Superselection Sectors in Asymptotic Quantization of Gravity

Alfredo E. Dominguez*, Carlos N. Kozameh†

FaMAF, Universidad Nacional de Córdoba, (5000) Cordoba, Argentina.

Malcolm Ludvigsen‡

Department of Mathematics Linkoping University S-581 83 Linkoping

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Abstract

Using the continuity of the scalar Ψ_2 (the mass aspect) at null infinity through i_o we show that the space of radiative solutions of general relativity can be thought of a fibered space where the value of Ψ_2 at i_o plays the role of the base space. We also show that the restriction of the available symplectic form to each "fiber" is degenerate. By finding the orbit manifold of this degenerate direction we obtain the reduced phase space for the radiation data. This reduced phase space posses a global structure, i.e., it does not distinguishes

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*e-mail: domingue@fis.uncor.edu

†e-mail: kozameh@fis.uncor.edu

‡e-mail: malud@mai.liu.se

between future or past null infinity. Thus, it can be used as the space of quantum gravitons. Moreover, a Hilbert space can be constructed on each "fiber" if an appropriate definition of scalar product is provided. Since there is no natural correspondence between the Hilbert spaces of different foliations they define superselection sectors on the space of asymptotic quantum states.

We discuss the physical relevance of the superselection sectors and show that the analogous construction for linearized gravity yields completely different results, thus emphasizing the need to use the full nonlinearity of the theory even when discussing asymptotic quantization.

I. INTRODUCTION

The Null Surface Formalism (NSF) shows that General Relativity can be viewed as a theory of hypersurfaces on a 4-dim manifold rather than a field theory for a metric with lorentzian signature. At a kinematic level NSF shows that if two (complex) conditions are imposed on these surfaces they become null hypersurfaces of a given metric. Field equations equivalent to the vacuum Einstein equations determine the dynamics of these characteristic hypersurfaces [1].

Within this formalism it is also possible to distinguish radiative solutions of the vacuum equations. It can be shown that the Bondi free data at $\mathcal{I}$ enters as a source term in the NSF field equations for null surfaces that are asymptotic planes at null infinity. Thus, for each Bondi data, the regular solutions to the field equations represent global null surfaces of an asymptotically flat, vacuum metric [1].

Recently, a paper extending the NSF to the quantum level was presented [2]. The starting point in that work is the (classical) field equation that yields global null surfaces, i.e. null surfaces associated with radiative solutions. Adopting Ashtekar's asymptotic quantization procedure [3], the Bondi free data of the NSF equations is promoted to a quantum operator that obeys commutation relations at null infinity. It then follows from the field equations that the null surfaces themselves become operators that obey non-trivial commutation relations [2]. Furthermore, since it is possible to identify points of the space-time as intersections of null surfaces, it can also be shown that the space-time points themselves become quantum operators.

There are however, technical difficulties in trying to construct a physically relevant Hilbert and associated Fock space of incoming or outgoing gravitons where these operators could act. As shown by Ashtekar, it is possible to define a Hilbert space at null infinity but doing so imposes severe restrictions on the free data of the associated phase space since one leaves out almost all physically relevant spaces [3].

In this paper we analyse again the phase space of asymptotic states. We show that, using an exact conservation law, it is possible to foliate this space in sectors that admit non-trivial scattering at a classical level. Furthermore, we also show that the restriction of the symplectic form to each foliation is degenerate and that a Hilbert space can be constructed in each sector if one factors out the degenerate direction.

In Section II we first review some results obtained in the context of asymptotically flat space-times and present a theorem that is very important for the main result of this work.

In Section III we define the phase space of radiative modes and introduce the notion of global structures on this space. We show that

1. the induced symplectic form on $\mathcal{I}$, and

2. a foliation on the phase space adopting the value of ψ_2 at i_o as the "base" space

are global structures.

In Section IV we show that the restricted symplectic form to each foliation is degenerate. We study the degeneracy direction and obtain the orbit space associated with this direction. We introduce a complex structure on the tangent space to each fibre and construct, via the symplectic form, a global, positive definite inner product.

Finally, in Section V we review the main results of this work, and discuss possible generalizations. The application of this formalism to the Maxwell field is given in the appendix.

II. RADIATIVE SPACETIMES

We define a radiative spacetime M to be a solution of the empty Einstein equations which is asymptotically flat at both future and past null-infinity and suitably regular at spacelike and time-like infinity. We also require that it contains no horizons and is topologically trivial in the sense that there exists a global coordinate system. This condition eliminates, for example, the Schwarzschild solution.

A word of caution is appropriate here: At the present moment no *explicit* solution of Einstein's equations satisfying these conditions has been found, and even the existence of such solutions has yet to be rigourously demonstrated. However, recent results, namely the Null Surface Formalism, though not in its final form, does indicate that radiative spacetimes do exist and that they can actually be constructed from a single function, σ, which can be taken to be the asymptotic shear on future null-infinity $\mathcal{I}^+$ (or $\mathcal{I}^-$). By taking σ to be the asymptotic shear on $\mathcal{I}^+$ with a Bondi scaling, this construction determines a radiative solution unique up to diffeomorphisms which preserve the structure at null-infinity.

Another difficulty concerns the regularity conditions at space-like infinity imposed in [6]. Although it would appear that these conditions are too restrictive for any physical spacetime to be included in this class, it has been shown by M. Herberthson [7] that the conditions can be relaxed and still allow sufficient regularity for the results of [6] to be valid. This class includes all known asymptotically flat spacetimes.

The physical interpretation of a radiative spacetime is that of a classical scattering event involving pure gravitational radiation, the in-state being defined by σ^- on $\mathcal{I}^-$ and the corresponding out-state by σ^+ on $\mathcal{I}^+$.

Asymptotic flatness at null and space-like infinity implies the existence of an extended conformally related space containing a null cone representing points at infinity. Its vertex i^o represents space-like infinity and its future and past parts, $\mathcal{I}^+$ and $\mathcal{I}^-$, past and future null infinity. Except for flat spacetime, the point i^o is not smoothly attached but has a direction dependent structure. This, however, allows the introduction of a stereographic coordinate function ζ on $I = I^+ \cup I^-$ which is constant along the null generators of I and continuous through i^o. This function effectively provides a one-to-one correspondence between the generators of $\mathcal{I}^+$ and $\mathcal{I}^-$.

In terms of this type of scaling, $\mathcal{I}^+$ is diverging and $\mathcal{I}^-$ is converging. A Bondi scaling, on the other hand, makes $\mathcal{I}^+$ and $\mathcal{I}^-$ divergence free and thus i^o becomes an infinitely removed point. Given a stereographic coordinate function ζ on I we introduce a particular Bondi

conformal factor Ω such that any space-like slice of I has an induced metric of the form

$$ds^2 = \frac{d\zeta d\bar{\zeta}}{P^2}$$

where $2P = 1 + \zeta\bar{\zeta}$. This is possible because all space-like slices are isometric by virtue of the divergence-free condition. On I the vector $n_a = -\nabla_a\Omega$ is null, non-zero, and points along the generators. On $\mathcal{I}^+$ it is future pointing and on $\mathcal{I}^-$ it is past pointing.

We now introduce a Bondi parameter function u on I satisfying $n^a\nabla_a u = 1$ which determines u up to

$$u \to u + \gamma$$

where γ is constant along the generators. Since I consists of two disconnected components, $\gamma = (\gamma^+, \gamma^-)$ where $\gamma^\pm$ are the restrictions of γ to $I^\pm$. These two functions may be chosen independently. Note that on both $\mathcal{I}^+$ and $\mathcal{I}^-$ $u = -\infty$ represents space-like infinity. We now have a global coordinate system $(u, \zeta, \bar{\zeta})$ which labels points on both $\mathcal{I}^+$ and $\mathcal{I}^-$.

Given a Bondi parameter u, we complete n^a to form a null-tetrad $(n^a, m^a, \bar{m}^a, l^a)$ on I by demanding that m^a is tangent the $u = $ constant slices. This determines l^a uniquely on I (it points out of I and is future-pointing on $\mathcal{I}^+$ and past-pointing on $\mathcal{I}^-$) but m^a only up to

$$m^a \to e^{i\lambda}m^a.$$

A tetrad dependent function η on I which transforms according to

$$\eta \to e^{is\lambda}\eta$$

is said to have spin-weight s. Though not strictly necessary, it is convenient to fix m^a by demanding $m^a\nabla_a\bar{\zeta} = 0$.

Two important tetrad dependent functions on I are the shear σ and the mass aspect ψ_2 which are defined by

$$\sigma(u, \zeta, \bar{\zeta}) = m^a m^b \nabla_a l_b \tag{1}$$

$$\psi_2(u, \zeta, \bar{\zeta}) = \Omega^{-1} C_{abcd} n^a \bar{m}^b l^c m^d, \tag{2}$$

Note that σ and ψ_2 have spin-weights 2 and 0 respectively. We demand smoothness in u and regularity in the angular coordinates $(\zeta, \bar{\zeta})$ in the sense that σ and ψ_2 are expandable in terms of the appropriate spin-weighted spherical harmonics. We also demand that the limits $\lim_{u \to \pm \infty} \sigma$ and $\lim_{u \to \pm \infty} \psi_2$ exist. Since I consists of two disconnected components, we have $\sigma = (\sigma^+, \sigma^-)$ where $\sigma^{\pm}$ are the restrictions of σ to $I^{\pm}$, and similarly for ψ_2.

By regularity at time-like infinity we mean

$$\lim_{u \to \infty} \psi_2 = 0 \quad (\lim_{u \to \infty} \psi_2^{\pm} = 0) \tag{3}$$

By an extension of the positive mass theorem [19] this implies that the initial data on an asymptotically null, space-like hypersurface, which intersects I at $u = $ constant, becomes flat for $u \to \infty$. Since we do not have flatness at space-like infinity

$$\lim_{u \to -\infty} \psi_2 \neq 0 \quad (\lim_{u \to -\infty} \psi_2^{\pm} \neq 0). \tag{4}$$

However, the results of [6] imply

$$\lim_{u \to -\infty} \psi_2^+ = \lim_{u \to -\infty} \psi_2^- = \chi(\zeta, \bar{\zeta}). \tag{5}$$

This equation is important since it provides a link between $\mathcal{I}^+$ and $\mathcal{I}^-$.

On I we have the spin-coefficient relations [17]:

$$\dot{\psi}_2 = -\eth^2 \dot{\bar{\sigma}} - \sigma \ddot{\bar{\sigma}} \tag{6}$$

and

$$\psi_2 - \bar{\psi}_2 = \bar{\sigma}\dot{\sigma} + \bar{\eth}^2 \sigma - c.c \tag{7}$$

Using eqs (7) and (3) we have

$$\lim_{u \to \infty} (\bar{\eth}^2 \sigma - \eth^2 \bar{\sigma}) = 0$$

which implies the existence of a Bondi frame (frames) such that

$$\lim_{u \to \infty} \sigma = 0 \quad (\lim_{u \to \infty} \sigma^{\pm} = 0) \tag{8}$$

7

Using equations (6), (8) and (5) we now have

$$\chi = \bar{\eth}^2 \sigma_o^{\pm} + \int_{-\infty}^{\infty} \dot{\sigma}^{\pm} \dot{\bar{\sigma}}^{\pm} du \tag{9}$$

where $\sigma_o^{\pm} = \lim_{u \to -\infty} \sigma^{\pm}$. This provides a very important relation between in-states described by σ^- and out-states described by σ^+. Classical scattering sends in-states to out-states associated with the same function χ. We exploit this feature in the next section to foliate the reduced phase space of radiative solutions.

Given a real function $\kappa(\zeta, \bar{\zeta})$ equation (9) gives

$$\chi_{\kappa} = \sigma_{\kappa}^{\pm} + P_{\kappa}^{\pm} \tag{10}$$

where

$$\sigma_{\kappa}^{\pm} = \int \kappa \bar{\eth}^2 \sigma_o^{\pm} dS \tag{11}$$

and

$$P_{\kappa}^{\pm} = \int_{I^{\pm}} \kappa \dot{\sigma} \dot{\bar{\sigma}} dI. \tag{12}$$

Here dS is the area element on the $u =$ constant slices and $dI = dSdu$. $P_{\kappa}^{\pm}$ is the total flux of super momentum through $\mathcal{I}^{\pm}$. If κ satisfies

$$\bar{\eth}^2 \kappa = 0 \tag{13}$$

$P_{\kappa}^{\pm}$ reduces to the total flux of Bondi momentum (in the direction defined by κ) through $I^{\pm}$ and, by integrating by parts, (10) gives the asymptotic conservation law

$$P_{\kappa}^{+} = P_{\kappa}^{-}. \tag{14}$$

It is interesting to consider the outline of another proof of this conservation law which brings out the necessity of condition (13) in a more geometrical way. By using a variation of the Ludvigsen-Vicker's proof of the positivity of Bondi mass [19], it can be shown that

there exists a spinor field κ_A in the interior of a radiative spacetime which induces a super-translation κ on I and an exact 3-form j_κ in the interior such that

$$P_\kappa^\pm = \int_{I^\pm} j_\kappa.$$

This does not lead directly to a conservation law $P_\kappa^+ = P_\kappa^-$ since j_κ is in general singular at i^o. However, in the special case where κ is a translation in that it satisfies (13), j_κ becomes sufficiently regular at i^o for Stokes' theorem to hold and this leads to the conservation law $P_\kappa^+ = P_\kappa^-$ for the total flux of Bondi momentum. The only way to obtain a conservation law for super momentum is to assume $j_\kappa = 0$ at i^o but this is true only for flat space. Neverthe-less, an interesting result for super momentum can be obtained for first-order perturbations of a radiative spacetime. In this case a perturbation can be chosen which does not affect space-like infinity in the sense that $\delta\chi = 0$. This gives $\delta j_\kappa = 0$ at i^o and a direct application of Stokes' theorem gives the perturbative conservation law for super momentum:

$$\delta P_\kappa^+ = \delta P_\kappa^-. \tag{15}$$

Combining this with equation (10) we obtain the following important result:

Lemma: *Given a perturbation such that $\delta\chi = 0$ and $\delta\sigma_o^- = 0$, then the same perturbation when propagated to $\mathcal{I}^+$ gives $\delta\sigma_o^+ = 0$.*

III. PHASE SPACE OF RADIATIVE STATES

We define the non-reduced phase space of radiative states to be the set Σ of all pairs $\sigma = (\sigma^+, \sigma^-)$ where σ^+ and σ^- are shear functions 'joined' by some radiative spacetime in the sense of the previous section. The reduced phase space can be obtained by factoring out the physically irrelevant structure provided by the choice of the coordinate functions u and ζ. This gives a space where each point corresponds to a radiative spacetime unique up to diffeomorphisms. Since σ transforms quite simply under a change of Bondi frame, such

9

a reduction can easily be obtained, but for the sake of simplicity we shall content ourselves with the non-reduced phase space Σ.

The Bondi shears σ^- and σ^+ determine an in-state and out-state respectively. By construction they satisfy

$$\lim_{u \to \infty} \sigma^{\pm} = 0,$$

together with the smoothness and regularity properties stated in the previous section. A given in-state σ^- determines a corresponding out-state σ^+ up to a BMS translation. Apart from this, all we know about the relation between σ^- and σ^+ is that given by equation (9). This provides a geometrically determined foliation on Σ where two points lie in the same leaf if they determine the same function $\chi(\zeta, \bar{\zeta})$. We thus have as many equivalence classes on Σ as there are regular complex functions on S^2.

A natural question to ask is whether σ^- may be chosen freely subject to the conditions already stated. The Null Surface Formalism shows that this is actually the case: given any function $\sigma^-(u, \zeta, \bar{\zeta})$ satisfying these conditions, a radiative spacetime can, in principle, be constructed together with a Bondi frame $(u, \zeta, \bar{\zeta})$ such that $\sigma^-(u, \zeta, \bar{\zeta})$ is its past shear.

By time-reversal symmetry, we see that $(\sigma^+, \sigma^-) \in \Sigma$ implies that $(\sigma^-, \sigma^+) \in \Sigma$. A geometrical structure on Σ which does not depend on a preference between σ^+ and σ^- will be said to be global. Global structures are particularly important as regards full quantization as apposed to asymptotic quantization. The foliation defined by the function χ is, for example, a global structure in this sense. The subspace of Σ defined by

$$\lim_{u \to -\infty} \sigma^- = \sigma_o^- = 0$$

is not, however, a global structure since this condition does not imply

$$\lim_{u \to -\infty} \sigma^+ = \sigma_o^+ = 0.$$

To obtain another type of global structure, namely a symplectic form, we must consider the space $T_\sigma(\Sigma)$ of tangent vectors a some point $\sigma \in \Sigma$. A tangent vector at σ (corresponding

to a spacetime M) may be viewed as a perturbation $\delta\sigma = (\delta\sigma^+, \delta\sigma^-)$ where $\sigma + \delta\sigma^+$ and $\sigma + \delta\sigma^-$ are joined by a perturbed space $M + \delta M$. Starting from the standard symplectic form defined on a Cauchy surface and using a conserved current in the interior of the spacetime M it is possible to show that [8]

$$
\begin{aligned}
\Omega(\delta\sigma_1, \delta\sigma_2) &= \int_{I^-} \mathrm{d}^3 I \ [\delta\sigma_1^- \delta\dot{\bar{\sigma}}_2^- - \delta\dot{\bar{\sigma}}_1^- \delta\sigma_2^-] + \ c.c. \\
&= \int_{I^+} \mathrm{d}^3 I \ [\delta\sigma_1^+ \delta\dot{\bar{\sigma}}_2^+ - \delta\dot{\bar{\sigma}}_1^+ \delta\sigma_2^+] + \ c.c.,
\end{aligned}
\tag{16}
$$

This defines a global, non-degenerate, symplectic form on Σ.

Our phase space Σ is thus a foliated, symplectic space. The symplectic form and foliation are preserved under reduction, where a point in the reduced space is now an equivalence class $\hat{\sigma}$ of related σ's and the foliation is determined by an equivalence class $\hat{\chi}$ of related χ's.

IV. TANGENT VECTOR SPACES

We shall now restrict attention to tangent vectors $\delta\sigma$ such that

$$
\lim_{u \to -\infty} \delta\sigma^- = 0.
\tag{17}
$$

We emphasise that this is not a global geometric condition since it does not imply

$$
\lim_{u \to -\infty} \delta\sigma^+ = 0.
\tag{18}
$$

but it is necessary for the construction of a hilbert space, which is normally the first step towards quantization. Our basic idea is to find a subspace of $T_\sigma(\Sigma)$ (subject to this condition) which is global in that both (17) and (18) are satisfied, and which admits a natural hilbert-space structure.

Let us first see how a natural, but non-global, hilbert-space structure can be defined on $T_\sigma(\Sigma)$ when subject to condition (17). By construction we have $\lim_{u \to \infty} \delta\sigma^- = 0$ and thus $\delta\sigma^-$ tends to zero at both ends of $\mathcal{I}^-$. This means that $\delta\sigma^-$ admits a Fourier decomposition with respect to u and, by means of this, we can find the positive and negative frequency

11

parts, $\delta\sigma_{pos}^-$ and $\delta\sigma_{neg}^-$, of $\delta\sigma^-$. The complex structure J corresponding to this decomposition is defined by

$$J\delta\sigma = ((J\delta\sigma)^+, (J\delta\sigma)^-) \tag{19}$$

where

$$(J\delta\sigma)^- = i(\delta\sigma_{pos}^- - \delta\sigma_{neg}^-) \tag{20}$$

With respect to this complex structure, multiplication by a complex number $z = x + iy$ is defined by

$$z\delta\sigma = x\delta\sigma + yJ\delta\sigma.$$

It can easily be seen that J is compatible with Ω in that

$$\Omega(J\delta\sigma_1, J\delta\sigma_2) = \Omega(\delta\sigma_1, \delta\sigma_2)$$

and positive in that

$$\Omega(\delta\sigma, J\delta\sigma) > 0$$

for non-trivial $\delta\sigma$. From these results we see that

$$\langle \delta\sigma_1, \delta\sigma_2 \rangle = \Omega(\delta\sigma_1, J\delta\sigma_2) + i\Omega(\delta\sigma_1, \delta\sigma_2) \tag{21}$$

is a positive-definite, non-degenerate hermitian product, i.e.,

$$\langle \delta\sigma_1, \delta\sigma_2 \rangle = \overline{\langle \delta\sigma_2, \delta\sigma_1 \rangle} \tag{22}$$

$$\langle \delta\sigma, z_1\delta\sigma_1 + z_2\delta\sigma_2 \rangle = z_1\langle \delta\sigma, \delta\sigma_1 \rangle + z_2\langle \delta\sigma, \delta\sigma_2 \rangle \tag{23}$$

$$\langle \delta\sigma, \delta\sigma \rangle > 0 \text{ if } \delta\hat{\sigma} \neq 0 \tag{24}$$

The tangent space $T_\sigma(\Sigma)$ thus has a natural hilbert space structure. This structure is also preserved under coordinate reduction but, as we have already pointed out, it is not global in that it is defined with respect to $\mathcal{I}^-$.

It is interesting to note at this point that $T_\sigma(\Sigma)$ possesses an even more natural complex structure defined simply by $J = i$. This is a rather curious fact because the basic elements we

12

are dealing with are radiative spacetimes which are essentially real objects, and yet we end up with a complex vector space, in fact a complex vector space with two complex structures.

By the final result in section (II) we see that the subspace H of $T_\sigma(\Sigma)$ consisting of vectors which satisfy $\delta\chi = 0$, and which therefore lie in the leaf containing σ, is global in that $\delta\sigma_o^- = 0$ implies $\delta\sigma_o^+ = 0$. From equation (9) we see that an element $\delta\sigma$ of $T_\sigma(\Sigma)$ is contained in H iff

$$\alpha(\delta\sigma) \equiv \int_{-\infty}^{\infty} (\delta\sigma\ddot{\bar{\sigma}} + \delta\bar{\sigma}\ddot{\sigma})du = 0. \tag{25}$$

The space H contains, in turn, a preferred subspace K consisting of vectors $\delta\sigma_k$ such that $\delta\sigma_k^- = \kappa\dot{\sigma}^-$ where κ is real and $\dot{\kappa} = 0$. Using (16) it can easily be checked that condition (25) is equivalent to

$$H = \{\delta\sigma : \Omega(\delta\sigma, \delta\sigma_k) = 0, \ \forall \, \delta\sigma_k \in K\}. \tag{26}$$

This defines H in terms of K, thus showing that K is global in spite of the fact that it is defined with respect to $\mathcal{I}^-$, and also shows that K contains all directions of degeneracy of Ω when restricted to H. [A vector $\delta\sigma_k$ is a direction of degeneracy of Ω restricted to H if $\Omega(\delta\sigma, \delta\sigma_k) = 0$ for all $\delta\sigma \in H$.]

In order to obtain a space with a non-degenerate symplectic product we factor out the directions of degeneracy and consider the space $\hat{H}$ of equivalence classes where $\delta\sigma$ and $\delta\sigma'$ belong to the same equivalence class $\delta\hat{\sigma}$ if $\delta\sigma - \delta\sigma' \in K$.

The symplectic product defined by

$$\hat{\Omega}(\delta\hat{\sigma}_1, \delta\hat{\sigma}_2) = \Omega(\delta\sigma_1, \delta\sigma_2)$$

is non-degenerate on $\hat{H}$.

To understand the physical meaning of this reduced phase space we study the integral lines of $\delta\sigma_k$, which are obtained by introducing a parameter s and solving

$$\frac{d\sigma_k}{ds} = \kappa\frac{d\sigma}{du}, \tag{27}$$

whose solution is given by

$$\sigma_s(u, \zeta, \bar{\zeta}) = \sigma(u + \kappa s, \zeta, \bar{\zeta}). \tag{28}$$

Thus, for each value of $(\zeta, \bar{\zeta})$ we have to factor out Bondi data that are supertranslations of a given function $\sigma(u, \zeta, \bar{\zeta})$ by an arbitrary distance along the timelike direction u.

The manifold of orbits of $\delta\sigma_k$ is the reduced phase space $\hat{\Sigma}$ and $\hat{H}$ its associated tangent space.

The complex structure J does not induce a complex structure on H since $\delta\sigma \in H$ does not imply $J\delta\sigma \in H$, or, equivalently, $\alpha(\delta\sigma) = 0$ does not imply $\alpha(J\delta\sigma) = 0$. In particular, we have

$$\alpha(J\delta\sigma_o) \neq 0 \tag{29}$$

for all non-trivial $\delta\sigma_o \in K$. Fortunately, we are not so much interested in H as its reduced version $\hat{H}$ consisting of equivalence classes of H. Using (29), we see that each such equivalence $\delta\hat{\sigma}$ class contains a unique representative $\delta\sigma'$ such that $\alpha(J\delta\sigma') = 0$ and hence $J\delta\sigma' \in H$. Since the space of all such vectors is clearly equivalent to $\hat{H}$ we see that J induces a complex structure $\hat{J}$ on $\hat{H}$. It is easily checked that $\hat{J}$ is compatible with $\hat{\Omega}$ and positive. It thus induces a hibert space structure on $\hat{H}$.

The observant reader will have noticed that a similar construction based on $\mathcal{I}^+$ rather than $\mathcal{I}^-$ leads to another compatible, positive, complex structure $\hat{J}'$ on $(\hat{H}, \hat{\Omega})$. However, since $\hat{H}$ and $\hat{\Omega}$ are globally defined in the sense that their definition is independent of the choice between $\mathcal{I}^+$ and $\mathcal{I}^-$, it seems reasonable to conjecture that $\hat{J} = \hat{J}'$. Whether or not this is true, we have at least shown the *existence* of a compatible, positive, complex structure on $(\hat{H}, \hat{\Omega})$.

V. SUMMARY AND CONCLUSIONS

We have shown that the phase space of the radiative degrees of freedom of General Relativity can be foliated using the continuity of the mass aspect (ψ_2) through i_o. It was

14

also shown that this foliation is a well defined global structure at $\mathcal{I}$

We have then shown that the induced symplectic form on each foliation is degenerate along a direction that represents a translation of the free data σ along each generator of $\mathcal{I}$. By factoring out this degenerate direction one obtains the restricted phase-space $\hat{\Sigma}$ associated with each foliation that also has a global meaning

Introducing a complex structure J one then defines an inner product on $\hat{H}$ that has a finite norm. This 1-graviton Hilbert space so constructed is the building block for the Fock space associated with each foliation. Since there is no natural relation between Fock spaces for different "fibres" this quantization procedure defines superselection sectors on the full phase space.

As we will show in the appendix, if we repeat the same construction for free Maxwell fields in flat space-time one can also find foliations associated with the continuity of a Maxwell scalar (ϕ_1) through i_o. However, the induced symplectic form on each foliation is now non-degenerate and thus, the procedure to construct a Hilbert space follows a different approach.

The differences between the Maxwell case and General Relativity has profound implications. If we had linearized the gravitational field equations to construct the asymptotic phase space we would have followed a similar approach to the Maxwell case and we would have missed the fact that the induced symplectic form on each foliation for full gravity is degenerate. The lesson being learned here is that the superselection sectors for linearized gravity are manifestly different from the ones that arise in the full theory.

At this time one could ask two questions:

1. Is this superselection rule physically significant?

2. Does the quantum S-matrix preserve the superselection sectors?

In Maxwell theory each superselection sector is associated with quantization of the radiation field in the presence of a classical distribution of electromagnetic charges. This

meaning extends to QED if we replace the classical distribution of charges with the density distribution of the Dirac field.

Likewise, in GR we associate superselection sectors with quantization of the fluctuations of a classical distribution of mass. It is not clear if a full quantum theory of gravity will admit superselection sectors, though the mass operator should be a conserved quantity.

Although a quantum theory is needed to construct the S-matrix and thus answer the second question, the answer is positive at a classical level since we have shown that these foliations are global structures. Thus, classical scattering preserves each foliation and the pushforward map sends tangent vectors to a foliation at $\mathcal{I}^-$ to tangent vectors to the corresponding foliation at $\mathcal{I}^+$. If the quantum S-matrix takes a vector belonging to a particular foliation at $\mathcal{I}^-$ and produces a vector that is not tangent to the same foliation at $\mathcal{I}^+$ then the S-matrix will fail to be unitary since in general a vector at $\mathcal{I}^+$ that is not tangent to the foliation will not have a finite norm. To make more assertive claims however, one needs the full quantum dynamical evolution.

The idea for future work is to use this kinematic quantization with the quantum NSF. As mentioned in the introduction, this quantization is done at the operator level and we are here introducing the appropriate Fock space where these operators act. The null cone quantization then gives the dynamical evolution of these operators and, by properly taking limits, one can construct the S-matrix of the theory.

APPENDIX A: SUPERSELECTION SECTORS IN MAXWELL THEORY

In this appendix we use the continuity of the scalar ϕ_1 through i_o to foliate the solution space of radiative solutions to the source free Maxwell's equations in Minkowski space. We show that the restricted symplectic form to each foliation is non-degenerate and that it yields a finite norm.

Since we work in the null tetrad formalism, instead of the Maxwell field F_{ab} we use the (complex) scalars

$$\phi_0 = F_{ab}l^a m^b$$
$$\phi_1 = \frac{1}{2}F_{ab}(l^a n^b + \bar{m}^a m^b)$$
$$\phi_2 = F_{ab}\bar{m}^a n^b,$$

where $(l^a, m^a, \bar{m}^a, n^a)$ is a null tetrad adapted to the geometry of compactified Minkowski space with null boundary $\mathcal{I}$.

Assuming the source free Maxwell field has finite energy one can show that the restriction of the scalar ϕ_1 to $\mathcal{I}$ satisfies

$$\lim_{x\to\rangle^+} \phi_1(x) = 0, \quad x \in \mathcal{I}^+$$
$$\lim_{x\to\rangle^-} \phi_1(x) = 0, \quad x \in \mathcal{I}^-. \tag{A.1}$$

Furthermore, a linearization of the approach presented in [6] shows that ϕ_1 is continuous through i_o, i.e.:

$$\lim_{x\to i_o} \phi_1(x) = \lim_{x'\to i_o} \phi_1(x'), \quad x \in \mathcal{I}^- \quad x' \in \mathcal{I}^+ \tag{A.2}$$

The idea is to use this continuity to foliate the solution space of radiative solutions.

Since we are following a similar approach to the gravitational case we will be interested in global structures. We construct the phase space $\mathcal{A}$ for Maxwell theory using the two degrees of freedom of the restricted maxwell potential A_a to $\mathcal{I}$. It can be shown that the complex function

$$A_{\pm} \equiv \lim_{x \to \mathcal{I}^{\pm}} A_a m^a \tag{A.3}$$

(with A_a a potential in the gauge $A_a n^a = 0$ at $\mathcal{I}^{\pm}$) captures the two degrees of freedom of the radiative solutions. As before, for simplicity we work on $\mathcal{I}^-$ and we drop the superscript on the scalars.

We define then the non-reduced phase space of radiation fields to be the set $\mathcal{A}$ of all pairs $A = (A^+, A^-)$ where A^+ and A^- are "joined" by a radiative solution of Maxwell's equations.

Using the relationship between the field and potential one can show that

$$\phi_1|_{\mathcal{I}} = \eth \bar{A}.$$

Furthermore, since the kernel of the $\eth$ operator acting on s.w. 1 functions vanishes, there is a one to one correspondence between ϕ_1 and A and thus, it follows from (A.2) that

$$\lim_{x \to i_o} A(x) = \lim_{x' \to i_o} A(x') = \Xi(\zeta, \bar{\zeta}), \quad x, \quad x' \in \mathcal{I}^- \tag{A.4}$$

As before, we have as many equivalence classes on $\mathcal{A}$ as there are regular complex functions on S^2. This equivalence relation introduces a foliation on $\mathcal{A}$. All A's belonging to a foliation are labelled by the function $\Xi(\zeta, \bar{\zeta})$.

We denote by $T_A(\mathcal{A})$ the tangent space at a point A and by $\delta A = (\delta A^+, \delta A^-)$ a tangent vector on this space. It can be easily shown that from (A.4) and the linearity of Maxwell's equations, any tangent vector δA is global. However, since our goal is to construct a Hilbert space on $T_A(\mathcal{A})$, we will only consider vectors such that $\delta \Xi = 0$ (only these vectors admit a Fourier decomposition along the u direction). On those vectors $\delta \hat{A}$ we introduce the following complex structure

$$J\delta\hat{A} = ((J\delta\hat{A})^+, (J\delta\hat{A})^-) \tag{A.5}$$

where

$$(J\delta\hat{A})^- = i(\delta\hat{A}^-_{pos} - \delta\hat{A}^-_{neg}) \tag{A.6}$$

18

We recall that the symplectic form defined in the canonical formalism induces a global, non-degenerate form on $\mathcal{I}$ given by.

$$\Omega(\delta A^1, \delta A^2) = \int d^3 I \left[\delta A^1 \, \delta \dot{\bar{A}}^2 - \delta \dot{A}^1 \, \delta \bar{A}^2 \right] + \ c.c, \tag{A.7}$$

where δA^1 and δA^2 are tangent vectors in the phase space. The idea now is to restrict the (weakly) non-degenerate symplectic form to each foliation, i.e., we evaluate (A.7) on vectors $\delta \hat{A}$. It follows from (A.4) that these vectors satisfy

$$\lim_{x \to i_o} \delta \hat{A}(x) = 0. \tag{A.8}$$

We want to show that this restricted form is non-degenerate. Assume there exists a $\delta \hat{A}_o$ that satisfies (A.8) and

$$\Omega(\delta \hat{A}_o, \hat{A}) = 0 \tag{A.9}$$

for all $\delta \hat{A}$'s that belong to $T_{A,\Xi}(\hat{\mathcal{A}})$. Integrating by parts this equation it then follows that

$$\delta \dot{\hat{A}}_o = 0, \tag{A.10}$$

or

$$\delta \hat{A}_o = f(\zeta, \bar{\zeta}), \tag{A.11}$$

which contradicts (A.8).

As before J is compatible with Ω

$$\Omega(J \delta A_1, J \delta A_2) = \Omega(\delta A_1, \delta A_2)$$

and is positive for non-trivial $\delta \sigma$. We thus define the following positive-definite, non-degenerate hermitian product

$$\langle \delta A_1, \delta A_2 \rangle = \Omega(\delta A_1, J \delta A_2) + i \Omega(\delta A_1, \delta A_2) \tag{A.12}$$

It is worth mentioning that vectors δA's that are not tangent to a foliation, i.e., that do not satisfy (A.8) will have infinite norm and thus will not belong to the Hilbert space associated with $\mathcal{A}$. These states belong to the infrared sector and are fully discussed in [3].

On the other hand, all vectors $\delta\hat{A}$ tangent to the foliations have finite norm and one thus defines a Hilbert space for each foliation. Since there is no natural connection between the different Hilbert spaces so constructed we call them superselection sectors.

What is the physical meaning of the superselection sectors?

We first observe that ϕ_1 yields the charge aspect of Maxwell fields. If we allow for the presence of bounded sources then each superselection sector yields the quantized radiation fields associated with a particular charge configuration. This relationship can be extended to full QED by constructing a fibre bundle where a point on the base space corresponds to a Dirac state and the fibre above this point is the corresponding superselection sector.

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