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Optimized-Variational Quark Mass Expansion and Dynamical Chiral Symmetry Breakdown [a]

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Abstract

A recently proposed variational mass expansion approach to dynamical chiral symmetry breakdown is reviewed. We briefly explain how a specific integral ansatz over the analytically continued, arbitrary Lagrangian mass parameter, resums the usual variational mass expansion ("delta-expansion".) The construction is then generalized to obtain non-perturbative expressions for the order parameters of the $SU(n_f)_L \times SU(n_f)_R$ breakdown ($n_f = 2$ or 3), in QCD. Emphasis is put on some general aspects as well as possible limitations of this approach.

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Optimized-Variational Quark Mass Expansion and Dynamical Chiral Symmetry Breakdown

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A recently proposed variational mass expansion approach to dynamical chiral symmetry breakdown is reviewed. We briefly explain how a specific integral ansatz over the analytically continued, arbitrary Lagrangian mass parameter, resums the usual variational mass expansion ("delta-expansion".) The construction is then generalized to obtain non-perturbative expressions for the order parameters of the $SU(n_f)_L \times SU(n_f)_R$ breakdown ($n_f =$ 2 or 3), in QCD. Emphasis is put on some general aspects as well as possible limitations of this approach.

1 Introduction

To calculate the low-energy properties of the QCD spectrum from "first principle" is clearly a desirable task, but obviously out of the reach of ordinary perturbation theory, due to the infrared growth of the coupling and the non-perturbative relevant dynamics. Chiral perturbation theory [1] gives a consistent low-energy effective description, but in terms of a set of parameters to be fixed from the data, whose precise connection with the basic QCD coupling and quark mass parameters is far from being resolved at present. Of particular relevance to the low-energy QCD dynamics are the order parameters of the $SU(n_f)_L \times SU(n_f)_R$ ($n_f =$ 2,3) chiral symmetry breakdown (CSB), such as the $\langle \bar{q}q \rangle$ condensate or the pion decay constant F_π, typically. Partially related to the previous aim, it has however been known for a long time that, at least in simplified field-theoretic models, definite information on specific non-perturbative quantities may be inferred from particular resummation properties[2] and/or appropriately modified perturbation series[3].

Recently [4,5] we have investigated a new approach to explore *how far* the basic QCD Lagrangian can provide non-zero quark condensates, pion decay constant, as well as *dynamical* quark masses, in the limit of vanishing Lagrangian (current) quark masses. The starting point is very similar to the idea developed long ago in refs. [3], where the convergence of ordinary perturbation was shown to be improvable by a variational procedure in which the separation of the action into "free" and "interaction" parts is made to depend on some set of auxiliary parameters[a].

[a] In one-dimensional field theories this optimized perturbation theory ("delta-expansion") has been shown [6] to lead to a rigorously convergent series of approximations, even in strong coupling cases.

An essential novelty is that our construction can reach *infinite* order [7,8] of such a variational-perturbative expansion, therefore presumably optimal, provided it converges. It also gives a consistent treatment of the renormalization, reconciling the variational expansion with the inherent infinities of most field theory of dimension higher than 1.

2 Mass Gap in the Gross-Neveu model

We shall first illustrate the basic ingredients of our construction with a determination [7,8] of the mass gap in the $O(2N)$ Gross-Neveu (GN) model.

From renormalization group (RG) resummation properties, one can infer the following form of a *bare*, RG-invariant resummed mass ($D = 2 + \epsilon$):

$$m_F(m_0) = m_0 \, (1 - 4\pi b_0 \, g_0^2 \, \tilde{\Gamma} m_F^{\epsilon})^{-\frac{\gamma_0}{2b_0}} \; , \qquad (1)$$

where m_0, g_0 are the bare mass and coupling, $\tilde{\Gamma} \equiv \Gamma[-\epsilon/2]/(4\pi)^{1+\epsilon/2}$, and b_0, γ_0 are the one-loop RG-coefficients of the running coupling and mass, respectively (we use a normalization such that $\beta(g) = -b_0 g^3 - b_1 g^5 - \cdots$, $\gamma_m(g) = \gamma_0 g^2 + \gamma_1 g^4 + \cdots$; where in the $O(2N)$ model $b_0 = (N-1)/(2\pi)$, $\gamma_0 = (N-1/2)/\pi$, etc [9].) As easily shown, eq. (1) gives, after renormalization, the exact mass gap result in the $N \to \infty$, $m \to 0$ limit: $m_F = \Lambda_{\overline{MS}}$, where $\Lambda_{\overline{MS}} \equiv \bar{\mu} \, e^{-1/(2b_0 g^2(\bar{\mu}))}$ is the basic scale in the $\overline{MS}$ scheme. Now, in the more complicated case of arbitrary N, according to the above-mentioned variational principle, we expect to obtain a series of approximants to the mass gap, by optimizing with respect to the *arbitrary* Lagrangian mass, at successive orders of the variational x-series, formally defined by substituting everywhere in the bare Lagrangian:

$$m_0 \to m_0\,(1-x); \qquad g_0^2 \to g_0^2\,x, \qquad (2)$$

and in particular for example in expression (1). However, in order to get a finite and non-trivial result (i.e. $m_F \neq 0$) one has to resum the resulting x-series by using a specific contour integration [8] of (1) over the (analytically continued) x parameter. Moreover, the one-loop RG-resummed expression (1), which includes only the leading dependence in m, has to be generalized to include both higher order RG-dependence upon m, *and* non-logarithmic purely perturbative corrections as well, in order to get a more realistic mass gap in the arbitrary N case. In this way we end up with the ansatz:

$$\frac{M_2^P(m'')}{\Lambda_{\overline{MS}}} = \frac{2^{-C} m''}{2i\pi} \oint dv \frac{e^{\,v}}{F^A(v)[C+F(v)]^B} \times \left(1 + \frac{\mathcal{M}_1}{F(v)} + \frac{\mathcal{M}_2}{F^2(v)} + \cdots\right), \qquad (3)$$

where the contour is around the negative real axis,

$$F(v) \equiv \ln[m'' v] - A\; \ln F - (B-C)\; \ln[C+F], \qquad (4)$$

with $A = \gamma_1/(2b_1)$, $B = \gamma_0/(2b_0) - \gamma_1/(2b_1)$, $C = b_1/(2b_0^2)$; $\Lambda_{\overline{MS}}$ is the (RG-invariant) scale at two-loop order; for convenience we introduced the scale-invariant, arbitrary (dimensionless) "mass" parameter, redefined in terms of m as

$$m'' \equiv \left(\frac{m(\bar\mu)}{\Lambda_{\overline{MS}}}\right)\; 2^C\; [2b_0\bar g^2]^{-\gamma_0/(2b_0)}\; \left[1 + \frac{b_1}{b_0}\bar g^2\right]^B . \qquad (5)$$

As mentioned, $F(1)$ in the integrand of (3) by construction resums the leading and next-to-leading logarithmic dependence in $m(\bar\mu)$ to all orders [5]. The two-loop non-logarithmic perturbative coefficients $\mathcal{M}_1$, $\mathcal{M}_2$, given explicitly in ref. [8], connect the *pole* mass with the running mass $m(\bar\mu)$.

We have shown [8] that expression (3) does resum the x-series [b] generated from the substitution (2). Moreover it is possible to choose a renormalization scheme (RS) such that $b_i = \gamma_i = 0$ for $i \geq 2$. In that sense, eq. (3) resums the full RG dependence in x and m''. In contrast, the purely perturbative (non-logarithmic) information, contained in $\mathcal{M}_1$, $\mathcal{M}_2$, is limited to the two-loop order. This is where the variational principle and optimization play their role, whereby we hope to obtain a sensible approximation to the exact mass gap. Accordingly, we may now look for extrema of expression (3) with respect to m'', using standard contour integration techniques. Observe in fact that, were we in a simplified theory where $\mathcal{M}_1 = \mathcal{M}_2 = \cdots = 0$ (as incidentally is the case in the large-N limit), (3) would have a very simple behaviour near its optimum (at $m'' \to 0$), giving a simple pole with residue $M_2 = (2C)^{-C}\; \Lambda_{\overline{MS}}$. Now, in the arbitrary N case where $\mathcal{M}_1$, $\mathcal{M}_2$,... cannot be neglected, one can construct a set of approximants of (3) in the $m'' \to 0$ limit, with some variants of Padé approximants (PA). The results [8] of two different optimal fifth-order PA are compared in the table below with exact results for the mass gap of the $O(2N)$ model (obtained alternatively from the Bethe ansatz in ref. [10]). As illustrated, an error of $\mathcal{O}(5\%)$ or less, depending on N, can be obtained [c].

N	Exact $m_F/\Lambda_{\overline{MS}}$	Padé 1 (error)	Padé 2 (error)
2	1.8604	1.758 (5.5%)	1.875 (0.8%)
3	1.4819	1.475 (0.5%)	1.486 (0.3%)
5	1.2367	1.284 (3.7%)	1.265 (2.3%)

3 Dynamical quark masses

Most of the previous construction (for arbitrary N) may be formally extended to the QCD case. In particular, expression (3), with the appropriate change to QCD values of the b_i and γ_i RG coefficients, provides a dynamical mass ansatz [4,5] as a function of the quark flavours n_f (the latter entering both $\Lambda_{\overline{MS}}$ and the b_i, γ_i expressions). It is important to note that expression (3) for arbitrary N in the GN model uses exactly the *same* amount of perturbative and RG information as is at our disposal at present for a QCD quark mass: namely, the *exact* two-loop RG-resummed plus purely perturbative $\mathcal{M}_1$, $\mathcal{M}_2$ [12] dependence. Since

[b] v in (3) is related to the original expansion parameter x as $x = 1 - v/q$, q being the order of the x-expansion.

[c] The results given in the table were obtained by further optimizing with respect to an arbitrary scale parameter, introduced from $\bar\mu \to a\,\bar\mu$. Results from PA of order lower than the (optimal) order 5 give a larger error.

our construction only relies on RG-properties (and analytic continuation techniques), passing from 2 to 4 dimensions is not expected to cause major changes, at least naively[d].

Actually, one complication *does* occur: as a more careful examination of relation (4) indicates, there are extra branch cuts in the v plane, with $\mathrm{Re}[v_{cut}] > 0$ for the relevant case of $n_f = 2$ or 3 in QCD. This prevents using the expansion near the origin, which would lead to ambiguities of $\mathcal{O}(\exp(+\mathrm{const.}/m''))$ for $m'' \to 0$. The origin of those singularities is rather similar to the renormalon ones[2], since they appear in a resummed expression relating a "reference" scale $M_{dyn} \simeq \Lambda_{\overline{MS}}$ to an infrared scale $m'' \simeq 0$. However, in the present construction it is possible[5] to move those extra cuts to the safe location $\mathrm{Re}[v'_{cut}] \leq 0$, by noting that the actual position of those cuts depend on the RS, via γ_1. Performing thus a second-order perturbative RS change in $m(\mu)$, $g(\mu)$, which changes $\gamma_1 \to \gamma_1'$, and looking for a flat optimum (plateau) of an appropriately constructed PA, with respect to the remnant RS arbitrariness[e], we obtain:

$$M^{Pad\acute{e}}_{opt}(m'' \to 0) \simeq 2.97\ \Lambda_{\overline{MS}}(2) \tag{6}$$

for $n_f = 2$, and a similar result for $n_f = 3$.

4 Order parameters of CSB: F_π and $\langle \bar{q}q \rangle$

The above dynamical quark mass, although it has some meaning as regards spontaneous CSB in QCD, hardly has a direct physical intepretation, e.g. as a pole in the S-matrix, due to confinement. In other words, it is not a properly defined order parameter. It is however possible to generalize the mass ansatz (3) to obtain a determination of the ratios $F_\pi/\Lambda_{\overline{MS}}$ and $\langle \bar{q}q \rangle(\mu)/\Lambda^3_{\overline{MS}}$. The latter gauge-invariant quantities are unambiguous order parameters, i.e. $F_\pi \neq 0$ *or* $\langle \bar{q}q \rangle \neq 0$ indicate spontaneous CSB. The appropriate generalization[f] of (3) for F_π is[5]

$$\frac{F_\pi^2}{\Lambda^2_{\overline{MS}}} = (2b_0)\ \frac{2^{-2C}(m'')^2}{2i\pi} \oint \frac{dv}{v}\ v^2 e^{\,v}\ \times \\ \frac{1}{F^{\,2A-1}[C+F]^{\,2B}}\ \delta_\pi \left(1 + \frac{\alpha_\pi}{F} + \frac{\beta_\pi}{F^2}\right) \tag{7}$$

in terms of the very same $F(v)$ defined in eq. (4) (therefore leading to the same extra cut locations as in the mass case), and where δ_π, α_π and β_π are fixed by matching the perturbative $\overline{MS}$ expansion, known to 3-loop order[13,14]. A numerical optimization with respect to the RS-dependence of an appropriate PA, along the same line as the mass case, gives e.g for $n_f = 2$:

$$F^{Pad\acute{e}}_{\pi,opt}(m'' \to 0) \simeq 0.55\ \Lambda_{\overline{MS}}(2)\ . \tag{8}$$

Concerning $\langle \bar{q}q \rangle$, an ansatz similar to (7) can be derived (with coefficients δ, α, β specific to $\langle \bar{q}q \rangle$ and obvious changes in the m'', F and v powers), but for the RG-invariant combination $m\langle \bar{q}q \rangle$ *only*, since our construction only makes sense for RG-invariant quantities. As it turns out, the limit $m \to 0$ simply gives $m\langle \bar{q}q \rangle \to 0$. To extract an estimate of the (scale-dependent) condensate $\langle \bar{q}q \rangle(\mu)$ in our framework is only possible by introducing[5] an explicit symmetry-breaking quark mass m_{exp} (independent from m), and expanding the $m\langle \bar{q}q \rangle$ ansatz to first order in m_{exp}. This gives e.g. for $n_f = 2$:

$$\langle \bar{q}q \rangle(\bar{\mu} = 1\ \mathrm{GeV}) \simeq 0.52\ \Lambda_{\overline{MS}}(2)\ . \tag{9}$$

Comparing the results (6), (8) and (9) gives a fairly small value of the quark condensate (and a fairly high value of the dynamical mass), as compared to other non-perturbative methods[15]. Although small values of $\langle \bar{q}q \rangle$ are not experimentally excluded at present[16], it is also clear that our relatively crude approximation deserves more refinements for more realistic QCD predictions.

5 Summary and conclusions

The variational expansion in arbitrary m, developed in the GN model[8], can thus be formally extended to the QCD case. This gives non-trivial

[d] From that point of view, there are *no* differences between QCD and the GN model (for arbitrary N), as compared e.g. with other two-dimensional (or a fortiori, one-dimensional) theories where particularly simple features are sometimes due to their super-renormalizable properties. Note also that a mass gap in the $O(2N)$ GN model does not contradict the Coleman theorem[11] on the non-breaking of the continuous chiral symmetry in two dimensions. This in a sense illustrates that the applicability of our method is more general than the physics of CSB.

[e] See ref.[5] for details.

[f] A non-trivial point in this generalization is the necessary additional (additive) renormalization of the axial vector-axial vector two-point correlator, whose first-order expansion term with respect to external momentum defines F_π[1]. For a fixed RS, this unambiguously leads to (7).

relationships between $\Lambda_{\overline{MS}}$ and the dynamical masses and order parameters, F_π and $\langle \bar{q}q \rangle$.

Let us conclude with some remarks on the possible generalizations (and limitations) of this approach. In principle, it may be applied to any (renormalizable or super-renormalizable) field theory, where the massless fermion (or scalar) limit is of interest, and most probably the basic idea could be applied to parameters other than mass. Now, what is at any rate limitative, is the relatively poor knowledge of the purely perturbative part of the expansion (only known to two-loop order in most realistic field theories). Our ansatzes (3), (7) would be exact if there were no such corrections, i.e. if the dependence on m was entirely dictated from RG properties. Therefore, our final numerical results crucially depend on the optimization[g]. Apart from a few models where the series is known to large orders (as in the anharmonic oscillator [3,17], or in the GN model for $N \to \infty$), we can hardly compare successive orders of the variational-optimized expansion to estimate, even qualitatively, its convergence properties. Invoking the "principle of minimal sensitivity" [3], although physically motivated, in a sense artificially forces the series to converge, with no guarantees that it is toward the right result.

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[g] For instance, PA results for (3), (7) are substantially different [5] in the (*unoptimized*) $\overline{MS}$ scheme.

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