View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by CERN Document Server

CERN-TH/96-254
hep-th/9609112

Open strings and D-branes in WZNW models

C. Klimčík[1]
Theory Division, CERN,
CH-1211 Geneva 23, Switzerland

and

P. Ševera
Department of Theoretical Physics, Charles University,
V Holešovičkách 2, CZ-18000 Praha, Czech Republic

Abstract

An abundance of the Poisson-Lie symmetries of the WZNW models is uncovered. They give rise, via the Poisson-Lie T-duality, to a rich structure of the dual pairs of D-branes configurations in group manifolds. The D-branes are characterized by their shapes and certain two-forms living on them. The WZNW path integral for the interacting D-branes diagrams is unambiguously defined if the two-form on the D-brane and the WZNW three-form on the group form an integer-valued cocycle in the relative singular cohomology of the group manifold with respect to its D-brane submanifold. An example of the $SU(N)$ WZNW model is studied in some detail.

CERN-TH/96-254
September 1996

[1] Address after 1 October 1996: IHES, 91440 Bures-sur-Yvette, France

The Poisson–Lie (PL) T-duality [1] is a generalization of the traditional non-Abelian T-duality [2]–[5] and it proved to enjoy [1], [7]–[13], at least at the classical level, all of the structural features of the traditional Abelian T-duality [14] and [15]. In particular, our so far last paper on the subject [13] has settled (at the classical level) the remaining big issue of the PL generalization: the momentum-winding exchange.

It is now of an obvious interest to promote the PL T-duality to the quantum world. Strictly speaking, a consistent quantum picture does not necessarilly imply that mutually dual quantum models have to be conformally invariant. However, we do wish to have *conformal* examples in order to apply the PL T-duality in string theory. In this paper we shall show that such conformal examples of PL dualizable σ-models are the standard WZNW models and we shall give the detailed classical account of the PL T-duality for them. The treatment of the first quantized strings we postpone to a forth-coming publication, where an emergence of a proliferation of quantum group structures seems unavoidable.

In what follows, we shall demonstrate that a PL dualizable σ-model satisfying only a certain mild algebraic condition is necessarily a WZNW model. This means that the WZNW models are not only 'some' conformal examples of the dualizable models but, in a sence, they are very characteristic for the structure of the PL T-duality. Moreover, for various Drinfeld doubles underlying the structure of PL T-duality one recovers the same WZNW model! Hence, there are many (in fact infinitely many) Poisson-Lie symmetries in WZNW models.

It turns out that the dual to the WZNW model is again the same WZNW model. This should not be interpreted as a drawback. After all, what really matters is the fact that this (self)-duality induces a non-trivial non-local map on the phase space of the model which, in particular, reshuffles zero modes of the string, much in the same way as in the Abelian T-duality. The fundamental groups of the compact non-Abelian groups[2] are rather small therefore the momentum-winding exchange for closed strings may be rather modest (cf. [13]). On the other hand, the duality transformation of the zero modes of open strings gives the rich and spectacular structure in the dual: the celebrated D-branes [16].

We have devoted one paper in our series to the PL T-duality between

[2]We wish to consider the compact groups for the string theory compactifications.

open strings and D-branes [11]. It describes the geometries of the D-branes for arbitrary perfect[3] Drinfeld double in terms of the symplectic leaves of the associated Poisson homogeneous spaces [11]. In this contribution, we have to describe the open strings - D-branes duality also for non-perfect doubles in order to account for the WZNW models. We shall again obtain a rich geometry of the D-branes dictated by a simple structure on the double. We wish to stress at this point that the PL D-branes are very different from the standard Abelian D-branes. In the latter case the D-branes are just points in the direction of the space-time coordinates with respect to which one performs the Abelian duality and they become extended objects only in the direction of the extra (spectator or Buscher) duality intact coordinates. In the PL case, however, the D-branes are not points even without extending the space-time by the spectator coordinates! They may posses quite a complicated geometry, as we shall see later on.

Our presentation will contain also the case of open strings in WZNW models. This is generally not a well defined system because it requires a choice of the two-form potential of the WZNW three-form. This potential is not only ambiguous but it must be also singular because the WZNW three-form is a nontrivial element of the third de Rham cohomology on the (compact) group manifold and, as such, it does not admit a globally defined potential. It may therefore seem that there is a lot of arbitrariness in defining open strings in WZNW models. One has to choose the singular points of the two-form potential and the potential itself with the condition that the dynamics will disallow the end-points of strings to hit the singularity[4].

Some CFT results have been already obtained for open strings in $SU(2)$ WZNW model in [19], however, we did not find a discuusion of the subtle issue of the meaning of the WZNW term for open strings. In our case the arbitrariness in defining the WZNW model for open strings is completely fixed by the requirement of the PL symmetry. By picking up one of the

[3] Every element of the 'perfect' Drinfeld double can be uniquely written as the product of two elements of the two groups forming the double. Since the semi-Abelian Drinfeld doubles, that correspond to the traditional non-Abelian duality, are perfect we did give the complete picture of the T-duality between open strings and D-branes in this traditional case. Later also works by two different groups [17, 18] appeared, dealing with open strings in traditional non-Abelian duality.

[4] The bulk of the string feels only the exterior derivative of the two-form potential which is nothing but the perfectly regular WZNW three-form.

Drinfeld doubles corresponding to a given WZNW model and by fixing one half-dimensional isotropic subalgebra in the algebra of the double, we fix uniquely the singular two-form potential of the WZNW three-form and ensure that the end-points of strings do not hit the singular points on the target. Moreover, there exists a dual D-brane configuration and its geometry is again given in terms of the simple data on the double.

In the first section of this paper, we provide a topological discussion of conditions when the WZNW path integral is well defined for interacting string diagrams corresponding to a given D-brane configuration. Then we give the description of the Poisson-Lie symmetries occuring in the WZNW models in terms of the underlying Drinfeld doubles. In the third section we describe the D-brane configurations for a particular PL symmetry also for the non-perfect doubles and describe the classical phase space of the system. We also formulate an easy *non*-cohomological criterion when the underlying data on the double give a well defined WZNW path integral for the interacting D-branes diagrams. In the fourth section we provide examples of the general construction: PL symmetries and the D-branes in the $SU(N)$ WZNW models.

1 D-branes and the WZNW path integral.

The standard WZNW action on a group manifold R reads

$$S(r) \equiv \frac{1}{4\pi}\int d\xi^+ d\xi^- \langle \partial_+ r\, r^{-1}, \partial_- r\, r^{-1}\rangle + \frac{1}{24\pi}\int d^{-1}\langle dr\, r^{-1}, [dr\, r^{-1}, dr\, r^{-1}]\rangle. \tag{1}$$

Here $\xi^\pm$ are the standard lightcone variables on the world-sheet

$$\xi^\pm \equiv \frac{1}{2}(\tau \pm \sigma), \quad \partial_\pm \equiv \partial_\tau \pm \partial_\sigma \tag{2}$$

and $\langle .,. \rangle$ denotes a non-degenerate invariant bilinear form on the Lie algebra $\mathcal{R}$ of R. The second term in the WZNW action is commonly referred to as the WZNW term and it provide the action with the antisymmetric tensor part. It is well-known that this antisymmetric tensor B of the WZNW background is not globally defined (for compact groups) because the WZNW form Ω is a non-trivial cocycle in the third de Rham cohomology $H^3(R)$ of the group manifold R. Inspite of this, the classical WZNW theory is well defined for the case of closed strings. The reason is simple: Consider an evolving loop which sweeps out a cylindrical world-sheet $g(\sigma, \tau)$ on the group manifold. The variational problem requires fixing of the initial and final position of the loop and slightly varying the position of the cylinder between: $r(\sigma, \tau) \to r(\sigma, \tau) + \delta r(\sigma, \tau)$; $\delta r|_{initial,final} = 0$. The antisymmetric tensor part of the variation of the action can be thus written as

$$\int (r + \delta r)^* B - \int r^* B = \oint dB = \oint \Omega. \tag{3}$$

The integral $\oint$ is taken over the volume interpolating between the world-sheets r and $r + \delta r$ and * means the pull-back of the map. We conclude that the variation of the action does indeed depend only on the WZNW three-from Ω and not on a choice of its potential B. Note that the interpolating volume is given unambiguously because the variation of the action is infinitesimal.

A well known additional topological problem may occur if we wish to define a path integral for the WZNW theory of closed strings [20]: Consider a set of fixed loops in R and all world-sheets interpolating among them. We wish to evaluate the WZNW action S of every world-sheet s, form an expression $\exp iS$ and sum up it over all interpolating world-sheets of arbitratry topology. Suppose we choose some reference interpolating world-sheet s_{ref} and calculate its WZNW action S_{ref} for some choice of the potential B. The action S of any other world-sheet s can be computed in the same way. It is tempting to conclude that the difference $S - S_{ref}$ does not depend on the choice of the potential B. Indeed, by using the same argument as in the variational problem, we easily see that the difference of the integral of B over the both world-sheets is given solely in terms of the integral $\oint \Omega$ over the three-surface which interpolates between the world-sheets[5]. But now the two world-sheets do not differ only infinitesimaly! It therefore seems that the interpolating three-surface is not given unambiguously. The way to get out of the trouble lies in comparing the quantity $S - S_{ref}$ for two non-homotopical three-surfaces interpolating between s and s_{ref}. This difference is obviously given in terms of the integral $\oint \Omega$ over a three-cycle obtained by taking the difference of (or the sum of oppositely oriented) non-homotopical

[5] We shall alway assume that the group manifold in question is simply connected. By Hurewicz isomorphism and the fact the second homotopy group of any Lie group vanishes we thus have that the second cohomology of the simply connected group manifold vanishes. This means that the interpolating three-surface always exists.

three-surfaces interpolating between s and s_{ref}. Fortunately, the WZNW three-form Ω is an integer-valued cocycle [20] in $H_3(R)$ hence it is enough to normalize action S properly in order to ensure that the quantities $S - S_{ref}$ differ by a term $2\pi k, k \in Z$ for any two interpolating three-surfaces. These $2\pi k$ terms do not contribute to the path integral and, moreover, a dependence on the reference surface s_{ref} results only in an unobservable change of the total phase of the path integral. We finish this little review by concluding that the WZNW path integral is well defined for the case of the interacting closed strings.

Consider now a D-branes configuration in the group target R. By this we simply mean that there are two given submanifolds D_i and D_f of R and open strings propagate on R in such a way that their end-points i and f stick on the D-branes D_i and D_f, respectively. We define the WZNW theory for this D-branes configuration by choosing two-forms α_i and α_f, living respectively on D_i and D_f such that

$$d\alpha_{i(f)} = \Omega|_{D_i(D_f)}. \tag{4}$$

In words: the exterior derivative of $\alpha_{i(f)}$ has to be equal to the restriction of the WZNW three form Ω to the D-brane $D_{i(f)}$.

The construction of the $WZNW$ theory based on the triplet $(\Omega, \alpha_i, \alpha_f)$ goes as follows: Pick up an open string $r(\sigma, \tau)$ with the topology of an open strip. The variational problem requires fixing of the initial and the final positions of the string on the target. Consider now such a variation $\delta r(\sigma, \tau), \delta r(\sigma, \tau_{i,f}) = 0$. The both original open strip and its variation form together a closed strip (a 'diadem'), whose edges lie on the opposite D-branes. We can define the variation δS_{WZNW} of the WZNW term of the WZNW action by choosing an interpolating surface $\Sigma_{i(f)} \subset D_{i(f)}$ between the edges of the original and the varied strip. This variation then reads

$$\delta S_{WZNW} = \oint \Omega - \int_{\Sigma_i} \alpha_i - \int_{\Sigma_f} \alpha_f, \tag{5}$$

where the $\oint \Omega$ is taken over the volume of the figure enclosed by Σ_i, Σ_f, the original strip and its variation. Note that this variation does not depend on the choice of the interpolating surface $\Sigma_{i(f)}$ because $d\alpha = \Omega|_D$ and all infinitesimal interpolating surfaces are mutually homotopic. Hence we conclude, that the classical WZNW theory of open strings with end-points on the D-branes is well defined in terms of the triplet $(\Omega, \alpha_i, \alpha_f)$.

The reader may wish to have a more concrete idea of how to compute the WZNW action of a single strip. For a particular choice of the potential B ($dB = \Omega$) the combination $\alpha - B$ on the D-brane is a closed form, hence, at least locally, it has a potential A on D. The WZNW action S for an open string configuration $r(\sigma, \tau)$ which sweeps out a two-surface s in the target R and respects the D-branes boundary conditions can now be written as follows

$$4\pi S(r) = \int \langle \partial_+ r\ r^{-1}, \partial_- r\ r^{-1} \rangle + \int_s B + \int_{\delta s \cap D} A. \tag{6}$$

Upon a change of

$$B \to B + d\lambda, \tag{7}$$

A has to be replaced by

$$A - \lambda|_D. \tag{8}$$

We may intepret the A-term of the action (6) as if there were equal and opposite charges on the end-points of the string which feel the electromagnetic fields A_i and A_f on the D-branes. This interpretation does not have an invariant meaning, however, because of the 'gauge invariance' (7) and (8). Moreover it holds only locally. We stress that the *global* invariant description of the WZNW model for D-branes configuration is given in terms of the triplet $(\Omega, \alpha_i, \alpha_f)$. We remark that in general there is no natural *closed* two-form living on the D-branes. This is true only in the case if the restriction of the WZNW three-form Ω on the D-brane vanishes. Note also that if the D-brane is as many dimensional as the whole group target R is, then the form α is nothing but some concrete choice of the potential B which, however, may be different for the different end-points of the string.

At the presence of the D-branes and open strings, the discussion of the string path integral is more involved as before. The group manifold will be always taken as simply connected and, for a while, we consider the case where also the D-branes are connected and simply connected. Now draw a general string diagram respecting the D-branes configurations. It is an interpolating world-sheet between a set of fixed open segments with end-points located on the D-branes and a fixed set of loops on the target R. Much as before, we can choose some reference interpolating world-sheet s_{ref} and calculate its WZNW part of the action S_{ref} for some choice of B and A according to the formula (6). Now we can take any other interpolating world-sheet s and calculate its action S in the same way. As in the case of the variational principle, the quantity $S - S_{ref}$ does not depend on the particular choice of B and A but only on the invariant globally defined triplet $(\Omega, \alpha_i, \alpha_f)$. The reason for this is the following: the union of the intersections $(\partial s_{ref} \cap D_{i(f)}) \cup (\partial s \cap D_{i(f)})$ is a contractible cycle in $D_{i(f)}$, hence it is a boundary of some two-surface $\Sigma_{i(f)}$. Now the union $s \cup s_{ref} \cup \Sigma_i \cup \Sigma_f$ is a two-boundary of some interpolating three-surface in the group manifold, because the second cohomology of the group manifold vanishes by assumption. Then the antisymmetric tensor (the WZNW term) part of $S - S_{ref}$ is defined by (5) where $\oint$ is taken over the interpolating three-surface.

There occurs the same problem as for the closed strings, namely, the interpolating three-surfaces between s and s_{ref} do not have to be homotopically equivalent. This means that the quantity $S - S_{ref}$ may depend on the homotopy of the chosen interpolating three-surface. But if the ambiguity in $S - S_{ref}$ is only of the form $2\pi k, k \in Z$ then the term $\exp i(S - S_{ref})$ is unambiguous and the path integral is well defined.

It is not difficult to find a cohomological formulation of the condition of the integer-valued ambiguity. All what we need is the notion of the relative singular homology $H_*(R, D_i \cup D_f)$ of the manifold R with respect to its submanifolds D_i and D_f (with real coefficients). The relative chains are the elements of the vector space of the standard chains in R factorized by its subspace of all chains lying in $D_i \cup D_f$. The operation of taking the boundary is the standard one. The corresponding homology is the relative singular homology $H_*(R, D_i \cup D_f)$. The triplet $(\Omega, \alpha_i, \alpha_f)$ can act on a relative cycle γ by the following prescription

$$\langle (\Omega, \alpha_i, \alpha_f), \gamma \rangle \equiv \int_\gamma \Omega - \int_{D_i \cap \partial\gamma} \alpha_i - \int_{D_f \cap \partial\gamma} \alpha_f. \tag{9}$$

If the cycle γ is itself a boundary then the pairing vanishes because Ω is closed. Hence our triplet $(\Omega, \alpha_1, \alpha_f)$ is an element (cocycle) of the relative singular cohomology $H^*(R, D_i \cup D_f)$ because it vanishes on the boundary of any relative chain.

Now we may conclude that if the cocycle $(\Omega, \alpha_1, \alpha_2)$ is integer-valued[6] the WZNW path integral is well-defined. Indeed, if we choose two non-

[6] The precise statement is as follows: The cocycle $(\Omega, \alpha_1, \alpha_2)$ is integer-valued, if it lies in the image of the natural map from the singular cohomology with integer coefficients to the singular cohomology with real coefficients.

homotopical three-surfaces interpolating between the world-sheets s and s_{ref} their oriented sum is a closed cycle in the relative singular homology and its pairing (9) with the triplet $(\Omega, \alpha_1, \alpha_2)$ is integer-valued.

It turns out that we can extend our discussion to the case of connected but not necessarily simply connected D-branes. The main problem to be addressed is the fact that now the union of the intersections $(\partial s_{ref} \cap D_{i(f)}) \cup (\partial s \cap D_{i(f)})$ is not necessarily a contractible cycle in $D_{i(f)}$ (which means that $s \cup s_{ref}$ is a relative two-cycle but not a relative two-boundary). Thus the two-surface $\Sigma_{i(f)}$ does not have to exist and we cannot in general use the formula (5) in order to determine $\exp i(S - S_{ref})$. It may seem that we may take some reference world-sheet for each homotopy class of the one-chain $\partial s \cap D_{i(f)}$ and assign it an arbitrary reference phase. But there is still a consistency condition that under summing of the relative two-cycles (unions of s and s_{ref}) the phases $\exp iS$ should be additive!

Recall that we can unambiguously assign the $\exp iS$ to every relative two-boundary in such a way that this mapping is homomorphism f from the group B of relative two-boundaries (with integer coefficients) into the group U of complex units (phases). The consistency condition means that there should exist an extension $\tilde{f} : Z \to U$ of this homomorphism defined on the group Z of all relative two-cycles. We now prove that such an extension always exists because U is the divisible group (this means that the equation $nx = a, a \in U, n \in \mathbf{N}$ has always a solution $x \in U$).

Consider the group $H_f = Z + U/\{b - f(b), b \in B\}$. We have an exact sequence

$$0 \to U \to H_f \to H \to 0, \tag{10}$$

where $H \equiv H_2(R, D_i \cup D_f) = Z/B$ and all homomorphisms are naturally defined. Suppose now that we do have an extension $\tilde{f} : Z \to U$ of the map $f : B \to U$. Such an extension enables us to write

$$H_f = H + U. \tag{11}$$

In words: H_f is a direct sum of H and U. Indeed, for $z + c, z \in Z, c \in U$ we have

$$z + c = (z - \tilde{f}(z)) + (0 + c + \tilde{f}(z)). \tag{12}$$

Evidently, the first term on the right hand side is from Z and the second from U. The decomposition (12) is consistent with the factorization by $\{b -$

$f(b), b \in B\}$ because $\tilde{f}$ is the homomorphism. The converse is also true: if we can write H_f as the direct sum $H + U$ then there exists an extension $\tilde{f} : Z \to U$ which is a homomorphism. Indeed, consider $z \in Z$ and embed it naturally into H_f i.e. $z \to z + 0 \in H_f$. $z + 0$ can be decomposed as $y + g, y \in H, g \in U$ by assumption, hence we obtain a natural homomorphism from Z into U: $z \to g$. This homomorphism is the extension of f which we look for.

Summarizing, if we prove that H_f is the direct sum of H and U, we are guaranteed that the extension $\tilde{f} : Z \to U$ always exists. But it is easy to prove this, by using the well-known result from the homological algebra that every extension of an (Abelian) group G by a divisible group X is necessarily the direct sum of G and X. In our case, we know from the exact sequence (10) that H_f is the extension of H by U. Therefore $H_f = H + U$, what was to be proved.

Notes:

1. We have a certain freedom in writing H_f as a direct sum of H and U which is described by the group of homomorphisms $Hom(H, U)$. The easiest way to see it is by noting that if we have an extension $\tilde{f} : Z \to U$ it can be modified by adding to it any homomorphism which vanishes on $b \in B$. Any such homomorphism is obviously from $Hom(H, U)$. The modified $\tilde{f}$ then gives another partition of H_f into the direct sum of H and U.

2. It may be instructive to relate the group H of the relative two-cycles with the fundamental groups π_1 of the D-branes. We have a natural exact sequence

$$0 = H_2(R) \to H_2(R, D_i \cup D_f) \to H_1(D_i) + H_1(D_f) \to 0 = H_1(R). \quad (13)$$

Hence

$$H = H_1(D_i) + H_1(D_f) \quad (14)$$

and

$$H_1(D_{i(f)}) = \pi_1(D_{i(f)})/[\pi_1(D_{i(f)}), \pi_1(D_{i(f)})]. \quad (15)$$

The last equality is the Hurewicz isomorphism which holds due to the assumption that the D-branes are connected.

2 PL symmetries of WZNW models

For the description of the PL T-duality, we need the crucial concept of the Drinfeld double, which is simply a Lie group D such that its Lie algebra $\mathcal{D}$ (viewed as a vector space) can be decomposed as the direct sum of two subalgebras, $\mathcal{G}$ and $\tilde{\mathcal{G}}$, maximally isotropic with respect to a non-degenerate invariant bilinear form on $\mathcal{D}$ [21]. It is often convenient to identify the dual linear space to $\mathcal{G}$ ($\tilde{\mathcal{G}}$) with $\tilde{\mathcal{G}}$ ($\mathcal{G}$) via this bilinear form.

From the space-time point of view, we have identified the targets of the mutually dual σ-models with the cosets D/G and $D/\tilde{G}$ [13]. Here D denotes the Drinfeld double, and G and $\tilde{G}$ two its mutually dual isotropic subgroups. In the special case when the decomposition $D = \tilde{G}G = G\tilde{G}$ holds globally, the corresponding cosets turn out to be the group manifolds $\tilde{G}$ and G, respectively [1].

The actions of mutually dual σ-models are encoded in a choice of an n-dimensional linear subspace $\mathcal{R}$ of the $2n$-dimensional Lie algebra $\mathcal{D}$ of the double D which is transversal to both $\mathcal{G}$ and $\tilde{\mathcal{G}}$. The σ-model actions on the targets D/G and $D/\tilde{G}$ have a similar structure; indeed, on D/G we have [13]

$$S = \frac{1}{2}I(f) - \frac{1}{4\pi}\int d\xi^+ d\xi^- \langle \partial_+ f\ f^{-1}, R^a_-\rangle (M_-^{-1})_{ab} \langle f^{-1}\partial_- f, T^b\rangle, \qquad (16)$$

where $f \in D$ is some local section of the D/G fibration which parametrizes the points of the coset. Recall [13] that

$$M^{ab}_\pm \equiv \langle T^a, f^{-1} R^b_\pm f\rangle \qquad (17)$$

and R^a_- (R^a_+) are vectors of an orthonormal basis of $\mathcal{R}$ ($\mathcal{R}^\perp$):

$$\langle R^a_\pm, R^b_\pm\rangle = \pm\delta^{ab}, \qquad \langle R^a_+, R^b_-\rangle = 0. \qquad (18)$$

The action of the dual σ-model on the coset $D/\tilde{G}$ has the same form; just the generators T^a of $\mathcal{G}$ are replaced by the generators $\tilde{T}_a$ of $\tilde{\mathcal{G}}$ and f will parametrize $D/\tilde{G}$ instead of D/G.

We have referred to the σ-models of the form (16) as those having a PL symmetry [13]. There is an important feature of such models, namely, their field equations can be written as the zero curvature condition valued in the algebra $\mathcal{G}$. Indeed,

$$d\lambda - \lambda^2 = 0, \qquad (19)$$

where

$$\lambda = \lambda_+ d\xi^+ + \lambda_- d\xi^- \tag{20}$$

and

$$\lambda_\pm = -\langle \partial_\pm f\ f^{-1}, R^a_\mp \rangle (M^{-1}_\mp)_{ab} T^b. \tag{21}$$

So far we have been reviewing the results of [13]; now a new observation comes: If the subspace $\mathcal{R}$ is itself a Lie algebra of a compact subgroup R of the double D then the model (16) is essentially the WZNW model on the target R for the both choices D/G and $D/\tilde{G}$! The argument goes in two steps:

1. $\mathcal{R}$ can be transported by the right action to the tangent space of every point of the double. Because $\mathcal{R}$ is the subalgebra, the distribution of the planes $\mathcal{R}$ in the tangent bundle of the double is integrable and it foliates the double into fibration with fibres R and basis $R\backslash D$. Since $\mathcal{R}$ is transversal to the both $\mathcal{G}$ and $\tilde{\mathcal{G}}$ (which means that it intersects $\mathcal{G}$ and $\tilde{\mathcal{G}}$ only in O) , any fiber of the R fibration either intersects the fiber G (or $\tilde{G}$) in some finite subgroup $R \cap G$ of R or does not intersect it at all. The latter cannot be true, however, if the group R is compact. Indeed, R acts on D/G by the left action. The R orbit of the element of D/G which has the unit element of D on its fiber is open. Since R is compact this orbit must be also closed which for connected doubles imply that this orbit is the whole D/G. In other words, there always exists an intersection of R and G. The argument for $D/\tilde{G}$ is the same.

If the finite subgroups $R \cap G$ and $R \cap \tilde{G}$ have only one element for both fibers G and $\tilde{G}$, respectively, it si not dificult to see that the both cosets D/G and $D/\tilde{G}$ can be globally identified with R. In general, the cosets D/G and $D/\tilde{G}$ can be identified with the discrete cosets $R/R \cap G$ and $R/R \cap \tilde{G}$, respectively.

2. For simplicity, consider only the case when R can be directly identified with D/G and $D/\tilde{G}$. In this case, we can choose the field $f(\sigma, \tau)$ in (16) to have values in R. Note that we can choose the basis R^a_- dependent on f in such a way that the combinations $f^{-1} R^a_- f$ are f independent. Then we can choose the basis T^a in such a way that $M_-(f)$ is the identity matrix. We have

$$\langle \partial_+ f\ f^{-1}, R^a_- \rangle = \langle f^{-1}\partial_+ f, f^{-1} R^a_- f \rangle \equiv (f^{-1}\partial_+ f)^a \tag{22}$$

and

$$\langle f^{-1}\partial_- f, T^a \rangle = \langle f^{-1}\partial_- f, f^{-1} R^c_- f \rangle M^{ca}_- = (f^{-1}\partial_- f)^a, \tag{23}$$

because M_- is the identity matrix. Putting (16),(22) and (23) together, we obtain

$$S = \frac{1}{2}I(f) - \frac{1}{4\pi}\int d\xi^+ d\xi^- (\partial_+ f\ f^{-1})^a \delta_{ab} (\partial_- f\ f^{-1})^b = -\frac{1}{2}I(f^{-1}). \quad (24)$$

We conclude, that the mutually dual σ-models on the cosets D/G and $D/\tilde{G}$ are the same, being equal to the WZNW model on R. In general, D/G ($D/\tilde{G}$) model is WZNW model on the target $R/R\cap G$ ($R/R\cap\tilde{G}$).
Notes:
1. The fact that the both models D/G and $D/\tilde{G}$ may be identical does not mean at all that the duality transformation is trivial. In fact, the PL T-duality always implies an existence of a non-trivial non-local transformation on the phase space of the $WZNW$ model. We shall explicitly describe this transformation in the next section.
2. It often happens (cf. section 4) that a compact group R can be embedded in many inequivalent ways into various Drinfeld doubles in such a way that the both cosets D/G and $D/\tilde{G}$ can be identified with R. In this case we have the abundance of the Poisson-Lie symmetries of the same WZNW model on the group manifold R, each of them corresponding to the double into which R is embedded.

3 D-branes in WZNW models

3.1 General discussion

For the further discussion of the D-branes, it is convenient to recall [13] the common 'roof' of the both models described by (16). They can be derived form the first order Hamiltonian action for field configurations $l(\sigma,\tau)\in D$:

$$S[l(\tau,\sigma)] =$$

$$= \frac{1}{8\pi}\int \Big\{ \langle \partial_\sigma l\ l^{-1}, \partial_\tau l\ l^{-1}\rangle + \frac{1}{6} d^{-1}\langle dl\ l^{-1}, [dl\ l^{-1}, dl\ l^{-1}]\rangle - \langle \partial_\sigma l l^{-1}, A\partial_\sigma l l^{-1}\rangle \Big\}. \quad (25)$$

Here A is a linear idempotent self-adjoint map from the Lie algebra $\mathcal{D}$ of the double into itself. It has two equally degenerated eigenvalues $+1$ and -1, and the corresponding eigenspaces are just $\mathcal{R}^\perp$ and $\mathcal{R}$ respectively.

As it stands, the action (25) is well defined only for the periodic functions of σ because of the WZNW term. This restriction corresponds to the case of closed strings [13] . The σ-model actions (16) are obtained from the duality invariant first order action (25) as follows: Consider the right coset D/G and parametrize it by the elements f of D [7]. With this parametrization of D/G we may parametrize the surface $l(\tau,\sigma)$ in the double as follows

$$l(\tau,\sigma) = f(\tau,\sigma)g(\tau,\sigma), \quad g \in G. \tag{26}$$

The action S then becomes

$$S(f,\Lambda \equiv \partial_\sigma g g^{-1}) = \frac{1}{2}I(f) - \frac{1}{2\pi}\int d\xi^+ d\xi^- \Big\{\langle \Lambda - \frac{1}{2}f^{-1}\partial_- f, \Lambda - \frac{1}{2}f^{-1}\partial_- f\rangle$$

$$+\langle f\Lambda f^{-1} + \partial_\sigma f f^{-1}, R_-^a\rangle\langle R_-^a, f\Lambda f^{-1} + \partial_\sigma f f^{-1}\rangle\Big\}. \tag{27}$$

Now it is easy to eliminate Λ from the action (27) and finish with the σ-model action (16). In the case of the coset $D/\tilde{G}$, the procedure is exactly analoguous.

Consider the case of open strings for a generic double D with vanishing second cohomology. In our previous paper on the subject [11], we have studied only the perfect doubles (cf. footnote 3) nevertheless we can easily generalize the construction.

Let F be a simply connected subgroup of the double D whose Lie algebra $\mathcal{F}$ is isotropic with respect to the bilinear form on $\mathcal{D}$. This subgroup, as a manifold, can be shifted by the right action of some element $d \in D$ (note that all non-equivalent shifts are parametrized by the coset $F\backslash D$). We declare that the manifolds $F \hookrightarrow D$ and $Fd \hookrightarrow D$ are D-branes in the double D. Consider now oriented open strings in D with the initial end-points on F and the final end-points on Fd. Their dynamics in the bulk is governed by the action (25) which contains the WZNW term. As we have learnt in the previous section such an action is well-defined provided we choose some two-forms on the D-branes such that the exterior derivative of them is equal to the restriction of the $WZNW$ three-form on the D-branes. In our present case, this restriction of the WZNW form vanishes in either of our D-branes because F and Fd are the isotropic surfaces in D. Thus we have to choose

[7]If there exists no global section of this fibration, we can choose several local sections covering the whole base space D/G.

some closed two forms on F and Fd; we choose them to vanish identically. We summarize that our open string dynamics is fully defined by the action (25), the D-branes boundary conditions and the vanishing two-forms on the D-branes.

Much as in the closed string case, we can derive the open string σ-model dynamics on the cosets D/G and $D/\tilde{G}$ from (25) and the D-branes data on the double; for concreteness let us consider the coset D/G:

As we have learnt in section 2, the WZNW model for open strings is fully defined if we manage to compute the WZNW action of the 'diadem'. Recall that the diadem is composed of two evolving open string world-sheets which are glued together at some initial and final times. The edges of the diadem , swept by the end-points of the open strings, lie in their corresponding D-branes.

Consider now the diadem in the double. We can choose some two-surface Σ (Σ_d) in the D-brane F (Fd) whose boundary is just the edge of the diadem lying in F (Fd). The diadem together with the surfaces Σ and Σ_d form a boundary of some three-dimensional domain γ. We may write the action S of the model (25) as

$$S = S_0 + S_{WZNW}, \tag{28}$$

where S_{WZNW} contains solely the term with the WZNW three-form c on D. Hence, the action of the diadem can be written as[8]

$$S = S_0 + \frac{1}{8\pi}\int_\gamma c. \tag{29}$$

Again, consider the parametrization of D/G by the elements f of D. A surface $l(\tau,\sigma)$ in the double (respecting the D-branes boundary conditions), can be written as

$$l(\tau,\sigma) = f(\tau,\sigma)g(\tau,\sigma), \quad g \in G. \tag{30}$$

The decomposition (30) induces two maps from D into D: $f(l) = f$ and $g(l) = g$. Consider now the Polyakov-Wiegmann (PW) formula [22]

$$(fg)^*c = f^*c + g^*c - d\langle f^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dl\ l^{-1})\rangle, \tag{31}$$

[8]Note that we have included the factor 1/6 from (25) in the definition of c.

where, as usual, $*$ denotes the pull-back of the forms under the mappings to the group manifold D. By using the PW formula, we can rewrite (29) as

$$S = S_0(fg) + \frac{1}{8\pi}\int_\gamma f^*c - \frac{1}{8\pi}\int_{diad\cup\Sigma\cup\Sigma_d} \langle f^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dl\ l^{-1})\rangle. \qquad (32)$$

Note that g^*c vanishes because of the isotropy of G. The action S now becomes

$$S(f,\Lambda\equiv\partial_\sigma gg^{-1}) = \frac{1}{2\pi}\int_{diad}\Big\{\frac{1}{4}\langle\partial_+f\ f^{-1},\partial_-f\ f^{-1}\rangle$$

$$-\langle\Lambda-\frac{1}{2}f^{-1}\partial_-f,\Lambda-\frac{1}{2}f^{-1}\partial_-f\rangle+\langle f\Lambda f^{-1}+\partial_\sigma ff^{-1},R_-^a\rangle\langle R_-^a,f\Lambda f^{-1}+\partial_\sigma ff^{-1}\rangle\Big\}$$

$$+\frac{1}{8\pi}\int_\gamma f^*c - \frac{1}{8\pi}\int_{\Sigma\cup\Sigma_d}\langle f^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dl\ l^{-1})\rangle. \qquad (33)$$

Of course, this is a similar expression as before (cf. (27)). However, the field f respects different boundary conditions. A configuration f is an open string configuration; its end-points stick on D-branes D_i and D_f in D/G which are obviously obtained just by projecting the D-branes F and Fd from the double D into the basis D/G parametrized by the section f.

Now we have to realize that upon varying Λ the last term in (33) vanishes! This follows from the isotropy of F, Fd and G. Indeed, if we have $fg\in F$ ($fg\in Fd$) and vary $g\to g\delta g$ at fixed f in such a way that $fg\delta g\in F$ ($fg\delta g\in Fd$), we observe that the last term in (33) does not change[9]. Hence we can eliminate the field Λ from (33) in the same way as from (27). The result is

$$S = \frac{1}{8\pi}\int_{diad}\Big\{\langle\partial_+f\ f^{-1},\partial_-f\ f^{-1}\rangle - 2\langle\partial_+f\ f^{-1},R_-^a\rangle(M_-^{-1})_{ab}\langle f^{-1}\partial_-f,T^b\rangle\Big\}$$

$$+\frac{1}{8\pi}\int_\gamma f^*c - \frac{1}{8\pi}\int_{\Sigma\cup\Sigma_d}\langle f^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dl\ l^{-1})\rangle.$$

Consider again the special situation in which the subspace $\mathcal{R}\equiv\mathrm{Span}R_-$ is the Lie algebra of the compact group R, moreover, R can be directly identified with D/G and $D/\tilde{G}$. Recall, that upon transporting $\mathcal{R}$ by the

[9]It is easy to see that $\delta g\in F\cap G$ ($\delta g\in F\cap dGd^{-1}$).

right action everywhere onto the double, we get the fibration of D with the fibers R and the basis $R\backslash D$. With some abuse of the notation, the fiber crossing the unit element of the double we shall also denote as R. We choose the parametrization of the double as follows

$$l = rg, \quad r \in R, \quad g \in G. \tag{34}$$

This parametrization holds for every element l of the double and is unique by the assumption. Note that the restriction of the WZNW three-form c gives just the WZNW three-form c_R on R.

It is easy to see that the D-branes D_i and D_f in R, being the projections of F and Fd to R, can be identified with the cosets $F/F \cap G$ and $F/F \cap dGd^{-1}$ respectively. On the other hand we have just seen (cf. footnote 9) that the variation $\delta g \in F \cap G$ ($\delta g \in F \cap dGd^{-1}$) leaves intact the two-form $\omega \equiv (1/8\pi)\langle r^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dl\ l^{-1})\rangle$ on F (Fd). This means that this two-form is a pull-back of some two-form α_i (α_f) from the D-brane D_i (D_f). Of course, the notation is not accidental; the two-forms $\alpha_{i(f)}$ are precisely those appearing in (5).

It is not difficult to find an explicit expression for $\alpha_{i(f)}$. For this, consider a map k_i (k_f) from D_i (D_f) into G such that

$$rk_{i(f)}(r) \in F(Fd), \quad r \in D_{i(f)}. \tag{35}$$

In general, the mapping k_i (k_f) is not defined unambiguously but it locally always exists since D_i (D_f) is just the projection of F (Fd) on R. Because two-form ω on F (Fd) is invariant under the variations from $F \cap G$ ($F \cap dGd^{-1}$) we can locally[10] write

$$\langle r^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dl\ l^{-1})\rangle|_{F(Fd)} = r^*\langle dr\ r^{-1} \stackrel{\wedge}{,} k_{i(f)}(r)^{-1}dk_{i(f)}(r)\rangle. \tag{36}$$

In other words, (36) is true independently of the choice of the map $k_{i(f)}$.

Thus in our special situation, the action of the diadem can be written as

$$S = -\frac{1}{8\pi}\int_{diad}\langle\partial_+ r\ r^{-1}, \partial_- r\ r^{-1}\rangle + \frac{1}{8\pi}\int_{r(\gamma)} c_R$$

$$-\frac{1}{8\pi}\int_{D_i}\langle dr\ r^{-1} \stackrel{\wedge}{,} k_i(r)^{-1}dk_i(r)\rangle - \frac{1}{8\pi}\int_{D_f}\langle dr\ r^{-1} \stackrel{\wedge}{,} k_f(r)^{-1}dk_i(r)\rangle. \tag{37}$$

[10] The two-form $\alpha_{i(f)}$ is defined *globally* on $D_{i(f)}$ only the explicit expression for it in terms of $k_{i(f)}$ may, in general, be written only locally.

We can read $\alpha_{i(f)}$ off directly from (37):

$$\alpha_{i(f)} = \frac{1}{8\pi}\langle dr\ r^{-1} \stackrel{\wedge}{,} k_{i(f)}(r)^{-1}dk_{i(f)}(r)\rangle. \tag{38}$$

It remains to prove that

$$d\alpha_{i(f)} = \frac{1}{8\pi}c_R|_{D_{i(f)}}. \tag{39}$$

This is easy: take the PW formula (31) and restrict all forms in it on the D-brane F (Fd) in the double. Then the form c vanishes by the isotropy of F (Fd). Hence

$$r^*c|_{F(Fd)} = d\langle r^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dl\ l^{-1})\rangle|_{F(Fd)} = r^*d\langle dr\ r^{-1} \stackrel{\wedge}{,} k_{i(f)}(r)^{-1}dk_{i(f)}(r)\rangle, \tag{40}$$

where the last equality follows from (36). Thus, upon removing the pull-back map r^*, we conclude that

$$\frac{1}{8\pi}c_R|_{D_{i(f)}} = \frac{1}{8\pi}d\langle r^{-1}dr \stackrel{\wedge}{,} dk_{i(f)}(r)k_{i(f)}(r)^{-1}\rangle = d\alpha_{i(f)}. \tag{41}$$

Remarks:
1. The model (37) has the 'wrong' sign in front of its first term. Upon the change of variables $r \to r^{-1}$ it gives the standard WZNW model on the group manifold R (cf. (1)). The D-branes $D_{i(f)}$ and the two forms $\alpha_{i(f)}$ on them have to be transformed correspondingly.
2. The geometry of the dual D-branes in $D/\tilde{G}$ is obtained in the same way as in the case D/G; it is enough to replace everywhere G by $\tilde{G}$.
3. We should mention that the Kiritsis-Obers duality [6] fits in our formalism. The double is the direct product of a compact group R with itself and the invariant bilinear form in the direct sum of the Lie algebras $\mathcal{R} + \mathcal{R}$ is the difference between the Killing-Cartan forms on each algebra. Hence, the diagonal embedding of R in $R \times R$ is isotropic. So it is the embedding in which second copy of $\mathcal{R}$ is twisted by some outer automorphism. The resulting duality is a D-branes D-branes duality, i.e. the D-branes have never the dimension of the group manifold.

3.2 The classical solvability

We wish to find the complete solution of the field equations of the model (25) submitted to the D-branes boundary conditions. It is not difficult to do that. The bulk equations following from (25) read

$$\langle \partial_\pm l \, l^{-1}, \mathcal{R}_\mp \rangle = 0. \tag{42}$$

We already know that after integrating away g from the decomposition (34) we get the WZNW model on R, hence, the solution l of (25) must look like

$$l(\sigma, \tau) = r_-(\xi^-) r_+(\xi^+) g(\xi^+). \tag{43}$$

The first two multiplicative terms on the right-hand-side follow from the known bulk solution of the WZNW model on R and the fact that g is only a function of ξ^+ follows from Eqs. (21).

Putting

$$h(\xi^+) \equiv r_+(\xi^+) g(\xi^+) \tag{44}$$

and inserting $l = r_-(\xi^-) h(\xi^+)$ into (37), we obtain

$$\partial_+ h \, h^{-1} \in \mathcal{R}_+ \equiv \mathcal{R}^\perp. \tag{45}$$

Here we have used the fact that $\mathcal{R}_-(\equiv \mathcal{R})$ is the Lie algebra of R. We conclude that every bulk solution of (25) look like

$$l = r_-(\xi^-) h(\xi^+), \qquad \partial_+ h \, h^{-1} \in \mathcal{R}_+. \tag{46}$$

It is important to note that $\mathcal{R}_+ \equiv \mathcal{R}^\perp$ does not have to be a Lie subalgebra of $\mathcal{D}$; in general it is just a linear subspace of $\mathcal{D}$.

Now we can take into account the effect of the boundary conditions. Recall that the initial point of the open string ($\sigma = 0$) should stick on the D-brane F in the double and the final point ($\sigma = \pi$) on the D-brane Fd; d is a fixed element of the double D. These two conditions can be rewritten as follows

$$r_-(\tau) h(\tau) = f_i(\tau), \qquad r_-(\tau - \pi) h(\tau) = f_f(\tau) d, \tag{47}$$

where f_i and f_f are some functions with values in the group F. It follows that

$$h^{-1}(\tau - \pi) h(\tau) = f_i^{-1}(\tau - \pi) f_f(\tau) d. \tag{48}$$

By differentiating Eq. (48) with respect to τ we obtain

$$-dh(\tau-\pi)h^{-1}(\tau-\pi)+dh(\tau)h^{-1}(\tau)=$$
$$=h(\tau-\pi)[-f_i^{-1}(\tau-\pi)df_i(\tau-\pi)+f_i^{-1}(\tau-\pi)df_f(\tau)f_f^{-1}(\tau)f_i(\tau-\pi)]h^{-1}(\tau-\pi). \tag{49}$$

Now we can bracket (49) with $\mathcal{R}$ which gives

$$df_f(\tau)f_f^{-1}(\tau)-df_i(\tau-\pi)f_i^{-1}(\tau-\pi)=0. \tag{50}$$

For deriving (50), we have used Eq. (47) and the fact that the Lie algebra $\mathcal{F}$ of F is transversal to $\mathcal{R}^\perp$.

By inserting (50) back in (49) we get a very important relation

$$dh(\tau+\pi)h^{-1}(\tau+\pi)=dh(\tau)h^{-1}(\tau). \tag{51}$$

It expresses the periodicity of the $\mathcal{R}^\perp$-valued 'connection' $dh\ h^{-1}$. The monodromy of this 'connection is also constrained; indeed, from (50) and (48) we conclude that

$$h^{-1}(\tau-\pi)h(\tau)=fd, \tag{52}$$

where f is some constant element of F. In words: the monodromy $h^{-1}(\tau-\pi)h(\tau)$ is an element of the double D which is equivalent to d in the sense of the coset $F\backslash D$.

Summary: The space of the solutions of the field equations (42) submitted to the D-branes boundary conditions (47) is given by an arbitrary element p of the double D and a periodic field $\rho(\xi^+)(\equiv dh\ h^{-1}(\xi^+))$ with values in the subspace $\mathcal{R}^\perp$ of $\mathcal{D}$ and with the monodromy

$$P\exp\int_{\tau-\pi}^{\tau} d\tau'\rho(\tau')\equiv h^{-1}(\tau-\pi)h(\tau) \tag{53}$$

equivalent to d in the sense of the coset $F\backslash D$. Of course, P in (53) means the ordered exponent. The full solution $l(\sigma,\tau)$ is then reconstructed as follows: take $\rho(\xi^+)$ and $p\in D$ and construct

$$h(\xi^+)=P\exp\{\int_{\xi_0^+}^{\xi^+} d\xi^{+'}\rho(\xi^{+'})\}\times p. \tag{54}$$

Obviously, the choice of ξ_0^+ is irrelevant and can be compensated by the corresponding change of p. Now we can reconstruct $r_-(\xi^-)$ by decomposing $h(\xi^-)$ as

$$h(\xi^-)=r_-^{-1}(\xi^-)f(\xi^-),\qquad r\in R,\quad f\in F. \tag{55}$$

This decomposition is unique, because R can be globally identified with D/F. Finally

$$l(\xi^+,\xi^-) = r_-(\xi^-)h(\xi^+). \tag{56}$$

It remains to recover from the solution (56) on the double the solutions of the σ-models on the cosets D/G and $D/\tilde{G}$. Recall that the both σ-models are the WZNW models on the group manifold R. In the D/G case we have to decompose h as

$$h(\xi^+) = r_+(\xi^+)g(\xi^+), \qquad r \in R, \quad g \in G, \tag{57}$$

while in the $D/\tilde{G}$ case as

$$h(\xi^+) = \tilde{r}_+(\xi^+)\tilde{g}(\xi^+), \qquad \tilde{r} \in R, \quad \tilde{g} \in \tilde{G}. \tag{58}$$

Because $r_+ \neq \tilde{r}_+$ we indeed obtain a nontrivial map from the phase space of the WZNW model with one set of the D-branes boundary conditions into the phase space of the same WZNW model but with the dual D-branes boundary conditions. The both phase spaces can be identified with the set of all solutions $l(\xi^+,\xi^-)$ on the double. The system (25) is already written in the Hamiltonian form, hence the mapping between the phase spaces is a canonical transformation.

Note: It is interesting that the both left movers $r_+(\xi^+)$ and right movers $r_-(\xi^-)$ are obtained from the master function $h(\xi^+)$ in a very similar way. Recall that

$$h(\xi^+) = r_+(\xi^+)g(\xi^+), \qquad h(\xi^-) = r_-^{-1}(\xi^-)f(\xi^-), \quad g \in G,\ f \in F. \tag{59}$$

In particular, if $G = F$ then the left and the right movers of the R WZNW model are given by the same function , i.e.

$$r(\sigma,\tau) = r_-(\xi^-)r_-^{-1}(\xi^+). \tag{60}$$

This means that the initial point $\sigma = 0$ of the string sits at the origin of the group R for all times. Indeed, the corresponding D-brane is just the group origin, being the projection of $F = G$ along G.

3.3 Interacting D-brane diagrams

Given a D-brane configuration on the target R we can in principle compute the WZNW path integral over all topologically non-trivial world-sheets interpolating between a set of fixed open string segments with end-points sitting on the D-branes and a set of fixed loops in the target R. We postpone the evaluation of some of such diagrams (like the open string propagator) to a forth-coming publication, here we just discuss whether there are some topological obstructions in doing that possibly coming from the WZNW term Ω and the two-forms α_i and α_f defined on the D_i and D_f by (38). We have learned in the section 2 that the WZNW path integral is well-defined if the triplet $(\Omega, \alpha_i, \alpha_f)$ is an integer-valued cocycle in the relative singular cohomology of the group manifold R with respect to its submanifold $D_i \cup D_f$. In general, we have found it to be a difficult topological problem to identify for which D-brane configuration $D_i \cup D_f$ and which choice of the two-forms α_i and α_f the cocycle $(\Omega, \alpha_i, \alpha_f)$ is integer-valued.

Fortunately enough, if the maximal compact subgroup of D is simple and simply connected, we have the key for solving our problem: we draw the interacting D-brane diagrams directly in the double and repeat the discussion of the section 2, using the duality invariant first-order action (25). The action (25) also contains the WZNW term but now the forms α vanish. This means that the pairing of the cocycle $(c, \alpha_F = 0, \alpha_{Fd} = 0)$ with any relative cycle γ is just

$$\langle (c, \alpha_F, \alpha_{Fd}), \gamma \rangle = \int_\gamma c. \tag{61}$$

Recall that we assumed $\pi_1(F) = 0$. By the Hurewicz isomorphism, we obtain $H^2(F) = 0$, hence every cycle in the relative singular homology $H_3(R, D_i \cup D_f)$ can be represented by a cycle in $H_3(R)$. This means that what matters is only whether c is the standard integer-valued three-cocycle in the third de Rham cohomology $H^3(D)$ of the Drinfeld double. But it is, because $H^3(D) = H^3(K)$, where K is the simple simply connected maximal compact subgroup of D, and it is known that the WZNW three-form restricted to K is the integer-valued cocycle.

We find quite appealing that the path integral for the D-branes configurations seems to be topologically more easily tractable by using the duality invariant formalism on the Drinfeld double. On the other hand, an account of the local world-sheet phenomena, like a short-distance behaviour, seems to be more difficult when working with the non-manifestly Lorentzian first order Hamiltonian action (25). We plan to study this issue in detail in a near future.

4 Example: $SU(N)$ WZNW model

Now we shall study examples of this general construction of the self-dual WZNW models. Consider the group $SL(N,\mathbf{C})$ viewed as the real group and the following invariant non-degenerate bilinear form on its algebra[11]

$$\langle X,Y\rangle = \mathrm{Im}[(a^*)^2\mathrm{Tr}XY], \qquad \mathrm{Im}a^2 = 4. \tag{62}$$

The group $SL(N,\mathbf{C})$ equipped with the invariant bilinear form is the Drinfeld double for every choice of the complex parameter a satisfying the normalization constraint in (62). Two isotropic subalgebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ of $\mathcal{D}$ are all upper and lower triangular matrices respectively with diagonal elements being $\lambda_k a,\ \lambda_k \in \mathbf{R}$ for $\mathcal{G}$ and $\tilde{\lambda}_k ia,\ \tilde{\lambda}_k \in \mathbf{R}$ for $\tilde{\mathcal{G}}$. Obviously, the index k denotes the position on the diagonal and lambdas are constrained by the tracelessness condition.

An example of $SL(2,\mathbf{C})$:

$$\mathcal{G} = \begin{pmatrix} \lambda a & z \\ 0 & -\lambda a \end{pmatrix}, \quad \tilde{\mathcal{G}} = \begin{pmatrix} \tilde{\lambda} ia & 0 \\ \tilde{z} & -\tilde{\lambda} ia \end{pmatrix}, \tag{63}$$

where $z, \tilde{z}$ are arbitrary complex numbers.

The dual pair of the σ-models is encoded in the choice of the half-dimensional subspace $\mathcal{R}$ of the Lie algebra $\mathcal{D}$ of the double. We choose $\mathcal{R}$ to be the $su(N)$ subalgebra of the algebra $sl(N,\mathbf{C})$. Following our discussion above, it is easy to find the principal fibrations of $SL(N,\mathbf{C}) \equiv D$, corresponding to the algebras $\mathcal{R}$, $\mathcal{G}$ and $\tilde{\mathcal{G}}$. The total space of the bundles is always the double D, the fibres are $SU(N)$, $\exp\mathcal{G} \equiv G$ and $\exp\tilde{\mathcal{G}} \equiv \tilde{G}$ and the bases are $SU(N)\backslash D$, D/G and $D/\tilde{G}$ respectively. Note that every fiber of all three fibrations can be obtained from the fiber crossing the unit element e of D by either the right (for $SU(N)$) or the left (for G and $\tilde{G}$) action of some

[11] The normalization of the bilinear form is always such that the resulting action of the $SU(N)$ WZNW model will be properly normalized in order to meet the requirement that the WZNW three-form is the integer-valued cocycle.

element of the double D. In the particular example of the double $SL(N,\mathbf{C})$, the intersection of a fibre $SU(N)$ with fibres G or $\tilde{G}$ occurs always precisely at one point. It is not difficult to prove this fact. We already know that the intersection always exists because $SU(N)$ is compact and $SL(N,\mathbf{C})$ is connected (cf. sec 3). If both fibers $SU(N)$ and G (or $\tilde{G}$) cross the unit element of the double (which is the intersection point), it is obvious that a non-unit element of G (or $\tilde{G}$) cannot be a unitary matrix. Thus the intersection is unique in this case. Also an intersection r of the $SU(N)$ fiber crossing the unit element of D with some fiber G must be unique. Indeed, the G fiber can be then written as rG where $r \in SU(N)$. By the left action of r^{-1} the G fiber can be transported to the origin of D where there is just one intersection.

Hence we conclude: for our data $D = SL(N,\mathbf{C}), G, \tilde{G}$ and $\mathcal{R} = su(N)$, the both models of the dual pair (16) are the standard $SU(N)$ WZNW models, because the restriction of the bilinear form (62) to $\mathcal{R}$ is nothing but the standard Killing-Cartan form on $su(N)$.

Now we may choose the subgroup F of D, which defines the D-branes in the double, to be equal to G. Thus we have given a concrete meaning to our so far abstract construction.

It may be of some interest to provide few explicit formulas for the $SL(2,\mathbf{C})$ Drinfeld double. The both cosets D/G and $D/\tilde{G}$ can be identified with the group $SU(2)$. Recall that the space of all D-branes corresponding to the choice F is parametrized by the *left* coset $F\backslash D$. In our case $F\backslash D$ can also be identified with $SU(2)$, hence a generic D-brane (Fd) in the double is a set of $SL(2,\mathbf{C})$ matrices of the form

$$Fd \equiv \begin{pmatrix} e^{\lambda a} & z \\ 0 & e^{-\lambda a} \end{pmatrix} \begin{pmatrix} C & -E^* \\ E & C^* \end{pmatrix}, \tag{64}$$

where C, E are fixed complex numbers satisfying $CC^* + EE^* = 1$, λ is a real and z a complex number. In order to get the D-branes in the cosets $D/G = D/F$ and $D/\tilde{G}$, we have to project Fd on $SU(2)$ along G and $\tilde{G}$, respectively:

$$Fd = \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix} \begin{pmatrix} e^{\eta a} & w \\ 0 & e^{-\eta a} \end{pmatrix}, \quad \eta \in \mathbf{R}, \quad w \in \mathbf{C}, \tag{65}$$

$$Fd = \begin{pmatrix} \tilde{A} & -\tilde{B}^* \\ \tilde{B} & \tilde{A}^* \end{pmatrix} \begin{pmatrix} e^{\tilde{\eta} i a} & 0 \\ \tilde{w} & e^{-\tilde{\eta} i a} \end{pmatrix}, \quad \tilde{\eta} \in \mathbf{R}, \quad \tilde{w} \in \mathbf{C}. \tag{66}$$

Here again $AA^* + BB^* = 1$ and the same constraint is of course true also for $\tilde{A}$ and $\tilde{B}$. If λ and z vary then A and B sweep a submanifold of $SU(2)$, which is just the D-brane in R and $\tilde{A}$ and $\tilde{B}$ sweep the dual D-brane in R.

There may occur three qualitatively different possibilities:

1. Both C and E do not vanish (a generic case).

Then $B \neq 0$ and it is convenient to parametrize D and B as

$$E = e^{E_1 a} e^{E_2 ia}, \qquad B = e^{B_1 a} e^{B_2 ia}, \quad B_i, E_i \in \mathbf{R}. \tag{67}$$

The original D-brane is then a two-dimensional submanifold of $SU(2)$ characterized by the condition

$$B_2 = E_2. \tag{68}$$

The dual D-brane is a three dimensional submanifold of $SU(2)$ which is complement of the circle $A = 0$.

2. $C = 0$.

The original D-brane is the same as in 1. but the dual D-brane is just the one-dimensional circle $A = 0$.

3. $E = 0$.

The original D-brane is a point $A = C$ and the dual D-brane is the same as in 1.

It is not difficult to compute also the two-form α on the D-brane (cf. (38)). For doing this, we have just to know the mapping $k(r)$ from the original D-brane in R to G and the dual mapping $\tilde{k}(r)$ from the dual D-brane in R to $\tilde{G}$ (cf. (35)). We do that for the case 3 choice $C = 1$ and $E = 0$. The original map k is trivial, since the D brane is just the point $A = 1$ but the dual map $\tilde{k}$ is nontrivial and it reads

$$\tilde{k}(A, B) = \begin{pmatrix} e^{iA_2 a} & 0 \\ -e^{-A_1 a} B & e^{-iA_2 a} \end{pmatrix}. \tag{69}$$

Here

$$0 \neq A^* \equiv e^{A_1 a} e^{iA_2 a}, \quad A_1, A_2 \in \mathbf{R}. \tag{70}$$

Now insert (69) in (38) and find

$$8\pi\tilde{\alpha}_f = \mathrm{Im}[a^{*2}\{(-BdB^* + B^*dB)a \wedge (A_1 + idA_2) + dB^* \wedge dB - 2ia^2 dA_1 \wedge dA_2\}]. \tag{71}$$

It is easy to compute the exterior derivative of $\tilde{\alpha}_f$:

$$8\pi d\tilde{\alpha}_f = 2\mathrm{Im}[a^{*2} a \; dB^* \wedge dB \wedge d(A_1 + iA_2)] = c_R|_{D_f}. \tag{72}$$

In words: the exterior derivative of α_f is equal to the restriction of the WZNW three-form c_R on the D-brane D_f.

It may be interesting to remark that in the case of the $SU(N)$ WZNW models there are no topological obstructions in quantizing the model on the topologically trivial open strip world-sheet. Thus, we do not have to lift the D-brane configuration to the double in order to make the argument but we can directly proceed at the level of the $D/G \equiv SU(N)$ target. Indeed, choose two different surfaces lying in the same D-brane and interpolating between the edges of the 'diadem'. Their oriented sum does not have a boundary and it is topologically the two-sphere. If we happen to show that the second homotopy group $\pi_2(D_{i(f)})$ of the D-brane $D_{i(f)}$ vanishes then the two interpolating surfaces are homotopically equivalent and there is no ambiguity coming from the WZNW term (cf. sec 2).

It is easy to prove that $\pi_2(D_f)$ vanishes if the D-brane D_f was obtained by our method of projecting the isotropic surface Fd from the double. Our basic tool is the long exact homotopy sequence [23]:

$$\pi_2(F) = 0 \to \pi_2(F/H) \to \pi_1(H) \to \pi_1(F) \to \pi_1(F/H) \to \pi_0(H) \to 0 = \pi_0(F), \tag{73}$$

which holds for a connected group F and its arbitrary subgroup H; note that π_2 of any Lie group vanishes. Now the D-brane on $SU(N)$ is gotten by projection of the surface Fd from the double to D/G. This means that topologically it can be identified with the coset $F/dGd^{-1} \cap F \equiv F/H$. We observe that in our $SL(N, \mathbf{C})$ context the group F can be topologically identified with its algebra $\mathcal{F}$ because the usual exponential mapping $\exp \mathcal{F} = F$ is one-to-one. So it is one-to-one for any its connected subgroup including the unity component of H. Hence $\pi_1(H) = 0$ and since $\pi_1(F) = 0$, from the sequence (73) we conclude that π_2 of the D-brane in $SU(N)$ vanishes.

We should mention that from the exact sequence (73) it also follows that $\pi_1(D_f) = \pi_0(H)$. In general, for our $SU(N)$ case the group H is not connected. This means that, strictly speaking, the diadem in our argument must be equivalent to the zero element of $H_2(R, D_f)$, or, in other, words it must be a relative two-boundary. If the diadem is a non-trivial relative two-cycle we use the results of section 2 and evaluate its contribution by choosing the extension $\tilde{f} : Z \to U$ of the homomorphism $f : B \to U$.

References

[1] C. Klimčík and P. Ševera, Phys. Lett. B351 (1995) 455, hep-th/9502122; P. Ševera, *Minimálne Plochy a Dualita*, Diploma thesis, Praha University, May 1995, in Slovak; C. Klimčík, Nucl. Phys. (Proc. Suppl.) 46 (1996) 116

[2] X. de la Ossa and F. Quevedo, Nucl. Phys. B403 (1993) 377

[3] B.E. Fridling and A. Jevicki, Phys. Lett. B134 (1984) 70

[4] E.S. Fradkin and A.A. Tseytlin, Ann. Phys. 162 (1985) 31

[5] A. Giveon and M. Roček, Nucl. Phys. B421 (1994) 173; E. Álvarez, L. Álvarez-Gaumé and Y. Lozano, Nucl. Phys. B424 (1994) 155; K. Sfetsos, Phys. Rev. D50 (1994) 2784; T. Curtright and C. Zachos, Phys. Rev. D49 (1994) 5408; O. Alvarez and C-H. Liu, Commun. Math. Phys. 179 (1996) 185; Y. Lozano, Phys. Lett. B355 (1995) 165

[6] E. Kiritsis and N. Obers, Phys. Lett. B334 (1994) 67

[7] E. Tyurin, *Non-Abelian Axial-Vector Duality: a Geometric Description*, hep-th/9507014

[8] A.Yu. Alekseev, C. Klimčík and A.A. Tseytlin, Nucl. Phys. B458 (1996) 430

[9] E. Tyurin and R. von Unge, *Poisson-Lie T-duality: the Path-Integral Derivation*, hep-th/9512025

[10] C. Klimčík and P. Ševera, Phys. Lett. B372 (1996) 65

[11] C. Klimčík and P. Ševera, Phys. Lett. B376 (1996) 82

[12] C. Klimčík and P. Ševera, Phys. Lett. B381 (1996) 56

[13] C. Klimčík and P. Ševera, *Non-Abelian Momentum-Winding Exchange*, preprint CERN-TH/96-142, hep-th/9605212, to appear in Phys. Lett. B

[14] K. Kikkawa and M. Yamasaki, Phys. Lett. B149 (1984) 357; N. Sakai and I. Senda, Prog. Theor. Phys. 75 (1986) 692

[15] T.H. Buscher, Phys. Lett. B194 (1987) 51; B201 (1988) 466; K. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33; A. Tseytlin, Nucl. Phys. B350 (1991) 395; M. Roček and E. Verlinde, Nucl. Phys. B373 (1992) 630; A.A. Tseytlin, Mod. Phys. Lett. A6 (1991) 1721; A.S. Schwarz and A.A. Tseytlin, Nucl. Phys. B399 (1993) 691; A. Giveon and M. Roček, Nucl. Phys. B380 (1992) 128; E. Kiritsis, Mod. Phys. Lett. A6 (1991) 2871, Nucl. Phys. B405 (1993) 109; E. Álvarez, L. Álvarez-Gaumé, J. Barbón and Y. Lozano, Nucl. Phys. B415 (1994) 71; A. Giveon, M. Porrati and E. Rabinovici, Phys. Rept. 244 (1994) 77; E. Álvarez, L. Álvarez-Gaumé and Y. Lozano, Nucl. Phys. Proc. Suppl. 41 (1995) 1; Phys. Lett. B336 (1994) 183; C. Klimčík and P. Ševera, Mod. Phys. Lett. A10 (1995) 323

[16] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724

[17] S. Forste, A. Kehagias and S. Schwager, *Non-Abelian Duality for Open Strings*, Munich preprint TUM-HEP-243-96, hep-th/9604013

[18] J. Borlaf and Y. Lozano, *Aspects of T-duality in Open Strings*, Princeton preprint PUPT-1633, hep-th/9607051

[19] G. Pradisi, A. Sagnotti and Ya.S. Stanev, Phys. Lett. B381 (1996) 97

[20] E. Witten, Commun. Math. Phys. 92 (1984) 455

[21] V.G. Drinfeld, *Quantum Groups*, in Proc. ICM, MSRI, Berkeley, 1986, p. 708; F. Falceto and K. Gawędzki, J. Geom. Phys. 11 (1993) 251; A.Yu. Alekseev and A.Z. Malkin, Commun. Math. Phys. 162 (1994) 147

[22] A. Polyakov and P. Wiegmann, Phys. Lett. B131 (1983) 121

[23] A. Schwarz, *Quantum Field Theory and Topology*, Springer-Verlag, Berlin, 1993