

Mirror Symmetry, $N = 1$ Superpotentials and Tensionless Strings on Calabi–Yau Four-Folds

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We study aspects of Calabi–Yau four-folds as compactification manifolds of F-theory, using mirror symmetry of toric hypersurfaces. Correlation functions of the topological field theory are determined directly in terms of a natural ring structure of divisors and the period integrals, and subsequently used to extract invariants of moduli spaces of rational curves subject to certain conditions. We then turn to the discussion of physical properties of the space-time theories, for a number of examples which are dual to $E_8 \times E_8$ heterotic $N = 1$ theories. Non-critical strings of various kinds, with low tension for special values of the moduli, lead to interesting physical effects. We give a complete classification of those divisors in toric manifolds that contribute to the non-perturbative four-dimensional superpotential; the physical singularities associated to it are related to the appearance of tensionless strings. In some cases non-perturbative effects generate an everywhere non-zero quantum tension leading to a combination of a conventional field theory with light strings hiding at a low energy scale related to supersymmetry breaking.

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1. Introduction

Dualities between perturbatively different string theories in various dimensions have led to a considerable improvement of the understanding of their non-perturbative aspects. In particular the duality [1],[2] between type II strings on Calabi–Yau three-folds and heterotic string on $K3 \times \mathbf{T}^2$, leading to $N = 2$ supersymmetric theories in four dimensions, makes possible the exact determination of string theory space-time instanton effects, reducing to the exact field theory result of [3] after taking appropriate limits [4]. The underlying duality (at present) is however now understood as the duality between F-theory in 8 dimensions on $K3$ and heterotic string on \mathbf{T}^2 [5][6]; four-dimensional type II/heterotic duality then follows from further fibration over $\mathbf{P}^1 \times \mathbf{T}^2$, using variants of the adiabatic argument introduced in [7].

Alternatively one can get theories with minimal $N = 1$ supersymmetry in four dimensions by fibering the eight dimensional duality such as to obtain a Calabi–Yau four-fold \mathbf{X} on the F-theory side and a Calabi–Yau three-fold \mathbf{Z} on the heterotic side [5][8]; Calabi–Yau four-fold compactifications of F-theory have been discussed recently in [9],[10][11]. While the geometrical data of the compactification manifolds are largely fixed by the adiabatic arguments, the choice of the appropriate vector bundle on the heterotic side - the generalization of the choice of the instanton numbers in $K3 \times \mathbf{T}^2$ compactifications - is not yet known in general, given \mathbf{X} on the F-theory side. Independently of this question one might ask to what extent more refined geometrical data of the four-fold - such as period integrals and the correlation functions calculated by the topological field theory [12] - will descend to relevant physical quantities of the $N = 1$ compactification. To address this question it is then natural to attempt to take advantage of the previously detailed studies of $N = 2$ dual pairs by choosing four-folds obtained as fibrations of three-folds over a further \mathbf{P}^1 .

It is useful to think about the various dual descriptions as obtained from limits of two-dimensional compactifications. Specifically, after compactification on \mathbf{S}^1 , F-theory on $\mathbf{X} \times \mathbf{S}^1$ is dual to M-theory on \mathbf{X} and after further compactification on \mathbf{S}^1 we have a duality between F-theory on $\mathbf{X} \times \mathbf{T}^2$ and type IIA on \mathbf{X} , which is the valid view for the discussion of periods and mirror symmetry in a geometrical string theory compactification on the four-fold. There are two particularly interesting limits to consider starting from this theory: first we can undo the \mathbf{S}^1 or \mathbf{T}^2 compactification by taking special limits in the Calabi–Yau moduli space. In this case we go back to the four-dimensional $N = 1$ theories, e.g. heterotic string on \mathbf{Z} . The second is to take the large base space limit of \mathbf{X} . In this case one flows to a theory which looks locally like $N = 2$ in four dimensions, e.g. heterotic string on $K3$ times the extra torus; in fact we will see that one obtains precisely the $N = 2$

periods in this limit. It is suggestive to think about the world sheet instantons associated to the base \mathbf{P}^1 departing from the large base space limit as $N = 2$ breaking corrections. In sect. 2 we discuss the behavior of the four-fold periods in the large base limit.

The derivation of the period integrals and the correlation functions of the topological field theory rely on methods of mirror symmetry between Calabi–Yau four-folds. A concept of mirror symmetry for Calabi–Yau d -folds for $d > 3$ has been defined in [13] for one moduli cases¹. For the more complicated four-folds which are relevant in the present context, we develop the appropriate framework in terms of toric geometry in sect. 3, defining the fundamental correlation functions of the topological field theory directly in terms of the period integrals and a natural ring structure present in the toric variety. Other than in $d = 3$ and in the one moduli cases considered in [13], the 3-pt functions calculate a whole set of invariants N_α , counting the Euler number of the moduli space of rational curves subjected to constraints on the location of the curves in the manifold, which arise from operators associated to codimension 2 submanifolds in \mathbf{X} .

In the second part we apply these methods to elliptically fibred four-folds which are fibrations of Calabi–Yau three-folds which have itself well-known $E_8 \times E_8$ heterotic $N = 2$ duals in four dimensions. In sect. 4 we determine the correlation functions and the invariants associated to them and describe the geometrical meaning of the Kähler moduli which relates them to the moduli of the heterotic dual. In sect. 5 we make some verifications on the numbers calculated in the four-fold by imposing appropriate constraints and comparing the result with the known numbers of rational curves in the fibre. We discuss some interesting properties of the couplings and their role in the space-time effective theory.

We then turn to the question of whether a superpotential is generated in the four-dimensional $N = 1$ supersymmetric F-theory compactification. In sect. 6 we analyze possible wrappings of five-branes in an M-theory compactification, following [8]. A complete classification of appropriate divisors (six-cycles) is given using intersection theory on the toric hypersurface; it turns out that a superpotential is indeed generated generically. We then ask about the physical effects related to these superpotential terms. As might be expected from the conjectured duality to $E_8 \times E_8$ heterotic string on a threefold, tensionless strings and compactifications of them play an important role. In sect. 7 we investigate singularities in the complex structure moduli space related to fibration singularities and possible gauge symmetry enhancement and describe the geometrical properties of the relevant divisors which provide the link to physical properties. In some cases the instanton

¹ For a discussion of mathematical aspects of mirror symmetry see [14][15].

generated superpotential can be interpreted as world sheet instantons of the magnetic non-critical string in six dimensions. Special singularities which appear in the moduli space when the five brane intersects or coincide with tensionless strings from three branes wrappings are discussed in sect. 8. A new kind of theories arises if non-perturbative effects generate a everywhere non-zero tension for the string with “classically” zero tension. In this case one obtains in the appropriate scaling limit a conventional field theory, however with a hidden string at a non-perturbatively generated low energy scale related to the scale of supersymmetry breaking.

2. Periods on the four-fold

One of the first questions about mirror symmetry of four-folds and its use to determine non-perturbative effects in F-theory compactifications is, which kind of non-perturbative effects are expected to be treated by the topological sigma model and which kind are not. In three-fold compactifications mirror symmetry allows to determine the exact Kähler moduli space \mathcal{M}_{KM} of the type IIA theory on \mathbf{Y} from the map to the complex structure moduli space \mathcal{M}_{CS} of the type IIB theory on the mirror \mathbf{Y}^* . From the brane point of view this is possible since the complex structure moduli are associated to 3-cycles on \mathbf{Y}^* , however there is no two-brane available in type IIB which can be wrapped on these 3-cycles to generate an instanton effect. Therefore the classical computation in the type IIB theory is exact and using the mirror map one obtains information about the world sheet instanton corrected Kähler moduli space of the type IIA theory on \mathbf{Y} . The same can not be said about the other moduli space - \mathcal{M}_{KM} of the type IIB theory on \mathbf{Y} or \mathcal{M}_{CS} of the type IIA theory on \mathbf{Y}^* - since the latter theory has Dirichlet two branes which do generate complex structure moduli dependent instanton effects. Moreover the string coupling constant is a hypermultiplet and there are perturbative corrections in the type IIA string theory.

We will be primarily interested in the Kähler moduli space of type IIA compactified on the four-fold \mathbf{X} , including the corrections to the correlation functions calculated by the isomorphisms of the two topological theories, called the A and the B model. It would be interesting to know possible factorization properties of the full non-perturbative moduli space, a problem which is of course closely related to a similar question about (0,2) moduli spaces. Generally we expect that different than in the three dimensional case there are corrections that are not taken into account by conventional mirror symmetry based on the isomorphism of two-dimensional topological theories. However the information provided

by the exact mirror map should be enough to pin down the individual origin of an instanton effect (thus counting D-branes states) from the scaling behavior whereas the exact contribution will contain an additional sum of corrections as e.g. in the case of D2 brane instantons in type IIA theory [16]. Moreover it is an interesting question, what is the freedom that is not fixed by the holomorphic bundle structure starting from the apparently rather complete information on the type IIA side. We will see later that the answer is related to the the cohomology $H^{2,2}$ of the four-fold, which is special in many respects.

In the following paragraph we consider F-theory compactification on four-folds M obtained from fibering elliptic Calabi–Yau three-folds \mathbf{Y} over a two-sphere, D , of volume t_D . If \mathbf{Y} has a K3 fibration in addition to the elliptic fibration, this theory is expected to have a heterotic $N = 1$ dual by fibre-wise application of the 8 dimensional duality between F-theory on K3 and heterotic string on T^2 [5].

2.1. Periods in the large base space limit

It is instructive to consider the large base space limit of F-theory on $\mathbf{X} \times T^2$; in this case one expects to recover $N = 2$ supersymmetric IIA on \mathbf{Y} in four dimensions or the dual representation, heterotic string on $K3 \times T^2$. Since we want to use mirror symmetry to extract physical couplings from the integrals over the holomorphic $(d, 0)$ form on a Calabi-Yau manifold \mathbf{X}^* it is useful to make precise this limit on the period integrals.

The observables of the B model on the mirror manifold \mathbf{X}^* are in correspondence with elements of the middle cohomology of \mathbf{X}^* , $\oplus_k H^{d-k,k}(\mathbf{X}^*)$, or rather a subspace of it in the case of four-folds as will be discussed in more detail in the next section. Mirror symmetry relates the correlation function of the B model on \mathbf{X}^* to those of the A model on \mathbf{X} . In the A model observables are associated to cohomology elements in $H^{k,k}(\mathbf{X})$ or equivalently dimension $2k$ homology cycles $\in H_{k,k}(\mathbf{X})$. The relation between the periods of the four-fold and the periods of the three-fold fibre \mathbf{Y} can be best understood in this last representation.

In the A model on the three-fold \mathbf{Y} , the $2 + h^{1,1}$ periods² are related to the 0 cycle $\mathcal{C}^{(0)'}$, $h^{1,1}$ 2-cycles $\mathcal{C}^{(2)'}$, $h^{1,1}$ 4-cycles $\mathcal{C}^{(4)'}$ and one 6-cycle $\mathcal{C}^{(6)'}$ = \mathbf{Y} in $\oplus_k H_{k,k}(\mathbf{Y})$. If we fibre \mathbf{Y} over a \mathbf{P}^1 to get a four-fold \mathbf{X} , we can think of the elements $\mathcal{C}^{(k)} \in H_{k,k}(\mathbf{X})$ as obtained from joining elements $\mathcal{C}^{(l)'}$ with 0 and 2 cycles in the base \mathbf{P}^1 . In this way we get $(1, h^{1,1} = h'^{1,1} + 1, h_V^{2,2} = 2h'^{1,1}, h^{3,3} = h'^{1,1} + 1, 1)$ homology cycles $\mathcal{C}^{(k)}$ of the four-fold, generating the so-called vertical primary subspace of $H^{k,k}(\mathbf{X})$.

² A prime refers to the fibre data in the following.

In the B model on \mathbf{Y}^* , expanded in special coordinates t_α around a large complex structure point, the $2 + 2h'^{1,3}(\mathbf{Y}^*) = 2 + 2h'^{1,1}(\mathbf{Y})$ periods in special coordinates are of the form

$$\begin{aligned} H_{3,0}(\mathbf{Y}^*) &\rightarrow 1, & H_{2,1}(\mathbf{Y}^*) &\rightarrow t_\alpha, \\ H_{1,2}(\mathbf{Y}^*) &\rightarrow \mathcal{F}_\alpha, & H_{0,3}(\mathbf{Y}^*) &\rightarrow \mathcal{F}_0 = 2\mathcal{F} - \sum_k^{h'^{1,1}} t_\alpha \mathcal{F}_\alpha. \end{aligned} \quad (2.1)$$

where $F_\alpha = \frac{\partial}{\partial t_\alpha} \mathcal{F}$ with \mathcal{F} the $N = 2$ prepotential. Note that in the A model the leading classical terms³ of the periods can be interpreted as the volume of the homology cycles. In the large base space limit world sheet instanton corrections from the base \mathbf{P}^1 are suppressed and integrating over the homology cycles $\mathcal{C}^{(k)}$ of the four-fold reduces to an integration over the three-fold cycles $\mathcal{C}^{(k)'}$, possibly multiplied by the classical volume of the base \mathbf{P}^1 , if $\mathcal{C}^{(k)}$ is obtained from $\mathcal{C}^{(k-1)'}$ by joining the whole base. The periods of the four-folds in the large base space limit are then simply given by combining these factors with the three-fold result (2.1):

$$(1, t_D) \times (1, t_\alpha, \mathcal{F}_\alpha, \mathcal{F}_0) \quad (2.2)$$

From the definition of the homology cycles on \mathbf{X} it is clear that non-vanishing intersections involve only pairs of elements which intersect on \mathbf{Y} ; more precisely there is a set of a cycles with $\int_{\gamma_a} \Omega \sim (1, t_\alpha, \mathcal{F}_\alpha, \mathcal{F}_0)$ and a set of b cycles with $\int_{\gamma_b} \Omega \sim t_D \times (1, t_\alpha, \mathcal{F}_\alpha, \mathcal{F}_0)$ with non-vanishing intersection only between a and b cycles and the intersection form given by that of the Calabi–Yau fibre. This implies in particular that the Kähler potential of the four-fold \mathcal{K} reduces that of the three-fold compactification plus a constant

$$\lim_{t_D \rightarrow \infty} e^{-\mathcal{K}} = \lim_{t_D \rightarrow \infty} \int_{\mathbf{X}^*} \Omega \wedge \bar{\Omega} = (t_D - \bar{t}_D)(2(\mathcal{F} - \bar{\mathcal{F}}) - \sum_\alpha (t_\alpha - \bar{t}_\alpha)(\mathcal{F}_\alpha + \bar{\mathcal{F}}_\alpha))$$

Starting from the precise relation in the large basis limit of the four-fold periods of the $N = 1$ compactification and the $N = 2$ structure of a compactification on the three-fold fibre, it is suggestive to treat the instanton corrections associated with the base space modulus as $N = 2$ breaking deformations of a $N = 2$ theory.

Note also that the periods of the four-fold (2.2) are algebraically dependent; this is not only true in the large base space limit but simply a consequence of

$$\int \Omega \wedge \Omega = 0 \quad (2.3)$$

³ That is powers of the t_α rather than instanton corrections $\sim q_\alpha = e^{2\pi i t_\alpha}$

which provides a non-trivial *algebraic* relation between the entries of the period vector. This is different than in the odd-dimensional case, where the first non-trivial equation derived from (2.3) involves a derivative acting on one Ω and leads to a differential equation relating the periods.

If instead of fibering the three-fold \mathbf{Y} over a \mathbf{P}^1 we consider a four-fold of the type $\mathbf{Y} \times \mathbf{T}^2$, eq. (2.2) becomes exact. Let $\mathbf{Y}_{\mathbf{F}_1}$ be the elliptically fibred three-fold with base \mathbf{F}_1 ; there is a point in the moduli space with the appearance of E_8 tensionless strings [6]. Then we have precisely the same situation as in [17], where a torus compactification of this string is considered, leading to $N = 2$ in four dimensions. In this $N = 2$ four-fold compactification the gauge coupling is determined from the Calabi–Yau periods in the usual way (taking into account the F-theory limit); moreover it is easy to see from the results in [18] that the relevant periods at the tensionless string point are precisely those over the shrinking del Pezzo inside \mathbf{Y} , implying its appearance in the final result of ref. [17].

3. Mirror map and Yukawa couplings

The description of moduli spaces of d -dimensional Calabi–Yau manifolds in terms of a holomorphic section Ω of the Hodge bundle and period integrals over this holomorphic $(d, 0)$ form has been given in [19],[20]. The concept of a mirror map relating n -point functions of A and B type topological field theories associated to a d -dimensional Calabi–Yau manifold \mathbf{X} and its mirror \mathbf{X}^* has been defined in [13], see also [21]. In this section we provide the general framework for the description of four-folds with an arbitrary number of moduli in terms of toric geometry.

3.1. Toric description of \mathbf{X} and \mathbf{X}^*

Batyrev [22] has given a construction of mirror pairs \mathbf{X} , \mathbf{X}^* of d -dimensional Calabi–Yau manifolds as hypersurfaces in $(d+1)$ -dimensional toric varieties $\mathbf{P}(\Delta)$, $\mathbf{P}(\Delta^*)$, where Δ and Δ^* denote the reflexive polyhedra $\subset \mathbf{R}^{d+2}$ defining the combinatorial data of $\mathbf{P}(\Delta)$ and $\mathbf{P}(\Delta^*)$. We will use this description of Calabi–Yau four-folds in the following.

Let ν_i^* denote the integral vertices of Δ^* . The toric variety $\mathbf{P}(\Delta^*)$ contains a canonical torus $(\mathbf{C}^*)^{d+1}$ with coordinates X_i . Then \mathbf{X}^* is defined as the zero set of the Laurent polynomial

$$f_{\Delta^*}(X, a) = \sum_{\nu_i^*} a_i X^{\nu_i^*} , \quad X^{\nu_i^*} \equiv \prod_k X_k^{\nu_i^*(k)} \quad (3.1)$$

where the coefficients a_i are parameters characterizing the complex structure of \mathbf{X}^* . In [22][23]⁴ Batyrev shows that the Hodge numbers $h^{p,1}$ are determined by the polyhedron data as

$$\begin{aligned}
h^{1,1}(\mathbf{X}) &= h^{d-1,1}(\mathbf{X}^*) = l(\Delta^*) - (d+2) - \sum_{\text{codim} S^*=1} l'(S^*) + \sum_{\text{codim} S^*=2} l'(S^*) \cdot l'(S) , \\
h^{p,1}(\mathbf{X}) &= \sum_{\text{codim} S^*=p+1} l'(S) \cdot l'(S^*) , \quad 1 < p < d-1 , \\
h^{d-1,1}(\mathbf{X}) &= h^{1,1}(\mathbf{X}^*) = l(\Delta) - (d+2) - \sum_{\text{codim} S=1} l'(S) + \sum_{\text{codim} S=2} l'(S) \cdot l'(S^*) ,
\end{aligned} \tag{3.2}$$

where S denotes faces of Δ and S^* the dual face of S . l and l' are the numbers of integral points on a face and in the interior of a face, respectively.

If the manifold has $SU(4)$ holonomy rather than a subgroup, then $h^{2,0} = h^{1,0} = 0$ and the remaining non-trivial hodge number $h^{2,2}$ is determined by $h^{2,2} = 12 + \frac{2}{3}\chi + 2h^{1,2}$ [9]. The Euler number χ is $\chi = 2(2 + h^{1,1} + h^{1,3} - 2h^{1,2}) + h^{2,2}$.

The target space toric variety $\mathbf{P}(\Delta^*)$ can be described as

$$\mathbf{P}(\Delta^*) = (\mathbf{C}^m - F) / (\mathbf{C}^*)^{m-d-2} ,$$

a generalization of projective space with $m - d - 2$ scaling symmetries $x^i \rightarrow \lambda^{\alpha_i} x^i$ acting on the m coordinates x_i of \mathbf{C}^m and a disallowed set F which consists of the unions of intersections of coordinate hyperplanes $D_i \equiv \{x_i = 0\}$ as determined by the so called primitive collections (see e.g. [25],[26]); e.g. for ordinary projective space one has $\lambda^{\alpha_i} = \lambda$ and $F = \{x_i = 0, \forall i\}$.

Apart from the combinatorial data Δ^* , a specific phase of the Calabi–Yau manifold \mathbf{X}^* depends on the choice of a regular triangularization of Δ^* . This defines in turn a choice of set of generators for the $h^{1,1}$ relations between the integral vertices⁵ ν_i^* , $l_j^{(i)} \nu_j^* = 0$, $i = 1 \dots h^{1,1}$, called the the Mori vectors $l^{(i)}$. The Kähler cone of the mirror \mathbf{X} is then the dual of the cone generated by the Mori generators. Starting from these data one obtains a system of differential equations, the Picard-Fuchs system, for the periods over the holomorphic $(d,0)$ form on \mathbf{X}^* [27]. The period integrals on \mathbf{X}^* are then given as linear combinations of the solutions to the Picard-Fuchs system.

⁴ See also [24].

⁵ Here we restrict ourselves to the set of vertices ν_i^* which lie on edges or faces of Δ^* ; in the general case it can be necessary to consider also vertices corresponding to those interior points on faces of Δ^* which represent automorphisms of $\mathbf{P}(\Delta^*)$ [26].

There is a natural ring structure on $\mathbf{P}(\Delta^*)$ from taking unions and intersections of toric divisors D_i . The intersection ring \mathcal{R} is defined as the quotient ring $\mathcal{R} = \mathbf{Z}[D_i]/\mathcal{I}$, where \mathcal{I} is the ideal generated by linear relations $\sum_i^m \langle m, \nu_i^* \rangle D_i = 0$ and a set of non-linear relations $\prod_i^m D_i^{\xi_i} = 0$; the latter is called the Stanley-Reisner ideal and determines the disallowed set F .

In the next section we will relate the elements of \mathcal{R} at degree k (where k is here the complex codimension of a homology element) to observables $\mathcal{O}^{(k)} \in H_V \subset \oplus_k H^{k,k}(\mathbf{X})$ of the A type model on \mathbf{X} ; here H_V is the so called primary vertical subspace of $\oplus_k H^{k,k}(\mathbf{X})$ [13] which is the subspace of $\oplus_k H^{k,k}(\mathbf{X})$ generated by wedge products of elements in $H^{1,1}(\mathbf{X})$. The ideal \mathcal{I} determines the dimension of the ring at degree k ; in fact, for Calabi–Yau fibered four-folds one has $\dim_k(\mathcal{R}) \equiv d_k(\mathcal{R}) = (1, h^{1,1}, 2h^{1,1} - 2, h^{1,1}, 1)$ for $k = 0 \dots 4$.

Another distinguished set of generators of \mathcal{R} is determined by the divisors as defined by the Kähler cone of \mathbf{X} . Let J_α be the $(1, 1)$ forms dual to the special flat coordinates t_α on the Kähler moduli space, centered at a large radius structure limit of maximal unipotent monodromy. Let $J = \sum_\alpha t_\alpha J_\alpha$ be the Kähler form, and K_α the divisors dual to the J_α . We can use equivalently K_α as generators of the intersection ring \mathcal{R} . In particular, if R_0 is the intersection form of \mathbf{X}

$$R_0 = \sum_{\alpha \geq \beta \geq \gamma \geq \delta} k_{\alpha\beta\gamma\delta} K_\alpha K_\beta K_\gamma K_\delta, \quad (3.3)$$

where the convention is that $k_{\alpha\beta\gamma\delta}$ is the value of the integral $\int_{\mathbf{X}} J_\alpha \wedge J_\beta \wedge J_\gamma \wedge J_\delta$, then the top element of dimension 4 of \mathcal{R} is simply R_0 while the volume of \mathbf{X} is obtained by replacing the divisors K_α by the coordinates t_α in (3.3) and relaxing the condition on the summation indices in (3.3).

Other topological invariants of \mathbf{X} are defined by integrating elements of \mathcal{R} wedged with the Chern classes c_i of \mathbf{X} , $i = 2 \dots 4$:

$$\begin{aligned} R_{22} &= \int_M c_2^2 \\ R_2 &= \int J \wedge J \wedge c_2, \\ R_3 &= \int J \wedge c_3, \\ R_4 &= \int_M c_4 = \chi \end{aligned} \quad (3.4)$$

with the obvious index structures. For $SU(4)$ holonomy, $R_{22} = 480 + \frac{\chi}{3}$ [9].

3.2. The A model

Mirror symmetry implies that the correlation functions of two topological field theories defined on a Calabi–Yau manifold \mathbf{X}^* and its mirror \mathbf{X} are isomorphic. The correlation functions of the first theory, the B model defined on \mathbf{X}^* , depend on the complex structure (CS) moduli of \mathbf{X}^* in a purely geometrical (classical) way and can be calculated straightforwardly. On the other hand the correlation functions of second theory, the A model defined on \mathbf{X} , depend on the Kähler moduli (KM) of \mathbf{X} in a complicated way due to the presence of world-sheet instanton corrections. Mirror symmetry allows to determine these A model correlation functions by construction of the explicit mirror map from the complex structure moduli space of the B model to the Kähler moduli space of the A model.

Choice of a basis for the A model

To match the moduli space of the CS moduli space of the B model to the Kähler moduli space of the A model we first chose a basis in the A model in the following way. The basis for the primary vertical subspace $H_V \subset \bigoplus_k H^{k,k}$ with the most natural geometrical interpretation is given by forms Poincare dual to submanifolds of complex codimension k [12]. Specifically we will chose a basis generated by the (1,1) forms J_k dual to the special coordinates on the KM space, t_α , and wedge products of them:

$$\begin{aligned} \mathcal{O}^{(0)} &= 1, & \mathcal{O}_\alpha^{(1)} &= J_\alpha, & \mathcal{O}_\alpha^{(2)} &= E_\alpha^{(2) \beta\gamma} J_\beta J_\gamma, \\ \mathcal{O}_\alpha^{(3)} &= E_\alpha^{(3) \beta\gamma\delta} J_\beta J_\gamma J_\delta, & \mathcal{O}_\alpha^{(4)} &= E_\alpha^{(4) \beta\gamma\delta\epsilon} J_\beta J_\gamma J_\delta J_\epsilon. \end{aligned} \tag{3.5}$$

As mentioned above the dimension of the basis $\mathcal{O}^{(i)}$, $i > 1$ is reduced by the intersection properties of the dual homology elements K_α determining the range of the lower index of the coefficients E_α . E.g., in the case of Calabi–Yau fibrations, where $\dim(H_V^{2,2}) = \dim(H_V^{3,3}) = 2h^{1,1} - 2$, the intersection of the divisor D dual to the base \mathbf{P}^1 with itself is empty, $D \cdot D = 0$, implying that there can appear at most one power of J_D in the definition of the $\mathcal{O}_\alpha^{(i)}$ (this is the same kind of argument that ensures the linear coupling of the Kähler coordinate identified as the dilaton in K3 fibrations). In general the $E^{(i)}$ are chosen such that the elements $\mathcal{O}^{(i)}$ generate the degree i subspace of \mathcal{R} .

Topological metric, operator product expansions and correlation functions

The 2-pt functions define the flat metric on H_V in terms of integrals of the basis elements $\mathcal{O}^{(i)}$ over \mathbf{X} :

$$\eta_{\alpha\beta}^{(i)} = \langle \mathcal{O}_\alpha^{(i)}, \mathcal{O}_\beta^{(d-i)} \rangle = \int \mathcal{O}_\alpha^{(i)} \mathcal{O}_\beta^{(d-i)} \quad (3.6)$$

In fact the metric is constant in the flat variables t_α [28] and non-zero only for pairs $\mathcal{O}^{(i)}, \mathcal{O}^{(j)}$ of operators with $i + j = d$.

The r.h.s. is then determined by the coefficients $E^{(i)}$ of a given basis (3.5) together with the intersection numbers $k_{\alpha\beta\gamma\delta}$.

The factorization properties of the topological field theory ensure that all correlation functions can be expressed in terms of the fundamental 2-pt and 3-pt functions. Similarly as in the case of three-folds there is only one independent type of 3-pt functions, namely $\langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle$, which contain the full information about the moduli dependence

$$Y_{\alpha\beta\delta} = \langle \mathcal{O}_\alpha^{(1)} \mathcal{O}_\beta^{(1)} \mathcal{O}_\gamma^{(2)} \rangle = \int \mathcal{O}_\alpha^{(1)} \mathcal{O}_\beta^{(1)} \mathcal{O}_\gamma^{(2)} + \text{inst.corr.}$$

The Y are determined in terms of the operator product coefficients $C^{(1)}, C^{(2)}$:

$$\mathcal{O}_\alpha^{(1)} \cdot \mathcal{O}_\beta^{(i)} = C_{\alpha\beta}^{(i)} \gamma \mathcal{O}_\gamma^{(i+1)} \quad (3.7)$$

to be

$$Y_{\alpha\beta\gamma} = C_{\alpha\beta}^{(1) \mu} \eta_{\mu\gamma}^{(2)} = C_{\alpha\gamma}^{(2) \mu} \eta_{\beta\mu}^{(1)}$$

While the 2-pt and 3-pt functions are the fundamental objects of the underlying topological theory, the simplest object on the four-fold which can be defined entirely in terms of the marginal operators $\mathcal{O}_\alpha^{(1)}$ are the 4-pt functions K

$$K_{\alpha\beta\gamma\delta} = \langle \mathcal{O}_\alpha^{(1)} \mathcal{O}_\beta^{(1)} \mathcal{O}_\gamma^{(1)} \mathcal{O}_\delta^{(1)} \rangle = \int_{X_4} \mathcal{O}_\alpha^{(1)} \mathcal{O}_\beta^{(1)} \mathcal{O}_\gamma^{(1)} \mathcal{O}_\delta^{(1)} + \text{inst.corr.}$$

whose classical piece is given by the intersection numbers $k_{\alpha\beta\gamma\delta}$ of \mathcal{R}_0 . Factorization in terms of 2-pt and 3-pt functions yields

$$K_{\alpha\beta\gamma\delta} = (C_\alpha^{(1)} \cdot \eta^{(2)} \cdot C_\gamma^{(1)T})_{\beta\delta} = (C_\alpha^{(1)} \cdot C_\gamma^{(2)} \cdot \eta^{(1)T})_{\beta\delta}, \quad (3.8)$$

where we use a matrix notation $(C_\alpha^{(i)})_\beta^\mu$. Non-trivial conditions on the ring coefficients $C^{(1)}, C^{(2)}$ follow from associativity of the operator products:

$$\begin{aligned} (C_\alpha^{(1)} \cdot \eta^{(2)} \cdot C_\gamma^{(1)T}) &= (C_\alpha^{(1)} \cdot \eta^{(2)} \cdot C_\gamma^{(1)T})^T, \\ C_{\alpha\beta}^{(1) \rho} C_{\gamma\rho}^{(2) \mu} &= C_{\alpha\gamma}^{(1) \rho} C_{\beta\rho}^{(2) \mu}, \end{aligned} \quad (3.9)$$

where the second identity follows from the first using $(C_\alpha^{(2)})^T = (\eta^{(1)})^{-1T} \cdot C_\alpha^{(1)} \cdot \eta^{(2)}$. This identity provides highly non-trivial relations between the instanton corrected correlation functions.

3.3. Basis for the B model and the mirror map

The next step to find the mirror map is the construction of a basis of the observables of the B model which matches the properties of the above chosen basis for the A model:

$$\mathcal{O}_\alpha^{(1)} \cdot \mathcal{O}_\beta^{(i)} = C_{\alpha\beta}^{(i)} \gamma \mathcal{O}_\gamma^{(i+1)}, \quad \langle \mathcal{O}_\alpha^{(i)}, \mathcal{O}_\beta^{(j)} \rangle = \delta_{i+j, d} \eta_{\alpha\beta}^{(i)} \quad \mathcal{O}_\alpha^{(1)} \cdot \mathcal{O}^{(4)} = 0 \quad (3.10)$$

The appropriate basis for the B model can be defined [13] using the Gauss–Manin connection ∇ , the flat metric-compatible connection on the Hodge bundle \mathcal{H} over the CS moduli space \mathcal{M}_{CS} . The following construction is a generalization of the procedure in [13]; we can therefore focus on the complications introduced by the higher dimensional moduli space as compared to the one moduli case considered in [13]. In particular we will define the 3-pt functions directly in terms of the period integrals over the holomorphic $(d, 0)$ form and the intersection ring \mathcal{R} .

The fundamental step in the construction of the B -model basis in [13] is the replacement of the operator product involving a charge one operator $\mathcal{O}^{(1)}$ with the action of the unprojected Gauss-Manin connection ∇

$$\mathcal{O}_\alpha^{(1)} \cdot \mathcal{O}_\beta^{(i)} \rightarrow \nabla_\alpha \mathcal{Q}_\beta^{(j)}$$

where now $\mathcal{Q}^{(i)}$ denote the basis elements of the B model and the directional derivative is defined in terms of the parametrization of the deformations corresponding to marginal operators $\mathcal{Q}_\alpha^{(1)}$ by the special flat coordinates t_α . This definition implies a holomorphic dependence of the basis $\mathcal{Q}^{(i)}$ as opposed to the other natural choice, a basis of elements of pure type $(d - k, k)$. We will now define a basis matching the property (3.10) of the A model basis using the intersection ring \mathcal{R} and a map $m : D_i \rightarrow \theta_i$ [26], where $\theta_i = z_i \frac{\partial}{\partial z_i}$ are the logarithmic derivatives with respect to the complex structure moduli of the B model [27]

$$z_i = (-)^{l_0^{(i)}} \prod_i a_i^{l_i^{(k)}} \quad (3.11)$$

Let $\gamma_\alpha^{(i)}$ be a basis of topological homology cycles spanning the primary horizontal subspace $H_H(\mathbf{X}) \subset \oplus_k H^{d-k, k}(\mathbf{X})$ and $\star \gamma_\alpha^{(i)}$ be the dual cohomology elements fulfilling

$$\int_{\gamma_\alpha^{(i)}} \star \gamma_\beta^{(j)} = \delta^{ij} \delta_{\alpha\beta}, \quad \int \star \gamma_\alpha^{(i)} \star \gamma_\beta^{(j)} = \begin{cases} 0, & i + j > d; \\ M_{\alpha\beta}^{(i, j)}, & i + j = d; \end{cases} \quad (3.12)$$

Furthermore let $\mathcal{Q}_\alpha^{(i)}$ be a set of elements of F^{d-i} , where $F^p = \bigoplus_{k \geq p} H^{k, d-k}(\mathbf{X}^*)$ are the holomorphic bundles of forms with anti-holomorphic degree at most $d - k$, fulfilling

$$\int_{\gamma_\alpha^{(i)}} \mathcal{Q}_\beta^{(i)} = \begin{cases} \delta_{\alpha\beta}, & i = j; \\ 0, & i < j \end{cases} \quad (3.13)$$

The following relations are elementary:

$$\begin{aligned} \mathcal{Q}_\alpha^{(i)} &= {}^* \gamma_\alpha^{(i)} + \sum_{k > i} \tilde{a}_{\alpha\beta}^{(i,k)} {}^* \gamma_\beta^{(k)}, & \tilde{a}_{\alpha\beta}^{(i,k)} &\equiv \int_{\gamma_\beta^{(k)}} \mathcal{Q}_\alpha^{(i)}, \\ \nabla_\alpha \mathcal{Q}_\beta^{(i)} &= (\nabla_\alpha \tilde{a}_{\beta\gamma}^{(i,i+1)}) \mathcal{Q}_\gamma^{(i+1)}, \\ \langle \mathcal{Q}_\alpha^{(i)}, \mathcal{Q}_\beta^{(j)} \rangle &= M_{\alpha\beta}^{(i,j)} \delta_{i+j,d}, \\ Y_{\alpha\beta\gamma} &= \langle \mathcal{Q}_\alpha^{(1)} \mathcal{Q}_\beta^{(1)} \mathcal{Q}_\gamma^{(2)} \rangle = (\nabla_\alpha \tilde{a}_{\beta\delta}^{(1,2)}) M_{\delta\gamma}^{(2,2)}. \end{aligned} \quad (3.14)$$

Comparing (3.14) with (3.6), (3.7) it follows that $\mathcal{Q}^{(i)}$ is the basis matching the properties (3.10) provided that

$$\nabla_\alpha \tilde{a}_{\beta\gamma}^{(i,i+1)} = C_{\alpha\beta}^{(i)} \gamma, \quad M_{\alpha\beta}^{(i,j)} = \eta_{\alpha\beta}^{(i)}. \quad (3.15)$$

We now construct the basis $\{\mathcal{Q}_\alpha^{(i)}, \gamma_\alpha^{(i)}\}$ in two steps:

a) First chose a basis for the $\mathcal{Q}_\alpha^{(i)}$. Of course we have $\mathcal{Q}^{(0)} = \Omega \in H^{d,0}(\mathbf{X})$. The $\mathcal{Q}_\alpha^{(i)}$ are then obtained by choosing $d_k(\mathcal{R})$ independent generators⁶ $\mathcal{O}_\alpha^{(k)} = E_\alpha^{(k)\alpha_1 \dots \alpha_k} J_{\alpha_1} \wedge \dots \wedge J_{\alpha_k} \in H^{k,k}(\mathbf{X})$ of the ring \mathcal{R} at degree k and defining

$$\mathcal{Q}_\alpha^{(k)} = L_\alpha^{(k)} \Omega, \quad L_\alpha^{(k)} = \mathcal{O}_\alpha^{(k)}(J_\gamma \rightarrow \theta_{z_\gamma})$$

where $L_\alpha^{(k)}$ are differential operators of degree k obtained from the map $\tilde{m} : K_\alpha \rightarrow \theta_{z_\alpha}$ which follows from $m : D_i \rightarrow \theta_{a_i}$ by transformation to the K_α basis. The topological metric in this basis is then given by $\eta_{\alpha\beta}^{(i)} = \int \mathcal{O}_\alpha^{(i)} \wedge \mathcal{O}_\beta^{(j)}$.

b) We have to find the basis of cycles $\{\gamma_\alpha^{(i)}\}$ that satisfies (3.13) with the given basis $\{\mathcal{Q}_\alpha^{(i)}\}$. This can be done by fixing the leading logarithmic behavior of the period integrals

$$\Pi_\alpha^{(i)} = \int_{\gamma_\alpha^{(i)}} \Omega = \int_{\gamma_\alpha^{(i)}} \mathcal{Q}^{(0)};$$

the exact periods $\Pi_\alpha^{(i)}$ are then determined by the solution of the Picard-Fuchs system with the appropriate leading behavior.

⁶ We use here $\mathcal{O}_\alpha^{(k)}$ to denote the element in $H^{k,k}(\mathbf{X})$ that corresponds 1-1 to the operator $\mathcal{O}_\alpha^{(k)}$.

At the large complex structure point of the CS moduli space \mathcal{M}_{CS} $z_i = 0 \forall i$, the solutions to the Picard–Fuchs system have the leading behavior $\sim (\ln z)^k$, $k = 0 \dots 4$, where z stands for any of the CS moduli z_i . In fact there are precisely $d_k(\mathcal{R})$ solutions $S_\alpha^{(k)}$ with leading behavior $(\ln z)^k$, as a consequence of the relation between the intersection ring \mathcal{R} and the ring of differential operators θ_i [27][26]. The basis $\{\gamma_\alpha^{(i)}\}$ with the property (3.13) is then fixed by the condition

$$\Pi_\alpha^{(k)} = S_\alpha^{(k)} + \dots, \quad L_\alpha^{(k)} S_\beta^{(k)} = \delta_{\alpha\beta} + \dots \quad (3.16)$$

where the ellipsis denote terms involving powers $(\ln z)^l$ with $l < k$ and polynomial corrections. For convenience we state explicitly the expression for the leading piece of $\Pi_\alpha^{(k)}$, as obtained from (3.16) by trivial matrix multiplication:

$$S_\alpha^{(k)} = \sum \frac{1}{k!} \tilde{E}_\alpha^{\beta_1 \dots \beta_k} \prod_{n=1}^k \ln(z_{\beta_n}), \quad \tilde{E}_\alpha^{\beta_1 \dots \beta_k} = E_\alpha^{\star \gamma_1 \dots \gamma_{d-k}} k_{\beta_1 \dots \beta_k \gamma_1 \dots \gamma_{d-k}}$$

where $k_{\beta_1 \dots \beta_d}$ is the intersection form given in (3.3), and $\mathcal{O}_\alpha^{\star(d-k)} \equiv E^{\star \gamma_1 \dots \gamma_{d-k}} J_{\gamma_1} \wedge \dots \wedge J_{\gamma_{d-k}}$ is the Poincare dual of $\mathcal{O}_\alpha^{(k)}$ (as is obvious from the relation $L_\alpha^{(k)} = \mathcal{O}_\alpha^{(k)}(J_\gamma \rightarrow \theta_{z_\gamma})$). The exact expressions are then determined as the linear combination of the solution to the Picard-Fuchs system with the appropriate leading behavior. For more details we refer to app. C.

3-pt functions

All the fundamental 3-pt correlators are then determined explicitly in terms of the period integrals on the middle dimensional cohomology of the Calabi–Yau 4-fold \mathbf{X}^\star . Namely, from

$$\nabla_\alpha \mathcal{Q}^{(0)} = \mathcal{Q}^{(1)}, \quad \nabla_\alpha \mathcal{Q}_\beta^{(1)} = C_{\alpha\beta}^{(1)} \gamma \mathcal{Q}_\gamma^{(2)}, \quad \nabla_\alpha \mathcal{Q}_\beta^{(2)} = C_{\alpha\beta}^{(2)} \gamma \mathcal{Q}_\gamma^{(3)}, \quad (3.17)$$

and (3.13) we obtain the final formula for the Yukawa coupling $Y_{\alpha\beta\gamma}$:

$$Y_{\alpha\beta\gamma} = \nabla_\alpha \nabla_\beta \tilde{\Pi}_\gamma^{(2)} \quad (3.18)$$

where $\tilde{\Pi}_\alpha^{(2)} = \Pi_\beta^{(2)} \eta_{\beta\alpha}^{(2)}$ with leading behavior $\tilde{\Pi}_\alpha^{(2)} = \frac{1}{2} C_{\beta\delta}^{(1)\gamma} \eta_{\gamma\alpha}^{(2)} \ln(z_\beta) \ln(z_\delta) + \mathcal{O}(z)$.

Integrating the relations (3.17) over the cycles $\gamma_\alpha^{(i)}$ we obtain the Picard-Fuchs equation satisfied by the periods:

$$\nabla_\lambda (C^{(3)})_{\nu\alpha}^{-1} \nabla_\alpha (C_{\hat{\epsilon}}^{(2)})_{\beta\delta}^{-1} \nabla_{\hat{\epsilon}} (C_{\hat{\rho}}^{(1)})_{\delta\mu}^{-1} \nabla_{\hat{\rho}} \nabla_\mu \Pi_\alpha^{(i)} = 0,$$

where hatted indices are not summed over. This is the holomorphic form of the differential equation reflecting the restricted Kähler structure of the CS moduli space \mathcal{M}_{CS} of the four-fold. The corresponding linear system is the system obtained by integrating (3.17) over the manifold.

Finally note that the full intersection matrix of the period vector $\Pi_\alpha^{(i)}$ is obtained from the topological metrics $\eta^{(i)}$ as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \eta^{(1)} & 0 \\ 0 & 0 & \eta^{(2)} & 0 & 0 \\ 0 & (\eta^{(1)})^T & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.19)$$

3.4. Counting of rational curves

One of the most striking aspects of the calculation of the world-sheet instanton corrected 3-pt couplings of the A model on a Calabi–Yau three-fold \mathbf{X} via mirror symmetry is the interpretation of the integral coefficients of the $q = e^{2\pi it}$ -expansion in terms of the number $N(\mathbf{n})$ of rational curves of multi-degree $\mathbf{n} = (n_1, \dots, n_{h^{1,1}})$ on \mathbf{X} [29]:

$$\begin{aligned} Y_{\alpha\beta\gamma} &= Y_{\alpha\beta\gamma}^{(0)} + \sum_{\mathbf{n}} N'(\mathbf{n}) \prod_{\delta=1}^{h^{1,1}} q_\delta^{n_\delta}, & q_\alpha &= e^{2\pi it_\alpha} \\ &= Y_{\alpha\beta\gamma}^{(0)} + \sum_{\mathbf{n}} N(\mathbf{n}) M_{\alpha\beta\gamma}(\mathbf{n}) \frac{\prod_{\delta=1}^{h^{1,1}} q_\delta^{n_\delta}}{(1 - \prod_{\delta=1}^{h^{1,1}} q_\delta^{n_\delta})}, & M_{\alpha\beta\gamma}(\mathbf{n}) &= n_\alpha n_\beta n_\gamma \end{aligned} \quad (3.20)$$

The factor $(1 - \prod_{\delta=1}^{h^{1,1}} q_\delta^{n_\delta})$ in the denominator of (3.20) takes into account the contribution of multiple coverings. This interpretation has been justified in the framework of the topological sigma model in [30]. The generalization of these argument to the case of $d > 3$ dimensional Calabi–Yau manifolds has been given in [13]; in particular it was shown that the multiple covers contribute in the analogous way as in three dimensions.

On the other hand, the additional factor $M_{\alpha\beta\gamma}(\mathbf{n}) = n_\alpha n_\beta n_\gamma$ in (3.20) gets modified. Let us recall the relevant fact of the definition of the correlation functions in the topological field theory. By definition the local operators $\mathcal{O}_\alpha^{(i)}(P)$, $P \in \Sigma$ have delta function support on maps $\Phi : \Sigma \rightarrow \mathbf{X}$ with the property $\Phi(P) \in H_\alpha^{(i)}$; here Σ is the 2d world sheet and $H_\alpha^{(i)}$ a codimension i homology cycle of the Calabi–Yau target space \mathbf{X} . For the case of a $\mathcal{O}^{(1)}$ operator, $H^{(i)}$ is a divisor, in fact in our choice of basis one of the divisors K_α as defined by the Kähler cone of \mathbf{X} . The n_α factors arise from the multiple intersections of a curve

C with that divisor and count the degree of the curve with respect to it. In the case of the 3-pt functions on the four-folds there are two charge one operators $\mathcal{O}_\alpha^{(1)}$, $\mathcal{O}_\beta^{(1)}$ involved, so one gets to factors of the relevant degrees, n_α and n_β , of the curve [31].

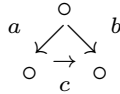
The third operator has charge two and is associated to a codimension two homology cycle - in our basis a linear combination of intersections of two Kähler divisors K_α . Differently from the previous case the condition $\Phi(P) \in H_\alpha^{(i)}$ is a real constraint on the curve C ; we have to adjust the position of the curve in the manifold to satisfy this condition. As a consequence the numbers $N(\mathbf{n})$ appearing in the 3-pt functions do count the appropriate Euler number of the moduli space of rational curves subject to a constraint. The constraint varies with the choice of $H_\alpha^{(2)}$ and thus with the choice of $\mathcal{O}_\alpha^{(2)}$. Therefore we do not expect to get the same numbers $N(\mathbf{n})$ from 3-pt functions involving different operators $\mathcal{O}_\alpha^{(2)}$.

This is actually a nice circumstance for the present case of Calabi–Yau fibrations. In general, rational curves of the fiber get moduli in the four-fold from “moving them over the base”. Therefore the Gromov–Witten invariant of a curve in the three-fold is generically not the same as the invariant of the same curve in the four-fold. However we will show that one can always fix the curves by choosing the appropriate operators $\mathcal{O}_\alpha^{(2)}$; in this case the numbers $N(\mathbf{n})$ of the four-fold indeed coincide with those of the Calabi–Yau fiber. The appropriate choice for the $\mathcal{O}^{(2)}$ is clear: one of the intersecting divisors will be the Calabi–Yau fiber D itself - tautologically this imposes no constraint on the curves of the fiber. The second divisor K_α which we intersect with D to obtain a codimension 2 cycle plays the role of the third charge one operator in the three-fold calculation and contributes naturally another factor of n_α , counting the number of intersections of C and K_α in the fiber.

4. Three-fold fibered four-folds: toric construction and calculation of invariants

In the following we use the above construction to analyze some examples of four-folds with four Kähler moduli which have at the same time phases which allow elliptic, K3 and Calabi–Yau three-fold fibrations. This structure will allow various interpretations in terms of compactifications of F -theory, type IIA/IIB and heterotic/type I theories. In particular much is known about the theories on elliptic three-folds $\mathbf{Y}_{\mathbf{F}_n}$ with base \mathbf{F}_n , $n = 0, 1, 2$ which can be fibered to four-folds with four Kähler moduli, one from the base and three from the Calabi–Yau fiber $\mathbf{Y}_{\mathbf{F}_n}$. In addition to the type of three-fold fiber there is the freedom to choose the bundle structure involving the base \mathbf{P}^1 ; this will determine in particular the Calabi–Yau three-fold $\mathbf{Z}_{\mathbf{F}_n}$ of the dual heterotic theory. In table 1 we collect examples of four-folds with four moduli arising in this way; the precise definition in terms of reflexive polyhedra is given in the text and in appendix D.

Schematically the fibration structure is of the form indicated in the first column of table 1. The three complex dimensional base of the elliptic fibration can be thought of a collection of three \mathbf{P}^1 factors with non-trivial bundle structure. The Chern classes c_1 of these bundles are described by the vector $(a,b;c)$. Here the first two entries refer to the bundle structure of the “top” \mathbf{P}^1 (the base of the elliptically fibered K3) over the other two \mathbf{P}^1 factors, and the third one to the structure of the remaining \mathbf{P}^1 over the base⁷.



In the following columns we give the hodge numbers and the Euler number. $h_{\text{np}}^{1,1}$ and $h_{\text{np}}^{1,2}$ denote the number of deformations which can not be realized as polynomial deformations in the toric description; they will play a role later on. In general there is more than one Calabi–Yau phase; in this case the stated fibration structure is present at least in one of those phases.

⁷ For a consistent notation, a choice has been made for those cases, where the base of the three-fold fibration can be chosen in different ways.

	type	$h^{1,1}$	$h_{\text{np}}^{1,1}$	$h^{1,3}$	$h_{\text{np.}}^{1,3}$	$h^{1,2}$	$h^{2,2}$	χ	
1	(0, 0; 0)	4	0	2916	0	0	11724	17568	IV
2	(1, 0; 0)	4	0	2916	0	0	11724	17568	
3	(2, 0; 0)	4	0	2916	1	0	11724	17568	
4	(1, 1; 0)	4	0	3156	0	0	12648	19008	II
5	(1, 2; 0)	4	0	3396	0	0	13644	20448	
6	(2, 2; 0)	4	0	3877	0	1	15566	23328	
7	(1, 0; 1)	4	0	2796	0	0	11244	16848	III
8	(1, 2; 2)	4	0	3156	1	0	12648	19008	
9	(2, 4; 2)	4	0	3877	1	1	15566	23328	
10	(2, 2; 1)	4	0	3396	0	0	13644	20448	V

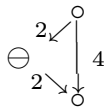
Table 1: Examples of 4 moduli four-folds from fibrations of elliptically fibered three-folds with base \mathbf{F}_n , $n = 0, 1, 2$.

In the following we consider in some detail the four-folds denoted by $I - V$ in the last column. The first one turns out to have a particularly simple singularity structure which we will investigate in sect. 8. Singularities in the three-fold compactification have been analyzed in [4],[32],[33],[6],[34],[35][36][37][38]. The other examples have as the fiber the elliptically fibered three-fold with base \mathbf{F}_1 ; the three-fold compactification has an interesting point in the moduli space with the appearance of tensionless strings with E_8 current algebra [39][6], which has recently attracted considerable interest [40][18][41].

For the invariants we will restrict to the first three cases; the superpotentials and physical singularities will be discussed in sects. 7 and 8 together with the geometrical properties which are relevant for the physical interpretation⁸. In the following we sketch the calculation for the first example. The character of the rest of this section is necessarily technical; the discussion of the results is therefore given in the next section.

4.1. The four-fold X_I

The first example we consider is a four-fold with a elliptic fibered three-fold with base \mathbf{F}_2 fibered over a \mathbf{P}^1 such that the last fibration has itself the structure of \mathbf{F}_2 :



	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
λ_1	0	0	0	2	3	1	0	0	0
λ_2	0	0	1	4	6	0	1	0	0
λ_3	0	1	2	8	12	0	0	1	0
λ_4	1	2	4	16	24	0	0	0	1

(4.1)

⁸ The relevant toric data are given in appendix D.

The line in the middle \mathbf{P}^1 denotes the codimension of the divisor $x_8 = 0$ which will play a role in the discussion of $N = 1$ superpotentials. The table on the right side specifies the C^* actions which is part of the definition of the toric variety; one can easily see the fibration structure

$$\mathbf{T}^2(x_4, x_5, x_6) \rightarrow \mathbf{P}_A^1(x_3, x_7) \rightarrow \mathbf{P}_B^1(x_2, x_8) \rightarrow \mathbf{P}_D(x_1, x_9)$$

where the last two bundles $\mathbf{P}_A^1 \rightarrow \mathbf{P}_B^1$ and $\mathbf{P}_B^1 \rightarrow \mathbf{P}_D^1$ have the structure of rational surfaces of type \mathbf{F}_2 . The dual heterotic theory has therefore \mathbf{F}_2 as the base.

*Basic data and properties*⁹

The dual polyhedron for the four-fold X_I is the convex hull of the negative unit vectors in \mathbf{R}^5 , ν_i^* , $i = 1 \dots 5$ and

$$\nu_6^* = (0, 0, 0, 2, 3), \nu_7^* = (0, 0, 1, 4, 6), \nu_8^* = (0, 1, 2, 8, 12), \nu_9^* = (1, 2, 4, 16, 24)$$

There is a single Calabi–Yau phase which is an elliptic and K3 fibration. The generators $l^{(\alpha)}$ of its Mori cone are

$$\begin{aligned} l^{(1)} &= (0, 0, 1, 0, 0, 0, -2, 1, 0), \quad l^{(2)} = (-6, 0, 0, 0, 2, 3, 1, 0, 0, 0) \\ l^{(3)} &= (0, 1, 0, 0, 0, 0, 0, 0, -2, 1), \quad l^{(4)} = (0, 0, 0, 1, 0, 0, -2, 1, 0, 0). \end{aligned} \quad (4.2)$$

The topological intersection numbers are:

$$\begin{aligned} R_0 &= (64K_2^4 + 2K_4K_2K_1^2 + 4K_2^2K_1^2 + 16K_2^3K_1 + 8K_4K_2^2K_1 + 32K_4K_2^3 + \\ &\quad 16K_2^2K_4^2 + 8K_4^3K_2 + 4K_4^2K_2K_1) + (2K_2^2K_1 + 2K_4^2K_2 + 4K_4K_2^2 + \\ &\quad K_4K_2K_1 + 8K_2^3)K_3 \\ R_2 &= (184K_1K_2 + 364K_2K_4 + 96K_1K_4 + 192K_4^2 + \\ &\quad 48K_1^2 + 728K_2^2) + (48K_4 + 24K_1 + 92K_2)K_3 \\ R_3 &= (-3856K_2 - 1920K_4 - 960K_1) - 480K_3. \end{aligned} \quad (4.3)$$

The coefficients of K_3 in the above expression are precisely the intersection invariants of the Calabi–Yau three-fold fiber.

Cohomology classes of genus zero curves

⁹ The calculation of the triangulations and the intersection calculus has been performed using the maple codes puntos [42] and schubert [43].

To relate physical properties to the geometry of the Calabi–Yau manifold one has to know the location of the relevant objects, e.g. divisors and curves, in the manifold. Let us first determine the classes of the curves C_i associated to the Mori generators $l^{(i)}$ in (4.2). This will be important for two reasons: firstly the volume of C_i is proportional to the special Kähler coordinate t_i . Because of the multiple fibration structure it is straightforward to establish a rough correspondence with some of the heterotic moduli: the base of the elliptically fibered K3 is related to the dilaton by the eight-dimensional duality, while the two-dimensional base of the K3 fibration on the F-theory side maps to the base of the heterotic side. Secondly this information will be necessary in order to understand the singularities which arise when this 2-cycle shrinks to zero size. Physical phenomena such as spectra and the behavior of non-perturbative corrections are closely related to the position and the embedding of the vanishing cycle in the Calabi–Yau four-fold.

Let C_α denote the curve dual to the Mori generator $l^{(\alpha)}$ and D_i denote the toric divisor $\{x_i = 0\}$. As explained in [44] the curve C_α can be determined from i) the intersection of C_α with the D_i 's, ii) the Stanley-Reisner ideal (SR) and iii) the explicit classes as defined by the polynomial constraint in the Batyrev-Cox variables. The Stanley Reisner ideal is given by

$$SR : \{x_1x_9, x_2x_8, x_3x_7, x_4x_5x_6\}$$

and defines the disallowed set $F = \{x_1 = x_9 = 0\} \cup \{x_2 = x_8 = 0\}, \{x_3 = x_7 = 0\}, \{x_4 = x_5 = x_6 = 0\}$ in the definition of the toric variety. The typical Batyrev-Cox polynomial is given by

$$x_7^{12}x_8^{24}x_6^6x_9^{48} + x_7^{12}x_8^{24}x_6^6x_1^{48} + x_7^{12}x_2^{24}x_6^6 + x_3^{12}x_6^6 + x_4^3 + x_5^2 \quad (4.4)$$

respecting the C^* actions (4.1). From the above data it is now straightforward to determine the volumes which are measured by the Kähler coordinates t_α : t_1 is the area of the fiber \mathbf{P}_B^1 , t_2 is the area of a rational curve in the elliptic fiber, C_3 is the the area of the base of the Calabi–Yau fibration \mathbf{P}_D^1 and C_4 is the the area of the fiber \mathbf{P}_A^1 .

Counting of rational curves on X_I

From the Mori generators (4.2) one obtains the Picard-Fuchs system as in [27]:

$$\begin{aligned} \mathcal{L}_1 &= z_1(2\theta_1 + 1 - \theta_4)(-\theta_4 + 2\theta_1) - \theta_1(\theta_1 - 2\theta_3) , \\ \mathcal{L}_2 &= 12z_2(5 + 6\theta_2)(1 + 6\theta_2) - \theta_2(\theta_2 - 2\theta_4) , \\ \mathcal{L}_3 &= z_3(\theta_1 - 2\theta_3)(\theta_1 - 2\theta_3 - 1) - \theta_3^2 , \\ \mathcal{L}_4 &= z_4(\theta_2 - 2\theta_4)(\theta_2 - 2\theta_4 - 1) + \theta_4(-\theta_4 + 2\theta_1) , \end{aligned} \quad (4.5)$$

where $\theta_i = z_i \partial_{z_i}$ and z_i are the algebraic coordinates on the complex structure moduli space (3.11). The period vector is generated by the 16 independent solution to this system; specifically, at $z_i = 0$ there is one series solution w_0 , four linear logarithmic solutions $w_1^i \sim \ln z_i$, six double logarithmic solutions w_2^i , four triple logarithmic solutions w_3^i and one quartic logarithmic solutions w_4 (see appendix C for more details). As described in sect. 3 we can determine the 3-pt functions from the intersection ring \mathcal{R} , whose basis we take to be

deg	generators
1	K_1, K_2, K_3, K_4
2	$K_1 K_2, 2K_1^2 + K_1 K_3, K_2 K_3,$ $K_4 K_1 + 2K_4^2, 2K_2^2 + K_2 K_4, K_3 K_4$
3	$2K_1 K_2^2 + 8K_2^3 + K_1 K_2 K_4 + 4K_2^2 K_4 + 2K_2 K_4^2,$ $2K_4 K_1^2 + K_3 K_4 K_1 + 4K_1 K_4^2 + 2K_3 K_4^2 + 8K_4^3,$ $2K_3 K_2^2 + K_3 K_2 K_4, 2K_2 K_1^2 + K_2 K_1 K_3$

(4.6)

The top element at degree 4 is obtained from the intersection form R_0 as in eq. (3.3). The leading logarithmic terms of the double periodic solutions $\Pi_\alpha^{(2)}$ are then determined by (3.16), while the 2-pt functions $\eta^{(1)}, \eta^{(2)}$ in the basis (4.6) follow from (4.3):

$$\eta^{(1)} = \begin{pmatrix} 178 & 0 & 5 & 0 \\ 712 & 89 & 20 & 10 \\ 89 & 0 & 0 & 0 \\ 356 & 0 & 10 & 5 \end{pmatrix} \quad \eta^{(2)} = \begin{pmatrix} 4 & 0 & 2 & 10 & 40 & 1 \\ 0 & 0 & 0 & 0 & 25 & 0 \\ 2 & 0 & 0 & 5 & 20 & 0 \\ 10 & 0 & 5 & 0 & 100 & 0 \\ 40 & 25 & 20 & 100 & 400 & 10 \\ 1 & 0 & 0 & 0 & 10 & 0 \end{pmatrix} \quad (4.7)$$

Using eq. (3.18) we obtain the 3-pt functions $Y_{\alpha\beta\gamma}$ as power series in the q_α where $q_\alpha = e^{2i\pi t_\alpha}$. The associativity relations (3.9) provide a set of quite non-trivial relations amongst these 3-pt functions, which can be seen to be satisfied. The results are displayed in the tables in app. F in terms of invariants $N_\alpha(\mathbf{n})$ explained in the next section.

5. Results and verification

We have collected the results for the 3-pt functions $Y_{\alpha\beta\gamma}$ for the four-fold examples under consideration in app. F in terms of invariants $N_\alpha(\mathbf{n})$ defined in the following way.

Taking into account the multiple covering formula and the two factors from the intersections of a curve C with the divisors $H_\alpha^{(1)}$, $H_\beta^{(1)}$ we write similar as in (3.20):

$$\begin{aligned}
Y_{\alpha\beta\gamma} &= Y_{\alpha\beta\gamma}^{(0)} + \sum_{\mathbf{n}} N'(\mathbf{n}) \prod_{\delta=1}^{h^{1,1}} q_\gamma^{n_\delta}, \\
&= Y_{\alpha\beta\gamma}^{(0)} + \sum_{\mathbf{n}} N_\gamma(\mathbf{n}) M_{\alpha\beta}(\mathbf{n}) \frac{\prod_{\delta=1}^{h^{1,1}} q_\delta^{n_\delta}}{(1 - \prod_{\delta=1}^{h^{1,1}} q_\delta^{n_\delta})}, \quad M_{\alpha\beta}(\mathbf{n}) = n_\alpha n_\beta
\end{aligned} \tag{5.1}$$

The result for the numbers $N_\gamma(\mathbf{n})$ obtained from the various $Y_{\alpha\beta\gamma}$ in this way depends then only on the index γ of the charge two operator $\mathcal{O}_\gamma^{(2)}$, reflecting the fact that the numbers $N_\gamma(\mathbf{n})$ contain a factor counting the intersection of the rational curve C with the codimension 2 cycle $H_\gamma^{(2)}$ associated to $\mathcal{O}_\gamma^{(2)}$; $N_\gamma(\mathbf{n})$ divided by this factor is expected to agree with $(-1)^{\dim \mathcal{M}}$ times the Euler number of the moduli space \mathcal{M} of rational curves on \mathbf{X} , subject to the constraint to intersect $H_\gamma^{(2)}$. To be more precise, there is a single normalization factor c_γ for each γ which we have not yet fixed by the constraints, which corresponds to multiplication of $\mathcal{O}_\gamma^{(2)}$ by a constant $\mathcal{O}_\gamma^{(2)} \rightarrow c_\gamma \mathcal{O}_\gamma^{(2)}$. We can fix these constant by explicit knowledge of one non-zero number $N_\gamma(\mathbf{n})$ for each γ , as we will do below by comparison with the fiber data.

Let us verify the above interpretation for the example X_I by comparing the invariants $N_\gamma(\mathbf{n})$ with the Gromov–Witten invariants of the three-fold fiber, calculated in [27]. As explained at the end of sect. 3, we can use the constraint that a curve C has to intersect $H_\gamma^{(2)}$ to fix possible moduli for the same curves in the three-fold fibered four-fold. Remind that $H_\gamma^{(2)}$ is obtained as the intersection of two divisors. The first divisor we chose is of course the fiber itself, which can be seen to be $D_1 = D_9 = K_3$. The second divisor will be one of the other K_i , $i = 1, 2, 4$. For the curves in the fiber the constraint to intersect $H_\gamma^{(2)}$ obtained in this way reduces to the constraint to intersect K_i , moreover the number of intersections is the degree n_i . From (4.6) we see that we have to consider 3-pt functions involving $\mathcal{O}_\gamma^{(2)}$ for $\gamma = 2, 3, 6$, respectively. The result agrees with the three-fold invariants in [27] for the normalization of $\mathcal{O}^{(2)}$ operators $c_\gamma = (\frac{1}{5}, 1, 1)$.

To be more precise, note that the operator $\mathcal{O}_2^{(2)}$ in (4.6) is not precisely $K_1 K_3$ but $K_1 K_3 + 2K_1^2$. Although it turns out to be sufficient in this geometrical simple example to chose the proper leading piece in the large base space limit, this is not the case in general. However we can determine the correct choice of $H_\alpha^{(2)}$ in the following way.

Consider the large base limit of the 4-pt functions $K_{\alpha\beta\gamma\delta}$ on \mathbf{X} . From their definition we expect that in the limit $t_D \rightarrow \infty$, where the instanton effects from the base are switched

off, the 4-pt functions with first index equal to D become the 3-pt functions on the fiber \mathbf{Y} . For the same reason, the operator product coefficients $C_{\alpha\beta}^{(1)\gamma}$ with first index D become constant. We have:

$$Y_{\alpha\beta\gamma}^{3f} = \lim_{q_D=0} K_{D\alpha\beta\gamma} = \lim_{q_D=0} Y_{\beta\gamma\mu} N_{\alpha}^{\mu}, \quad N_{\alpha}^{\mu} = \lim_{q_D=0} C_{D\alpha}^{(1)\mu} \quad (5.2)$$

stating that the coefficients of the q_D independent terms in the new basis obtained by multiplication of $Y_{\alpha\beta\mu}$ with the constant matrix N_{α}^{μ} agree with those of the three-fold fiber. The corresponding operators $O_{\alpha}^{(2)}$ follow then from transforming them into the same basis. The matrices N_{α}^{μ} determining the appropriate choice of $O_{\alpha}^{(2)}$ in the cases X_{II} and X_{III} are given in the appendix.

5.1. Space-time effective action

The limiting behavior (5.2) can also be understood from the relation of the correlation functions $K_{\alpha\beta\gamma\delta}$ and $Y_{\alpha\beta\gamma}$ to couplings in the space time effective action. In the following we will perform a preliminary analysis of the properties of these couplings and discuss some of their possible implications; we defer a more complete analysis and the precise relation to the four-dimensional effective theory of the heterotic dual to a future publication.

Consider the eleven dimensional supergravity limit of [45], with field content the eleven-dimensional vielbein e_M^a , the gravitino Ψ_M and the antisymmetric gauge tensor $A_{MNR}^{(3)}$, augmented by a term taking care of the five-brane sigma model anomaly [46]:

$$\delta S = \frac{1}{2} \int A^{(3)} \wedge X_8(R) \quad (5.3)$$

where $X_8(R)$ is the eight-form anomaly polynomial, quartic in the Riemann tensor R . The equation of motion for $F^{(4)}$, including a source term for the membrane and five brane reads

$$d \star F^{(4)} = -\frac{1}{2} F^{(4)} \wedge F^{(4)} + X_8(R) + \sum_i Q_{2B}^i \delta_i^{(8)} + \sum_i Q_{5B}^i \delta_i^{(5)} \wedge T^{(3)} \quad (5.4)$$

where $T^{(3)}$ is the self-dual three-form field strength on the five brane, $\delta^{(n)}$ is a n -form integrating to one on the transverse directions to a brane and Q_{2B}, Q_{5B} the charges of the branes.

Implications of the term (5.3) for M-theory compactifications to three dimensions have been discussed in [47][9][48]. Integrating (5.4) over the four-fold imposes the constraint that the integral on the rhs has to be zero. This integral is also the coefficient of the $A^{(3)}$

tadpole in the three dimensional space-time which has again to be zero for a consistent vacuum [9]. The integral of the anomaly polynomial $X_8(R)$ over \mathbf{X} is proportional to the Euler number, $\int_{\mathbf{X}} X_8(R) = -\frac{1}{24}\chi$ [47], and gives generically a non-zero contribution. It can be cancelled by the contributions from membranes filling space-time and $F^{(4)}$ flux on the manifold. In fact it has been shown in [48] that there has to be non-zero $F^{(4)}$ flux in the case that the first Pontryagin class p_1 of \mathbf{X} is not divisible by four.

Different supersymmetric vacua on \mathbf{X} are then characterized by the values of the non-trivial flux of the internal components $F_{m\bar{m}n\bar{n}}$ of $F^{(4)}$ which can be expanded as

$$F^{(4)} = \sum_{\gamma} \nu^{\gamma} \omega_{\gamma}^{2,2}, \quad (5.5)$$

where $\omega_{\gamma}^{i,j}$ denote a basis of harmonic (i, j) forms and the number ν^{γ} are integers (or possibly half-integers, if $\frac{p_1}{4}$ is not an integral class [48]). The only other non-vanishing component of $F^{(4)}$ is $F_{\mu\nu\rho m} \sim \epsilon_{\mu\nu\rho}$ [47], where roman (greek) indices refer to compact (non-compact) dimensions.

Expanding the fields Ψ_M and $A_{MNP}^{(3)}$ in harmonics on \mathbf{X} one obtains terms in the three-dimensional action which are classically proportional to integrals over the internal manifold. In particular, terms related to the couplings $K_{\alpha\beta\gamma\delta}$ and $Y_{\alpha\beta\gamma}$ should have an internal structure

$$K_{\alpha\beta\gamma\delta}^{\text{cl.}} = \int \omega_{\alpha}^{1,1} \wedge \omega_{\beta}^{1,1} \wedge \omega_{\gamma}^{1,1} \wedge \omega_{\delta}^{1,1}, \quad Y_{\alpha\beta\gamma}^{\text{cl.}} = \int \omega_{\alpha}^{1,1} \wedge \omega_{\beta}^{1,1} \wedge \omega_{\gamma}^{2,2}.$$

The relevant four field coupling in eleven dimensions related to $K^{\text{cl.}}$ are the four-fermi interactions of the gravitino Ψ_M in the action of [45] which yield four-fermi terms in three dimensions from zero modes of Ψ_M which are related to harmonic $h^{1,1}$ forms similar as in the three-fold case [49].

On the other hand, terms proportional to $Y^{\text{cl.}}$ arise from the three field couplings

$$\int A^{(3)} \wedge F^{(4)} \wedge F^{(4)}, \quad \int (\bar{\Psi}_{\mu} \Gamma^{\mu\nu\alpha\beta\gamma\delta} \Psi_{\nu} + 12 \bar{\Psi}^{\alpha} \Gamma^{\gamma\delta} \Psi^{\beta}) F_{\alpha\beta\gamma\delta}^{(4)}, \quad (5.6)$$

From (5.5) it follows that the 3pt couplings $Y_{\alpha\beta\gamma}$ are related to Chern-Simons couplings in three dimensions

$$c_{\alpha\beta} A^{\alpha} \wedge F^{\beta}, \quad c_{\alpha\beta} \equiv Y_{\alpha\beta\gamma} \nu^{\gamma}, \quad (5.7)$$

whereas the second coupling in (5.6) provides the $N = 2$ supersymmetric completion, a mass term for the fermionic superpartners.

M-theory on \mathbf{X} is dual to heterotic string on $\mathbf{Z} \times S^1$. It is interesting to compare the above result with the compactification of the heterotic string to three dimensions on Calabi–Yau three-fold times S^1 performed in [50]. Starting from the dual formulation in ten dimensions involving a six-form gauge potential instead of the usual two-form potential one obtains similarly gauge and gravitational Chern-Simons type of couplings in three dimensions¹⁰. In fact it was argued there that these couplings break supersymmetry. We will see momentarily that tree-level supersymmetry indeed implies the vanishing of the coefficients $c_{\alpha\beta}$ in the four-fold compactification. However non-zero $c_{\alpha\beta}$ will be generated by instanton corrections to the classical periods.

$h^{2,2}$ cohomology and $F^{(4)}$ flux

The four-form cohomology $H^{2,2}(\mathbf{X})$ plays a special role in many respects. From the point of topological field theory it is special since it is part of both the vertical primary space $H_V^{k,k}(\mathbf{X})$ as well as its horizontal primary space $H_H^{d-k,k}(\mathbf{X})$. Similarly there exist mixed boundary conditions in the construction of supersymmetric 4-cycles [52]. Moreover recall that the elements of $H_V^{2,2}(\mathbf{X})$ considered so far are only a tiny subspace of the full $H^{2,2}(\mathbf{X})$ cohomology, which has a dimension of the order of 10^4 in the simple four-folds we consider (see table 1).

Elements of $H^{2,2}(\mathbf{X})$ are related to topological degrees of freedom of the vacuum, the $F^{(4)}$ flux on \mathbf{X} characterized by the flux numbers ν^γ . This is in agreement with the relation between $H^{2,2}(\mathbf{X})$ and the operators of $\mathcal{O}^{(2)}$ in the topological field theory. They are obtained by fusion of $\mathcal{O}^{(1)}$ operators and have twice the $U(1)$ charge of a marginal operator, representing massive perturbations of the topological background rather than massless fields. It is also in agreement with the expected spectrum for the type IIA theory after compactification on a further circle. In particular, starting from the dual five form field strength in ten dimensions one obtains vector gauge fields in two dimensions which can carry only topological degrees of freedom. Since the $\mathcal{O}^{(1)}$ operators are naturally related to the fundamental space-time fermions there is also the possibility that the $\mathcal{O}^{(2)}$ operators could also correspond to condensates quadratic in space-time fields.

The actual choice of the fluxes ν^γ is related directly to both internal and space-time properties of the vacuum, namely the $F^{(4)}$ flux and the resulting Chern-Simons couplings (5.7). From the internal point of view there are several immediate questions concerning a non-zero value of a flux $F^{(4)}$ on a 4-cycle: i) what are the possible consistent configurations

¹⁰ The analysis of the five-dimensional duality between M-theory and the heterotic string in [51] suggests that the choice of representation is actually not essential.

of flux on the internal manifold given the cohomology of \mathbf{X} ; ii) how does the choice of flux influence wrappings of branes and therefore the spectrum and physical properties of the vacuum; iii) what is its origin.

Concerning the first question consider the equation fulfilled by the components $F_{m\bar{m}n\bar{n}}$ derived in [47]:

$$F_{m\bar{m}n\bar{n}}g^{n\bar{n}} = 0, \quad (5.8)$$

where $g_{m\bar{m}}$ is the Kähler metric on \mathbf{X} . It is straightforward to show that this equation implies the vanishing of the integral $I(F) = \int_{\mathbf{X}} J \wedge J \wedge F$, where J is the Kähler form. Moreover $F^{(4)}$ has to be self-dual. $I(F)$ is a quadratic polynomial in the $h^{1,1}$ special coordinates t_α , $I(F) = \sum_{\alpha\beta\gamma} t_\alpha t_\beta \nu^\gamma Y_{\alpha\beta\gamma}^{\text{cl}}$. For generic values of these moduli the $h^{1,1}(h^{1,1} + 1)/2$ coefficients have to vanish. Since $\dim(H_V^{2,2}(\mathbf{X})) = 2h^{1,1} - 2$, there is in general no non-trivial solution for the ν^γ within $H_V^{2,2}(\mathbf{X})$ and $I(F) = 0$ imposes a constraint on the moduli. In fact $I(F)$ measures the volume of a 4-cycle if F is dual to a codimension two submanifold as is the case for the individual terms in the expansion in (5.5) for a properly chosen basis $\omega_\gamma^{2,2}$. Moreover $I(F) = 0$ implies that the classical Chern-Simons couplings (5.7) vanish. One may ask about the stability of such a vacuum, which is consistent only for special values of the Kähler moduli. The following observation indicates that vacua of this kind indeed exist. Mathematically one expects a reduced quantum moduli space associated to the slice of fixed t_α . This kind of quantum moduli spaces can be realized as that of toric manifolds which describe the modding of a basic toric manifold (the physical theory without flux) by an identification of Kähler moduli, that is a modding on moduli space. For an example of this kind we refer the reader to the discussion of the 4-fold $P_{1,1,1,1,4,4}$ in sect.6.

Allowing for general elements of $H^{2,2}(\mathbf{X})$ in the expansion of $F^{(4)}$ in (5.5), there will be non-trivial solutions for the ν^γ for which all coefficients of the quadratic polynomial $I(F)$ vanish. The classical coefficient $c_{\alpha\beta}$ of the Chern-Simons couplings then still are zero for the same reasons as before. However since such a non-trivial solution involves relations between classical geometrical data which are generically not respected by the quantum corrections, instantons will generate non-zero $c_{\alpha\beta}$. These instanton generated Chern-Simons interactions in the three-dimensional space-time effective action with a potential of breaking supersymmetry deserve certainly further study. A more detailed analysis is necessary to determine the vacuum structure which could also involve condensates arising from the 4-pt couplings $K_{\alpha\beta\gamma\delta}$.

Phase transitions and disconnected vacua

An interesting interplay between the flux conditions ν^γ and possible five-brane wrappings

inside \mathbf{X} is implied by the Bianchi identity of the three-form field strength $T^{(3)}$ on the five-brane, $dT^{(3)} = F^{(4)}$ [53][34]. This equation has been used in [54] to argue that the flux of $F^{(4)}$ associated to a four-cycle C_4 has to be zero in order that the five-brane can be wrapped around C_4 .

Said differently, a vacuum can be stabilized by putting appropriate $F^{(4)}$ flux on 4-cycles in divisors which can support superpotential terms generated by wrappings of five branes. It seems likely that this mechanism plays a role in the situation where a point is blown up in a four-fold compactification \mathbf{X} which did not generate a superpotential before the blow up [8][9]. In this case one faces the contradiction that the two vacua, \mathbf{X} and \mathbf{X}' , which are apparently connected by a phase transition, differ in that a superpotential W is generated only in the theory \mathbf{X}' with the point blown up. The above comments suggest the following resolution to this problem: the moduli space $\mathcal{M}(\mathbf{X})_{W=0}$ where $W = 0$ is connected to a version of \mathbf{X}' with moduli space $\mathcal{M}(\mathbf{X}')_{W=0}$, where flux on the exceptional divisor E prevents a superpotential (alternatively one could also consider the case where some of the space-time filling membranes are located on E and change the zero mode structure of a five brane wrapped on E). On the other hand the moduli space of that version of \mathbf{X}' with $W \neq 0$ is disconnected from \mathbf{X} . Indeed the superpotential depends now non-trivially on the volume of E and is therefore no longer a flat direction.

Also the Bianchi identity for the three form field strength $T^{(3)}$ has to be augmented by a source term for a membrane boundary in a more complicated configuration of branes, $dT = F^{(4)} + Q_{1B}\delta^{(4)}$, where $\delta^{(4)}$ is a four-form which integrates to one in the directions transverse to the one-dimensional intersection of the membrane with the five brane. Clearly it will be interesting to understand the possible consistent configurations and transitions amongst them.

Let us finally also mention the homology class of 3-cycles on \mathbf{X} . 3-cycle classes are relatively rare objects in the four-fold we have encountered as can be seen from the values $h^{1,2} = 0, 1$ in table 1. If a three cycle exists it can contribute to the tadpole in three dimensions via the interaction $\int_{5B} A^{(3)} \wedge T^{(3)}$ in the presence of further membranes.

6. $N = 1$ superpotentials

In the next section we will look in some detail at the singularities in the moduli space of the Calabi–Yau four-folds. Physical phenomena associated to these singularities may lead to the generation of a superpotential in the 4d $N = 1$ theories obtained from compactification of F-theory on \mathbf{X} or heterotic string on \mathbf{Z} . It was shown in [8] that superpotential terms can be generated by wrapping the M-theory five brane on divisors in \mathbf{X} which fulfill a certain set of conditions. In the following we give a *complete classification of divisors of the appropriate type from intersection theory on the toric variety*. Here we use the fact that the hypersurfaces $D_i : \{x_i = 0\}$ of the toric variety provide a complete basis of divisors on \mathbf{X} [22]. Note that intersection theory is anyway the minimal framework that is needed to make any sensible statement about the superpotential, since it determines the instanton action $\sim e^{-V_D}$.

The first condition derived in [8] from anomaly cancellation requires the arithmetic genus $\chi(\mathcal{O}_D)$ of D to be one, where $\chi(\mathcal{O}_D) = h^{0,0}(D) - h^{1,0}(D) + h^{2,0}(D) - h^{3,0}(D)$. More precisely, to have the correct number of fermionic zero modes to generate a superpotential, we consider divisors with $h^{1,0} = h^{2,0} = h^{3,0} = 0$. As explained in [8], $h^{3,0}(D)$ is the number of complex structure deformations in the class D ; therefore D should have no moduli.

The contribution of $h^{1,0}$ and $h^{2,0}$ is zero for a divisor given by the section of a positive line bundle [8] by the Lefschetz theorem, which would imply non zero Hodge numbers $h^{1,0}$, $h^{2,0}$ for the Calabi–Yau in contradiction with $SU(4)$ holonomy. On the other hand exceptional divisors introduced by the resolution of singularities can have non zero $h^{i,0}$, $i = 1, 2$; this happens e.g. in the case of resolutions of curve singularities in three-folds with an exceptional divisor of the form $C \times \mathbf{P}^1$, where C is a genus g curve. In this case g adjoint hypermultiplets arise in the twisted world brane theory from the g holomorphic one forms on the singular curve C [33].

A further condition on D arises from the scaling behavior of its area when the four-dimensional limit of M theory on \mathbf{X} is taken using the relation to F theory on $\mathbf{X} \times \mathbf{S}^1$ [8]. Contributions from horizontal divisors - that is divisors which are sections of the elliptic fibration π , $\pi(D) = B$ - are suppressed compared to vertical divisors - that is divisors which project on divisors D' in B , $\pi(D) = D' \subset B$. Said differently, the action of the five brane wrapped around the elliptic fiber is smaller in the limit of vanishing fiber volume than that of a five brane wrapped only in the base. In the following let \hat{D} denote a divisor fulfilling the above conditions; that is it is vertical and has $h^{i,0} = 0$, $i = 1, 2, 3$.

We can determine the classes of divisors D which can contribute to the superpotential in F-theory compactification on \mathbf{X} represented as before as a hypersurface in a toric variety,

from the intersection data on \mathbf{X} . $\chi(\mathcal{O}_D)$ can be expressed in terms of the intersections on \mathbf{X} as [55]:

$$\chi(\mathcal{O}_D) \equiv \sum (-)^n h^{n,0}(D) = -\frac{D^4 + D^2 c_2}{24} = 1 \quad (6.1)$$

This determines $\chi(\mathcal{O}_D)$ for any toric divisor $D = \sum a_i D_i$ in terms of the topological intersection numbers in R_2 and R_0 in (3.4). To avoid the moduli problem we require a divisor D to be the single divisor with a certain weight under the C^* actions of \mathbf{X} . Vertical divisors are obtained by taking linear combinations which do not impose a constraint on the fiber.

In the examples in [8], the arithmetic genus of D was always less or equal than one. It was pointed out that in the case that $\chi(\mathcal{O}_D) = n$ with $n > 1$, where naively there are too many zero modes to allow for a non zero superpotential, strong infrared dynamics might lead to the generation of fractional instantons with the right quantum numbers. In the more general four-folds we consider there are points in the moduli space where strong infrared behavior is expected and indeed we will also find the case with $\chi(\mathcal{O}_D) > 1$.

There is a subtlety in the calculation of $\chi(\mathcal{O}_D)$ in the toric variety which arises from the fact that the map from the divisor classes in the toric variety to divisor classes in the Calabi–Yau hypersurface might not be one to one. In particular this happens if there are points on a face in the dual polyhedron Δ^* and points in the interior of the dual face in the original polyhedron Δ . These correspond to perturbations, which can not be realized as polynomial deformations. Geometrically what happens is that the divisor introduced by the resolution of a singularity in the toric variety intersects the Calabi–Yau hypersurface more than one time. As a consequence, a priori independent divisors associated to resolutions of the Calabi–Yau hypersurface are not independent in the toric description. In this case the arithmetic genus calculated from the toric intersection numbers sums up the contribution of several divisors in the Calabi–Yau which all correspond to the same class in the toric variety. As a simple example, consider the degree 12 hypersurface \mathbf{X} in $P_{1,1,1,1,4,4}$ which was also considered in [8]. There are three exceptional divisors \mathbf{P}^2 in \mathbf{X} from blowing up the \mathbf{Z}_4 quotient singularities $x_i = 0$, $i = 1, \dots, 4$. They all correspond to the same divisor class in the standard description as a toric variety where only 2 of 4 deformations can be represented as polynomial deformations. As a result one obtains $\chi(\mathcal{O}_D) = 3$ from the intersections in the toric variety and eq. (6.1). From the definition of $\chi(\mathcal{O}_D)$ and $h^{i,0} = 0$, $i = 1, 2, 3$ it is clear that the arithmetic genus three arises from the three disconnected components of the toric divisor in \mathbf{X} , contributing each with a one to $h^{0,0}$.

It was also explained in [8] how the location of the complex codimension of \hat{D} in \mathbf{X} is related to the physical interpretation of the effect in the heterotic theory on \mathbf{Z} : vertical divisors \hat{D} in \mathbf{X} map to divisors \hat{D}' in \mathbf{Z} . If \hat{D}' is vertical with respect to the elliptic fibration $\pi_H : \mathbf{Z} \rightarrow B_H$, it is localized in complex dimension one in the base and corresponds to a world sheet instanton effect on the heterotic side. On the other hand horizontal divisors are interpreted as space time instantons.

Note that by classifying the divisor classes which can support five brane compactifications we have not counted the actual representatives of a given class¹¹. In the case of rational curves this counting is possible due to mirror symmetry; clearly it would be important to get a similar control over the representatives of a divisor in a given class.

In the following we determine the divisors \hat{D} for the four-folds X_{I-V} using intersections and describe their geometrical location and the fibration structure of \mathbf{X} . They will serve as examples for various physical phenomena one encounters , namely

- o space-time/world-sheet instanton generated superpotentials
- o compactifications of six-dimensional strings
- o singularities associated with tensionless strings and generation of a quantum tension

Divisors on X_I

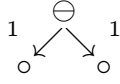
The intersection data for X_I are given in (4.3). From (4.1) we see that a D_i , $i = 6 \dots 9$ provide a basis for the classes of divisors $D = \sum a_i D_i$ on X_I . In order that D has no complex structure moduli it has to be the single divisor with a certain weight Λ under the C^* symmetries (4.1). Moreover D_6 represents a restriction on the elliptic fiber. We therefore consider divisors $D = a_7 D_7 + a_8 D_8$. From the intersections we find $\chi(\mathcal{O}_D) = 1 = 2a_8^2$, which has no integer solutions¹². So naively there is no divisor of the type \hat{D} to generate a superpotential; one would have to wrap half a five brane on D_8 to get the correct number of fermionic zero modes. We will come back to this question in the next section when we discuss the singularities in the moduli space of this model. Finally note that D_8 maps to a vertical divisor in \mathbf{Z} and would therefore be associated to a *world sheet* instanton effect in the heterotic theory.

¹¹ Such a counting has been performed in [55] for a special four-fold.

¹² By the same reasonings one can see that there are horizontal divisors with $\chi(\mathcal{O}_D) = 1$. Thus M-theory in three dimensions develops a superpotential; this is a consequence of the existence of a global section of the elliptic fibration and is true for all of the four-folds below.

Divisors on X_{II}

The fibration structure of X_{II} is



	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
λ_1	0	0	0	2	3	0	0	1	0
λ_2	0	0	1	4	6	1	0	0	0
λ_3	1	0	1	6	9	0	0	0	1
λ_4	0	1	1	6	9	0	1	0	0

(6.2)

The C^* actions (6.2) make transparent the fibration structure

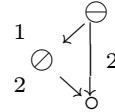
$$\mathbf{T}^2(x_4, x_5, x_8) \rightarrow \mathbf{P}_A^1(x_3, x_6) \rightarrow (\mathbf{P}_{D_1}^1(x_2, x_7) \times \mathbf{P}_{D_2}(x_1, x_9))$$

where the fiber structure of $\mathbf{P}_A^1 \rightarrow \mathbf{P}_{D_i}^1$ is that of of the rational surfaces \mathbf{F}_1 . In particular the dual heterotic string has as compactification manifold the elliptically fibered Calabi–Yau with base $\mathbf{P}^1 \times \mathbf{P}^1$ [6]. From (6.2) we see that we have to consider divisors $D = a_6 D_6$, with $\chi(\mathcal{O}_D) = 1 = a_6^2$. So in this case there is a divisor, namely simply D_6 . Since D_6 maps to the base of the three-fold on the heterotic side, the physical interpretation in the heterotic theory is that this contribution from the superpotential arises due to a *space-time* instanton effect, rather than a world sheet instanton.

Divisors on X_{III}

X_{III} differs from X_{II} only by the fibration structure over the base

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
λ_1	0	0	0	2	3	0	0	1	0
λ_2	0	0	1	4	6	1	0	0	0
λ_3	0	1	1	6	9	0	1	0	0
λ_4	1	2	2	12	18	0	0	0	1



(6.3)

that is $\mathbf{T}^2(x_4, x_5, x_8) \rightarrow \mathbf{P}_A^1(x_3, x_6) \rightarrow \mathbf{P}_B^1(x_2, x_7) \rightarrow \mathbf{P}_D(x_1, x_9)$ where the last two sequences $\mathbf{P}_A^1 \rightarrow \mathbf{P}_B^1$ and $\mathbf{P}_B^1 \rightarrow \mathbf{P}_D$ have the structure of rational surfaces of type \mathbf{F}_1 and \mathbf{F}_2 , respectively. The heterotic dual has again \mathbf{F}_2 as the base of the elliptic fibration. Divisors $D = a_6 D_6 + a_7 D_7$. have $\chi(\mathcal{O}_D) = 1 = 2a_7^2 + a_6^2$. So D_6 is a divisor with the correct properties. On the other hand D_7 has similar properties as the divisor D_8 in the first example. Similar to the previous cases D_6 maps to the base on the heterotic side while D_7 maps to a vertical divisor in \mathbf{Z} .

Divisors on X_{IV}

X_{IV} has been considered already in sect. 3 of [8]; the fibration structure is

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
λ_1	0	0	0	0	0	0	2	3	1
λ_2	0	1	0	0	1	0	4	6	0
λ_3	0	0	1	1	0	1	6	9	0
λ_4	1	0	0	0	0	1	4	6	0



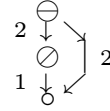
(6.4)

reflecting the fibration structure $\mathbf{T}^2(x_7, x_8, x_9) \rightarrow (\mathbf{P}_A^1(x_2, x_5) \times (\mathbf{P}_B^1(x_1, x_6) \rightarrow \mathbf{P}_D(x_3, x_4))$. There are two heterotic duals, one with base \mathbf{F}_1 , the other with base \mathbf{F}_0 . A divisor $D = a_1 D_1$ has $\chi(\mathcal{O}_D) = a_1^2$ thus D_1 has the deliberate properties. D_1 maps to a horizontal divisor in the fibration with \mathbf{F}_0 as the base and to a vertical one in the \mathbf{F}_1 fibration.

Divisors on X_V

The last example has a fibration structure

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
λ_1	0	0	0	2	3	1	0	0	0
λ_2	0	0	1	4	6	0	1	0	0
λ_3	0	1	2	8	12	0	0	1	0
λ_4	1	1	2	10	15	0	0	0	1



(6.5)

reflecting the fibration structure $\mathbf{T}^2(x_4, x_5, x_6) \rightarrow \mathbf{P}_A^1(x_3, x_7) \rightarrow \mathbf{P}_B^1(x_2, x_8) \rightarrow \mathbf{P}_D(x_1, x_9)$. From (6.5) we see that we have to consider divisors $D = a_7 D_7 + a_8 D_8$; such a divisor has $\chi(\mathcal{O}_D) = a_8^2 + a_7^2 - a_8 a_7$; divisors with $\chi(\mathcal{O}_D) = 1$ are therefore D_7 , D_8 and $D_7 + D_8$. D_7 maps to the heterotic base B_H whereas D_8 maps to a divisor in B_H .

7. Singularities in the moduli space

For special values of the moduli, the Calabi–Yau manifold \mathbf{X} develops singularities. At these points in the moduli space the periods and the correlation functions derived from them may vanish or acquire singularities. The singularities of the $N = 2$ theories appearing at the discriminant locus of three-folds can often be understood in terms of the presence of massless BPS states at these special submanifolds in the moduli space [56].

The elliptic/K3/Calabi–Yau fibration structure of the four-folds we consider is primarily a property of the four-fold \mathbf{X} ; although in many cases also the mirror manifold \mathbf{X}^* will have a similar structure for appropriate choice of complex structure. Therefore the singularities analyzed by the singularity structure of the correlation functions are associated to Kähler moduli. Since the dimension of the cycles associated to the Kähler moduli is even and the type IIB theory branes have odd dimension these singularities are generically associated to tensionless strings. Differently than in the three-fold case also the complex structure moduli are naturally related to even dimensional cycles. One possibility to get massless point like states is to shrink a p cycle to an extended cycle rather than to a point; this is how e.g. gauge symmetry enhancement can be related to the variation of Kähler moduli in the conventional type IIB compactifications [57],[32],[33]. On the other hand the fiber plays a special role in the F-theory compactification; singularities in the fiber are interpreted as seven-branes located at complex codimension two rather than in terms of vanishing cycles. Therefore massless point-like states associated to gauge symmetries can arise from singularities in the fiber reached by varying the complex structure.

It is known that non-vanishing superpotentials require singularities if the compact moduli space is compact [58]. Therefore some information about the physical origin of non-vanishing superpotentials from divisors \hat{D} as determined in the previous section can be gained by looking for the singularities associated to them. Five branes dual to small $SO(32)$ instantons with enhanced $SU(2)$ gauge symmetries at zero size [59] have been argued to generate a superpotential in [11] by analyzing the singularities in the elliptic fibration of the dual F-theory compactification. We will perform a similar analysis of the singularities in the fibration structure of our four-folds below, with the result that singularities in the fibration appear only at high codimension. This is not too surprising in the light of the fact that the four-folds we consider are expected to be dual to $E_8 \times E_8$ heterotic compactifications; small instanton dynamics of E_8 instantons however involves tensionless strings [39] rather than enhancement of gauge symmetries. The fact that the existence of a T-duality between $SO(32)$ and $E_8 \times E_8$ heterotic strings in three dimensions for a given compactification would imply the relevance of tensionless string dynamics has been pointed out in [11].

7.1. Weierstrass model and fiber singularities

Singularities in the elliptic fibrations have been analyzed in detail [6],[60] using a Weierstrass model for \mathbf{X} . The Weierstrass model for X_I is $y^2 + x^3 + xz^4f + z^6g$ where $y = x_5$, $x = x_4$, $z = x_6$ and

$$\begin{aligned} X_I : f &= \sum_{\delta, \epsilon} f_{(2\epsilon; 2\delta - \epsilon; 8 - \delta; \delta; \epsilon)}(x_1, x_9; x_2; x_3; x_7; x_8), \\ g &= \sum_{\delta, \epsilon} g_{(2\epsilon; 2\delta - \epsilon; 12 - \delta; \delta; \epsilon)}(x_1, x_9; x_2; x_3; x_7; x_8) \end{aligned} \quad (7.1)$$

where the sections f and g are polynomials in the x_i with the degrees denoted by the subscripts. The fibration becomes singular when the discriminant of the elliptic fiber $\Delta_{\mathbf{T}^2} = 4f^3 + 27g^2$ vanishes. From (7.1) we see that above $x_8 = 0$, f and g are of the form

$$f \sim \sum_{\delta=0}^8 x_2^{2\delta} x_3^{8-\delta} x_7^\delta f_\delta, \quad g \sim \sum_{\delta=0}^{12} x_2^{2\delta} x_3^{12-\delta} x_7^\delta g_\delta, \quad (7.2)$$

whereas $\Delta_{\mathbf{T}^2}=0$ requires $f \sim \Phi(x_i)^2$, $g \sim \Phi(x_i)^3$ for some function $\Phi(x_i)$. Counting the number of parameters we see that a singular fiber appears at the first time at codimension nine. Similar arguments apply to the other four-folds. For further reference we collect their Weierstrass forms:

$$\begin{aligned} X_{II} : & x_5^2 + x_4^3 + x_4 x_8^4 \sum_{\delta, \epsilon} f_{(4+\delta; 4+\delta; 8-\delta; \delta)}(x_1, x_9; x_2, x_7; x_3; x_6) \\ & + x_8^6 \sum_{\delta, \epsilon} g_{(6+\delta; 6+\delta; 12-\delta; \delta)}(x_1, x_9; x_2, x_7; x_3; x_6) \end{aligned} \quad (7.3)$$

$$\begin{aligned} X_{III} : & x_5^2 + x_4^3 + x_4 x_8^4 \sum_{\delta, \epsilon} f_{(2\epsilon; 4+\delta-\epsilon; 8-\delta; \delta; \epsilon)}(x_1, x_9; x_2; x_3; x_6; x_7) \\ & + x_8^6 \sum_{\delta, \epsilon} g_{(2\epsilon; 6+\delta-\epsilon; 12-\delta; \delta; \epsilon)}(x_1, x_9; x_2; x_3; x_6; x_7) \end{aligned} \quad (7.4)$$

$$\begin{aligned} X_{IV} : & x_8^2 + x_7^3 + x_7 x_9^4 \sum_{\delta} f_{(8-\delta; 8, 12-\delta, \delta)}(x_1; x_2, x_5; x_3, x_4; x_6) \\ & + x_9^6 \sum_{\delta} g_{(12-\delta; 12, 18-\delta, \delta)}(x_1; x_2, x_5; x_3, x_4; x_6) \end{aligned} \quad (7.5)$$

$$\begin{aligned} X_V : & x_5^2 + x_4^3 + x_4 x_6^4 \sum_{\delta, \epsilon} f_{(4+\epsilon; 2\delta-\epsilon; 8-\delta; \delta; \epsilon)}(x_1, x_9; x_2; x_3; x_7; x_8) \\ & + x_6^6 \sum_{\delta, \epsilon} g_{(6+\epsilon; 2\delta-\epsilon; 12-\delta; \delta; \epsilon)}(x_1, x_9; x_2; x_3; x_7; x_8) \end{aligned} \quad (7.6)$$

Analogously as in the previous case one can verify that the condition $\Delta_{\mathbf{T}^2} \sim x_i$ for the relevant x_i requires the restriction to a locus of high codimension in the complex moduli space.

7.2. Tensionless strings and their compactifications

In six dimensions there are several types of non-critical strings with different number of supersymmetries and world sheet properties [61][39][62][6]. Clearly F-theory compactification on three-fold fibered four-folds will involve compactification of these non-critical strings to four dimensions. In particular non-critical strings in type IIB or F-theory arise if a 2-cycle C_2 or 4-cycle C_4 shrinks to zero size from wrapping the 3-brane and 5-brane around these cycles, respectively¹³. The world sheet properties of the string depend on the normal bundle of C_i in the manifold. Additional structure may arise if the cycle wrapped by the brane is itself intersected by branes or if lower dimensional branes live on it.

We will call the two relevant types of normal bundles of vanishing 2-cycles and strings from wrapping three branes around them type i): $\mathcal{O}(-1) \times \mathcal{O}(-1)$ and type ii): $\mathcal{O}(-2)$.

In the present context - fibrations of elliptically fibered threefolds with base \mathbf{F}_n - these types of vanishing cycles arise e.g. when the exceptional section E of \mathbf{F}_n in the fiber \mathbf{Y} shrinks to zero size; we will restrict to the cases $n = 1, 2$ in the following.

In six dimensions the type i string has $N = 1$ supersymmetry. E is intersected by eight seven branes, leading to a current algebra of rank eight on the world sheet of the string [34]. This is the same tensionless string as it appears if a M-theory fivebrane comes close to a nine brane and becomes a small E_8 instanton [39]. In fact in F-theory compactification to six dimensions, where one has to take the limit of small elliptic fiber, the vanishing of E coincides with the vanishing of a whole 4-cycle S in the Calabi–Yau [6]. After compactification on a circle this theory becomes dual to M-theory on \mathbf{Y} and develops a new exotic phase [18]. In this five dimensional theory one can now shrink E and S independently; in the first case one obtains only a massless hypermultiplet [34], whereas for small volume of S one obtains a low tension magnetic string from wrapping the five brane around S and infinitely many massless electric states from wrapping membranes around 2-cycles in S which are all equally relevant in the limit of zero volume [6],[34],[18].

For $n = 2$, E is not intersected by any seven brane and consequently there is no variation of the type IIB coupling constant above it; shrinking E one obtains a tensionless string with $N = 2$ supersymmetry, which becomes in the small volume limit the same string as in type IIB on K3 with vanishing 2-cycle [34]. More precisely the vanishing tension requires also the adjustment of four real scalars corresponding to a hypermultiplet in the $N = 2$ spectrum.

In compactifications of F-theory to four dimensions, new quantum corrections may arise and lead to the appearance of new phases in the lower dimensional theories.

¹³ For a detailed discussion of aspects of shrinking 4-cycles see e.g. [34],[6].

7.3. Five branes and the superpotential

We will see now that the divisors determined in sect. 6 in the four-folds can be divided two classes: a) divisors \hat{D}_a with $\chi = 1$ arise from fibering the special 4-cycle S of del Pezzo type over a further \mathbf{P}^1 in different ways. From the five-dimensional point of view the superpotential arises therefore from compactifications of the magnetic non-critical string to three dimensions; b) divisors \hat{D}_b with $\chi = 2$ have the structure $K3 \times C$, where C is the base of the four-fold. C can be shrunk to zero size in the four-fold; at this point in the moduli space a \mathbf{Z}_2 symmetry is restored.

4-cycle fibrations

As an example for the first type let us identify the singularities in the Kähler moduli space of X_{III} associated to the contribution to the superpotential from D_6 . F-theory compactification on the fiber $\mathbf{Y}_{\mathbf{F}_1}$ has been discussed in detail in [6],[18]. This three-fold compactification has a point in the moduli space where the 4-cycle S in the Calabi–Yau shrinks to zero size. S is a del Pezzo surface and has a remarkable geometric structure; in particular there is a natural action of the Weyl group of E_8 on the representatives in $H^{1,1}$. After compactification on a circle one can count the BPS winding states which in the dual M-theory arise from membranes wrapping 2-cycles in S [40][18]. In addition to these “electric” winding states there is a “magnetic” string from wrapping the five brane around S . To see that this five brane is the origin of the superpotential term arising from D_6 , consider the Weierstrass model of X_{III} , eq. (7.4), expanded around $x_6 = 0$:

$$y^2 + x^3 + xz^4 \sum_{\epsilon=0}^4 f_{2\epsilon}(x_1, x_9)x_2^{4-\epsilon}x_7^\epsilon + z^6 \sum_{\epsilon=0}^6 f_{2\epsilon}(x_1, x_9)x_2^{6-\epsilon}x_7^\epsilon, \quad (7.7)$$

where $y = x_5$, $x = x_4$, $z = x_8$, f_α is a homogeneous polynomial of degree α and we have set x_3 to one. Choosing a point on the base \mathbf{P}^1 , say $x_1 = 0$, $x_9 = 1$, this is nothing but the E_8 type del Pezzo surface. The complex surface D_6 in the four-fold is a fibration of the vanishing 4-cycle over the base \mathbf{P}^1 . The superpotential in four dimensions arises from the compactification of the magnetic string in six dimensions on the base \mathbf{P}^1 of the fibration.

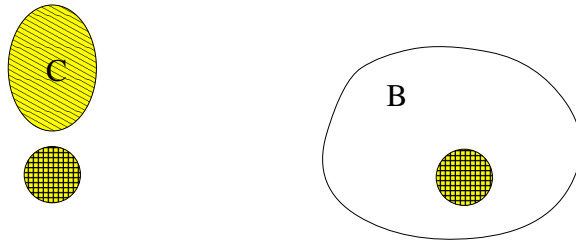


Fig.1: Generation of the superpotential by worldsheet instantons of a solitonic string. On the F-theory side (left) wrapping the five brane on the 4-cycle C one obtains a solitonic string. Wrapping the world-sheet further on the base \mathbf{P}^1 of the 6-cycle produces an instanton contribution to the superpotential. The dual heterotic theory observes a world-sheet instanton from wrapping the *fundamental* string on the same \mathbf{P}^1 in the base B of the elliptic fibration $\pi : \mathbf{Z} \rightarrow B$, where B common to F-theory and heterotic compactifications.

Similar arguments apply to the divisors \hat{D} with $\chi = 1$ found in the other examples, as can be verified from the Weierstrass form given in eqs. (7.1)-(7.6); the divisors \hat{D} are fibrations of the 4-cycle S over a further \mathbf{P}^1 . However there are also interesting differences: first as mentioned previously the divisors can map to horizontal and vertical divisors on the heterotic base with the consequence that they can appear as world sheet or space time instanton effects in the heterotic case. The situation that geometrically similar configurations appear as either world sheet or space time instanton effects in the dual heterotic theory is particularly marked in the four-fold X_{IV} with two heterotic duals [8], where the same five brane wrapping in the M-theory picture is interpreted differently in the two dual theories. Additional differences arise due to various configurations of 2-cycles contained in or intersecting \hat{D} , which support further brane wrappings and can get zero size.

A peculiar situation is found in the four-fold X_V where the divisor $\hat{D} = D_7$ is of the form $S \times \mathbf{P}^1$, as can be seen from eq. (7.6). Let ϵ^2 be the volume of S and ϵ' the volume of the \mathbf{P}^1 . In the M-theory compactification we obtain the following states: a) the five-brane instanton with an action $\sim \epsilon^2 \epsilon'$; b) the magnetic string with tension $\sim \epsilon^2$ from wrapping S ; c) a new string with tension $\sim \epsilon \epsilon'$ from wrapping $\mathbf{P}^1 \times C$, where C is a holomorphic curve in S ; three-dimensional states from wrapping d) the \mathbf{P}^1 leading to a mass $\sim \epsilon'$ or e) a curve C in S providing a state with mass $\sim \epsilon$. In this case we can actually count the number of representatives of *four*-cycles contributing to c) by counting holomorphic curves in S ; this can be done using mirror symmetry as in [18]. The result is that the number of curves C of degree n is given by the coefficient of q^n in the “string partition function”, $\Theta_{E_8} \eta^{-12}$ where Θ_{E_8} is the E_8 lattice sum. Each contributes a different string state. This

is similar as in the case of IIB on K3 with a A_N singularity where each vanishing 2-cycle contributes a tensionless string in the six-dimensional theory.

The other divisor with $\chi = 1$, D_8 , has also an interesting structure. Again from (7.6) one finds that it can be either interpreted as a fibration of S over \mathbf{P}^1 or as a fibration of K3 over another \mathbf{P}^1 . The M-theory five-brane wrapped on K3 gives a string with the world-sheet structure of the heterotic string [63]. Viewing the fibration from the one or the other perspective the five-brane instanton arises in the compactification of the "S"-string or the solitonic heterotic string. In fact the K3 fibration has the same base \mathbf{P}^1 as the dual heterotic theory, where it is interpreted as a world-sheet instanton, too (since D_8 maps to a vertical divisor in Z); this provides a quite explicit picture of how the instanton effects are generated in the dual theory. It would be interesting to have a similar understanding for the "S"-type string seen in the second fibration.

Divisors with $\chi = 2$

The divisors with $\chi = 2$ have a different structure. From (7.1),(7.2) we see that D_8 in X_I is given by an equation

$$y^2 + x^3 + xz^4 f_8(x_3, x_7) + z^6 g_{12}(x_3, x_7) ,$$

where f_8 and g_{12} are again homogeneous polynomials and x_2 has been set to one. This equation defines a K3 hypersurface of degree twelve in $\mathbf{WP}(1, 1, 4, 6)$. Together with the base \mathbf{P}^1 D_8 is of then of the form $K3 \times C$ with C a rational curve. The individual contributions to the arithmetic genus are $\chi(\mathcal{O}_D) = 1 - 0 + 1 - 0 = 2$, where the contribution to $h^{2,0}$ comes from the holomorphic two form of K3 ¹⁴. C is a curve with normal bundle of type ii which can get zero size; the singularity associated to this point in the moduli space where a tensionless string appears and a \mathbf{Z}_2 symmetry is restored will be discussed momentarily.

¹⁴ The same argument applies also to the models and exceptional divisors considered in [11]; however in these cases the K3 is singular. We did not check with eq. (6.1) since the toric construction is quite involved due to the large number of moduli in these models.

8. Non-critical four-dimensional strings and quantum tension

In the previous section we have argued that different than in the cases considered in [11] the generation of a superpotential is linked to singularities in the moduli space due to tensionless strings rather than strong coupling dynamics of non-abelian gauge symmetries. Apart from the singularity which is associated to the vanishing 4-cycle producing the E_8 type of tensionless string, additional singularities with light strings arise if one can shrink 2-cycles; in fact the divisors \hat{D} have in common that there are points at finite distance in the moduli space where one can shrink 2-cycles C_2 contained in them, at least to a very small size. These can be either curves of type i which can be flopped or curves of type ii which come together with a restoration of a \mathbf{Z}_2 symmetry. At the point in the moduli space where the volume V of C_2 vanishes, the effective action becomes singular due to the appearance of new light modes. At least in some cases one can decouple (fundamental) string and gravity states and obtains genuine new theories which involve strings of an arbitrary light scale and can not be described in terms of conventional field theory.

Since the tension of the string is related to the volume of the 2-cycle we have to be very careful about the definition of this volume V . It is well-known that the so called sigma model measure of V [64], which is the definition of the volume related to the scale of the BPS states in three-folds, is corrected by world-sheet instantons. In general there can be of course also space-time instanton contributions.

In the cases where the instanton corrections can be calculated using mirror symmetry, the quantum size V can be inferred from arguments related to the topology of the discriminant in the moduli space and the leading classical behavior. For other instanton corrections we will have to use some more indirect arguments.

In any case there are two possible scenarios which will arise: starting from a “classical” singularity in the moduli space which is due to a tensionless string, the quantum corrections might leave the string tensionless or give it a non-vanishing tension proportional to a new dynamically generated scale Λ due to non-perturbative corrections to the quantum volume V of C_2 . However the scale of Λ may be still very small compared to all other scales in the theory and a limit can exist where one can decouple the fundamental string and gravity effects similar as in the “classical” theory. In this case one obtains a theory which at lowest energies looks like a conventional field theory but at very small scale $\Lambda \ll M_{pl}$, generated by non-perturbative effects, has a string with tension $\sim \Lambda^2$ bounded from below by the instanton effects which give it a non-zero quantum tension. In fact we will see that, for certain geometries, Λ characterizes at the same time the scale of supersymmetry breaking. This new combination of a field theory and a massive but light string like

spectrum seems to be quite interesting and it is encouraging to find it in the present context.

Space-time instanton corrected tension

World-sheet instantons can be partially analyzed using mirror symmetry, as will be done in some detail later on. A case with space time instanton corrected tension can be found using the recent results on the exact metric for hypermultiplet moduli spaces in four and three dimensional string and field theories [65][66][16][67].

Consider type IIA compactified on a three-fold at a point in the Kähler moduli space with 2-cycle C_2 of vanishing volume $V = 0$, even after taking into account world-sheet instantons as determined by mirror symmetry. There is a massless hypermultiplet from wrapping the D2 brane on C_2 and a singularity in the moduli space [56]. After compactification to three dimension on a circle of radius R the singularity is smoothed out by D2 brane instantons on $C_2 \times S^1$; this follows from the exact known moduli space of $U(1)$ with $N_f = 1$ in three dimensions [65][66]. T-duality on the circle transforms type IIA to type IIB, the D2 brane on C_2 in a D3 brane wrapped on $C_2 \times S^1$ and the D2 brane wrapped on $C_2 \times S^1$ into a D1 brane wrapped on C_2 . The light state in three dimensions in IIB is the 3 brane wrapped on $C_2 \times S^1$. Although we can use the world sheet instanton corrected prepotential of the type IIA theory on \mathbf{X} to describe also the hypermultiplet moduli space of IIB on \mathbf{X} [68], in the latter case this is not the exact answer; in particular the sigma model measure of V is zero in type IIA, but not an exact quantity for the type IIB theory. This is just good since the above picture implies that making S^1 large in the IIB theory does not suppress instanton effects; therefore if the tension of the string is not zero in three dimensions then it should be non-zero already in four dimensions. In measuring the tension we have neglected instanton effects which depend on the space time coupling [69]. Mirror symmetry - now on \mathbf{X} rather than the S^1 - transforms the situation to type IIA with a vanishing 3-cycle which has been solved in [16] with the result that there is indeed no singularity in four dimensions, the relevant effect being D2 brane instantons wrapped on the vanishing 3-cycle.

For $R \rightarrow \infty$ when the instanton effects are exponentially suppressed, the hypermultiplet of type IIA becomes very light. The state in the type IIB theory is the four-dimensional low tension string wrapped around the cycle with radius $1/R \rightarrow 0$. On the other hand for $R \rightarrow 0$, the type IIB theory becomes four-dimensional. For some value of R the state obtained from winding the string around the large circle becomes less relevant than the excitations of the low tension string itself; thus the three dimensional effective $N_f = 0$ theory related to the type IIA side develops a relevant light string like object.

8.1. World sheet instanton corrections

We want now to use mirror symmetry to determine the quantum corrections to V which are calculated by the two-dimensional topological field theory. In particular we ask about the fate of the 2-cycles of type i and ii and the del Pezzo type 4-cycle which appear in the present context. For the 2-cycles the strategy is to first ensure that the volume of a 2-cycle C_2 is zero taking into account instantons which wrap C_2 itself and then to see how instanton effects associated to the global embedding change the picture. Below \mathbf{X} stands collectively for a Calabi–Yau manifold without specifying the dimension.

Let t denote the special Kähler modulus associated to C_2 of \mathbf{X} ; t is also the sigma model measure of the volume V . There exists an ordinary differential equation describing the relation between t and the complex structure modulus z associated to it in the mirror \mathbf{X}^* of \mathbf{X} [64]. In the complex structure moduli space of \mathbf{X}^* there are singularities if the hypersurface becomes singular itself; this happens if a solution to the equations $f_{\Delta^*} = df_{\Delta^*}$ (see eq. (3.1)) exists. This locus is the discriminant locus $\Delta_{\mathbf{X}^*}$ and is given by a polynomial in the complex structure moduli z_i in a given Calabi–Yau phase. From the above map $z = z(t)$ and its inverse one can determine the value of t for a value of $z = z_0$ on the discriminant, $\Delta(z_0) = 0$.

To put the above in a concrete physical context, let t be e.g. a modulus which plays the role of the scalar field in a vector multiplet parameterizing the Coulomb branch of $N = 2$ four-dimensional $SU(2)$ theory as in [3] (i.e. we consider type IIA on a threefold \mathbf{X}). The period related to the 2-cycle C_2 , namely t , and the dual 4-cycle become the periods of the field theory in a certain limit. There is always a locus in the compactified moduli space where one can shrink C_2 to zero size - and therefore $a = 0$. This is due to the fact that one of the Calabi–Yau moduli plays the role of the quantum scale Λ of the theory; one can therefore switch off any quantum effects by taking the limit $\Lambda \rightarrow 0$ (corresponding usually to restriction to a boundary divisor in the Calabi–Yau moduli space). The precise question about the quantum volume of C_2 is then whether it stays zero for generic values of all the other z_i ; it can be answered by analyzing the topology of a discriminant factor associated to the singularity.

It is clear that the answer depends on how the cycle is embedded in \mathbf{X} , said differently, how the modulus t couples to the other moduli. We are lead therefore to consider a system with two moduli and to ask whether the new modulus y introduces quantum corrections to the volume of C_2 and give it a everywhere non-vanishing size (away from $\Lambda(y) = 0$). In the present context the question can be addressed by considering two moduli systems

associated to the Hirzebruch surfaces \mathbf{F}_1 and \mathbf{F}_2 which provide the appropriate types of 2-cycles and embeddings when used as a part of a Calabi–Yau three-fold or four-fold.

The one moduli systems associated to type i and type ii cycles are given in [64] and have solutions which can be written in terms of the elementary functions:

$$i) : t = \frac{1}{2\pi i} \ln z \quad ii) : t = \frac{1}{2\pi i} \ln \left(\frac{1 - 2z - 2\sqrt{1 - 4z}}{2z} \right) \quad (8.1)$$

The relevant singularities are at $z = 1, \frac{1}{4}$, respectively, in both cases $t = 0$. In the second case there is a \mathbf{Z}_2 monodromy $t \rightarrow -t$ under a loop of z around $z = 1$. Physics wise, this monodromy can play the role of a Weyl reflection of an enhanced gauge symmetry or the \mathbf{Z}_2 symmetry at the critical point of the type ii tensionless string.

To treat the quantum corrections we consider the system of differential equations associated to \mathbf{F}_1 and \mathbf{F}_2 , which have a toric description defined by the vertices of the dual polyhedron, $\nu_i^* = (0, 0), (-1, 0), (0, -1), (0, 1), (1, n)$ with Mori generators:

$$\begin{aligned} \mathbf{F}_1 : l^{(1)} &= (-2, 0, 1, 1, 0), \quad l^{(2)} = (-1, 1, 0, -1, 1) \\ \mathbf{F}_2 : l^{(1)} &= (-2, 0, 1, 1, 0), \quad l^{(2)} = (0, 1, 0, -2, 1) \end{aligned} \quad (8.2)$$

The homology of \mathbf{F}_n consists of the two rational curves associated to these Mori generators; the curves associated to $l^{(2)}$ are the base of the \mathbf{P}^1 fibration. For \mathbf{F}_1 , $C_2^{(2)}$ is of type i, the other 2-cycles are of type ii. However they are differently coupled to the interior point $\nu_0^* = (0, 0)$ which corresponds to the hyperplane section. The differential equations obtained from these generators coincide with those obtained from the Picard-Fuchs system of the Calabi–Yau \mathbf{X} when restricted to the two moduli describing a \mathbf{F}_n surface which is part of \mathbf{X} , $z_3 = z_4 = 0$ in (D.11) and $z_2 = z_3 = 0$ in (4.5). The locus in the moduli space where a cycle vanishes can be detected as previously by the vanishing of the discriminants, which are given by:

$$\begin{aligned} \Delta_{\mathbf{F}_1} : (1 - z_2) + (36z_2 - 8 - 27z_2^2)z_1 + 16z_1^2 = \\ (1 - 4z_1)^2 + (-1 + 36z_1)z_2 + (-27z_1)z_2^2 \\ \Delta_{\mathbf{F}_2} : (1 - 4z_2) \times [(1 - 4z_1)^2 - (64z_1^2)z_2] \end{aligned} \quad (8.3)$$

From the above we see that the discriminant factors $z_2 = \text{const.}$ associated to the singularity of the base \mathbf{P}^1 do not change topology under deformation with z_1 whereas the factors associated to $z_1 = \text{const.}$ are quadratic factors splitting for $z_2 \neq 0$. This split is associated to a non-zero quantum volume of the fiber \mathbf{P}^1 's $C_2^{(1)}$, as can be understood

from the following argument. Consider a hyperplane H in the properly resolved and compactified complex structure moduli space \mathcal{M}_{CS} . The discriminant Δ has in general several factors describing codimension one loci which intersect H in points. If H is generic, the monodromies in \mathcal{M}_{CS} are generated by loops around the intersection points of Δ with H . Clearly the monodromies can not change under variations of the position of H unless H becomes non-generic; in particular this means that the vanishing cycle causing a singularity stays the same under smooth variations of the position of H . On the other hand if a point splits into two points, as it happens if $H : z_2 = 0$ is moved to $H' : z_2 \neq 0$ in (8.3), the combined monodromy around the now two singular points will be still the same as the one around the single point at $z_2 = 0$, however the individual monodromies are different and related to new states with different quantum numbers and associated to new vanishing cycles.

In the above case we see the cycles corresponding to the base \mathbf{P}^1 of $\mathbf{F}_n, C_2^{(2)}$, can still get zero size whereas the fibers always have a non-zero quantum volume. The singularities associated to vanishing cycles which cause the singularities $\Delta = 0$ in the first case are therefore related to the vanishing of 2-cycles, whereas in the second case the singularities are due to the *dual* cycles; the former singularity due to $V_{C_2} = 0$ has been wiped out by the quantum effects.

As a concrete example consider again type IIA on a three-fold where t corresponds to the scalar a in the vector multiplet of $N = 2$ Yang-Mills with $SU(2)$ gauge group and no matter. This theory arises as a certain limit in the moduli space of the elliptically fibered three-fold with base \mathbf{F}_2 [4], which is also a K3 fibration with a base \mathbf{P}^1 identical to the base of \mathbf{F}_2 . If we take the large base space limit which by heterotic/type IIA duality in four dimensions corresponds to switching off quantum effects, we have to take therefore $z_2 = 0$ in (8.3) and recover the gauge symmetry enhancement of the six-dimensional theory on K3 from wrapping the 2 brane around the fiber of \mathbf{F}_2 . For finite base size however the volume of the fiber \mathbf{P}^1 never becomes zero again. The two singularities of the four-dimensional theory - which reduce to the monopole states of the field theory in the appropriate limit - are given by wrapping different vanishing cycles. In fact one can show that the vanishing periods are related to 4-cycles instead of 2-cycles¹⁵.

¹⁵ This might seem natural, since the dual cycles of 2-cycles in the Calabi–Yau are 4-cycles; however from a simple dimensional counting of the moduli spaces it is clear that one needs all kind of even dimensional cycles and combinations of them to reproduce the singularity structure of the “mirror theory”. Some consequences of the above reasonings for the dual heterotic theory are collected in appendix E.

Although vanishing 2-cycles are replaced by vanishing 4-cycles they still have an arbitrarily small size compared to the other scales of the theory and in particular pass the decoupling limit in which one obtains the field theory - supporting the moderately massive gauge bosons of the field theory. If we put type IIB on the same manifold, states are replaced by strings and the singularities might get wiped out by the space-time perturbative and instanton effects as in [16]. In fact in the four-dimensional theories with $N = 2$ supersymmetries it follows from the results in [65][66] that the singularity is smoothed for the curves of type i corresponding to $U(1)$ with $N_f = 1$. For the curves of type ii the situation depends of whether they appear as the base or as a fiber. In the first case one gets enhanced $SU(2)$ with adjoint matter multiplets in a type IIA theory compactification as in [32][33]; there is also a singularity in the three-dimensional compactification of the theory implying in turn a true zero tension string in the four-dimensional type IIB theory compactified on the same manifold.

Hidden strings at the supersymmetry breaking scale

On the other hand if the type ii curve appears as the fiber, world sheet instantons split the singularity and one ends up again with singularities with a single hypermultiplet. In these cases one is left only with a low energy range described by a field theory modes together with very light strings, with a tension entirely due to non-perturbative effects. More precisely, the scale is of the order of e^{-V_B} , where V_B is the volume of the base \mathbf{P}^1 . An attractive picture emerges, if this geometry appears in the context of 3-fold fibered 4-folds, with B at the same time the base of the 3-fold fibration. As explained in sect. 2.1., any supersymmetry breaking effect is necessarily of the order e^{-V_B} - as is the minimal string tension.

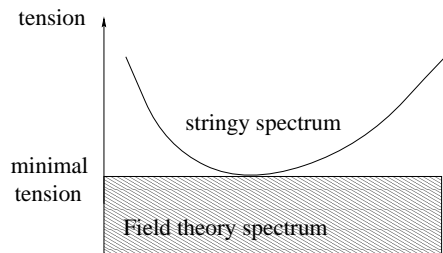


Fig.2: *In the case with non-zero quantum tension, the spectrum at lowest energies is that of a conventional field theory. Above a non-perturbatively generated scale $\Lambda \sim M_{SUSY}$ the spectrum is enriched by states of the solitonic string*

Quantum tension for the E_8 string

Let us now consider the generation of a quantum volume in the case of the vanishing del Pezzo 4-cycle in four dimensions. There are two interesting situations to consider: a) a four-dimensional $N = 1$ four-fold F-theory compactification with a zero size fiber; b) a four-dimensional $N = 2$ three-fold type IIA compactification with finite size fiber. As mentioned at the beginning, the latter case is dual to the the situation considered in [17].

In the case a) one can use the fact that if the elliptic fiber of the del Pezzo is identical to that of the Calabi–Yau, the base of the del Pezzo fibration agrees with the base of a F_1 Hirzebruch surface embedded in the Calabi–Yau. After taking the zero size fiber limit the problem therefore reduces to that of a type i curve considered before; we have seen that there are no quantum correction in this case. Therefore we expect a tensionless string as far as world sheet instantons corrections are concerned.

In the case b) we have to determine the quantum corrections to both, the volume of the 4-cycle S and the 2-cycles embedded in it. A simple argument shows that neither of the two volumes can be corrected at the singularity. First note that the volume of the 2-cycle, supporting the Kähler class which is varied to reach the singularity, remains zero. This is already almost clear from the fact that the singular locus in the moduli space is at a boundary where one can either flop to a non-geometric phase [6] or make a transition to the elliptically fibered three-fold with base \mathbf{P}^2 [34][6]. Alternatively one can use a similar argument as in the previous cases based on the fact that the discriminant factor does not change the topology under the new deformation. It is then straightforward to see that the monodromy under a loop around the singular locus requires either both volumes and periods to be zero or both to be non-zero. This is because the monodromy mixes non-trivially the vanishing period associated to the 2-cycle with the period corresponding to the dual 4-cycle as can be e.g. inferred from the index structure of the periods. On the other hand the 2-cycle period has still to vanish after the monodromy transformation; together this implies that the dual period vanishes, too. Therefore there are no world sheet quantum corrections from world sheet instantons to the volumes of both the 4-cycle and the 2-cycles in it in the four-dimensional theory at the singularity.

8.2. Tensionless string singularities and the superpotential

The local geometry of vanishing 2-cycles described above can lead to non-critical strings with low tension in the four-dimensional F-theory compactification¹⁶. Apart from the new physical phenomena associated to these theories in itself, it is interesting to study their interplay with the generation of a superpotential. In particular singularities associated to tensionless strings can provide the poles in the superpotential required by holomorphicity [58].

In fact, the divisors considered before contain always 2-cycles or 4-cycles which can be shrunk to zero size. The divisors \hat{D} with $\chi = 1$ contain a curve C of type i with normal bundle $\mathcal{O}^{(1)} \times \mathcal{O}^{(1)}$ which can be flopped in the Calabi–Yau. This is the 2-cycle which one has to flop out of the del Pezzo \mathbf{B}_9 , which is \mathbf{P}^2 blown up at nine points, to be able to shrink the remaining 4-cycle in the Calabi–Yau¹⁷. As mentioned already, in the small fiber limit this flop coincides with the collapse of the whole 4-cycle, whereas for finite size fiber (e.g in three dimensions) C can be shrunk separately. In particular C is always interior to the base of the K3 fibration and can also be seen on the heterotic side.

The fact that C can be flopped indicates already that it can be shrunk to zero size also after including world sheet instantons, since it is replaced on the other side by a different homology class and should no longer exist. In fact it can be seen from the previously given C^* actions, that the situation is locally described by the analysis of the \mathbf{F}_1 case and the arguments of the previous section apply.

A special case where C can be shrunk without shrinking the 4-cycle (at zero size fiber), thus leading to a simple type i string appears in the 4-fold X_{II} which admits two heterotic duals on the threefold with base \mathbf{F}_1 . In this case $\hat{D} = D_6$ contains two curves C_i of type i which can be flopped. Performing a flop on one of them, say C_1 one passes to a Calabi–Yau phase in which one can shrink the homology 2-cycle which is related to C_2 in the first phase, C'_2 , without shrinking the 4-cycle. The heterotic dual is compactified on the elliptically fibered 3-fold with base $F_0 = \mathbf{P}_1^1 \times \mathbf{P}_2^1$. The flop maps to a point in the moduli space of F_0 where the volumes of the two \mathbf{P}^1 are equal. According to the previous arguments, in a four-dimensional $N = 2$ theory we would expect space time instantons to give this string a non-zero quantum tension. It seems very plausible that the same happens in the $N = 1$ theory.

¹⁶ A different representation of the $N = 2$ supersymmetric string as a string living on the 3+1 dimensional intersection of two M-theory five branes has been analyzed in [70].

¹⁷ See [6],[18][41] for more details.

Two-cycles of type ii can also be contained in the divisors \hat{D} , in both variants discussed in sect. 8.1. In fact there is always a curve C_2 of the uncorrected type contained in the exceptional divisors with $\chi(\mathcal{O}_D) = 2$; it is at the same time the base of the Calabi–Yau three-fold fibration. From the results in sect. 8.1. we know that world sheet instantons do not lead to a non-zero quantum volume (this can be also seen from the singularity structure of the full Calabi–Yau moduli space); moreover there is a restoration of a \mathbf{Z}_2 symmetry as in the six-dimensional theory. From the local geometry the string we obtain is identical to the one obtained in the six-dimensional F-theory compactification on the threefold with base \mathbf{F}_2 , dual to the heterotic string on K3 with instanton embedding (10,14) [34][6]. In fact there is one more analogy: recall that the six dimensional string required the adjustment of a further hypermultiplet due to the fact that it is a $N = 2$ string in a $N = 1$ theory. This hypermultiplet is represented by a non-polynomial deformation which is frozen in the toric description. We find that the same happens in the 4-fold case: there is again the same kind of non-polynomial deformation frozen to zero at the \mathbf{Z}_2 symmetric point.

In summary, the divisors \hat{D} with $\chi(\mathcal{O}_D) = 2$ are of the form $K3 \times C$, where C can get zero size. The tensionless string obtained from wrapping the three brane comes together with a \mathbf{Z}_2 symmetry restored at that point and which in turn can be spontaneously broken by switching on the deformation frozen in the toric description. In fact the local geometry is that of a resolved A_1 surface singularity which resembles a \mathbf{Z}_2 orbifold singularity. It is tempting to suggest that in this situation instanton configurations with the appropriate number of fermionic zero modes exist.

It is an interesting question of what is the precise theory in the scaling limit where one shrinks cycles within a divisor supporting the superpotential. Recall that both types of divisors we have found involve a 4-cycle either of the del Pezzo or K3 type. The 2-cycle cohomology of these 4-cycles - which in both cases involves a factor of the intersection form proportional to the E_8 Cartan matrix - is expected to generate world brane degrees of freedom which become light in the limit of vanishing 2-cycles.

As an illustration of how infinitely many new contributions to the superpotential become relevant when the volume of a 2-cycle shrinks to zero size, consider the four-fold compactification analyzed in ref. [55], where a precise counting of the individual representatives in the divisor classes contributing to the superpotential has been done. The four-fold X in this case is an elliptic fibration over the base $B: S \times \mathbf{P}^1$, where, as before, S denotes the elliptically fibered del Pezzo with $\chi = 12$. Apart from the elliptic fibration $\pi : X \rightarrow B$ there is a second elliptic fibration of X , $\tilde{\pi} : X \rightarrow \tilde{B}$ where the fiber

is the elliptic fiber of the del Pezzo and the base is a K3 fibration. In [55] the divisors \hat{D} , vertical with respect to the first fibration π , have been determined with the result that they are sections¹⁸ of the second fibration $\tilde{\pi}$.

Now consider F-theory compactified on X but with the elliptic fibration defined by $\tilde{\pi}$ instead of π . While the five brane wrappings on the divisors \hat{D} have of course still the correct numbers of zero modes to contribute to the superpotential in the M-theory compactification on X , they fail to be vertical since the intersection of the divisor with the del Pezzo factor is a curve C with volume proportional to $nt_E + t_B$, $n \geq 1$, where t_E is the size of the elliptic fiber and t_B the size of the base B of the elliptically fibered del Pezzo. Therefore in the limit of small fiber, instantons associated to these divisors fail to produce the necessary factor $\sim t_E$ to render their action finite and none of the divisors \hat{D} contributes to the superpotential of F-theory on X at a generic point of the moduli space. However it is clear now that if, at the same time as one takes the four-dimensional limit $R_{S^1} \sim 1/\epsilon \rightarrow \infty$, one adjusts the size of B to be of the order of ϵ , all divisors \hat{D} contribute to the superpotential despite of the fact that they are not vertical. Said different, infinitely many new contributions to the superpotential become relevant in the limit, where the size of this 2-cycle becomes comparable to the inverse of the radius of the fourth dimension.

9. Discussion

Mirror symmetry together with the powerful non-renormalization theorems separating the hyper and vector multiplet moduli spaces have played an important role in the determination of exact non-perturbative quantities of four-dimensional $N = 2$ supersymmetric string theories. It has to be seen to which extent these concepts can be generalized and replaced in the case of less supersymmetry. The calculation of the correlation functions of the topological sigma model in the first part is apart from its mathematical interest only a first step in the determination of the $N = 1$ effective action in four dimensions. It remains to determine the precise map to the physical quantities in the heterotic theory and also to get control over effects which can not be encoded in the geometrical four-fold compactification of the F-theory. This involves a better knowledge of the moduli space of $(2, 0)$ compactifications and properties of it, such as factorization properties as in the $N = 2$ case. Moreover even to the moduli dependence as calculated from the F-theory compactifications one should expect corrections which can not be included in the correlation functions of the two-dimensional topological field theory.

¹⁸ Note that also in this case the divisors \hat{D} contain a K3 factor.

At first sight it seems difficult to generalize the concepts which have been useful in the $N = 2$ case. However things might be not as bad. Firstly the geometrical concepts which have been shown to be intimately linked to physical quantities seem to be too general and too strong not to play an important role. We have seen that taking a certain geometrical limit in the four-fold reproduces physical sensible limits where the correspondence is known. Moreover recall that in $N = 1$ orbifold compactifications - even in $(0, 2)$ cases - the perturbative moduli dependence of the holomorphic quantities arises from so-called $N = 2$ subsectors and is determined by the one-loop result. And although orbifolds are special constructions, they have turned out to have remarkably generic properties in the past. Our results on the large base limit imply that the full world-sheet instanton corrections to the gauge couplings are encoded properly in the four-fold dual to such $N = 1$ orbifolds. It will be very interesting to see to which extent these properties generalize to heterotic compactifications on smooth Calabi–Yau manifolds.

An omnipresent feature of the four-fold compactifications dual to $E_8 \times E_8$ heterotic strings is the presence of points in the moduli space where non-critical strings obtain classically zero tension. We have seen that divisors generating the superpotential involve precisely the vanishing cycles which support these strings. An interesting issue is the non-perturbative generation of a minimal tension for extended BPS states which are classically tensionless at certain points in the moduli space. As discussed, such an effect can result effectively in field theories with a (non-critical) string spectrum starting at a new low energy scale. The generation of a small, exponentially suppressed scale related to the quantum size of a geometric cycle might also be relevant for providing a lower cut-off on the size of those six-cycles which generate superpotential terms from wrapped five-branes and the scale of supersymmetry breaking.

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Appendix A. Holomorphic structure

Let $\gamma_\alpha^{(i)}$ be a basis of the middle cohomology with intersection form $\eta_{\alpha\beta}^{(i)}$ as in (3.12). The holomorphic (4,0) can be decomposed as

$$\Omega = \gamma^{(0)} Z^0 + \gamma_\alpha^{(1)} Z^\alpha + \gamma_\alpha^{(2)} H^\alpha + \gamma_\alpha^{(3)} \mathcal{G}^\alpha + \gamma^{(4)} \mathcal{G}^0$$

Z^M , $M = 1, \dots, h^{1,1}$ are coordinates on the moduli space while H^α and \mathcal{G}^α are sections depending on the Z^M . From the usual orthogonality relations

$$0 = \langle \Omega, \Omega \rangle = \langle \Omega, \partial_M \Omega \rangle = \langle \Omega, \partial_M \partial_N \Omega \rangle = \langle \Omega, \partial_M \partial_N \partial_Q \Omega \rangle$$

one obtains

$$\begin{aligned} 2Z \cdot \mathcal{G} + H \cdot H &= 0, \\ (\eta^{(1)} \mathcal{G})_M + Z \cdot \partial_M \mathcal{G} + H \cdot \partial_M H &= 0, \\ H \cdot \partial_M \partial_N H + Z \cdot \partial_M \partial_N \mathcal{G} &= 0, \\ H \partial_M \partial_N \partial_P H + Z \cdot \partial_M \partial_N \partial_P \mathcal{G} &= 0, \\ H \partial_M \partial_N \partial_P \partial_Q H + Z \cdot \partial_M \partial_N \partial_P \partial_Q \mathcal{G} &= K_{MNPQ}, \end{aligned}$$

where a dot denotes contraction with the appropriate metric $\eta_{\alpha\beta}^{(i)}$ and $K_{MNPQ} = \langle \Omega, \partial_M \partial_N \partial_P \partial_Q \Omega \rangle$. The first equation reflects the algebraic dependence of the components of the period vector which is always the case for even (complex) dimensions. The remaining equations can be considered as differential equations determining the sections \mathcal{G}_M in terms of H . Taking derivatives of these relations we find

$$\begin{aligned} \partial_R (\eta^{(1)} \mathcal{G})_M + \partial_M (\eta^{(1)} \mathcal{G})_R + \partial_M H \cdot \partial_R H &= 0, \\ \partial_M \partial_N (\eta^{(1)} \mathcal{G})_R + \partial_R H \cdot \partial_M \partial_N H &= 0, \\ K_{RMNP} + \partial_M \partial_N \partial_P (\eta^{(1)} \mathcal{G})_R + \partial_R H \cdot \partial_M \partial_N \partial_P H &= 0, \\ \partial_S \partial_M \partial_N (\eta^{(1)} \mathcal{G})_R + \partial_S \partial_R H \cdot \partial_M \partial_N H + \partial_R H \cdot \partial_S \partial_M \partial_N H &= 0, \end{aligned}$$

and therefore

$$K_{RMNP} = \partial_R \partial_M H \cdot \partial_N \partial_P H. \quad (\text{A.1})$$

The last equation reflects the factorization of four point functions in terms of the fundamental three point couplings, see eqs. (3.8),(3.18),(3.19).

Appendix B. Non-holomorphic equations

Conventions are as in [19]. \mathcal{D} is the covariant derivative acting on $\mathcal{H} \oplus \mathcal{L}$:

$$\mathcal{D}_m = \nabla_m + q \mathcal{K}_m$$

where ∇ is the metric compatible covariant derivative on \mathcal{H} and q is the charge w.r.t. the $U(1)$ line bundle with the Kähler class as its first Chern class. Define $G_{m_1, m_2, \dots} = \mathcal{D}_{m_1} \mathcal{D}_{m_2} \dots \Omega$ and as above $\langle A, B \rangle = \int A \wedge B$. We have

$$\begin{aligned} \langle \Omega, \bar{\Omega} \rangle &= e^{-\mathcal{K}} , \\ \langle G_m, \bar{\Omega} \rangle &= 0 , \\ \langle G_{mn}, \bar{\Omega} \rangle &= \langle G_{mn}, G_{\bar{m}} \rangle = 0 , \\ \langle G_{mnp}, \bar{\Omega} \rangle &= \langle G_{mnp}, G_{\bar{m}} \rangle = 0, \quad \langle G_{\bar{m}\bar{n}}, G_{mnp} \rangle = e^{-\mathcal{K}} \mathcal{D}_m R_{n\bar{m}p\bar{n}} \end{aligned}$$

and

$$\begin{aligned} \langle G_m, G_{\bar{m}} \rangle &= -e^{-\mathcal{K}} g_{m\bar{m}} \equiv -e^{-\mathcal{K}} \partial_m \partial_{\bar{m}} \mathcal{K} , \\ \langle G_{mn}, G_{\bar{m}\bar{n}} \rangle &= e^{-\mathcal{K}} (R_{m\bar{m}n\bar{n}} - g_{m\bar{m}} g_{n\bar{n}} - g_{m\bar{n}} g_{n\bar{m}}) . \end{aligned}$$

Appendix C. Solutions to the Picard–Fuchs equations

The solutions to the Picard-Fuchs system for Calabi–Yau 3-folds periods with several moduli have been considered in detail in [27], to which we refer for details. Picard-Fuchs equations related to the case of $d > 3$ have been considered in [71].

In an expansion around the large complex structure point $z_i = 0$, $\forall i$, there is a unique power series solution w_0 given by

$$\begin{aligned} w_0 &= W(x; \mathbf{r})|_{\mathbf{r}=0} , \quad w(x; \mathbf{r}) = \sum c(\mathbf{n}, \mathbf{r}) \prod_{i=1}^{h^{1,3}} z_i^{n_i} , \\ c(\mathbf{n}, \mathbf{r}) &= \frac{\Gamma(1 - \sum_{i=1}^{h^{1,3}} l_0^{(i)}(n_i + r_i))}{\prod_{j=1}^{(d+1)+h^{1,3}} \Gamma(1 + \sum_{i=1}^{h^{1,3}} l_j^{(i)}(n_i + r_i))} , \end{aligned}$$

where a subscript j , $j = 0 \dots h^{1,3} + d + 1$ at the Mori generator $l_j^{(i)}$ denotes the j -th entry. Single, double, triple and quartic logarithmic solutions are obtained from the system of derivatives of $w(x; \mathbf{r})$ at $\mathbf{r} = 0$:

$$\partial_{r_i} w(x; \mathbf{r})|_{\mathbf{r}=0}, \quad \partial_{r_i} \partial_{r_j} w(x; \mathbf{r})|_{\mathbf{r}=0}, \quad \dots$$

such that the leading logarithmic pieces are annihilated by the principal parts of the Picard-Fuchs operators, as in (3.16). The mirror map is defined by the relation between the special coordinates t_α by $t_i = \frac{\omega_i}{\omega_0}$, where ω_0 is the unique power series solution and ω_i the single logarithmic solutions. It was observed in the second paper in [27] that the solutions obtained in this way for 3-folds, contain the information about the topological intersections defined as in (3.4) for 3-folds, that is χ and $\int J_i \wedge c_2$. More precisely the constant piece of the double logarithmic solutions corresponding to $\mathcal{F}_i = \partial_{t_i} \mathcal{F}$ coincides with $-24 \int J_i \wedge c_2$ and the constant piece of the triple logarithmic solution obtained in this way is $-i\zeta(3)\chi/(2\pi)^3$. An analogous phenomenon can be observed in the 4-fold case. The leading piece of the quartic logarithmic solution $\Pi^{(4)}$ is obtained from R_0 in (3.3) by multiplying each coefficient with the number of possible different permutations of its indices and replacing K_α by $\frac{1}{2\pi i} \ln(z_\alpha)$. In a basis for $\Pi_\alpha^{(3)}$ where the leading cubic pieces represent derivatives of $\Pi^{(4)}$ w.r.t. $t_\alpha \sim \ln(z_\alpha)$, the linear and quadratic pieces of $\Pi^{(3)}$ are proportional to $\zeta(3)R_3$ and $\pi^2 R_2$ in (3.4), again with the coefficients multiplied by the number of possible different permutations of the index structure.

Appendix D. Toric data for the other 4-folds

D.1. The 4-fold X_{II}

Basic data and properties

The dual polyhedron for the 4-fold X_{II} is the convex hull of the negative unit vertices ν_i^* , $i = 1 \dots 5$ and

$$\nu_6^* = (0, 0, 1, 4, 6), \nu_7^* = (0, 1, 1, 6, 9), \nu_8^* = (0, 0, 0, 2, 3), \nu_9^* = (1, 0, 1, 6, 9)$$

There are three Calabi–Yau phases, related by flops. First there is one phase in which corresponds to the K3 phase in the fiber, with Mori generators

$$\begin{aligned} l^{(1)} &= (0, 1, 0, 0, 0, 0, -1, 0, -1, 1), l^{(2)} = (0, 0, 0, 1, 0, 0, 1, 0, -2, 0), \\ l^{(3)} &= (-6, 0, 0, 0, 2, 3, 0, 0, 1, 0), l^{(4)} = (0, 0, 1, 0, 0, 0, -1, 1, -1, 0). \end{aligned} \tag{D.1}$$

By flops of the curves associated to the first or fourth Mori generator one reaches two other phases, where one can shrink 4-cycles in the \mathbf{F}_1 fiber. These two phases are actually

isomorphic and related by the exchange of the \mathbf{P}^1 's associated to $l^{(1)}$ and $l^{(4)}$. The Mori generators for one of the two are

$$\begin{aligned} l^{(1)} &= (0, 0, 1, 1, 0, 0, 0, 1, -3, 0), l^{(2)} = (-6, 0, 1, 0, 2, 3, -1, 1, 0, 0), \\ l^{(3)} &= (0, 0, -1, 0, 0, 0, 1, -1, 1, 0), l^{(4)} = (0, 1, -1, 0, 0, 0, 0, -1, 0, 1). \end{aligned} \quad (\text{D.2})$$

The topological intersection numbers are:

$$\begin{aligned} R_0 &= (3K_3^2K_2 + K_3K_2^2 + K_4K_3K_2 + 8K_3^3 + 2K_4K_3^2)K_1 + K_4K_3K_2^2 + \\ &\quad 6K_3^2K_2^2 + 3K_4K_3^2K_2 + 8K_4K_3^3 + 2K_3K_2^3 + 52K_3^4 + 18K_3^3K_2 \\ R_2 &= (24K_4 + 92K_3 + 36K_2)K_1 + 92K_3K_4 + 36K_2K_4 + 72K_2^2 + \\ &\quad 206K_2K_3 + 596K_3^2 \\ R_3 &= -480K_1 - 480K_4 - 1080K_2 - 3136K_3 \end{aligned} \quad (\text{D.3})$$

and

$$\begin{aligned} R_0 &= (K_2K_1^2 + 9K_3^3 + 8K_2^3 + 3K_3K_2K_1 + 9K_3^2K_2 + 3K_3^2K_1 + 9K_3K_2^2 \\ &\quad + K_3K_1^2 + 3K_2^2K_1)K_4 + 18K_2^3K_1 + 18K_3K_2^2K_1 + 6K_3K_2K_1^2 \\ &\quad + 18K_3^2K_2K_1 + 2K_2K_1^3 + 6K_3^2K_1^2 + 54K_3^2K_2^2 + 18K_3^3K_1 + \\ &\quad 54K_3^3K_2 + 2K_3K_1^3 + 6K_2^2K_1^2 + 52K_2^4 + 54K_3K_2^3 + 54K_3^4 \\ R_2 &= K_1^2 + 52K_2^4 + 54K_3K_2^3 + 54K_3^4(102K_3 + 92K_2 + 36K_1)K_4 + 596K_2^2 \\ &\quad + 206K_1K_3 + 618K_2K_3 + 72K_1^2 + 206K_1K_2 + 618K_3^2 \\ R_3 &= -1080K_1 - 3258K_3 - 480K_4 - 3136K_2, \end{aligned} \quad (\text{D.4})$$

respectively.

Cohomology classes of genus zero curves on X_{II}

For the physical interpretation of the Kähler moduli let us determine the cohomology classes of rational curves associated to the Kähler moduli for the first phase. The Stanley Reisner ideal is given by

$$SR : \{x_1x_9, x_2x_7, x_3x_6, x_4x_5x_8\}$$

while the typical Batyrev-Cox polynomial reads

$$\begin{aligned} &x_9^{18}x_8^6x_6^{12}x_2^{18} + x_9^{18}x_8^6x_6^{12}x_7^{18} + x_8^6x_6^{12}x_2^{18}x_1^{18} + x_8^6x_6^{12}x_1^{18}x_7^{18} + x_9^6x_3^{12}x_8^6x_2^6 \\ &+ x_9^6x_3^{12}x_8^6x_7^6 + x_3^{12}x_8^6x_2^6x_1^6 + x_3^{12}x_8^6x_1^6x_7^6 + x_4^3 + x_5^2 \end{aligned} \quad (\text{D.5})$$

respecting the C^* actions (6.2). It follows that t_1 is the area of the base $\mathbf{P}_{D_2}^1$, t_2 is area of the fiber \mathbf{P}_A^1 , t_3 is the area of a rational curve in the elliptic fiber and t_4 is area of the base $\mathbf{P}_{D_1}^1$.

Counting of rational curves on X_{II}

From the Mori generators (D.1) one obtains the Picard-Fuchs system

$$\begin{aligned}\mathcal{L}_1 &= z_1(-\theta_2 + \theta_1 + \theta_4)(-\theta_3 + \theta_4 + \theta_1 + 2\theta_2) - \theta_1^2 \\ \mathcal{L}_2 &= z_2(\theta_1 + 2\theta_2 + 1 - \theta_3 + \theta_4)(-\theta_3 + \theta_4 + \theta_1 + 2\theta_2) + \theta_2(-\theta_2 + \theta_1 + \theta_4) \\ \mathcal{L}_3 &= 12z_3(5 + 6\theta_3)(1 + 6\theta_3) + \theta_3(-\theta_3 + \theta_4 + \theta_1 + 2\theta_2) \\ \mathcal{L}_4 &= z_4(-\theta_2 + \theta_1 + \theta_4)(-\theta_3 + \theta_4 + \theta_1 + 2\theta_2) - \theta_4^2\end{aligned}$$

We chose the following basis for the operators $\mathcal{O}_\alpha^{(2)}$:

$$K_3^2 + K_4K_3, K_1K_4, K_2^2 + K_2K_4, K_2^2 + K_1K_2, K_1K_3 + K_3^2, K_2K_3 + 2K_3^2 \quad (\text{D.6})$$

The results for the 3-pt functions are collected in the tables F.II.i. A different basis for the operators $\mathcal{O}_\alpha^{(2)}$ can be chosen such as to reproduce the Gromov–Witten invariants of the 3-fold fiber; they are determined by the matrix N_α^μ in (5.2) which in the present case is given by

$$N_\alpha^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -\frac{1}{7} & 0 & 0 & 0 & \frac{6}{7} & -\frac{2}{7} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

D.2. The 4-fold X_{III}

Basic data and properties

The dual polyhedron for the 4-fold X_{III} is the convex hull of the negative unit vertices ν_i^* , $i = 1 \dots 5$ and

$$\nu_6^* = (0, 0, 1, 4, 6), \nu_7^* = (0, 1, 1, 6, 9), \nu_8^* = (0, 0, 0, 2, 3), \nu_9^* = (1, 2, 2, 12, 18)$$

There are two Calabi–Yau phases corresponding to the two phases of the 3-fold fiber discussed in detail in [18]. The first phase admits a K3 fibration and has the following Mori generators $l^{(\alpha)}$:

$$\begin{aligned} l^{(1)} &= (0, 0, 1, 0, 0, 0, -1, 1, -1, 0), \quad l^{(2)} = (0, 0, 0, 1, 0, 0, 1, 0, -2, 0), \\ l^{(3)} &= (-6, 0, 0, 0, 0, 2, 3, 0, 0, 1, 0), \quad l^{(4)} = (0, 1, 0, 0, 0, 0, 0, 0, -2, 0, 1) \end{aligned} \quad (\text{D.7})$$

The second phase has its Mori cone spanned by the generators

$$\begin{aligned} l^{(1)} &= (-6, 0, 1, 0, 2, 3, -1, 1, 0, 0), \quad l^{(2)} = (0, 0, 1, 1, 0, 0, 0, 1, -3, 0), \\ l^{(3)} &= (0, 0, -1, 0, 0, 0, 1, -1, 1, 0), \quad l^{(4)} = (0, 1, 0, 0, 0, 0, 0, 0, -2, 0, 1) \end{aligned} \quad (\text{D.8})$$

We will restrict our discussion of the first phase in the following. For the topological invariants (3.4) of the first phase with Mori generators (D.7) we find

$$\begin{aligned} R_0 &= (8K_3^3 + K_3K_2K_1 + K_3K_2^2 + 2K_3^2K_1 + 3K_3^2K_2)K_4 + 6K_3^2K_2K_1 + 2K_3K_2K_1^2 \\ &\quad + 2K_3K_2^2K_1 + 16K_3^3K_1 + 2K_3K_2^3 + 4K_3^2K_1^2 + 6K_3^2K_2^2 + 18K_3^3K_2 + 52K_3^4 \\ R_2 &= (92K_3 + 36K_2 + 24K_1)K_4 + 72K_1K_2 + 72K_2^2 + 184K_1K_3 + 48K_1^2 \\ &\quad + 596K_3^2 + 206K_2K_3 \\ R_3 &= -960K_1 - 3136K_3 - 1080K_2 - 480K_4 \end{aligned} \quad (\text{D.9})$$

Again the coefficients of K_4 in the above expression are precisely the intersection invariants of the Calabi–Yau fiber [18].

Genus zero curves on X_{III}

The Stanley Reisner ideal is given by

$$SR : \{x_1x_9, x_2x_7, x_3x_6, x_4x_5x_8\}$$

and the Batyrev-Cox polynomial reads

$$\begin{aligned} &x_9^{36}x_8^6x_7^{18}x_6^{12} + x_8^6x_7^{18}x_6^{12}x_1^{36} + x_9^{12}x_8^6x_7^6x_3^{12} + \\ &x_8^6x_7^6x_1^{12}x_3^{12} + x_8^6x_6^{12}x_2^{18} + x_8^6x_3^{12}x_2^6 + x_4^3 + x_5^2 \end{aligned} \quad (\text{D.10})$$

respecting the C^* actions (6.3). From the above data we can determine the areas associated to the Kähler moduli t_α : t_1 is the area of the fiber \mathbf{P}_B^1 , t_2 is the area of the fiber \mathbf{P}_A^1 , t_3 is the area of a curve in the elliptic fiber and t_4 is area of the base of the Calabi–Yau fibration.

Counting of rational curves on X_{III}

From the Mori generators (D.7) one obtains the Picard-Fuchs system

$$\begin{aligned}
\mathcal{L}_1 &= z_1(-\theta_2 + \theta_1)(-\theta_3 + \theta_1 + 2\theta_2) - \theta_1(\theta_1 - 2\theta_4) , \\
\mathcal{L}_2 &= z_2(\theta_1 + 2\theta_2 + 1 - \theta_3)(-\theta_3 + \theta_1 + 2\theta_2) + \theta_2(-\theta_2 + \theta_1) , \\
\mathcal{L}_3 &= 12z_3(5 + 6\theta_3)(1 + 6\theta_3) + \theta_3(-\theta_3 + \theta_1 + 2\theta_2) , \\
\mathcal{L}_4 &= z_4(\theta_1 - 2\theta_4)(\theta_1 - 2\theta_4 - 1) - \theta_4^2 .
\end{aligned} \tag{D.11}$$

The results for the 3-pt functions are collected in the tables F.III.i in a basis with $\mathcal{O}^{(2)}$ operators corresponding to

$$K_1K_2 + K_2^2, \quad -2K_1K_3 + K_2K_3, \quad K_1K_3 + K_3^2, \quad 2K_1^2 + K_1K_4, \quad K_2K_4, \quad K_3K_4 . \tag{D.12}$$

The matrix N_α^μ in (5.2) is given by

$$N_\alpha^\mu = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

D.3. The 4-folds X_{IV} , X_V

The dual polyhedron for the 4-fold X_{IV} is the convex hull of the negative unit vertices ν_i^* , $i = 1 \dots 5$ and

$$\begin{aligned}
X_{IV} &: (0, 0, 1, 4, 6), (0, 1, 0, 4, 6), (1, 0, 1, 6, 9), (0, 0, 0, 2, 3) \\
X_V &: (0, 0, 0, 2, 3), (0, 0, 1, 4, 6), (0, 1, 2, 8, 12), (1, 1, 2, 10, 15)
\end{aligned}$$

In both cases there is a K3 fibered Calabi–Yau phase; by the same methods as before the cohomology classes of curves dual to the Mori generators can be seen to be related in this phase to the volumes of the \mathbf{P}^1 factors and a curve in the elliptic fiber.

Appendix E. Heterotic U-duality symmetries

The discussion in sect. 8.1. applies also to the $N = 2$ compactifications of type II on threefolds and their heterotic duals and implies some general properties of the non-perturbative duality group of the heterotic theory. The duality between type II compactified on a Calabi–Yau 3-fold and heterotic string on $K3 \times \mathbf{T}^2$ maps the monodromy group of the Calabi–Yau compactification to the non-perturbative U-duality group of vector moduli space of the heterotic theory. Perturbatively the heterotic T-duality group is generated by monodromies in the vector moduli space around singular points, where there are classically enhanced gauge symmetries or massless hypermultiplets, together with the monodromies around infinity reflecting the axionic shift symmetries of the imaginary parts of the scalar fields.

From the analysis in sect 8.1 we can infer some general properties of these non-perturbative U-duality groups. Firstly, for generators of the U-duality group which have a perturbative origin we can go backwards and determine the type of the quantum monodromy - uncorrected or deformed due to a non-zero volume of the classical vanishing cycle - from continuously connecting to the field theory limit switching off string and gravity effects. Therefore elements of the perturbative T-duality symmetries which are generated perturbatively by Weyl reflections of enhanced gauge symmetries with non asymptotic free spectrum descend to exact non-perturbative U-dualities; on the other hand perturbative T-duality symmetries associated to enhanced gauge symmetries with asymptotic free spectrum are broken in the non-perturbative theory as in the case of the mirror symmetry $T \leftrightarrow U$ of the torus in standard compactifications observed in the examples in [72]¹⁹.

On the other hand we can also make precise the circumstances under which the heterotic theory has non-perturbative duality symmetries exchanging the dilaton with geometric moduli²⁰ which appears in the same examples. The dual type IIA Calabi–Yau compactification is given by a K3 fibration with the dilaton being the size of the base \mathbf{P}^1 of the fibration. If this base \mathbf{P}^1 arises as part of the resolution of a A_N curve singularity, which in fact is a canonical construction representing a large class of K3 fibrations,

¹⁹ This breaking is not merely the known quantum shift of the one-loop correction [73] but involves a true topology change of the discriminant locus as is clear from the relation to the field theory

²⁰ These symmetries have been explained geometrically in the F-theory context in [6] as the exchange of base \mathbf{P}^1 's and the volumes associated to it. However they appear similarly in 3-folds which do not admit an elliptic fibration; in these cases a similar geometrical origin and the relation to the six dimensional heterotic/heterotic duality of [54] is not clear. See also [74] for related works.

it is of the uncorrected type ii and can be shrunk to zero size with the restoration of a \mathbf{Z}_2 symmetry representing the exchange element. More precisely the full group of Weyl reflections of A_N generate a whole subgroup of exact non-perturbative duality symmetries of the heterotic theory exchanging the dilaton with $N - 1$ geometric moduli.

Appendix F. Invariants for the examples

F.1. Invariants for X_I

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0		480	960	-1	0	0	0	0	0
0	1	0	480	565776	0	480	676656	0	0	0
0	2	0	0	960	0	1440	-452160	0	0	960
1	0	0	0	0	-1	0	0	-4	0	0
1	1	0	0	0	0	480	676656	0	1440	2362608
1	2	0	0	0	0	1440	-452160	0	0	8640
2	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0	0	0

$n_3 \quad n_4$

Table I.2: Numbers of invariants $\frac{1}{2}N_1(\mathbf{n})$ for X_I

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	-2	480	282888	0	0	0
0	2	0	0	0	-4	1440	-226080	0	0	960
1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	-2	480	282888	-4	960	565776
1	2	0	0	0	-4	1440	-226080	4	-1920	895680
2	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0	0	0

$n_3 \quad n_4$

Table I.2: Numbers of invariants $\frac{1}{5}N_2(\mathbf{n})$ for X_I

n_1	0			1			2		
n_2	0	1	2	0	1	2	0	1	2
0	0	480	960	0	0	0	0	0	0
0	1	0	480	565776	0	480	565776	0	0
0	2	0	0	960	0	1440	-452160	0	0
1	0	0	0	0	-2	0	0	-4	0
1	1	0	0	0	0	480	787536	0	1440
1	2	0	0	0	0	1440	-452160	0	0
2	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0	0

$n_3 \quad n_4$

Table I.3: Numbers of invariants $N_3(\mathbf{n})$ for X_I

n_1	0			1			2		
n_2	0	1	2	0	1	2	0	1	2
0	0	0	0	0	0	0	0	0	0
0	1	-2	480	282888	-2	480	282888	0	0
0	2	0	0	960	-11	3840	-676080	0	0
1	0	0	0	0	0	0	0	0	0
1	1	0	0	0	-2	480	282888	-6	1440
1	2	0	0	0	-11	3840	-676080	-4	0
2	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0	0

$n_3 \quad n_4$

Table I.4: Numbers of invariants $\frac{1}{10}N_4(\mathbf{n})$ for X_I

n_1	0			1			2			
n_2	0	1	2	0	1	2	0	1	2	
0	0		480	960	0	0	0	0	0	0
0	1	0	480	621216	0	480	621216	0	0	0
0	2	0	0	960	0	1440	-452160	0	0	960
1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	480	621216	0	1440	1863648
1	2	0	0	0	0	1440	-452160	0	0	8640
2	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0	0	0

n_3 n_4

Table I.5: Numbers of invariants $\frac{1}{20}N_5(\mathbf{n})$ for X_I

n_1	0			1			2			
n_2	0	1	2	0	1	2	0	1	2	
0	0		0	0	0	0	0	0	0	
0	1	-2	480	282888	-2	480	282888	0	0	0
0	2	0	0	960	-8	2880	-452160	0	0	960
1	0	0	0	0	0	0	0	0	0	
1	1	0	0	0	-2	480	282888	-6	1440	848664
1	2	0	0	0	-14	4800	-900000	-4	0	895680
2	0	0	0	0	0	0	0	0	0	
2	1	0	0	0	0	0	0	0	0	
2	2	0	0	0	0	0	0	0	0	

n_3 n_4

Table I.6: Numbers of invariants $N_6(\mathbf{n})$ for X_I

F.2. Invariants for X_{II}

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0		0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0
1	0	3600	3600	0	876	-7200	-14400	0	0	18000
	1	996	-7200	-14400	0	0	144000	0	0	0
	2	0	0	18000	0	0	0	0	0	0
2	0	7200	5535504	7200	34380	1729980	5086800	-69132	-68760	-6934320
	1	37980	1733580	5086800	-384372	-144720	-60056640	0	0	1565100
	2	-82752	-75960	-6948720	0	0	1691100	0	0	

$n_3 \quad n_4$

Table II.1: Numbers of invariants $N_1(\mathbf{n})$ for X_{II}

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0		0	0	1	3	5	0	0	-12
	1	1	3	5	1	-4	-101	1	-4	28
	2	0	0	-12	1	-4	28	2	-12	72
1	0	0	0	0	252	-960	-1920	0	0	4800
	1	252	-960	-1920	-120	1320	46200	-120	1320	-14040
	2	0	0	4800	-120	1320	-14040	-360	4440	-37440
2	0	0	0	0	5130	118170	339120	-18504	-20520	-947280
	1	5130	118170	339120	-107604	-404640	-11674080	20520	-376920	4579470
	2	-18504	-20520	-947280	20520	-376920	4579470	55200	-1261440	

$n_3 \quad n_4$

Table II.2: Numbers of invariants $N_2(\mathbf{n})$ for X_{II}

n_1	0			1			2		
n_2	0	1	2	0	1	2	0	1	2
0	0	-6	0	0	13	42	0	0	-58
	1	0	17	54	0	0	-387	0	0
	2	0	0	-80	0	0	0	0	0
1	0	0	1440	0	-3960	-15480	0	0	22320
	1	0	-5040	-19440	0	0	178920	0	0
	2	0	0	30240	0	0	0	0	0
2	0	0	848664	2880	0	636390	2765160	0	-30780 -4343580
	1	0	918270	3495600	0	-144720	-46296360	0	0 1290060
	2	0	-30780	-5845320	0	0	1517940	0	0

$n_3 \quad n_4$

Table II.3: Numbers of invariants $N_3(\mathbf{n})$ for X_{II}

n_1	0			1			2		
n_2	0	1	2	0	1	2	0	1	2
0	0	-6	0	0	17	54	0	0	-80
	1	0	13	42	0	0	-387	0	0
	2	0	0	-58	0	0	0	0	0
1	0	0	1440	0	0	-5040	-19440	0	0 30240
	1	0	-3960	-15480	0	0	178920	0	0 0
	2	0	0	22320	0	0	0	0	0 0
2	0	0	848664	2880	0	918270	3495600	0	-30780 -5845320
	1	0	636390	2765160	0	-144720	-46296360	0	0 1517940
	2	0	-30780	-4343580	0	0	1290060	0	0

$n_3 \quad n_4$

Table II.4: Numbers of invariants $N_4(\mathbf{n})$ for X_{II}

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0		0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0
1	0	3600	3600	0	996	-7200	-14400	0	0	18000
	1	876	-7200	-14400	0	0	144000	0	0	0
	2	0	0	18000	0	0	0	0	0	0
2	0	7200	5535504	7200	37980	1733580	5086800	-82752	-75960	-6948720
	1	34380	1729980	5086800	-384372	-144720	-60056640	0	0	1691100
	2	-69132	-68760	-6934320	0	0	1565100	0	0	

n_3 n_4

Table II.5: Numbers of invariants $N_5(\mathbf{n})$ for X_{II}

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0		0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0
1	0	7320	7320	0	1248	-14640	-29280	0	0	36600
	1	1248	-14640	-29280	0	0	292800	0	0	0
	2	0	0	36600	0	0	0	0	0	0
2	0	14640	11858544	14640	48240	3495960	10343160	-101256	-96480	-14013120
	1	48240	3495960	10343160	-512496	-192960	-121912560	0	0	2170800
	2	-101256	-96480	-14013120	0	0	2170800	0	0	

n_3 n_4

Table II.6: Numbers of invariants $N_6(\mathbf{n})$ for X_{II}

F.3. Invariants for X_{III}

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0		-8	0	0	20	64	0	0	-92
0	1	0	0	0	0	20	64	0	0	-516
0	2	0	0	0	0	0	0	0	0	-92
1	0	0	1920	0	0	-6000	-23280	0	0	35040
1	1	0	0	0	0	-6000	-23280	0	0	238560
1	2	0	0	0	0	0	0	0	0	35040
2	0	0	1131552	3840	0	1036440	4173840	0	-41040	-6792600
2	1	0	0	0	0	1036440	4173840	0	-192960	-61728480
2	2	0	0	0	0	0	0	0	-41040	

$n_3 \quad n_4$

Table III.1: Numbers of invariants $N_1(\mathbf{n})$ for X_{III}

n_1		0			1			2		
n_2		0	1	2	0	1	2	0	1	2
0	0		0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0
1	0	-840	-840	0	-1248	1680	3360	0	0	-4200
1	1	0	0	0	-1248	1680	3360	0	0	-33600
1	2	0	0	0	0	0	0	0	0	-4200
2	0	-1680	-344016	-1680	-48240	-443880	-1186920	101256	96480	1778880
2	1	0	0	0	-48240	-443880	-1186920	512496	192960	14331600
2	2	0	0	0	0	0	0	101256	96480	

$n_3 \quad n_4$

Table III.2: Numbers of invariants $N_2(\mathbf{n})/5$ for X_{III}

n_1	0			1			2			
	n_2	0	1	2	0	1	2	0	1	2
0	0		0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0
1	0	4080	4080	0	1248	-8160	-16320	0	0	20400
1	1	0	0	0	1248	-8160	-16320	0	0	163200
1	2	0	0	0	0	0	0	0	0	20400
2	0	8160	6101280	8160	48240	1969920	5765040	-101256	-96480	-7 896000
2	1	0	0	0	48240	1969920	5765040	-512496	-192960	-68122080
2	2	0	0	0	0	0	0	-101256	-96480	

n_3 n_4

Table III.3: Numbers of invariants $N_3(\mathbf{n})$ for X_{III}

n_1	0			1			2			
	n_2	0	1	2	0	1	2	0	1	2
0	0	0	0	1	3	5	0	0	-12	
0	1	0	0	0	1	3	5	1	-4	-101
0	2	0	0	0	0	0	0	0	0	-12
1	0	0	0	0	252	-960	-1920	0	0	4800
1	1	0	0	0	252	-960	-1920	-120	1320	46200
1	2	0	0	0	0	0	0	0	0	4800
2	0	0	0	0	5130	118170	339120	-18504	-20520	-947280
2	1	0	0	0	5130	118170	339120	-107604	-404640	-11674080
2	2	0	0	0	0	0	0	-18504	-20520	

n_3 n_4

Table III.4: Numbers of invariants $\frac{1}{5}N_4(\mathbf{n})$ for X_{III}

n_1	0			1			2			
n_2	0	1	2	0	1	2	0	1	2	
0	0	-2	0	0	3	10	0	0	-12	
0	1	0	0	0	7	22	0	0	-129	
0	2	0	0	0	0	0	0	0	-34	
1	0	0	480	0	-960	-3840	0	0	4800	
1	1	0	0	0	-2040	-7800	0	0	59640	
1	2	0	0	0	0	0	0	0	12720	
2	0	0	282888	960	0	118170	678240	0	-10260	-947280
2	1	0	0	0	0	400050	1408680	0	-48240	-15432120
2	2	0	0	0	0	0	0	0	-10260	

n_3 n_4

Table III.5: Numbers of invariants $N_5(\mathbf{n})$ for X_{III}

n_1	0			1			2			
n_2	0	1	2	0	1	2	0	1	2	
0	0	0	0	0	0	0	0	0	0	
0	1	0	0	0	0	0	0	0	0	
0	2	0	0	0	0	0	0	0	0	
1	0	480	480	0	252	-960	-1920	0	0	2400
1	1	0	0	0	372	-960	-1920	0	0	19200
1	2	0	0	0	0	0	0	0	0	2400
2	0	960	565776	960	10260	236340	678240	-18504	-20520	-947280
2	1	0	0	0	13860	239940	678240	-128124	-48240	-8065440
2	2	0	0	0	0	0	0	-32124	-27720	

n_3 n_4

Table III.6: Numbers of invariants $N_6(\mathbf{n})$ for X_{III}

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