian subspace of $\mathbb{K}$ and by $\mathcal{H}$ the Cartan subalgebra of $\mathbb{G}$. It can be proven [14] that $\mathcal{H}_K = \mathcal{H} \cap \mathbb{K}$, that is it consists of all non compact elements of $\mathcal{H}$. Furthermore let h_{α_i} denote the elements of $\mathcal{H}_K$, $\{\alpha_i\}$ being a subset of the positive roots of $\mathbb{G}$ and Φ^+ the set of positive roots β not orthogonal to all the α_i (i.e. the corresponding "shift" operators E_β do not commute with $\mathcal{H}_K$). It can be demonstrated that the solvable algebra $\mathbb{G}_s$ defined by the Iwasawa decomposition may be expressed in the following way:

$$\mathbb{G}_s = \mathcal{H}_K \oplus \{ \sum_{\alpha \in \Phi^+} E_\alpha \cap \mathbb{G} \} \tag{5}$$

where the intersection with $\mathbb{G}$ means that $\mathbb{G}_s$ is generated by those suitable complex combinations of the "shift" operators which belong to the real form of the isometry algebra $\mathbb{G}$.

The *rank* of an homogeneous coset manifold is defined as the maximum number of commuting semisimple elements of the non compact subspace $\mathbb{K}$. Therefore it coincides with the dimension of $\mathcal{H}_K$, i.e. the number of non compact Cartan generators of $\mathbb{G}$. A coset manifold is *maximally non compact* if $\mathcal{H} = \mathcal{H}_K \subset \mathbb{G}_s$. The relevance of maximally non compact coset manifolds relies on the fact that they are spanned by the scalar fields in the maximally extended supergravity theories.

As an example of the procedure just described we will work out the solvable Lie algebra corresponding to the manifold [12]

$$\mathcal{M} = \frac{SU(1,n+2)}{U(1) \otimes SU(n+2)} \tag{6}$$

whose rank is one. Indeed, if we express the roots α of $SU(1,n+2)$ as $\epsilon_i - \epsilon_j$, $1 \leq i < j \leq n+3$, the only non compact element of the Cartan subalgebra of $SU(1,n+2)$ is $H_{\epsilon_1 - \epsilon_{n+3}}$, (H_α, E_α) denoting the canonical basis of the $SU(1,n+2)$ algebra. The positive roots of Φ^+ are the $(2n+3)$ of the form $\epsilon_1 - \epsilon_i, \epsilon_j - \epsilon_{(n+3)}$ ($i = 2, \cdots, n+3$, $j = 2, \cdots, n+2$). According to (5), the generators of $\mathbb{G}_s$ are:

$$\{H_{\epsilon_1 - \epsilon_{n+3}}, (E_{\epsilon_1 - \epsilon_i}, E_{\epsilon_j - \epsilon_{(n+3)}}) \cap SU(1,n+2)\} \tag{7}$$

Defining $x_i = E_{\epsilon_1-\epsilon_i}$, $y^i = E_{\epsilon_i-\epsilon_{(n+3)}}$, $i = 2, \cdots, n+2$, $h = H_{\epsilon_1-\epsilon_{n+3}}/2$ and $z = E_{\epsilon_1-\epsilon_{n+3}}$, one can check that these generators fulfill the commutation relations in equations (9), (10) of next section, which characterize the action of the isometries of a special quaternionic manifold on the R–R scalars.

3 c–map, special quaternionic manifolds and their solvable Lie algebra

The simplest example of solvable Lie algebra occuring in an effective supergravity theory is found while considering the c–map [13] of special Kähler manifolds in the context of string compactifications on Calabi–Yau threefolds. In this context the classical quaternionic geometry of $N = 2$ hypermultiplets for type II strings is given by *special quaternionic manifolds* $\mathcal{SQM}$ of real dimension $\dim_{\mathbf{R}} \mathcal{SQM} = 4\, h_{(2,1)} + 4$ of which half are Ramond–Ramond scalars C_Λ, $(\Lambda = 0, 1 \cdots h_{(2,1)})$ and the other half are Neveu–Schwarz scalars. The latter include the axion–dilaton degree of freedom $\mathcal{S}$ and the $h_{(2,1)}$ moduli z^i $(i = 1 \cdots h_{(2,1)})$ of the Calabi–Yau threefold complex structures.

In [25] it was observed that a generic special quaternionic metric has always a $4 + 2h_{(2,1)}$ dimensional group of isometries which act on the R–R (complex) scalars C_Λ and the axion–dilaton system S as follows:

$$\begin{aligned}
\mathcal{S}' &= \mathcal{S} + \mathrm{i}\alpha - 2\, C_\Lambda\, \gamma^\Lambda - \gamma^\Lambda\, \mathcal{N}_{\Lambda\Sigma}\, \gamma^\Sigma \\
C'_\Lambda &= C_\Lambda + \mathrm{i}\beta_\Lambda + \mathcal{N}_{\Lambda\Sigma}\, \gamma^\Sigma \\
\mathcal{S}' &= \lambda\, \mathcal{S} \\
C'_\Lambda &= \lambda^{1/2}\, C_\Lambda
\end{aligned} \tag{8}$$

where α, λ, $\gamma^\Lambda, \beta_\Lambda$, $(\Lambda = 0, \cdots, h_{(2,1)})$ are real parameters and $\mathcal{N}_{\Lambda\Sigma}$ is a symmetric matrix depending on the moduli $z^i, \overline{z}^i$. For infinitesimal α, λ, $\gamma^\Lambda, \beta_\Lambda$ transformations the corresponding generators z, h, y^Λ, x_Λ satisfy the following Lie algebra:

$$\begin{aligned}
\left[x_\Lambda\, ,\, y^\Sigma\right] &= \delta^\Sigma_\Lambda\, z \\
\left[x_\Lambda\, ,\, z\right] &= \left[y^\Lambda\, ,\, z\right] = \left[x_\Lambda\, ,\, x_\Sigma\right] = \left[y^\Lambda\, ,\, y^\Sigma\right] = 0
\end{aligned} \tag{9}$$

$$[h\, ,\, z] = z; \quad [h\, ,\, x_\Sigma] = \frac{1}{2}\, x_\Sigma; \quad \left[h\, ,\, y^\Sigma\right] = \frac{1}{2}\, y^\Sigma \tag{10}$$

where eq. 9 define a $2\,h_{(2,1)}+3$ nilpotent Lie algebra. When extended with the h generator it becomes a $2\,h_{(2,1)}+4$ solvable Lie algebra with a one–dimensional Cartan subalgebra $\mathcal{H}_S = h$.

The Lie algebra in eq.s (9), (10) is nothing else but the solvable Lie algebra $Solv\left(\mathcal{F}_{h_{(2,1)}}\right)$ generating the coset

$$\mathcal{F}_{h_{(2,1)}} \equiv \frac{SU(1,h_{(2,1)}+2)}{U(1)\otimes SU(h_{(2,1)}+2)} \tag{11}$$

This is is simply a consequence of the fact that the special quaternionic manifolds can be viewed as a $\mathcal{F}_{h_{(2,1)}}$–fibration over the $h_{(2,1)}$ dimensional Special Kählerian moduli space. In other words, the fiber above each point in moduli space is diffeomorphic and isometric to $\mathcal{F}_{h_{(2,1)}}$. This is the pointwise splitting into the special Kähler base manifold and the R–R + axion–dilaton fiber. The maximal abelian ideal of $Solv\left(\mathcal{F}_{h_{(2,1)}}\right)$ has therefore dimension $h_{(2,1)}+2$ of which $h_{(2,1)}+1$ are Ramond generators and 1 is a Neveu–Schwarz generator.

4 Maximal rank solvable Lie algebras: N–S and R–R scalars for maximal supergravities in diverse dimensions

Let us consider the list of maximally extended supergravities that are obtained dimensionally reducing $D=11$ supergravity [26] on a $(11-D)$–torus, and keeping all the massless modes. In this case the U-duality algebra is $E_{11-D(11-D)}$ [7], namely that real section of the complex Lie algebra E_{11-D} which is maximally non-compact. To explain the notations: by $E_{n(r)}$ we denote the real form of the rank n complex Lie algebra E_n, where $r \le n$ Cartan generators are non–compact: when $r=n$ all Cartan generators are non compact and from section 2 we know that this is the case where the total number of non–compact generators is maximum. Indeed, when $\mathcal{H}_K = \mathcal{H}$ all the positive roots are included in the solvable Lie algebra. This latter has therefore the universal simple form:

$$\mathbb{G}_s = \mathcal{H} \oplus \sum_{\alpha\in\Phi^+} E^\alpha \tag{12}$$

where $\mathcal{H}$ is the Cartan subalgebra, E^α is the root–space corresponding to the root α and Φ^+ denotes the set of positive roots of the U–duality group ($E_{11-D(11-D)}$). The scalar fields parametrize the coset manifold $E_{11-D(11-D)}/H$ where H is the maximal compact subgroup $H \subset E_{11-D(11-D)}$. The number $r = 11-D$, which is the rank of both the U–duality algebra and of the scalar manifold, is by its own definition the number of compactified dimensions.

In fact the Cartan semisimple piece $\mathcal{H} = O(1,1)^{11-D}$ of the solvable Lie algebra has the physical meaning of [3] diagonal moduli for the T_{11-D} compactification torus (roughly speaking the radii of the $11-D$ circles) (in modern language, this is the M–theory interpretation) [27].

From a stringy (type IIA) perspective one of them is the dilaton and the others are the Cartan piece of the maximal rank solvable Lie algebra generating the moduli space $\frac{O(10-D,10-D)}{O(10-D)\otimes O(10-D)}$ of the T_{10-D} torus.

This trivially implies that the Cartan piece is always in the N–S sector.

We are interested in splitting the maximal solvable subalgebra (12) into its N–S and R–R parts. To obtain this splitting, as already mentioned in the introduction, we just have to decompose the U–duality algebra U with respect to its ST–duality subalgebra $ST \subset U$ [29], [27].[4] We have:

$$\begin{array}{rcrl} 5 \le D \le 9 & : & ST = & O(1,1) \otimes O(10-D, 10-D) \\ D = 4 & : & ST = & Sl(2, \mathbb{R}) \otimes O(6,6) \\ D = 3 & : & ST = & O(8,8) \end{array} \tag{13}$$

Correspondingly we obtain the decomposition:

$$\begin{array}{rclcl} 5 \le D \le 9 & : & \mathrm{adj}\, E_{11-D(11-D)} & = & \mathrm{adj} O(1,1) \oplus \mathrm{adj} O(10-D, 10-D) \\ & & & & \oplus \Big(2, \mathrm{spin}_{(10-D,10-D)}\Big) \\ D = 4 & : & \mathrm{adj}\, E_{7(7)} & = & \mathrm{adj} Sl(2,\mathbb{R}) \oplus \mathrm{adj} O(6,6) \oplus \Big(2, \mathrm{spin}_{(6,6)}\Big) \\ D = 3 & : & \mathrm{adj}\, E_{8(8)} & = & \mathrm{adj} O(8,8) \oplus \mathrm{spin}_{(8,8)} \end{array} \tag{14}$$

From (14) it follows that:

$$\begin{array}{rcl} 5 \le D \le 9 & : & \mathrm{dim} E_{11-D(11-D)} = 1 + (10-D)(19-2D) + 2^{(10-D)} \\ D = 4 & : & \mathrm{dim} E_{7(7)} = \mathrm{dim}[(\mathbf{66}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{32})] \\ D = 3 & : & \mathrm{dim} E_{8(8)} = \mathrm{dim}[\mathbf{120} \oplus \mathbf{128}] \end{array} \tag{15}$$

[3]Similar reasonings appear in refs.[21][22]

[4]Note that at $D = 3$, ST–duality merge in a simple Lie algebra [30][31].

The dimensions of the maximal rank solvable algebras are instead:

$$
\begin{aligned}
5 \le D \le 9 \quad &: \quad \dim \mathbb{G}_s = (10-D)^2 + 1 + 2^{(9-D)} = \dim \frac{U}{H} \\
D = 4 \quad &: \quad \dim \mathbb{G}_s = 32 + 37 + 1 = \dim \frac{U}{H} \\
D = 3 \quad &: \quad \dim \mathbb{G}_s = 64 + 64 = \dim \frac{U}{H}
\end{aligned} \tag{16}
$$

The above parametrizations of the dimensions of the cosets listed in Table 1 can be traced back to the fact that the N–S and R–R generators are given respectively by:

$$
\text{N–S} = \text{Cartan generators} \oplus \text{positive roots of adj}\, ST \tag{17}
$$

and

$$
\text{R–R} = \text{positive weights of spin}_{ST} \tag{18}
$$

In this way we have:

$$
\begin{aligned}
\dim(\text{ N–S}) \quad &= \quad \begin{cases} (10-D)^2 + 1 \qquad (5 \le D \le 9) \\ 38 = 7 + 1 + 30 \qquad (D = 4) \\ 64 = 8 + 56 \qquad (D = 3) \end{cases} \\
\dim(\text{ R–R}) \quad &= \quad \begin{cases} 2^{(9-D)} \qquad (5 \le D \le 9) \\ 32 \qquad (D = 4) \\ 64 \qquad (D = 3) \end{cases}
\end{aligned} \tag{19}
$$

For $D = 3$ we notice that the ST–duality group $O(8,8)$ is a non compact form of the $\mathbb{H}$ maximal compact subgroup $O(16)$ of the U–duality group $E_{8(8)}$.

This explains why R–R = N–S = 64 in this case. Indeed, 64 are the positive weigths of $\dim(\text{spin}_{16}) = 128$. This coincides with the counting of the bosons in the Clifford algebra of $N = 16$ supersymmetry at $D = 3$.

In Table 2 we give, for each of the previously listed cases, the dimension of the maximal abelian ideal $\mathcal{A}$ of the solvable algebra and its N–S, R–R content [7], [2].

$D=9$	$E_{2(2)} \equiv SL(2,\mathrm{I\!R}) \otimes O(1,1)$	$H=O(2)$	$\dim_{\mathbf{R}}(U/H) = 3$
$D=8$	$E_{3(3)} \equiv SL(3,\mathrm{I\!R}) \otimes Sl(2,\mathrm{I\!R})$	$H=O(2)\otimes O(3)$	$\dim_{\mathbf{R}}(U/H) = 7$
$D=7$	$E_{4(4)} \equiv SL(5,\mathrm{I\!R})$	$H=O(5)$	$\dim_{\mathbf{R}}(U/H) = 14$
$D=6$	$E_{5(5)} \equiv O(5,5)$	$H=O(5)\otimes O(5)$	$\dim_{\mathbf{R}}(U/H) = 25$
$D=5$	$E_{6(6)}$	$H=Usp(8)$	$\dim_{\mathbf{R}}(U/H) = 42$
$D=4$	$E_{7(7)}$	$H=SU(8)$	$\dim_{\mathbf{R}}(U/H) = 70$
$D=3$	$E_{8(8)}$	$H=O(16)$	$\dim_{\mathbf{R}}(U/H) = 128$

Table 1: U–duality groups and maximal compact subgroups of maximally extended supergravities.

D	dim $\mathcal{A}$	N–S	R–R
3	36	14	22
4	27	11	16
5	16	8	8
6	10	6	4
7	6	4	2
8	3	2	1
9	1	0	1

Table 2: Maximal abelian ideals.

5 Electric subgroups

In view of possible applications to the gauging of isometries of the four dimensional U–duality group, which may give rise to spontaneous partial supersymmetry breaking with zero–vacuum energy [15], [16], it is relevant to answer the following question: what is the electric subgroup[5] of the solvable group? Furthermore, how many of its generators are of N–S type and how many are of R–R type? Here as an example we focus on the maximal $N=8$ supergravity in $D=4$. To solve the problem we have posed we need to con-

[5]By "electric" we mean the group which has a lower triangular symplectic embedding, i.e. is a symmetry of the lagrangian [32], [33].

sider the splitting of the U–duality symplectic representation pertaining to vector fields, namely the **56** of $E_{7(7)}$, under reduction with respect to the ST–duality subgroup. The fundamental **56** representation defines the symplectic embedding:

$$E_{7(7)} \longrightarrow Sp(56, \mathrm{I\!R}) \tag{20}$$

We have:

$$\mathbf{56} \stackrel{Sl(2,R)\otimes SO(6,6)}{\longrightarrow} (\mathbf{2},\mathbf{12}) \oplus (\mathbf{1},\mathbf{32}) \tag{21}$$

This decomposition is understood from the physical point of view by noticing that the 28 vector fields split into 12 N–S fields which, together with their magnetic counterparts, constitute the $(\mathbf{2},\mathbf{12})$ representation plus 16 R–R fields whose electric and magnetic field strenghts build up the *irreducible* **32** spinor representation of $O(6,6)$. From this it follows that the T–duality group is purely electric only in the N–S sector [27]. On the other hand the group which has an electric action both on the N–S and R–R sector is $Sl(8,R)$. This follows from the alternative decomposition of the **56** [4], [34]:

$$\mathbf{56} \stackrel{Sl(8,R)}{\longrightarrow} \mathbf{28} \oplus \mathbf{28} \tag{22}$$

We can look at the intersection of the ST–duality group with the maximal electric group:

$$SL(2,\mathrm{I\!R}) \otimes O(6,6) \cap Sl(8,\mathrm{I\!R}) = Sl(2,R) \otimes Sl(6,\mathrm{I\!R}) \otimes O(1,1). \tag{23}$$

Consideration of this subgroup allows to split into N–S and R–R parts the maximal electric solvable algebra. Let us define it. Let $Solv\left(E_{7(7)}/SU(8)\right)$ be the complete solvable algebra. The electric part is defined by:

$$Solv_{el} \equiv Solv\left(E_{7(7)}/SU(8)\right) \cap Sl(8,\mathrm{I\!R}) = Solv\left(Sl(8,\mathrm{I\!R})/O(8)\right) \tag{24}$$

Hence we have that:

$$\dim_{\mathbf{R}} Solv_{el} = 35 \tag{25}$$

One immediately verifies that the non–compact coset manifold $Sl(8,\mathrm{I\!R})/O(8)$ has maximal rank, namely $r = 7$, and therefore the electric solvable algebra has once more the standard form as in eq.12 where $\mathcal{H}$ is the Cartan subalgebra of $Sl(8,\mathrm{I\!R})$, which is the same as the original Cartan subalgebra of $E_{7(7)}$ and the sum on positive roots is now restricted to those that belong to $Sl(8,R)$.

These are 28. On the other hand the adjoint representation of $Sl(8,\mathbb{R})$ decomposes under the $Sl(2,\mathbb{R})\otimes Sl(6,\mathbb{R})\otimes O(1,1)$ as follows

$$\mathbf{63} \overset{Sl(2,\mathbf{R})\otimes Sl(6,\mathbf{R})\otimes O(1,1)}{\longrightarrow} (\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{6},\mathbf{2}) \quad (26)$$

Therefore the N–S generators of the electric solvable algebra are the 7 Cartan generators plus the $16 = 1\oplus 15$ *positive roots* of $Sl(2,\mathbb{R})\otimes Sl(6,\mathbb{R})$. The R–R generators are instead the *positive weights* of the $(\mathbf{2},\mathbf{6},\mathbf{2})$ representation. We can therefore conclude that:

$$\dim_{\mathbf{R}} Solv_{el} = 35 = 12\text{R–R} \oplus [(15+1)+7]\,\text{N–S} \quad (27)$$

Finally it is interesting to look for the maximal abelian subalgebra of the electric solvable algebra. It can be verified that the dimension of this algebra is 16, corresponding to 8 R–R and 8 N–S.

6 Considerations on non–maximally extended supergravities

Considerations similar to the above can be made for all the non maximally extended or matter coupled supergravities for which the solvable Lie algebra is not of maximal rank. Indeed, in the present case, the set of positive roots entering in formula (5) is a proper subset of the positive roots of U, namely those which are not orthogonal to the whole set of roots defining the non–compact Cartan generators. As an example, let us analyze the coset $\frac{O(6,22)}{O(6)\otimes O(22)} \otimes \frac{Sl(2,\mathbf{R})}{U(1)}$ corresponding to a $D=4$, $N=4$ supergravity theory obtained compactifying type IIA string theory on $K_3\times T_2$ [35], [36], [2]. The product $Sl(2,\mathbb{R})\otimes O(6,22)$ is the U–duality group of this theory, while the ST–duality group is $Sl(2,\mathbb{R})\otimes O(4,20)\otimes O(2,2)$. The latter acts on the moduli space of $K_3\times T_2$ and on the dilaton–axion system. Decomposing the U–duality group with respect to the ST duality group $Sl(2,\mathbb{R})\otimes O(4,20)\otimes O(2,2)$ we get:

$$\begin{aligned}\mathrm{adj}(Sl(2,\mathbb{R})\otimes O(6,22)) &= \mathrm{adj}Sl(2,\mathbb{R}) \\ &+ \mathrm{adj}O(4,20) + \mathrm{adj}O(2,2) + (\mathbf{1},\mathbf{24},\mathbf{4}) \quad (28)\end{aligned}$$

The R–R fields belong to the subset of positive roots of U contributing to $\mathbb{G}_s$ which are also positive weights of the ST–duality group, namely in this case those defining the $(\mathbf{1},\mathbf{24},\mathbf{4})$ representation. This gives us 48 R–R fields. The N–S fields, on the other hand, are selected by taking those positive roots of U entering the definition of $\mathbb{G}_s$, which are also positive roots of ST, plus those corresponding to the non–compact generators $(\mathcal{H}_k)$ of the U–Cartan subalgebra.

In our case we have:

$$
\begin{aligned}
&\mathrm{dim}U = \mathrm{dim}O(6,22) + \mathrm{dim}Sl(2,\mathbb{R}) = 381 \\
&\#\text{ of positive roots of U } = 183 \\
&\#\text{ of positive roots of U not contributing to } \mathbb{G}_s \\
&\qquad\qquad = 183 - (\mathrm{dim}U/H - \mathrm{rank}U/H) = 56 \\
&\mathrm{dim}ST = \mathrm{dim}O(4,20) + \mathrm{dim}O(2,2) + \mathrm{dim}Sl(2,\mathbb{R}) = 285 \\
&\#\text{ of positive roots of ST } = \frac{1}{2}(285-15) = 135 \\
&\#\text{ of positive roots of ST contributing to } \mathbb{G}_s \ = 135 - 56 = 79 \\
&\#\text{ of N–S } = 79 + \mathrm{rank}U/H = 79 + 7 = 86 \\
&\mathrm{dim}(U/H) = \mathrm{dim}\mathbb{G}_s = 48 + 86 = 134.
\end{aligned} \tag{29}
$$

The maximal abelian ideal $\mathcal{A}$ of $\mathbb{G}_s$ has dimension 64 of which 24 correspond to R–R fields while 40 to N–S fields.

In an analogous way one can compute the number of N–S and R–R fields for other non maximally extended supergravity theories.

7 Conclusions

In this note we used a particular parametrization of non compact coset spaces underlying various duality symmetries in terms of solvable Lie algebras. In this way we found a natural splitting between R–R and N–S scalars. For maximal supergravities the associated cosets, and therefore the solvable algebras, have maximal rank while this is not the case for non maximal and/or matter coupled supergravities.

The generators of the maximal abelian ideal of solvable Lie algebras correspond to the Peccei–Quinn symmetries of the theory.

Part of them pertain to the R–R scalars and part to the N–S scalars. Contrary to naive reasoning R–R scalars do not always correspond to translational symmetries. This can be traced back to Chern–Simons couplings in the original theory.

Partial supersymmetry breaking with vanishing cosmological constant appears also to be related to the gauging of nilpotent generators of the solvable Lie algebra.

It is hoped that some of the aspects of solvable Lie algebra discussed in this paper may unreveal some nonperturbative properties underlying superstring dynamics.

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