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NEW SINGULARITIES IN NON-RELATIVISTIC COUPLED CHANNEL SCATTERING I: SECOND ORDER

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Abstract

New and surprising singularities are found in the forward scattering amplitude for non-relativistic potential scattering with coupled channels. In the simplest case of two coupled channels, these singularities appear when the energy difference between the two channels is larger than the inverse range of the potential. They are similar to singularities recently discovered by one of us for potential scattering on $R^3 \otimes S^1$.

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I. Introduction

In this paper we will demonstrate that, in coupled-channel non-relativistic potential scattering, new and surprising singularities (even simple poles) can appear in the forward amplitude. It has been generally believed that in such cases, as well as in quantum field theory, a common feature is the absence of singularities in the forward amplitudes on the physical sheet (except possibly for bound state poles).

Most of this paper deals with the case of two coupled channels where $T_{nm}(\vec{k}, \vec{k'})$ is a 2x2 matrix, and the potential V_{mn} has Yukawa off-diagonal entries. We explicitly calculate $T_{11}(k)$ and $T_{22}(k)$ for the forward case in second order perturbation theory and show that, while $T_{11}^{(2)}(k)$ is analytic on the physical sheet as expected, $T_{22}^{(2)}(k)$ has a pole at $k = i(a^2 + \mu^2)/2\mu$, where a^2 is the energy difference between the two channels, $E_2 - E_1 = a^2$, and μ^{-1} is the Yukawa range with $a^2 > \mu^2$.

The pole in $T_{22}^{(2)}(k)$ on the physical sheet is unexpected. It definitely is not a bound state since it appears in perturbation theory. It is interesting to have a better understanding of these singularities and to look for physical systems where their presence may lead to detectable effects.

Our main task is to understand this new singularity better. It would be helpful to know if it appears in the full amplitude. This so far we have not achieved. The next best thing is to look at higher orders. It turns out that the next order where this same singularity could appear is the fourth order in perturbation. This is already a complicated problem, and we tackle it in an accompanying paper. It is found that the singularity appears there too and at the same point, $k = +i(a^2 + \mu^2)/2\mu$.

A historical remark is now in order, namely what motivated us to consider this problem. The path, as is often the case in physics, is quite indirect. More than two years ago, one of us studied the problem of non-relativistic potential scattering on a space with an additional internal compact dimension[1], more specifically $R^3 \otimes S^1$. The motivation for examining such a problem arose in the context of whether the proposed existence [2] of a new compact internal dimension, with radius, R, and $R^{-1} = O(1TeV)$. The question is whether such a new compact dimension would lead to a violation of the forward dispersion relations, and hence could be detected experimentally. This led one of us to look at a simple well defined model, non-relativistic quantum mechanics on $R^3 \otimes S^1$. The results were surprising. The analyticity properties[3,4,5] which are true for R^3 do not hold in $R^3 \otimes S^1$. Indeed new poles appeared on the physical sheet in second order perturbation theory whenever $R^{-1} > \mu$, where μ^{-1} is the range of the potential.

In section II, we give a brief review of this $R^3 \otimes S^1$ case. In the following section we show that the same result occurs when S^1 is replaced by N discrete equally spaced points on a circle of radius R, i.e. $R^3 \otimes Z_N$. The new poles are there for all $N > 1$, including $N = 2$, which is just a two coupled channel problem. The only requirement being again that $1/R > \mu$.

In section IV we define a simple two-channel problem, and proceed to calculate the second order forward scattering amplitude. We show that, under similar conditions, the same pole appears in the forward amplitude.

II. The $R^3 \otimes S^1$ Case

We sketch in this section the main result of ref. 1.

The Schrodinger equation on $R^3 \otimes S^1$, written in dimensionless form is

$$[\vec{\nabla}^2 + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} + K^2 - V(r;\phi)]\Psi(\vec{r};\phi) = 0, \tag{2.1}$$

where $\vec{r}\epsilon R^3$, R is the fixed radius of S^1, and ϕ is the angle on S^1. The potential, $V(r,\phi)$, is taken to be periodic in ϕ,, $V(r,\phi) = V(r,\phi+2\pi)$. One also assumes from the beginning that there are two scales, $\frac{1}{R} > \mu$, where μ^{-1} is the range of the force in R^3. The normalized free solutions of (2.1) are

$$\psi_o(\vec{x},\phi) = \frac{1}{(2\pi)^2}e^{i\vec{k}.\vec{x}}e^{in\phi}, n = 0, \pm1, \pm2, ..., \tag{2.2}$$

and the total energy is $K^2 = k^2 + n^2/R^2$.

The free Green's function is given by

$$G_o(K;\vec{x},\phi;\vec{x'},\phi') = -\frac{1}{(2\pi)^4}\sum_{n=-\infty}^{+\infty}\int d^3p\frac{e^{i\vec{p}.(\vec{x}-\vec{x'})}e^{in(\phi-\phi')}}{p^2 + n^2/R^2 - K^2 - i\epsilon}, \tag{2.3}$$

The main surprising result of reference 1 is the appearance of a pole in the second order calculation of $T^{(2)}_{nn}(K)$, the forward scattering amplitude for $n \geq 1$. It suffices to consider the simple potential

$$V(r,\phi) = 2\frac{e^{-\mu r}}{r} cos\phi. \tag{2.4}$$

A straightforward calculation then gives us

$$T^{(2)}_{nn}(K) = -\frac{1}{2\pi^2} F_1(k;a^2) \mid_{a^2=\frac{2n-1}{R^2}} - \frac{1}{2\pi^2} F_2(k;a^2) \mid_{a^2=\frac{2n+1}{R^2}}, \tag{2.5}$$

where F_1 and F_2 are given by

$$F_1(k,a^2) = \int d^3p \frac{1}{[(\vec{p}-\vec{k})^2+\mu^2]^2[p^2-(k^2+a^2)-i\epsilon]},$$

$$F_2(k,a^2) = \int d^3p \frac{1}{[(\vec{p}-\vec{k})^2+\mu^2]^2[p^2-(k^2-a^2)-i\epsilon]}. \tag{2.6}$$

The above integrations were performed in ref. 1, see Eqs. (4.10) and (4.11). In particular we have

$$F_1(k,a^2) = \frac{\pi^2 i}{4k\mu^2}\left[\frac{\sqrt{k^2+a^2}+k-i\mu}{k-i\frac{(a^2+\mu^2)}{2\mu}} + \frac{\sqrt{k^2+a^2}-k-i\mu}{k+i\frac{(a^2+\mu^2)}{2\mu}}\right] \tag{2.7}$$

Continuing the above expression into the region $Imk \geq 0$, there is a pole at

$$k = +i\frac{a^2+\mu^2}{2\mu}. \tag{2.8}$$

The residue, $(\sqrt{k^2+a^2}+k-i\mu)$, does not vanish at this pole. In addition to the above pole, $F_1(k,a^2)$ has a branch points at $k=\pm ia$. The branch cut is taken to join these two points since we know that $T^{(2)}_{nn}(K)$ is analytic for large enough $\mid k \mid, Imk > 0$. Also, with $1/R > \mu, a > \mu$, and hence $(a^2+\mu^2)/2\mu > \mu$, and the pole lies above the branch point.

Finally, both F_2 and $T^{(2)}_{oo}$ have no unusual singularities.

III. The Case $R^3 \otimes Z_N$

In order to shed more light on the origin of the unusual pole found in ref. 1, and reviewed in the preceding section, we consider a discrete version of the previous model. Physically, such a discrete model is also of more relevance to atomic and nuclear systems. Instead of S^1 we take a set of N discrete points on a circle of radius R, i.e. $R^3 \otimes Z_N$.

There are N discrete internal states $n = 0, 1, ..., N-1$. The previous model corresponds to $N \to \infty$. Thus we have $\phi \to \phi_j, j = 0, ..., N-1; \phi_j = \frac{2\pi j}{N}$. If we define the shift operator $D : j \to j+1$, then we replace $\frac{\partial^2}{\partial \phi^2}$ by

$$\frac{\partial^2}{\partial \phi^2} \to (\frac{N}{2\pi})^2 (D - 2 + D^{-1}) \tag{3.1}$$

The Schrodinger equation, after making the above replacement, becomes

$$[\nabla^2 + \frac{1}{R^2}(\frac{N}{2\pi})^2 (D - 2 + D^{-1}) + K^2 - V(r,j)]\Psi(r,j) = 0. \tag{3.2}$$

More explicitly, we have

$$[\nabla^2 - \frac{2}{R^2}(\frac{N}{2\pi})^2 + K^2 - V(r,j)]\Psi(r,j) + \frac{1}{R^2}(\frac{N}{2\pi})^2 [\Psi(r,j+1) + \Psi(r,j-1)] = 0. \tag{3.3}$$

The free solution is now

$$\psi_o(\vec{x}, \phi_j) = (\frac{1}{2\pi})^2 e^{i\vec{k}.\vec{x}} e^{in\phi_j}; \phi_j = \frac{2\pi j}{N}, \tag{3.4}$$

The total energy of this state is

$$K^2 = k^2 - \frac{1}{R^2}(\frac{N}{2\pi})^2 [e^{in\frac{2\pi}{N}} - 2 + e^{-in\ frac2\pi N}] = k^2 + \frac{1}{R^2}(\frac{N}{\pi})^2 sin^2 \frac{n\pi}{N}. \tag{3.5}$$

Thus the replacement is $\frac{n^2}{R^2} \to \frac{1}{R^2}(\frac{N}{\pi})^2 sin^2 \frac{n\pi}{N}$. The Green's function now is

$$G_o(K;\vec{x},j;\vec{x'},j') = -\frac{1}{(2\pi)^4}\sum_{n=0}^{N-1}\int d^3p \frac{e^{i\vec{p}.(\vec{x}-\vec{x'})}e^{in(j-j')}}{p^2+\frac{1}{R^2}(\frac{N}{\pi})^2 sin^2\frac{n\pi}{N} - K^2 - i\epsilon}. \tag{3.6}$$

From this point onwards the calculation of $T^{(2)}_{nn}$ is very similar to the preceding case. In F_1, a^2 is now given through the replacement $\frac{n^2}{R^2} \to \frac{1}{R^2}(\frac{N}{\pi})^2 sin^2\frac{n\pi}{N}$, and since

$$a^2 = \frac{2n-1}{R^2} \equiv \frac{n^2}{R^2} - \frac{(n-1)^2}{R^2}, \tag{3.7}$$

the new a^2 for F_1 is

$$a^2 = \frac{1}{R^2}(\frac{N}{\pi})^2[sin^2\frac{\pi n}{N} - sin^2\frac{(n-1)\pi}{N}] = \frac{1}{R^2}(\frac{N}{\pi})^2 sin\frac{\pi}{N} sin\frac{(2n-1)\pi}{N}. \tag{3.8}$$

The final result for $n \geq 1$ is

$$\begin{aligned} T^{(2)}_{nn}(K) \quad &= \quad -\frac{1}{2\pi^2}F_1(k;a^2)\mid_{a^2=\frac{1}{R^2}(\frac{N}{\pi})^2 sin\frac{\pi}{N} sin\frac{(2n-1)\pi}{N}} \\ &\quad -\frac{1}{2\pi^2}F_2(k;a^2)\mid_{a^2=\frac{1}{R^2}(\frac{N}{\pi})^2 sin\frac{\pi}{N} sin\frac{(2n+1)\pi}{N}} \ . \end{aligned} \tag{3.9}$$

Here, F_1 and F_2 are given by (2.6) and the result for F_1 explicitly given by (2.7). Again we have a pole in $T^{(2)}_{nn}, n \geq 1$, at

$$k = +i\frac{a^2+\mu^2}{2\mu}, \tag{3.10}$$

but with a^2 given in (3.8).

The only condition we need to satisfy is:

$$\frac{1}{R^2}(\frac{\pi}{N})^2 sin\frac{\pi}{N} sin\frac{(2n-1)\pi}{N} > \mu^2. \tag{3.11}$$

Next we expand the wave functions $\Psi(\vec{r}, j)$ in terms of the internal eigenstates of the internal momentum operator, $\frac{1}{i}\partial/\partial\phi_j$, which are given by $\frac{1}{\sqrt{N}}\exp(\frac{i2\pi nj}{N})$, for internal state, n. We define $\Phi_n(\vec{r})$ by

$$\Psi(\vec{r}; j) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\Phi_n(\vec{r})e^{\frac{i2\pi jn}{N}}, \tag{3.12}$$

where now $\Phi_n(\vec{r})$ represents a wave function in a specific internal eigenstate, n. Inverting the sum in (3.12) we get

$$\Phi_n(\vec{r}) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\Psi(\vec{r}; j)e^{\frac{-i2\pi jn}{N}}. \tag{3.13}$$

Multiplying the Eq. (3.3) by $\frac{1}{\sqrt{N}}e^{\frac{i2\pi jn}{N}}$ and summing over j, we get

$$[\nabla^2 + K^2 - \frac{1}{R^2}(\frac{N}{\pi})^2\sin^2\frac{\pi n}{N}]\Phi_n(\vec{r}) - \sum_{n'=0}^{N-1}U(r, n-n')\Phi_{n'}(\vec{r}) = 0, \tag{3.14}$$

where

$$U(r, n) \equiv \frac{1}{N}\sum_{j-0}^{N-1}V(r, j)e^{\frac{i2\pi jn}{N}}, \tag{3.15}$$

and

$$U(r, n) = U(r, n+N). \tag{3.16}$$

Finally, in ref. 1 we used the explicit example where

$$V(r, \phi) = 2\frac{e^{-\mu r}}{r}\cos\phi \tag{3.17}$$

We can replace this by

$$V(r, j) = 2\frac{e^{-\mu r}}{r}\cos\frac{2\pi j}{N} \tag{3.18}$$

Using Eq. (3.16) this gives

$$U(r,n) = \frac{e^{-\mu r}}{r}(\delta_{n,1} + \delta_{n,-1}). \tag{3.19}$$

The phenomenon of reference 1 is already present for $N = 2$, and the Schrodinger equation in that case is

$$\begin{pmatrix} \nabla^2 + K^2 & \frac{-e^{-\mu r}}{r} \\ -\frac{e^{-\mu r}}{r} & \nabla^2 + K^2 - \frac{1}{R^2}(\frac{2}{\pi})^2 \end{pmatrix} \begin{pmatrix} \Phi_1(r) \\ \Phi_2(r) \end{pmatrix} = 0. \tag{3.20}$$

This is just a 2×2 coupled channel problem with $E_1 = K^2$, $E_2 = k^2$, and $K^2 = k^2 + a^2$, and in this case $a^2 = \frac{1}{R^2}(\frac{2}{\pi})^2$.

IV The Two Channel Problem

We consider the following two channel Hamiltonian,

$$H = \begin{pmatrix} -\nabla^2 & V(r) \\ V(r) & -\nabla^2 + a^2 \end{pmatrix} \tag{4.1}$$

where a is real and fixed. The free eigenstates of H and the internal state, with momentum, $\vec{k}$, are

$$\phi_1 = \begin{pmatrix} e^{i\vec{k}\cdot\vec{x}} \\ 0 \end{pmatrix} \ ; \qquad \phi_2 = \begin{pmatrix} 0 \\ e^{i\vec{k}\cdot\vec{x}} \end{pmatrix} \ ; \tag{4.2}$$

with energies $E_1 = k^2$, and $E_2 = K^2 = k^2 + a^2$. The fact that no potential terms appear along the diagonal in Eq. (4.1) is not relevant to the rest of this paper. One could easily add $U_{11}(r)$ and $U_{22}(r)$ without affecting the main result.

For this problem there are two forward scattering amplitudes, $T_{11}(K)$ and $T_{22}(K)$, with $\sqrt{E}=K$.

To demonstrate that the phenomenon of ref. 1 already exists in this simple problem, we will calculate explicitly $T_{22}^{(2)}(K)$ in second order perturbation theory for the case

$$V(r)=e^{-\mu r}/r. \tag{4.3}$$

We will find later that $T_{11}(K)$ has no unusual singularities for $ImK>0$. The free Green's function associated with H defined in Eq. (4.1) is, for energy $E=K^2$,

$$G(K)=\begin{pmatrix} G_o(K) & 0 \\ 0 & G_o(k), \end{pmatrix} \tag{4.4}$$

where $(H-K^2)G(K;\vec{x}-\vec{y})=\delta^3(\vec{x}-\vec{y})$, and $G_o(q)=-\frac{1}{4\pi}e^{iq|\vec{x}-\vec{y}|}/\mid\vec{x}-\vec{y}\mid$.

For forward scattering from an initial state $\begin{pmatrix} 0 \\ e^{i\vec{k}\cdot\vec{x}} \end{pmatrix}$ to the same outgoing state we have

$$T_{22}^{(2)}(K)=\frac{1}{\pi}\int d^3x\int d^3x' e^{-i\vec{k}.\vec{x}}V(x)G_o(K;\mid\vec{x}-\vec{x}'\mid)V(x')e^{i\vec{k}.\vec{x}'}. \tag{4.5}$$

The off diagonal nature of V makes $G_o(K)=G_{11}$ appear in $T_{22}^{(2)}$ and not $G_o(k)=G_{22}$. This is the main feature leading to the new singularity.

Going to momentum space. we obtain

$$T_{22}^{(2)}=\frac{-1}{2\pi^2}F(K), \tag{4.6}$$

where $F(K)$ are given by

$$F(K)=\int d^3p\frac{1}{(p^2+\mu)^2[(\vec{p}+\vec{k})^2-K^2-i\epsilon]}. \tag{4.7}$$

This is represented by figure 1. With a change of variables, this is identical to the expression for F_1 in Eq. (2.6), with $K^2 = k^2 + a^2$. For k and a real, this integral was calculated in reference 1, and is the same as Eq. (2.7), i.e.

$$F(K) = \frac{\pi^2 i}{4k\mu^2}\left[\frac{\sqrt{k^2+a^2}+k-i\mu}{k-i(a^2+\mu^2)/2\mu} + \frac{\sqrt{k^2+a^2}-k-i\mu}{k+i(a^2+\mu^2)/2\mu}\right]. \tag{4.8}$$

With $a^2 > \mu^2$ we can now continue F into the region $Imk > 0$. There is again the pole at $k = +i[(a^2+\mu^2)/2\mu]$. The residue of this pole does not vanish, since

$$\sqrt{k^2+a^2}\mid_{pole} = \sqrt{-\frac{(a^2+\mu^2)}{4\mu^2}+a^2} = i\frac{(a^2-\mu^2)}{2\mu}. \tag{4.9}$$

and,

$$[\sqrt{k^2+a^2}+k-i\mu]_{pole} = i(\frac{a^2}{\mu}-\mu) \neq 0. \tag{4.10}$$

F has also two branch points at $k = \pm ia$. These must be joined by a branch cut, since we know a priori that F is analytic for large $\mid k \mid$, $Imk > 0$. Finally, with $a^2 > \mu^2$,

$$\frac{a^2+\mu^2}{2\mu} > a, \tag{4.11}$$

and the pole is above the branch point at $k = +ia$. Using the variable, K, $E = K^2$, we get

$${K_{pole}}^2 = -\frac{(a^2+\mu^2)^2}{4\mu^2} + a^2 = \frac{-(a^2-\mu^2)^2}{4\mu^2} \tag{4.12}$$

and

$$K_{pole} = +i\frac{(a^2-\mu^2)}{2\mu} \tag{4.13}$$

For $a^2 > \mu^2$; the r.h.s. of (4.13) is positive and the pole is on the physical energy sheet. The calculation of $T_{11}^{(2)}$ is straightforward and leads to

$$T_{11}^{(2)} = \frac{-\pi^2}{4k\mu}[1 + \frac{-i\mu}{k + i\frac{\mu}{2}}] \tag{4.14}$$

The only pole is at $k = -i\mu/2$, on the unphysical sheet.

Finally, we should stress the obvious fact that only for a pure Yukawa,$V(r)$, do we get a simple pole. If we choose $V(r) = \int_{\mu_o}^{\infty} C(\mu)e^{-\mu r} d\mu$, i.e. a superposition of Yukawa, then the pole becomes a branch cut but still on the physical sheet.

V Remarks

The results of this paper present us with two problems, one mathematical and the other physical.

The mathematical question concerns this new singularity and whether it appears in the full amplitude, not just in second order. This has not yet been accomplished. However, in the accompanying paper we study the fourth order two channel problem. We find that a singularity appears at the same point $k = +i(a^2 + \mu^2)/2\mu$, and its strength is the same as a pole. The position of the singularity remains unchanged in fourth-order. This suggests strongly that the full forward amplitude indeed has a singularity located at $k = +i(a^2 + \mu^2)/2\mu$.

The second question relates to the relevance of our result to physics. Is it just a mathematical oddity, or does there exist real physical models where our result manifests itself? The nuclear physicists deal normally with the situation where $a^2 < \mu^2$ not $a^2 > \mu^2$. In other words, they deal with coupled channels whose energy difference is small compared to the mass of the pion. The splitting of the levels is produced by the same forces and cannot be too large. A more promising approach is to look for a coupled channel problem where the channel splitting is generated by an external factor. For example, in atomic physics, one could look for a situation where the splitting is enhanced by a strong magnetic field while the potentials are essentially unchanged by the field. There also could be such examples in condensed matter physics. At present we have no concrete examples, but we are continuing our search.

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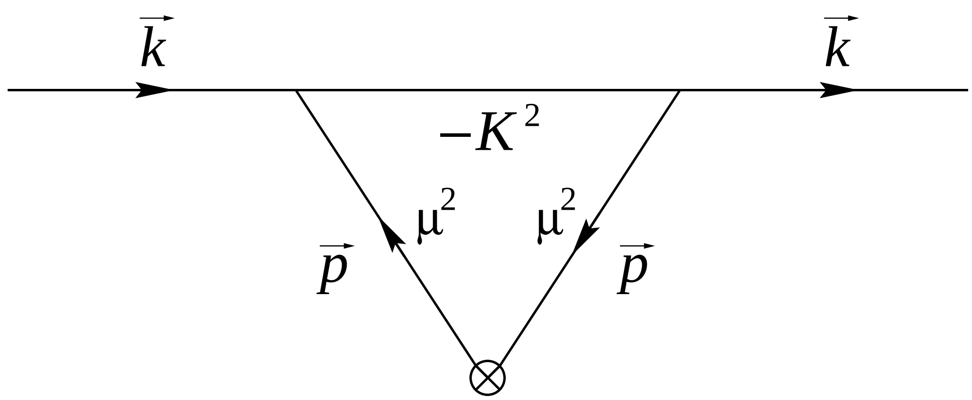

Figure 1: Feynman diagram for the second order amplitude. The propagator for the horizontal internal line has imaginary mass $= -iK = -i\sqrt{k^2 + a^2}$.