Periods, Coupling Constants and Modular Functions

in N=2 SU(2) SYM with Massive Matter*

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Abstract

We determine the mass dependence of the coupling constant for N=2 SYM with $N_f = 1, 2, 3$ and 4 flavours. All these cases can be unified in one analytic expression, given by a Schwarzian triangle function. Moreover we work out the connection to modular functions which enables us to give explicit formulas for the periods. Using the form of the J–functions we are able to determine in an elegant way the couplings and monodromies at the superconformal points.

1   Introduction

Initiated by the seminal work of Seiberg and Witten [1, 2] N=2 supersymmetric gauge theories have received a lot of interest during the last two years. The moduli space of the Coulomb branch characterized by the scalar fields of N=2 vector multiplets and the masses of the matter fields

receive quantum corrections which are fully under control. The Wilsonian effective action is completely determined in terms of certain elliptic curves. Although these theories can be treated analytically they have interesting strong coupling behaviour such as confinement, chiral symmetry breaking. Moreover, at certain points in the moduli space, they provide examples of non–trivial interacting N=2 superconformal theories in four dimensions [3, 4]. The objects of interest in the exact solution are the period integrals over homology cycles on these elliptic curves which can alternatively be determined using Picard–Fuchs equations. Up to now this has been accomplished for the theories with $SU(2)$ gauge group with massless matter [5, 6, 7] and higher gauge groups [8]. More recently, it has been generalized to the massive cases of the $SU(2)$ theories [9].

The purpose of the present article is to determine the periods for the $SU(2)$ theories for different numbers N_f of matter fields using a unique differential equation. We find that the coupling constant is given in terms of a Schwarzian triangle function. This function depends only on the J–invariant of the elliptic curve and thus unifies all cases – both massless and massive, $N_f = 1, \ldots, 4$. This is shown in section 2.

In section 3 we work out the relation between periods and certain modular functions of $SL(2, \mathbb{Z})$ or subgroups thereof. For several values of N_f we calculate the discriminant, the modulus u and finally the periods as modular functions of the coupling constant τ. We demonstrate the unifying power of the J invariant as a function of u, the masses and the scale Λ. That allows us in section 4 to read off the monodromies around the singularities in the moduli space and to obtain the effective coupling constants and the monodromies at the superconformal points in a very simple fashion.

2   The coupling constant τ

Let us consider the quartic curve $y^2 = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$ written in *Weierstrass form*

$$y^2 = 4x^3 - g_2 x - g_3 \ , \tag{2.1}$$

with

$$
\begin{aligned}
g_2 &= ae - 4bd + 3c^2 \\
g_3 &= ace + 2bcd - ad^2 - b^2 e - c^3 \ .
\end{aligned}
\tag{2.2}
$$

This curve corresponds to the one–parameter family of curves embedded in CP^2

$$X_s : x_1^3 + x_2^3 + x_3^3 - s x_1 x_2 x_3 = 0 \ , \tag{2.3}$$

with $g_2 = 3s(8 + s^3)$ and $g_3 = 8 + 20s^3 - s^6$ [10]. In particular, this curve describes all cases $N_f = 0, 1, 2, 3, 4$ of [1, 2]. The variation of the period of the holomorphic one–form $\frac{dx}{y}$ along a homology cycle Γ

$$\tilde{\omega}_\Gamma = \oint_\Gamma \frac{dx}{y} \tag{2.4}$$

is described by the second–order ODE [11]

$$\frac{d^2\Omega_\Gamma}{dJ^2} + \frac{1}{J}\frac{d\Omega_\Gamma}{dJ} + \frac{31J - 4}{144J^2(1 - J)^2}\Omega_\Gamma = 0 \ , \tag{2.5}$$

with the J function defined as

$$J = \frac{g_2^3}{\Delta} = \frac{g_2^3}{g_2^3 - 27g_3^2} \tag{2.6}$$

and the normalized periods

$$\tilde{\omega}_\Gamma = \sqrt{\frac{g_2}{g_3}}\Omega_\Gamma \ . \tag{2.7}$$

The solutions $\Omega_\Gamma(J)$ of this equation arise from the general case of the hypergeometric differential equation corresponding to the *Riemann P-function* [12]

$$P(J) = \begin{bmatrix} 0 & \infty & 1 \\ -\frac{1}{6} & 0 & \frac{1}{4} & J \\ \frac{1}{6} & 0 & \frac{3}{4} \end{bmatrix} \equiv (1 - J)^{\frac{1}{4}} J^{-\frac{1}{6}} \begin{bmatrix} 0 & \infty & 1 \\ 0 & \frac{1}{12} & 0 & J \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{2} \end{bmatrix} \ . \tag{2.8}$$

Therefore two linear independent solutions of (2.5), appropriate for the regime $|J| > 1$, $|arg(1 - J)| < \pi$ for $|J| > 1$, read e.g.

$$\begin{aligned} \Omega_0(J) &= 2\pi i(1 - J)^{\frac{1}{4}} J^{-\frac{1}{4}} {}_2F_1\left[\frac{1}{12}, \frac{5}{12}, 1, \frac{1}{J}\right] \ , \\ \Omega_1(J) &= -(\ln J + 3\ln 12) \Omega_0(J) - w_1(J) \end{aligned} \tag{2.9}$$

with:

$$\begin{aligned} w_1(J) &= (1 - J)^{\frac{1}{4}} J^{-\frac{1}{4}} \sum_{n=1}^\infty \frac{\left(\frac{1}{12}\right)_n \left(\frac{5}{12}\right)_n}{(n!)^2} h_n J^{-n} \\ h_n &= 2\psi(n + 1) - \psi\left(\frac{1}{12} + n\right) - \psi\left(\frac{5}{12} + n\right) + \psi\left(\frac{1}{12}\right) + \psi\left(\frac{5}{12}\right) - 2\psi(1) \ . \end{aligned} \tag{2.10}$$

Here $(a)_n \equiv \Gamma(a + n)/\Gamma(n)$ is the Pochhammer symbol. Of course, there may be constructed many other pairs of independet solutions both by taking different linear combinations and expanding at the other singular points $J = 0, 1$. The specific choice (2.9) will

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become clear in a moment and we also will have to say more about that at the end of this section.

Since the zeros of the discriminant $\Delta(u_i) = g_2^3 - 27 g_3^2 = 0$ correspond to $J(u_i) = \infty$, from the $J \to \infty$ limit we can extract the strong coupling behaviour around the points u_i. But we also have $J \to \infty$ for $u \to \infty$. Therefore a property of the solutions (2.9) is that they have the appropriate form for both the weak–coupling and the strong–coupling expansions. In particular we obtain:

$$
\begin{aligned}
\tilde{\omega}_0 &\longrightarrow (-1)^{\frac{1}{4}} \sqrt{\frac{g_2(u,m)}{g_3(u,m)}} \ , \\[2mm]
\tilde{\omega}_1 &\longrightarrow -(-1)^{\frac{1}{4}} \sqrt{\frac{g_2(u,m)}{g_3(u,m)}} \ (\ln J + 3 \ln 12) \ .
\end{aligned}
\tag{2.11}
$$

The factor $\sqrt{\frac{g_2}{g_3}}$ is the leading piece and it is that part which is responsible for the different leading behaviours at $u \to u_i$ and $u \to \infty$. Let us present two examples[1]:

- $\underline{N_f = 0}$:

  For the weak–coupling patch $u \sim \infty$:

$$
\begin{aligned}
\frac{da_D}{du} &= \frac{1}{12\pi} \sqrt{\frac{g_2(u,m)}{g_3(u,m)}} \Omega_1(J) \ , \\[2mm]
\frac{da}{du} &= \frac{(-1)^{\frac{1}{4}}}{24\pi} \sqrt{\frac{g_2(u,m)}{g_3(u,m)}} \Omega_0(J) \ ,
\end{aligned}
\tag{2.12}
$$

  and for the strong–coupling regime $u \sim \Lambda^2$:

$$
\begin{aligned}
\frac{da_D}{du} &= \frac{1}{12\pi} \sqrt{\frac{g_2(u,m)}{g_3(u,m)}} \Omega_0(J) \ , \\[2mm]
\frac{da}{du} &= \frac{(-1)^{-\frac{1}{4}}}{24\pi} \sqrt{\frac{g_2(u,m)}{g_3(u,m)}} \Omega_1(J) \ .
\end{aligned}
\tag{2.13}
$$

- $\underline{N_f = 1}$:

  For the weak–coupling patch $u \sim \infty$:

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[1]While this paper was being typed some related work appeared in [14]. Here the integral (2.4) was performed explicitly for the three cases $N_f = 1, 2, 3$, however in a separate way. Therefore, neither the unifying rôle of J nor the connection to elliptic functions could be seen.

$$\frac{da_D}{du} = -\frac{i}{24\pi}\sqrt{\frac{g_2(u,m)}{g_3(u,m)}}\Omega_1(J) \; ,$$

$$\frac{da}{du} = -\frac{i}{24\pi}\sqrt{\frac{g_2(u,m)}{g_3(u,m)}}\Omega_0(J) \; ,$$

(2.14)

and for the strong–coupling regime $u \sim u_1$:

$$\frac{da_D}{du} = \frac{i}{24\pi}\sqrt{\frac{g_2(u,m)}{g_3(u,m)}}\Omega_0(J) \; ,$$

$$\frac{da}{du} = \frac{i}{24\pi}\sqrt{\frac{g_2(u,m)}{g_3(u,m)}}\Omega_1(J) \; .$$

(2.15)

Indeed, using several hypergeometric identities [13], known as quadratic and cubic identities, and the explicit expressions for the J–functions (2.6)

$$1728J = \frac{(3\Lambda^4 + u^2)^3}{\Delta_0(u)} \;\; , \;\; N_f = 0 \; ,$$

$$1728J = \frac{256(3\Lambda_1^3 m - 4u^2)^3}{2^{14}\Delta_1(u,m)} \;\; , \;\; N_f = 1 \; ,$$

$$1728J = \frac{(3\Lambda_2^4 - 48\Lambda_2^2 m^2 + 64u^2)^3}{2^{18}\Delta_2(u,m)} \;\; , \;\; N_f = 2 \; ,$$

$$1728J = \frac{(-\Lambda_3^4 - 576\Lambda_3^2 m^2 + 3072\Lambda_3 m^3 + 256\Lambda_3^2 u - 4096u^2)^3}{2^{36}\Delta_3(u,m)} \;\; , \;\; N_f = 3 \; ,$$

(2.16)

with the discriminants:

$$\Delta_0(u) = 2^{-6}\Lambda^4(u^2 - \Lambda^4)^2$$

$$\Delta_1(u,m) = 2^{-20}\Lambda_1^6(27\Lambda_1^6 + 256\Lambda_1^3 m^3 - 288\Lambda_1^3 mu - 256m^2 u^2 + 256u^3)$$

$$\Delta_2(u,m) = 2^{-24}\Lambda_2^4(\Lambda_2^2 + 8m^2 - 8u)^2(\Lambda_2^4 - 64\Lambda_2^2 m^2 + 16\Lambda_2^2 u + 64u^2)$$

$$\Delta_3(u,m) = 2^{-32}\Lambda_3^2(-3\Lambda_3^3 m - 24\Lambda_3^2 m^2 - 2048\Lambda_3 m^3 - 8\Lambda_3^2 u + 768\Lambda_3 mu + 2048u^2)$$
$$\times \; (-\Lambda_3 m - 8m^2 + 8u)^3 \; ,$$

(2.17)

we arrive at the periods $(\tilde{\omega}_D, \tilde{\omega})$ given in [8, 6, 7] for $m = 0$. The above functions refer to the curves [1, 2]:

$$y^2 = (x^2 - \Lambda^4)(x - u) \;\; , \;\; N_f = 0 \; ,$$

(2.18)

and

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$$y^2 = x^2(x-u) + \frac{1}{4}m\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6 \quad , \quad N_f = 1$$

$$y^2 = \left(x^2 - \frac{1}{64}\Lambda_2^4\right)(x-u) + \frac{1}{4}m_1 m_2 \Lambda_2^2 x - \frac{1}{64}(m_1^2 + m_2^2)\Lambda_2^4 \quad , \quad N_f = 2$$

$$y^2 = x^2(x-u) - \frac{1}{64}\Lambda_3^2(x-u)^2 - \frac{1}{64}(m_1^2 + m_2^2 + m_3^2)\Lambda_3^2(x-u) +$$
$$+ \frac{1}{4}m_1 m_2 m_3 \Lambda_3 x - \frac{1}{64}(m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2)\Lambda_3^2 \quad , \quad N_f = 3 .$$

$$(2.19)$$

with $m_1 = m_2 = m_3 = m$, respectively. Choosing (2.13) and (2.15) for the periods of the monopole patch the limits (2.11) can be compared with the monopole expansions in [8, 5, 6, 7] for $m = 0$.

Since the coupling constant of the underlying theory

$$\tau(J) = \frac{d^2\mathcal{F}}{da^2} = \frac{da_D}{da} = \frac{da_D}{du} \Big/ \frac{da}{du} \tag{2.20}$$

is the quotient of two solutions of the hypergeometric equation (2.5), it satisfies the *Schwarzian differential equation* associated to eq. (2.5):

$$\{\tau, J\} \equiv \frac{\tau'''}{\tau'} - \frac{3}{2}\left(\frac{\tau''}{\tau'}\right)^2 = \frac{4}{9J^2} + \frac{3}{8(1-J)^2} + \frac{23}{72J(1-J)} . \tag{2.21}$$

Its solution is known as the *Schwarzian triangle function* $\omega(J) \equiv s[\frac{1}{2}, \frac{1}{3}, 0, J]$ which represents an infinite–valued map from the J–plane to the complex plane [11]. Notice that this provides an analytic expression for the coupling constant τ

$$\tau(J) = \omega[J] \equiv s\left[\frac{1}{2}, \frac{1}{3}, 0, J\right] , \tag{2.22}$$

modulo a subgroup of $SL(2, \mathbb{Z})$ transformations. This function maps the points $J = 0, J = 1$ and $J = \infty$ to the edges of a triangle with angles $\pi/3, \pi/2$ and 0 at the points $\tau = \rho, i, i\infty$, respectively (with $\rho = e^{2\pi i/3}$). From

$$\left\{\frac{A\tau + B}{C\tau + D}, J\right\} = \{\tau, J\} , \quad AD - BC = 1 , \tag{2.23}$$

it follows that any linear combination $\Omega_D' = A\Omega_D + B\Omega$ and $\Omega' = C\Omega_D + D\Omega$ of the two solutions (2.9) also satisfies (2.21). It can be proven that all solutions of (2.21) are of this type. To give an analytic expression for $\omega(J)$ we have to select a particular expansion in J. When encircling the three singular points $J = \infty, 0, 1$ the two solutions Ω_0 and Ω_1 have a certain monodromie behaviour which becomes clear when we look at the related points in the $\tau = \Omega_0/\Omega_1$–plane:

- $J = \infty$. The corresponding point $\tau = i\infty$ in the τ–plane is invariant under $T : \tau \to \tau + 1$. Therefore at $J = \infty$:

$$\begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix} .$$

- $J = 1$. The corresponding point $\tau = i$ in the τ–plane is invariant under $S : \tau \to -1/\tau$. Therefore at $J = 1$:

$$\begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix} .$$

- $J = 0$. The corresponding point $\tau = \rho$ in the τ–plane is invariant under $T^{-1}S^{-1}$: $\tau \to \tau + 1$. Therefore at $J = 0$:

$$\begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix} \to \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix} .$$

As expected, altogether the monodromies close: $T^{-1}S^{-1}ST = 1$. By these conditions the form of the solutions Ω_0, Ω_1 is fixed up to normalization factors. To make this more precise we give two ratios for τ, one valid for $|J| < 1$ and the other for $|J| > 1$. Since we know the monodromie behaviour of the hypergeometric functions, we may write at $J = 0$:

$$\begin{aligned} \Omega_0(J) - \rho^2 \Omega_1(J) &= A_0 \; {}_2F_1 \left[\frac{1}{12}, \frac{1}{12}, \frac{2}{3}, J \right] \\ \Omega_0(J) - \rho \Omega_1(J) &= A_1 \; {}_2F_1 \left[\frac{5}{12}, \frac{5}{12}, \frac{4}{3}, J \right] . \end{aligned} \tag{2.24}$$

The combinations on the l.h.s. are eigenvectors with eigenvalues ρ, ρ^2 under the monodromie action of $T^{-1}S^{-1}$. The normalizations $A_0 = \rho(1 - \rho)$ and $A_1 = -\rho\lambda\sqrt{3}$ with

$$\lambda = (2 - \sqrt{3}) \left[\frac{\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{7}{12}\right)} \right]^2 \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)} \tag{2.25}$$

are determined when comparing the solutions (2.24) with the actual integral (2.4). Therefore for $|J| < 1$

$$\tau(J) = e^{\frac{2\pi i}{3}} \frac{{}_2F_1\left[\frac{1}{12}, \frac{1}{12}, \frac{2}{3}, J\right] - \lambda e^{\pi i/3} J^{\frac{1}{3}} \; {}_2F_1\left[\frac{5}{12}, \frac{5}{12}, \frac{4}{3}, J\right]}{{}_2F_1\left[\frac{1}{12}, \frac{1}{12}, \frac{2}{3}, J\right] - \lambda e^{-\pi i/3} J^{\frac{1}{3}} \; {}_2F_1\left[\frac{5}{12}, \frac{5}{12}, \frac{4}{3}, J\right]} \tag{2.26}$$

and for $|J| > 1$, $|arg(1 - J)| < \pi$:

$$\begin{aligned} 2\pi i \tau(J) = \frac{\Omega_1(J)}{\Omega_0(J)} &= -\ln J - 3\ln 12 - \frac{\sum_{n=1}^{\infty} \frac{\left(\frac{1}{12}\right)_n \left(\frac{5}{12}\right)_n}{(n!)^2} h_n J^{-n}}{{}_2F_1\left[\frac{1}{12}, \frac{5}{12}, 1, \frac{1}{J}\right]} \\ &= -\ln J - 3\ln 12 + \frac{31}{72} J^{-1} + \frac{13157}{82944} J^{-2} + \dots . \end{aligned} \tag{2.27}$$

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Indeed, these expressions imply $\tau(0) = e^{\frac{2\pi i}{3}}$, $\tau(1) = i$ and $\tau(\infty) = i\infty$. Altogether, we map the J–plane onto half of the fundamental region of $SL(2, \mathbb{Z})$, representing a degenerate triangle. Eqs. (2.26) and (2.27) allow us to determine the coupling constant τ in the whole J–plane and by (2.16) we obtain its full u and m dependence. Of course, as soon as we specify a region in the u–plane, the $SL(2, \mathbb{Z})$ transformations are reduced to the corresponding monodromy group of the relevant periods in this patch. Inserting $J = j/1728$ in the above expressions and using [15], we immediately see that (2.26) and (2.27) become the coupling constant of $N_f = 4$.

3 Modular functions and periods

Let us focus on the first solution of (2.9), which gives us the expression:

$$\frac{da}{du} = \frac{\pi}{12}\sqrt{\frac{g_2}{g_3}} \, (J-1)^{\frac{1}{4}} J^{-\frac{1}{4}} \, {}_2F_1\left[\frac{1}{12}, \frac{5}{12}, 1, \frac{1}{J}\right] \, . \tag{3.1}$$

We apply the powerful identity [16]

$$ {}_2F_1\left[\frac{1}{12}, \frac{5}{12}, 1, \frac{1728}{j(\tau)}\right]^4 = E_4(\tau) \tag{3.2}$$

and

$$J = \frac{j(\tau)}{1728} = \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} \tag{3.3}$$

to arrive at

$$\frac{da}{du} = \frac{\pi}{12}\sqrt{\frac{g_2 E_6(\tau)}{g_3 E_4(\tau)}} \, . \tag{3.4}$$

For the massless $N_f = 4$ case with the curve $y^2 = x^3 - \frac{1}{4}g_2(\tau)xu^2 - \frac{1}{4}g_3(\tau)u^3$ and $g_2(\tau) = \frac{4\pi^4}{3}E_4(\tau)$, $g_3(\tau) = \frac{8\pi^6}{27}E_6(\tau)$, we just have to determine its J–function

$$J = \frac{j(\tau)}{1728} \, , \quad N_f = 4 \, , \tag{3.5}$$

to pass from (3.4) to

$$\frac{da}{du} = \frac{1}{2}\frac{1}{\sqrt{2u}} \, . \tag{3.6}$$

Expression (3.4) is a modular function of weight $+1$. Let us remark that g_2 and g_3 are *invariants* when considered as functions of u, since $u(\tau)$ is a modular invariant function in τ. The relation (3.2) indicates the existence of a deeper connection of periods to modular functions.

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Indeed, recently, it was shown in [17] how to construct the weight -1 modular function $c(\tau) = a_D - \tau a$ once its singularity structure in the τ–moduli space is known. Let us present some more connections for the $N_f = 0$ case by writing J of (2.16) completely in terms of modular functions. Using the identity [18]

$$\frac{j(\tau)}{1728} = \frac{4}{27} \frac{[1 - \lambda(\tau) + \lambda^2(\tau)]^3}{\lambda^2(\tau)[1 - \lambda(\tau)]^2} \quad \text{with} \quad \lambda(\tau) = \frac{\theta_1^4(\tau)}{\theta_3^4(\tau)} \tag{3.7}$$

and eq. (2.16)

$$J = \frac{(3\Lambda^4 + u^2)^3}{27\Lambda^4(u^2 - \Lambda^4)^2} = -\frac{1}{27} \frac{(z-4)^3}{z^2} \quad , \tag{3.8}$$

we realize that the choice $z = 1 - \frac{u^2}{\Lambda^4} = 4\frac{1}{\lambda}(1 - \frac{1}{\lambda})$ matches eq. (3.7) with (3.8). This leads to

$$\frac{u(\tau)}{\Lambda^2} = -1 + \frac{2}{\lambda(\tau)} \quad , \tag{3.9}$$

which indeed has the correct behaviour at the cusps $\tau = 0, 1, i\infty$ corresponding to $u = 1, -1, i\infty$, respectively. This gives an invariant expression for $u(\tau)$, since $\lambda(\tau)$ is a modular function of $\Gamma(2)_\tau$ [18], which is the monodromy group of the curve discussed in [1].

$\underline{N_f = 0 :}$

We want to discuss the curve $y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda^4 x$ discussed in [2]. It has:

$$
\begin{aligned}
J &= \frac{(-3\Lambda^4 + 4u^2)^3}{27\Lambda^8(u^2 - \Lambda^4)} \\[2mm]
g_2 &= \frac{1}{48}(4u^2 - 3\Lambda^4) \\[2mm]
g_3 &= \frac{1}{1728}(8u^3 - 9\Lambda^4 u) \\[2mm]
\triangle_0(u) &= \frac{\Lambda^{12}}{4096}\left(\frac{u^2}{\Lambda^4} - 1\right) .
\end{aligned}
\tag{3.10}
$$

Its moduli space is $\Gamma^0(4)$ with the three cusps points $\tau = 0, 2, i\infty$. The behaviour of $u(\tau)$ at the three cusps can be worked out:

$$
\begin{aligned}
\frac{u^4}{\Lambda^8} &= \frac{27}{64}e^{-2\pi i\tau} \quad , \quad \tau \longrightarrow i\infty \\[2mm]
\frac{u}{\Lambda^2} - 1 &= \frac{1}{54}e^{-2\pi i\frac{1}{\tau}} \quad , \quad \tau \longrightarrow 0 + i\epsilon \\[2mm]
\frac{u}{\Lambda^2} + 1 &= -\frac{1}{54}e^{2\pi i\frac{1}{2-\tau}} \quad , \quad \tau \longrightarrow 2 + i\epsilon.
\end{aligned}
\tag{3.11}
$$

These conditions may be deduced by writing $J(\tau)$ with a different argument $J\left(\frac{a\tau+b}{c\tau+d}\right)$ depending on which power series one is interested in. For example for the expansion

around the monopole point one chooses $J\left(-\frac{1}{\tau}\right)$, since $\tau_D = -\frac{1}{\tau}$ is the coupling of the dual theory which becomes weak $\tau_D \to i\infty$ at the monopole point. Similarly, $J\left(\frac{1}{\tau-2}\right)$ at the dyon point $\tau = 2$. Of course, since $J\left(-\frac{1}{\tau}\right) = J(\tau) = J\left(\frac{1}{\tau-2}\right)$ this makes no difference for the J–function. This is the effect we have already encountered in the previous section, that $J = \infty$ holds for the monopole, the dyon and weak coupling point. Eqs. (3.10) are enough to determine the discriminant:

$$\triangle_0(\tau) = \kappa_0\Lambda^{12}\frac{\eta^{24}\left(\frac{\tau}{2}\right)}{\eta^{24}(\tau)} \quad , \quad \kappa_0 = -2^{-18}\ 1728^{-2}\ , \tag{3.12}$$

which is also invariant under $\Gamma_0(2)_{\tilde\tau}$ with $\tilde\tau = \frac{\tau}{2}$. If we were interested to match only the monopole behaviour at $\tau = 0$ and the weak coupling behaviour at $\tau \to i\infty$ we would obtain:

$$\frac{u(\tau)}{\Lambda^2} = \frac{1}{2^3\ 1728}\frac{\eta^8\left(\frac{\tau}{4}\right)}{\eta^8(\tau)} + 1\ . \tag{3.13}$$

This expression has also been found in [17]. Similarly, when one only matches the dyon point $\tau = 2$ and the weak coupling point $\tau = i\infty$, we get

$$\frac{u(\tau)}{\Lambda^2} = -\frac{1}{2^3\ 1728}\frac{\eta^{24}\left(\frac{\tau}{2}\right)}{\eta^{16}(\tau)\eta^8\left(\frac{\tau}{4}\right)} - 1\ . \tag{3.14}$$

These expressions are manifestly invariant only under $\Gamma^0(4)\tau$. Using

$$\begin{aligned}
g_{2,N_f}^3(\tau) &= \frac{1}{1728}\frac{E_4^3(\tau)}{\eta^{24}(\tau)}\triangle_{N_f}(\tau) \\
g_{3,N_f}^2(\tau) &= \frac{1}{27\ 1728}\frac{E_6^2(\tau)}{\eta^{24}(\tau)}\triangle_{N_f}(\tau)\ ,
\end{aligned} \tag{3.15}$$

we determine:

$$\begin{aligned}
g_2(\tau) &= \Lambda^4\frac{\kappa_0^{\frac{1}{3}}}{12}\ E_4(\tau)\frac{\eta^8\left(\frac{\tau}{2}\right)}{\eta^{16}(\tau)} \\
g_3(\tau) &= \Lambda^6\frac{\kappa_0^{\frac{1}{2}}}{216}\ E_6(\tau)\frac{\eta^{12}\left(\frac{\tau}{2}\right)}{\eta^{24}(\tau)}\ .
\end{aligned} \tag{3.16}$$

Combining the two expressions for g_2 in eqs. (3.10) and (3.16) gives us the full τ–dependence for $u(\tau)$, in contrast to eqs. (3.13) and (3.14):

$$\frac{u^2(\tau)}{\Lambda^4} = \kappa_0^{\frac{1}{3}}\ E_4(\tau)\frac{\eta^8\left(\frac{\tau}{2}\right)}{\eta^{16}(\tau)} + \frac{3}{4}\ . \tag{3.17}$$

Finally, with (3.4) we obtain:

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$$\frac{da}{du} = \frac{\sqrt{2}\pi\kappa_0^{-\frac{1}{12}}}{4\Lambda} \frac{\eta^4(\tau)}{\eta^2\left(\frac{\tau}{2}\right)} . \tag{3.18}$$

$\underline{N_f = 2 :}$

The massless $N_f = 2$ case[2] is described by the moduli space of $\Gamma(2)$ with the three cusps points $\tau = 0, 1, i\infty$. At these points we obtain for $u(\tau)$

$$\begin{aligned}
\frac{u^2}{\Lambda_2^4} &= \frac{27}{64}e^{-2\pi i\tau} \ , \quad \tau \longrightarrow i\infty \ , \\
\left(\frac{u}{\Lambda_2^2} - u_{12}\right)^2 &= \frac{1}{108}e^{-2\pi i\frac{1}{\tau}} \ , \quad \tau \longrightarrow 0 + i\epsilon \ , \quad u_{12} = \frac{1}{8} \\
\left(\frac{u}{\Lambda_2^2} - u_{34}\right)^2 &= \frac{1}{108}e^{2\pi i\frac{1}{1-\tau}} \ , \quad \tau \longrightarrow 1 + i\epsilon \ , \quad u_{34} = -\frac{1}{8} \ ,
\end{aligned} \tag{3.19}$$

from which we get

$$\triangle_2(\tau) = \kappa_2\Lambda_2^{12} \frac{\eta^{48}(\tau)}{\eta^{48}(2\tau)} \ , \quad \kappa_2 = 2^{-36} \ 1728^{-2} . \tag{3.20}$$

It is interesting that this expression has an even bigger invariance, namely $\Gamma_0(2)_\tau$, as it would be dictated by the monodromy group $\Gamma(2)_\tau$. Using (3.15) we obtain for g_2 and g_3

$$\begin{aligned}
g_2(\tau) &= \Lambda_2^4 \frac{\kappa_2^{\frac{1}{3}}}{12} E_4(\tau) \frac{\eta^8(\tau)}{\eta^{16}(2\tau)} \ , \\
g_3(\tau) &= \Lambda_2^6 \frac{\kappa_2^{\frac{1}{2}}}{216} E_6(\tau) \frac{\eta^{12}(\tau)}{\eta^{24}(2\tau)} .
\end{aligned} \tag{3.21}$$

From the explicit form of $g_2 = \frac{1}{768}(3\Lambda_2^4 + 64u^2)$ we are able to extract $u(\tau)$. With (3.4) we arrive at:

$$\frac{da}{du} = \frac{\sqrt{2}\pi\kappa_2^{-\frac{1}{12}}}{4\Lambda_2} \frac{\eta^4(2\tau)}{\eta^2(\tau)} . \tag{3.22}$$

$\underline{N_f = 3 :}$
The massless $N_f = 3$ case is described by the moduli space of $\Gamma_0(4)$ with the three cusps points $\tau = 0, \frac{1}{2}, i\infty$. From the behaviour at the three cusps

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[2]The massive case is more involved, since the cusps move.

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$$
\frac{u}{\Lambda_3^2} = -\frac{27}{64} e^{-2\pi i \tau} \quad , \quad \tau \longrightarrow i\infty \ ,
$$

$$
\left(\frac{u}{\Lambda_3^2} - u_{1234} \right)^4 = \frac{1}{2^{22} 3^3} e^{-2\pi i \frac{1}{\tau}} \quad , \quad \tau \longrightarrow 0 + i\epsilon \quad , \quad u_{1234} = 0 \tag{3.23}
$$

$$
\frac{u}{\Lambda_3^2} - u_5 = \frac{-1}{2^{10} 3^3} e^{-2\pi i \frac{\tau}{2\tau - 1}} \quad , \quad \tau \longrightarrow \frac{1}{2} + i\epsilon \quad , \quad u_5 = \frac{1}{256} \ .
$$

we are able to determine the discriminant

$$
\triangle_3(\tau) = \kappa_3 \Lambda_3^{12} \frac{\eta^{24}(\tau) \eta^{24}(2\tau)}{\eta^{48}(4\tau)} \quad , \quad \kappa_3 := -2^{-132} \, 1728^{-5} \ . \tag{3.24}
$$

which is the modular invariant expression for $\triangle_3(u, m)$ in (2.17). This function must be a modular invariant under $\Gamma_0(4)_\tau$. In addition, it is also invariant under $\Gamma(2)_{\tilde\tau}$ for $\tilde\tau = 2\tau$ arising from $\Gamma_0(4) \simeq \Gamma(2)$. In other words, this just reflects the fact that we can also describe the monodromies of the $N_f = 3$ case in the $\tilde\tau$ moduli space. Let us also mention the identity [19]

$$
\triangle_3(\tau) = \kappa_3 \Lambda_3^{12} e^{-\frac{\pi i}{3}} \frac{\eta^{32}(\tau)}{\eta^{32}(4\tau)} \frac{\eta^8(\tau + \frac{1}{2})}{\eta^8(4\tau)} \ , \tag{3.25}
$$

to realize that our discriminant is the product of two single functions describing the correct behaviour at $\tau = 0$ and $\tau = \frac{1}{2}$, respectively. Furthermore, using (3.15), we may evaluate:

$$
\begin{aligned}
g_2(\tau) &= \Lambda_3^4 \frac{\kappa_3^{\frac{1}{3}}}{12} E_4(\tau) \frac{\eta^8(2\tau)}{\eta^{16}(4\tau)} \ , \\
g_3(\tau) &= \Lambda_3^6 \frac{\kappa_3^{\frac{1}{2}}}{216} E_6(\tau) \frac{\eta^{12}(2\tau)}{\eta^{24}(4\tau)} \ .
\end{aligned} \tag{3.26}
$$

With $g_2 = \frac{1}{49152}(\Lambda_3^4 - 256\Lambda_3^2 u + 4096 u^2)$ we are able to deduce $u(\tau)$. Finally, from (3.4) we obtain:

$$
\frac{da}{du} = \frac{\sqrt{2}\pi \kappa_3^{-\frac{1}{12}}}{4\Lambda_3} \frac{\eta^4(4\tau)}{\eta^2(2\tau)} \ . \tag{3.27}
$$

Let us now come to a case with $m \neq 0$.

$\underline{N_f = 3 \ , \ m = \frac{\Lambda_3}{8}}$:

This case corresponds to a superconformal point (see sect. 4 for discussions). Relating the J–function (3.3) with its explicit form (2.17) gives two branches for $u(\tau)$:

$$
\frac{u(\tau)}{\Lambda_3^2} = \frac{\pm E_4(\tau)^3 \left[-23 + 27\sqrt{\frac{E_6(\tau)^2}{E_4(\tau)^3}} \right] - 4 E_6(\tau)^2}{128 [E_4(\tau)^3 - E_6(\tau)^2]} \ . \tag{3.28}
$$

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Similar expressions may be found for $g_2(\tau)$, $g_3(\tau)$ to derive from (3.4):

$$\frac{da}{du} = \frac{4\sqrt{3}\pi}{9\Lambda_3} E_4(\tau)^{1/4} \sqrt{\pm 1 + \frac{E_6(\tau)}{E_4(\tau)^{3/2}}} \ . \tag{3.29}$$

To summarize, we have explicitly demonstrated for $N_f = 0, 2, 3$ that one may express $\frac{da}{du}$ and therefore also $\frac{da_D}{du}$ by a modular function $c(\tau)$ of weight -1:

$$\begin{aligned}
\frac{da}{du} &= c^{(-1)}(\tau) \\
\frac{da_D}{du} &= \tau \, c^{(-1)}(\tau) \ .
\end{aligned} \tag{3.30}$$

However, from these expressions we are able to extract a_D and a up to an integration constant containing possible residua in the massive case. From [20] we have:

$$a_D - \tau a = \left(\frac{da}{du}\right)^{-1} \left[\frac{i}{2\pi}(4 - N_f) - \sum_i m_i \int_{x_i}^\infty \frac{dx}{y}\right] \ . \tag{3.31}$$

The x_i are the locations of the residua on the hyperelliptic curve. Using (3.30) we obtain

$$a_D - \tau a = c(\tau) \left[\frac{i}{2\pi}(4 - N_f) - \sum_i m_i \int_{x_i}^\infty \frac{dx}{y}\right] \ , \tag{3.32}$$

from which we immediately get expressions for the periods a_D and a in the case $m = 0$:

$$\begin{aligned}
a &= -\frac{i}{2\pi}(4 - N_f) \frac{dc}{d\tau} \\
a_D &= \frac{i}{2\pi}(4 - N_f) \left[c - \tau \frac{dc}{d\tau}\right] \ .
\end{aligned} \tag{3.33}$$

4 J Invariants and monodromies

In this section we will use the explicit form of the J–functions (2.16) of the Seiberg–Witten curves in terms of the u, m and Λ, to obtain the behaviour near the singularities of the moduli space. We are mainly interested in the pole structure of the J–function because from that we can derive the monodromies (up to conjugation). Near the cusp $\tau = i\infty$ the J–invariant takes the following form

$$J(\tau) \sim q^{-1} = e^{-2\pi i \tau} \ , \tag{4.1}$$

as can be seen from (2.27). A loop in J–space $J \to Je^{2\pi i}$ corresponds to the shift $\tau \to \tau - 1$ or in other words the associated monodromy is T^{-1}. Furthermore we will find points in the moduli space of the massive theories where $J = 0$ or $J = 1$. We will show that these points correspond to superconformal points, which have been studied in the literature [4].

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We find that the coupling constant at these points is ρ or i and the associated monodromy is conjugate to $(ST)^{-1}$ or S^{-1}.

Let us begin with the case $N_f = 0$. The J–invariant (3.10) has three poles at $u = \pm\Lambda^2$ and $u = \infty$, which correspond to the two strong coupling singularities and the weak coupling singularity. At these poles J shows the following behaviour:

$$
\begin{aligned}
J &= \pm\frac{\Lambda^2}{54}(u \mp \Lambda^2)^{-1} \ , \quad u = \pm\Lambda^2 \\
J &= \frac{64}{27\Lambda^4}u^4 \ , \quad u = \infty \ .
\end{aligned}
\tag{4.2}
$$

If u moves around the singularity at infinity $u \to ue^{2\pi i}$, J loops four times, which means that the weak coupling monodromy is conjugate to T^{-4}. Along the same line we conclude that the monodromies at the strong coupling singularities are conjugate to T, in agreement with [2]. At the monopole point $u = \Lambda^2$ we have $\tau = 0$, which corresponds to strong coupling, and we have to perform a duality transformation, which takes τ to infinity and introduce a dual coupling $\tau_D = -1/\tau$ with $\tau_D \to i\infty$ for $\tau \to 0$. In a more physical language: we go from a strongly coupled description of the theory to a weakly coupled description with coupling constant τ_D. Therefore we find

$$
J = e^{-2\pi i\tau_D} = e^{2\pi i/\tau} \sim c_+(u - \Lambda^2)^{-1}
\tag{4.3}
$$

$$
\Rightarrow u = \Lambda^2 + c_+e^{-2\pi i/\tau} \ ,
\tag{4.4}
$$

and similarly at the dyon point $u = -\Lambda^2$, where $\tau = 2$, we a duality transformation $\tau_{D'} = 1/(2 - \tau)$ and find:

$$
u = -\Lambda^2 + c_-e^{\frac{2\pi i}{2-\tau}} \ .
\tag{4.5}
$$

The behaviour at the singularities fixes u uniquely to be a $\Gamma^0(4)$ modular function. Since the poles of J correspond to τ being either ∞ or a rational number, we can always transform this point to $i\infty$ by an $SL(2, \mathbb{Z})$ duality transformation. Furthermore these points are always cusps of infinite order of the corresponding fundamental domain.

We will repeat the same arguments for the case $N_f = 1$, where we have three strong coupling singularities and a weak coupling singularity for generic values of m. By $u_i, i = 1, \ldots, 3$ we denote the three zeros of the $N_f = 1$ discriminant (2.17). The behaviour at the singularities is

$$
\begin{aligned}
J &= c_i(u - u_i)^{-1} \ , \quad u = u_i \\
J &= \frac{-64}{27\Lambda_1^6}u^3 \ , \quad u = \infty \ ,
\end{aligned}
\tag{4.6}
$$

where $c_i, i = 1, \ldots, 3$ are factors depending on m and Λ_1 to be determined from J in (2.16). The monodromies at the three dyon points are conjugate to T and the weak coupling monodromy is conjugate to T^{-3}, as expected [2]. As was shown in [4] the moduli space of the $N_f = 1$ theory contains a superconformal point at $u = 3/4\Lambda_1^2, m = 3/4\Lambda_1$ where two mutually non–local particles become massless. In the notation of [4] this

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corresponds to a $(1,1)$ superconformal point, which is equivalent to the superconformal point of the $SU(3)$ $N = 2$ SYM theory studied in [3]. At an $(n,1)$ superconformal point there are n mutually local massless particles together with one particle that is non–local with respect to them. Indeed if we set $m = 3/4\Lambda_1$ the discriminant (2.17) develops a simple zero at $u = -15/16\Lambda_1^2$ and a double zero at $u = 3/4\Lambda_1^2$. The J–invariant takes the following form:

$$J_{(1,1)}(u) = \frac{4(3\Lambda_1^2 - 4u)(3\Lambda_1^2 + 4u)}{27\Lambda_1^6(15\Lambda_1^2 + 16u)}. \tag{4.7}$$

For the simple zero J has a simple pole and a monodromy conjugate to T, whereas for $u = 3/4\Lambda_1^2$ we find that $J = 0$. Locally $J \sim -128(u - 3/4\Lambda_1^2)/(27\Lambda_1^2)$. This means that at the superconformal point $\tau_{eff} = \rho = (1 + i\sqrt{3})/2$, since $J(\rho) = 0$ and the monodromy is conjugate to $(ST)^{-1}$, which is of order 3. This implies that the coupling constant has no log-dependence at the superconformal point. We expect to happen this precisely for a conformal theory. Away from these special points all singularities are cusps of infinite order, which is responsible for the log–dependence of the effective coupling constant. This value of the coupling was also found for the superconformal point of [3]. The order 3 is natural in the sense that there are in fact three possibilities that two of three strong coupling singularities can collide.

For the $N_f = 2$ case we will restrict ourselves to the case $m_1 = m_2 = m$ to keep formulas simpler. The $N_f = 2$ discriminant (2.17) has two simple zeros at $u_{1,2} = -\Lambda_2^2/8 \mp \Lambda_2 m$ and one double zero at $u_3 = \Lambda_2^2/8 + m^2$. The J–function near the singularities behaves as follows:

$$\begin{aligned}
J &= \mp\frac{\Lambda_2(\pm\Lambda_2 + 2m)^2}{216m}(u - u_i)^{-1}, \quad u = u_{1,2} \\
J &= \frac{(-\Lambda_2^2 + 4m^2)^4}{108\Lambda_2^4}(u - u_3)^{-2}, \quad u = u_3 \\
J &= \frac{64}{27\Lambda_2^4}u^2, \quad u = \infty.
\end{aligned} \tag{4.8}$$

At $u = u_{1,2}$ we have $\tau = 1$, which is one of the three cusps of the fundamental domain of the massless $N_f = 2$ theory $\Gamma(2)$, which has cusps of infinite order at $\tau = 0, 1$ and $i\infty$. We find two monodromies conjugate to T, which collide for $m = 0$ where they produce a monodromy conjugate to T^2. They correspond to two mutually local particles and therefore they cannot generate a superconformal point at $m = 0$. On the other hand there is a monodromy conjugate to T^2 at $u = u_3$ and $\tau = 0$, which splits into two singularities with monodromies conjugate to T for generic values of m_1 and m_2. There exist two $(2,1)$ superconformal points $m = \pm\Lambda_2/2$, $u = 3/8\Lambda_2^2$ and two more if we choose $m_1 = -m_2$. Near this point J takes the following form:

$$J_{(2,1)} = \frac{(3\Lambda_2^2 + 8u)^3}{27\Lambda_2^4(5\Lambda_2^2 + 8u)}. \tag{4.9}$$

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At $u = 3/8\Lambda_2^2$ we have $J = 1$, which implies that $\tau_{eff} = i$ at the superconformal point and the respective monodromy is conjugate to S^{-1}. Notice, at this point it is not u_1 and u_2 which collide (they correspond to mutually local dyons which contribute to the same cusp), but u_2 and u_3 or u_1 and u_3 for $m = \pm\Lambda_2/2$, respectively, where two mutually local and one mutually non–local particles become massless.

Finally, we consider the $N_f = 3$ case. Again, we choose all masses to be equal $m_1 = m_2 = m_3 = m$. In the massless case there are two strong coupling monodromies, one at $u_1 = 0$, which corresponds to four monopoles and one at $u_2 = \Lambda_3^2/256$ with a dyon of quantum numbers $(2, 1)$. The leading terms of the J–invariant at the singularities are for $m = 0$:

$$
\begin{aligned}
J &= \frac{\Lambda_3^8}{2^{22}3^3} u^{-4} \ , \quad u = u_1 \\
J &= -\frac{\Lambda_3^2}{2^{10}3^3}(u - u_2)^{-1} \ , \quad u = u_2 \\
J &= -\frac{64}{27\Lambda_2^4} u \ , \quad u = \infty \ .
\end{aligned}
\tag{4.10}
$$

It is easy to see that this is in agreement with the form of the monodromies, i.e. they are conjugate to T^4, T and T^{-4}, respectively. If we turn on the mass m the global flavour symmetry $SU(4)$ is broken to $SU(3) \times U(1)$ and the four–fold singularity at $u = 0$ splits up into a simple singularity and a threefold singularity with monodromies conjugate to T and T^3. As pointed out in [4] there are several possibilities to get superconformal points of type $(1,1)$ and $(2,1)$ for varying masses, but we only treat the special points of type $(3,1)$. This point occurs at $m = \Lambda_3/8$, where the $(2, 1)$ dyon singularity coincides with the triple singularity $u = \Lambda_3^2/32$ and there is another dyon singularity at $u = -19\Lambda_3^2/256$. For this special value of m the J–function takes the form:

$$
J_{(3,1)}(u) = \frac{-16(32u - \Lambda_3^2)^2}{27\Lambda_3^2(19\Lambda_3^2 + 256u)} \ .
\tag{4.11}
$$

At the superconformal point J vanishes, which means that $\tau_{eff} = \rho$. With $E_4(\rho) = 0$, we recover that very easily from (3.28). The monodromy is conjugate to $(ST)^{-2}$ and of order 3.

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