CERN-TH/96-311
hep-th/9611011

Two-Dimensional Chiral Matrix Models and String Theories

Ivan K. Kostov *†
C.E.A. - Saclay, Service de Physique Théorique
F-91191 Gif-Sur-Yvette, France

Matthias Staudacher◇
CERN, Theory Division
CH-1211 Geneva 23, Switzerland

We formulate and solve a class of two-dimensional matrix gauge models describing ensembles of non-folding surfaces covering an oriented, discretized, two-dimensional manifold. We interpret the models as string theories characterized by a set of coupling constants associated to worldsheet ramification points of various orders. Our approach is closely related to, but simpler than, the string theory describing two-dimensional Yang-Mills theory. Using recently developed character expansion methods we exactly solve the models for target space lattices of arbitrary internal connectivity and topology.

SPhT-96/123

CERN-TH/96-311
November 1996

* member of CNRS

† kostov@spht.saclay.cea.fr

◇ matthias@nxth04.cern.ch

1. Introduction

The idea that the strong interactions are described by a string theory which is in some sense dual to perturbative QCD is a major challenge for high energy theory. More generally, a D-dimensional confining Yang-Mills theory is expected to define a string theory with D-dimensional target space stable in the interval $2 \leq D \leq 4$.

This "YM string" has been constructed so far only for $D = 2$ [1],[2]. It has been shown that the partition function of pure $U(N)$ gauge theory defined on a two-dimensional manifold of given genus and area can be represented in terms of a weighted sum over maps from a worldsheet Σ_W to spacetime Σ_T. The allowed worldsheet configurations represent *minimal area maps* $\Sigma_W \to \Sigma_T$. The latter condition is equivalent to the condition that the embedded surfaces are not allowed to have *folds*. Ramification points are however allowed, as it has been suggested in earlier studies [3]. In other words, the path integral of the "YM_2 string" is over all *branched covers* of the target manifold [1], [2].

Unfortunately, this construction is highly involved and does not easily reduce to a system of simple geometrical principles[1]. One can nevertheless speculate, using the analogy with the random-walk representation of the $O(N)$ model, that there exists an underlying $2D$ string theory with clear geometrical interpretation. The YM_2 string is obtained from the latter by tuning the interactions due to ramification points and adding new contact interactions via microscopic tubes, etc.

One is thus led to look for the most general $2D$ string theory whose path integral is given by branched covers of the target space. Any such string theory is invariant under area-preserving diffeomorphisms of the target space Σ_T. Its partition function depends on Σ_T only through its genus and area. In addition to the topological coupling constant N^{-1} there is a set of couplings t_n associated with the interactions due to ramification points of order n.

In this letter we propose a discretization of the path integral for such string theories, in which the target space Σ_T represents a two-dimensional simplicial complex with given topology. The branch points (the images of the ramification points) are thus located at the vertices of the target lattice Σ_T. We will not try to establish a worldsheet description but instead construct and solve a class of equivalent matrix models. Our approach is

[1] The random surface representation of YM_2 on a lattice found in [4] is geometrically clear but has the inconvenience of being highly redundant. For example, the folds are not forbidden but their contribution vanishes as a result of cancellations.

thus analogous to quantizing Polyakov string theory by employing randomly triangulated surfaces.

The matrix model associated with the target space Σ_T will be formulated as a $2D$ lattice gauge theory whose local link variables are complex $N \times N$ matrices. However, the matrices are no longer unitary: the Haar measure on $U(N)$ is replaced by a Gaussian measure. The model resembles very much the $2D$ Weingarten model [5] with the important difference that the lattice action represents a sum over the *positively* oriented cells only. The latter restriction eliminates from the string path integral all surfaces containing folds (which are believed to cause the trivial critical behavior of the standard Weingarten model [6]). We will demonstrate that these $2D$ matrix models are exactly solvable for any target space lattice and any N by employing the character expansion methods recently developed in [7], [8], [9], [10]. We pay special attention to the cases of spherical and toroidal topology. In the first case we observe that the string theory exists only for a target space with sufficiently large area. At the critical area a third-order transition takes place due to the entropy of the ramification points. In the second case we find the same partition function as the chirally perturbed conformal field theory considered recently by R. Dijkgraaf [11]. In this letter we restrict our attention to the discrete case since it involves combinatorial problems interesting on their own. The continuum limit of an infinitely dense target lattice will be considered in detail elsewhere [12].

2. Definition of the models

By target space Σ_T we will understand an oriented triangulated surface (two-dimensional simplicial complex) of genus G containing $\mathcal{N}_0$ points, $\mathcal{N}_1$ links and $\mathcal{N}_2$ two-dimensional cells. These numbers are related by the Euler formula

$$\mathcal{N}_0 - \mathcal{N}_1 + \mathcal{N}_2 = 2 - 2G. \tag{2.1}$$

In addition, each cell c is characterized by its area A_c.

We will first consider the simplest string theory with this target space, in which all ramification points have Boltzmann weight one, and the corresponding matrix model. The generalized model, to be considered in sect. 6, will depend on a set of external field variables associated with the points of Σ_T.

At each link ℓ is defined a field variable Φ_ℓ representing an $N \times N$ matrix with complex elements. The partition function of the matrix model is defined as

$$\mathcal{Z} = \int \prod_{\ell} [\mathcal{D}\Phi_\ell] \prod_{c} \ \exp(\beta_c \ N \ \mathrm{Tr}\Phi_c), \tag{2.2}$$

where $\beta_c = e^{-A_c}$, Φ_c denotes the ordered product $\prod_{\ell \in \partial c} \Phi_\ell$ of link variables along the oriented boundary ∂c of the cell c, and the integration over the link variables is performed with the Gaussian measure

$$[\mathcal{D}\Phi_\ell] = (N/\pi)^{N^2} \Pi_{ij} d\Phi_{ij} d\Phi^*_{ij} e^{-N \ \mathrm{Tr} \ \Phi_\ell \Phi_\ell^\dagger}. \tag{2.3}$$

Note that we omit the complex conjugate $\Phi^\dagger_c$ from the plaquette action, eliminating thereby orientation reversing plaquettes. This is why we call the model with partition function (2.2) a *chiral* matrix model. The model is invariant under complex conjugation of the matrix variables *and* reversing the orientation of the target space.

The perturbative expansion of (2.2) results in a representation of the free energy $\mathcal{F} = \ln \mathcal{Z}$ in terms of connected lattice surfaces Σ_W embedded in the target surface Σ_T. Each of these surfaces is obtained by placing plaquettes on the faces of the lattice Σ_T and gluing any two together along the edges. All plaquettes should have the same orientation, which means that the surfaces cannot have folds. It is geometrically evident that the number of plaquettes, say n, covering each cell is constant throughout Σ_T. Thus the surface Σ_W is wrapping n times Σ_T and its area is $A_W = nA_T$ where $A_T = \sum_c A_c$ is the total area of the target space. The surface Σ_W may have ramification points whose images are points of Σ_T. The map $\Sigma_W \to \Sigma_T$ defines at each point $p \in \Sigma_T$ a branching number B_p. The branch points are the points with $B_p \neq 0$. By the Riemann-Hurwitz formula $2g - 2 = n(2G - 2) + \sum_p B_p$ where g is the genus of the surface Σ_W. The free energy $\mathcal{F} = \ln \mathcal{Z}$ of our chiral model can be written as

$$\mathcal{F} = \sum_{n=1}^{\infty} e^{-nA_T} \sum_{g=G}^{\infty} N^{2-2g} \mathcal{F}(n, g|G, \mathcal{N}_0) \tag{2.4}$$

where $\mathcal{F}(n, g|G, \mathcal{N}_0)$ is the number of no-fold surfaces of genus g covering n times Σ_T. This number depends on Σ_T only through its genus G and the number of the $\mathcal{N}_0$ of allowed locations of the branched points.

3. Exact solution by the character expansion method

Applying to (2.2) the same strategy as in ref.[10], we expand the exponential of the action for each cell c as a sum over the characters χ_h of the polynomial representations of $GL(N)$. We use the shifted weights $h = \{h_1, h_2, \ldots, h_N\}$ where h_i are related to the lengths $m_1, ..., m_N$ of the rows of the Young tableau by $h_i = N - i + m_i$ and are therefore subjected to the constraint $h_1 > h_2 > \ldots > h_N \geq 0$. We will denote by $|h| = \Sigma_i m_i$ the total number of boxes of the Young tableau and by

$$\Delta_h = \prod_{i<j} \frac{h_i - h_j}{i - j}. \tag{3.1}$$

the dimension of the representation h. The character expansion of the exponential then reads

$$e^{\beta_c N \mathrm{Tr} \Phi_c} = \sum_h \beta_c^{|h|} \, \frac{\Delta_h}{\Omega_h} \, \chi_h(\Phi_c), \tag{3.2}$$

where we have defined

$$\Omega_h = N^{-|h|} \prod_{i=1}^{N} \frac{h_i!}{(N-i)!}. \tag{3.3}$$

Now the model can be diagonalized in h-space by using the following two simple but powerful facts. Firstly, one easily proves [7],[10]

$$\int [\mathcal{D}\Phi] \; \chi_h(\Phi\Phi^\dagger) = \Omega_h \; \Delta_h, \tag{3.4}$$

where Ω_h is given as above by (3.3). Secondly, one uses that complex matrices may be diagonalized by a bi-unitary transformation $U \Phi_D V^\dagger$, where U, V are unitary and Φ_D is a positive definite and diagonal matrix. Then one uses the formula (3.4) together with the fission and fusion rules for unitary matrices

$$\begin{aligned} \int \mathcal{D}U \; \chi_h(AUBU^\dagger) &= \frac{1}{\Delta_h} \; \chi_h(A) \; \chi_h(B), \\ \int \mathcal{D}U \; \chi_h(AU) \; \chi_{h'}(U^\dagger B) &= \delta_{h,h'} \; \frac{1}{\Delta_h} \; \chi_h(AB), \end{aligned} \tag{3.5}$$

to derive fission and fusion rules for complex matrices corresponding to the measure (2.3):

$$\begin{aligned} \int [\mathcal{D}\Phi] \; \chi_h(A\Phi B\Phi^\dagger) &= \frac{\Omega_h}{\Delta_h} \; \chi_h(A) \; \chi_h(B), \\ \int [\mathcal{D}\Phi] \; \chi_h(A\Phi) \; \chi_{h'}(\Phi^\dagger B) &= \delta_{h,h'} \; \frac{\Omega_h}{\Delta_h} \; \chi_h(AB). \end{aligned} \tag{3.6}$$

Therefore, complex matrices behave exactly like unitary matrices apart from the extra local factor Ω_h for each integration[2].

Having at hand (3.6), we can proceed with the exact solution of (2.2), just as in the gauge theory case [13],[14]. Using the fusion rule one progressively eliminates links between adjoining cells until one is left with a single cell. The remaining links along that plaquette are integrated out by applying the fission rule. One arrives at the formula

$$\mathcal{Z} = \sum_h \Delta_h^{2-2G}\ \Omega_h^{\mathcal{N}_0+2G-2}\ e^{-|h|A_T}. \tag{3.7}$$

We observe that, in accord with the surface representation (2.4), the solution's only dependence on the original target space lattice data is through the genus G, the number of vertices $\mathcal{N}_0$, and the total area A_T of the target space. It is therefore clear that a given character expansion (3.7) corresponds to many different complex multi matrix models (2.2) with the same partition function: As many, as we can form simplicial complexes differing in their internal connectivity but constrained by the global data G, $\mathcal{N}_0$ and A_T. (This is a lattice version of the invariance with respect to area-preserving automorphisms.) In particular we can represent the target space by a one-cell complex and write down a matrix model with a minimal number $\mathcal{N}_1 = \mathcal{N}_0 - 1 + 2G$ of link variables Φ_ℓ giving the expansion (3.7):

$$\mathcal{Z} = \int \prod_{\ell=1}^{\mathcal{N}_0-1+2G} [\mathcal{D}\Phi_\ell]\ \exp\left(e^{-A_T}\ N\ \mathrm{Tr}\Big[\prod_{s=1}^{G} \Phi_{2s-1}\Phi_{2s}\Phi_{2s-1}^\dagger\Phi_{2s}^\dagger \prod_{k=2G+1}^{\mathcal{N}_0-1+2G} \Phi_k\Phi_k^\dagger \Big] \right). \tag{3.8}$$

Note that the trace in the exponent of (3.8) corresponds to the element of the homotopy group of a surface with G handles and N_0 punctures which is equivalent to the identity (Fig. 1).

---

[2] The fission and fusion rules actually hold for any $U(N)$ invariant measure $[\mathcal{D}\Phi]$, if we define the factor Ω_h by the integral (3.4). In particular one could consider potentials higher than linear in $\Phi\Phi^\dagger$ in (2.3).

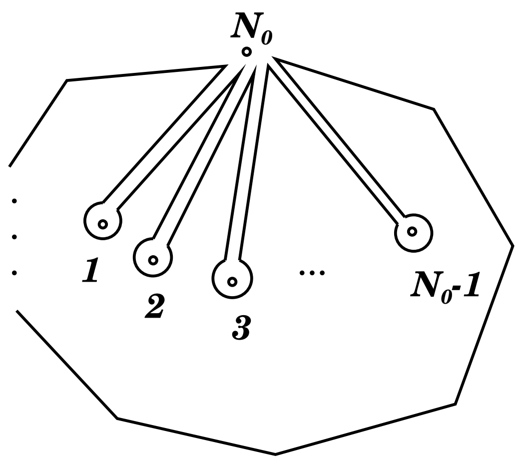

Fig.1: The target space of the effective one-plaquette model.

4. Spherical target space

In the case of a spherical target space ($G = 0$) the partition function (3.7) becomes:

$$\mathcal{Z}_{G=0} = \sum_h \Delta_h^2 \ \Omega_h^{\mathcal{N}_0 - 2} \ e^{-|h|A_T}. \tag{4.1}$$

The simplest example of $\mathcal{N}_0 = 2$ is described by a Gaussian integral. There is only one surface covering n times a sphere with two punctures and it contributes $\frac{1}{n} e^{-nA_T}$ to the free energy (the factor $\frac{1}{n}$ coming from the cyclic symmetry). No higher-genus surfaces are contributing. Summing on n gives therefore

$$\mathcal{F} = -N^2 \ln(1 - e^{-A_T}), \tag{4.2}$$

which is also the result of the Gaussian integration.

If we have $\mathcal{N}_0 \geq 3$ the situation becomes non-trivial. Now there are contributions from non-spherical ($g > 0$) world sheets as well. Concentrating on the leading ($g = 0$, i.e. $N = \infty$) term, we can proceed by using the saddlepoint techniques of [7]. Here the sum (4.1) describes a "gas" of mutually repulsive weights at thermal equilibrium and is dominated by the contribution of a "classical" configuration $\{h\}$. One introduces continuum variables $h = \frac{h_i}{N}$ and a density $\rho(h) = \frac{1}{N} \sum_i \delta(h - h_i)$. The saddlepoint density is found by functionally varying (4.1) with respect to h; this gives

$$⨍_b^a dh' \frac{\rho(h')}{h - h'} = \frac{2 - \mathcal{N}_0}{2} \log h + \frac{1}{2} A_T - \log \frac{h}{h - b}. \tag{4.3}$$

It is straightforward to explicitly solve this equation and calculate the free energy. Here we will merely write out the equation determining the critical behavior; one finds, for the variable $u = \frac{1}{4}(\sqrt{a} + \sqrt{b})^2$, the algebraic equation

$$(\mathcal{N}_0 - 1)\ e^{-A_T}\ u^{\mathcal{N}_0 - 1} = u - 1. \tag{4.4}$$

This equation predicts for $\mathcal{N}_0 \geq 3$ a critical area $A_T = A_T^c > 0$: The sum over surfaces diverges if the area of the target manifold falls below A_T^c. A quite similar transition has already observed in YM_2 by Douglas and Kazakov [15]. One finds for the free energy near the critical area

$$\mathcal{F} \sim N^2 (A_T - A_T^c)^{\frac{5}{2}} \tag{4.5}$$

i.e., the same critical behavior as the Polyakov string in zero dimensions. We can understand the critical behavior by considering e.g. $\mathcal{N}_0 = 3$

$$\mathcal{Z} = \int [\mathcal{D}\Phi_1][\mathcal{D}\Phi_2]\ e^{qN\ \mathrm{Tr}\Phi_1\Phi_1^\dagger\Phi_2\Phi_2^\dagger}. \tag{4.6}$$

where $q = e^{-A_T}$. Eq.(4.6) generates planar diagrams which can be interpreted as abstract plaquettes without embedding. Its critical behavior is in the university class of pure gravity.

For $\mathcal{N}_0 \geq 4$ the models (3.8) are intractable without our character expansion techniques. The qualitative behavior remains the same: As seen from (4.4), we can always find the behavior of pure $2D$ gravity. The critical size of the target sphere is given by

$$A_T^c = 2\ln(\mathcal{N}_0 - 1) + (\mathcal{N}_0 - 2)\ln\Big(\frac{\mathcal{N}_0 - 1}{\mathcal{N}_0 - 2}\Big). \tag{4.7}$$

and tends to infinity in the continuum limit $\mathcal{N}_0 \to \infty$. It is therefore evident that the theory should be modified in order to have a sensible continuum limit.

5. Toroidal target space

If the target space is a torus ($G = 1$) the Vandermonde determinants in (3.7) cancel, the free energy is now order $\mathcal{O}(N^0)$ and the saddlepoint method is no longer applicable. However, we can do even better in this case and immediately give a result to all orders in $\frac{1}{N^2}$. Each term in the sum (3.7) factorizes into a product over the rows of the Young tableau; the sum remains nevertheless non-trivial due to the ordering constraint on the weights. Using (3.3), rewrite $\mathcal{Z}$ as

$$\mathcal{Z}_{G=1} = \sum_h e^{-|h|A_T} \prod_{\{i,j\}\in h} \Big(1 - \frac{i-j}{N}\Big)^{\mathcal{N}_0}. \tag{5.1}$$

Here $\{i,j\}$ denotes the a box in row i and column j and the product goes over all boxes in the tableau h. Slicing the tableau through the diagonal and counting the fraction of boxes in the rows of the upper half and the columns of the lower half we can elegantly express (5.1) as

$$\mathcal{Z}_{G=1} = \oint \frac{dz}{2\pi i z} \prod_{n=0}^{\infty} [1 + zq^{n+\frac{1}{2}} \prod_{k=1}^{n} \left(1 + \frac{k}{N}\right)^{\mathcal{N}_0}] [1 + z^{-1} q^{n+\frac{1}{2}} \prod_{k=1}^{n} \left(1 - \frac{k}{N}\right)^{\mathcal{N}_0}] \quad (5.2)$$

where we have denoted $q = e^{-A_T}$. The contour integral ensures that an equal number of rows and columns emanate from the diagonal; it eliminates configurations that cannot be interpreted as a tableau[3]. The leading order of the free energy gives the partition function of the noninteracting string. It is immediately extracted from either (5.1) or (5.2) and is identical to what one finds for the continuum Yang-Mills theory on a torus [1]:

$$\mathcal{F} = -\sum_{n=1}^{\infty} \ln(1 - q^n) + \mathcal{O}(N^{-2}). \quad (5.3)$$

The differences start with the $\frac{1}{N^2}$ terms. These are in principle computable in terms of quasi modular forms as in [16],[18],[17],[11].

Similarly to the YM_2 theory, the matrix models with toroidal target spaces can be interpreted as chiral deformations of a two-dimensional free field theory defined by the action [16],[17],[11]

$$S = \int \frac{d^2 z}{2\pi} \partial\varphi \bar{\partial}\varphi + \oint \sum_n \frac{s_n}{n+1} (\partial\varphi)^{n+1}. \quad (5.4)$$

The partition function (5.2) is equal to the zero $U(1)$ charge sector of the partition function of (5.4) computed on a torus, with an appropriate choice of the couplings $\{s_n\}$.

Let us mention an interesting interpretation of the model with $\mathcal{N}_0 = 1$ whose target space is a torus with one puncture. Explicitly, we have

$$\mathcal{Z} = \int [\mathcal{D}\Phi_1][\mathcal{D}\Phi_2] \; e^{qN \; \mathrm{Tr}\Phi_1\Phi_2\Phi_1^\dagger\Phi_2^\dagger}. \quad (5.5)$$

The model is superficially very similar to (4.6) of the last section. Again a network of square plaquettes (corresponding to the vertices of (5.5)) is generated. But here a local

[3] It is interesting to point out that a very similar result was first found by Douglas [16] in the context of continuum $2D$ Yang-Mills theory on the torus. The combinatorial derivation of (5.2) is due to Dijkgraaf [17].

rule suppresses all *positive* curvature: The coordination numbers at the vertices of such a surface can take values $4, 8, 12, \ldots$, which correspond to zero or negative curvature. No local curvature fluctuations, leading to pure quantum gravity behavior, are possible: For fixed genus g the surfaces contain only a finite number of quantized, negative curvature defects of total curvature $-4\pi(g-1)$. The model is therefore similar in spirit to the "almost flat planar graphs" of [8], where discrete surfaces of vanishing or positive local curvature were considered.

6. The general case

Let us now consider the discretized string theory in the general case when a ramification point of order $n = 2, 3, 4, \ldots$ associated with the point $p \in \Sigma_T$ is weighted by a factor $t_n^{(p)}$. It is straightforward to appropriately modify the original matrix theory and solve it by repeating the same steps explained in section 3.

We will parametrize the couplings $t_m^{(p)}$ by an $N \times N$ external matrix field

$$t_n^{(p)} = \frac{1}{N} \mathrm{Tr}(B_p)^n \tag{6.1}$$

and insert the matrix B_p into the r.h.s. of (2.2). This should be done in the following way: For each point p we choose one of the cells c such that $p \in \partial c$ and insert the matrix B_p into the corresponding trace. For example, the one-plaquette theory (3.8) becomes

$$\mathcal{Z} = \int \prod_{\ell=1}^{\mathcal{N}_0-1+2G} [\mathcal{D}\Phi_\ell] \ \exp\left(e^{-A_T} \ N \ \mathrm{Tr}\Big[B_{\mathcal{N}_0} \prod_{s=1}^{G} \Phi_{2s-1}\Phi_{2s}\Phi^\dagger_{2s-1}\Phi^\dagger_{2s} \prod_{k=2G+1}^{\mathcal{N}_0-1+2G} B_k \Phi_k \Phi_k^\dagger \Big] \right). \tag{6.2}$$

Then one can proceed as in sect. 3, arriving at the following final expression replacing (3.7):

$$\mathcal{Z} = \sum_h \ e^{-|h|A_T} \ \left(\frac{\Delta_h}{\Omega_h}\right)^{2-2G} \ \prod_{k=1}^{\mathcal{N}_0} \left(\chi_h(B_k) \frac{\Omega_h}{\Delta_h} \right). \tag{6.3}$$

The model (6.3) can be solved in the spherical limit using the methods developed in [7]-[10]. Let us note that in the case of a sphere with three punctures ($G = 0, \mathcal{N}_0 = 3$) one reproduces the model of the dually weighted planar graphs in the most general formulation presented in [10]:

$$\mathcal{Z}_{(A_T=0, G=0, \mathcal{N}_0=3)} \ = \sum_h \ \frac{\Omega_h}{\Delta_h} \ \chi_h(B_1)\chi_h(B_2)\chi_h(B_3). \tag{6.4}$$

7. Concluding remarks

1. We have constructed a discretization of the most general string field theory of nonfolding surfaces immersed in a two-dimensional compact spacetime. In the continuum limit the sum over the positions of a branch point should be replaced by an integral with respect to the area. This means that the couplings t_n should scale as

$$t_n = \frac{A_T}{\mathcal{N}_0}\tilde{t}_n \tag{7.1}$$

where $\tilde{t}_n$ are the interaction constants of the continuum theory. The properties of the expansion (6.3) in this limit will be studied elsewhere [12].

2. The YM-string should be obtained by a special tuning of the coupling constants and by introducing contact (tube-like) interactions. These interactions can be implemented by considering the B-matrices as dynamical fields.

3. We point out an interesting equivalence between ensembles of coverings of a sphere with fixed number of punctures and an ensemble of abstract (i.e., without embedding) planar graphs. It is therefore not a miracle that we find a third order phase transition as in the case of pure $2D$ gravity. In our case the transition is due to the entropy of surfaces with a large number of ramification points. A very similar transition has already observed in YM_2 by Douglas and Kazakov [15].

4. The chiral matrix models defined on a torus give a nice geometrical interpretation of the general chiral deformation of topological theory studied by R. Dijkgraaf [11]. The sum over surfaces can again be interpreted in terms of abstract planar graphs describing random surfaces with non-positive local curvature.

Acknowledgements

One of us (I.K.) thanks the CERN Theory Division for hospitality.

References

[1] D. Gross, Nucl. Phys. B B 400 (1993) 161; D. Gross and W. Taylor, Nucl. Phys. B 400 (1993) 181; Nucl. Phys. B 403 (1993) 395.

[2] S. Cordes, G. Moore and S. Ramgoolam, *Large N 2-D Yang-Mills Theory and Topological String Theory*, hep-th/9402107, and *Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories*, hep-th/9411210, 1993 Les Houches and Trieste Proceedings; G. Moore, *2-D Yang-Mills Theory and Topological Field Theory*, hep-th/9409044.

[3] V.A. Kazakov and I. Kostov, Nucl. Phys. B 176 (1980) 199; V.A. Kazakov, Nucl. Phys. B 179 (1981) 283.

[4] V. Kazakov, Phys. Lett. B 128 (1983); K. O'Brien and J.-B. Zuber, Phys. Lett. B 144 (1984) 407; I. Kostov, Nucl. Phys. B 265 (1986), 223, B 415 (1994) 29.

[5] D. Weingarten, Phys. Lett. B 90 (1980) 280.

[6] B. Durhuus, J. Fröhlich and T. Jónsson, Nucl. Phys. B 240 FS[12] (84) 453.

[7] V.A. Kazakov, M. Staudacher and T. Wynter, Comm. Math. Phys. 177 (1996) 451.

[8] V.A. Kazakov, M. Staudacher and T. Wynter, Comm. Math. Phys. 179 (1996) 235.

[9] V.A. Kazakov, M. Staudacher and T. Wynter, Nucl. Phys. B 471 (1996) 309.

[10] V.A. Kazakov, M. Staudacher and T. Wynter, *Advances in Large N Group Theory and the Solution of Two-Dimensional R^2 Gravity*, hep-th/9601153, 1995 Cargèse Proceedings.

[11] R. Dijkgraaf, *Chiral Deformations of Conformal Field Theories*, hep-th/9609022.

[12] I. Kostov, M. Staudacher and T. Wynter, in preparation.

[13] A.A. Migdal, Zh. Eksp. Teor. Fiz. 69 (1975) 810 (Sov. Phys. JETP 42 (413)).

[14] B. Rusakov, Mod. Phys. Lett. A5 (1990) 693.

[15] M.R. Douglas and V.A. Kazakov, Phys. Lett. B 312 (1993) 219.

[16] M.R. Douglas, *Conformal Field Theory Techniques in Large N Yang-Mills Theory*, hep-th/9311130, 1993 Cargèse Proceedings.

[17] R. Dijkgraaf, *Mirror Symmetry and Elliptic Curves*, in *The Moduli Space of Curves*, Progress in Mathematics 129 (Birkhäuser, 1995), 149.

[18] R. Rudd, *The String Partition Function for QCD on the Torus*, hep-th/9407176.