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SHORT DISTANCE PHYSICS OF FRACTIONALLY QUANTIZED HALL FLUIDS

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ABSTRACT

In order to obtain a local description of the short-distance physics of fractionally quantized Hall states for realistic (e.g. Coulomb) interactions, I propose to view the zeros of the ground state wave function, as seen by an individual test electron from far away, as particles. I then present evidence in support of this interpretation, and argue that the electron effectively decomposes into quark-like constituent particles of fractional charge.

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1 Introduction

When asked what happens in a superconductor on a very naive level, most learned physicists would answer that electrons form pairs which then condense into a superfluid. Very few, however, could offer an equally insightful explanation of the fractional quantum Hall effect. One might say that two-dimensional electrons in a strong perpendicular magnetic field condense into an incompressible quantum fluid described by an ingenious wave function due to Laughlin, but all of us would probably agree that such an explanation falls short of the simplicity Bardeen, Cooper and Schrieffer could offer.

In this lecture, I will attempt to make up for this deficiency. I will argue that electrons in fractionally quantized Hall fluids effectively decompose into smaller, quark-like particles, which then bind together to form—electrons.

This is not to say that electrons cease to be the fundamental degrees of freedom in these systems—a quantum mechanical description of all the electrons in the liquid is as complete as any description can be—but rather that the hierarchy of effective field theories is reversed. While we usually assume that constituent particles are more fundamental than composite particles—quarks are thought as more fundamental than hadrons in the standard model, or electrons as more fundamental than Cooper pairs in superconductors—fractionally quantized Hall liquids provide us with an example where the composite particles, the electrons, are fundamental while the smaller constituent particles, which I call *quantum Hall quarks*, are fictitious or effective degrees of freedom induced by the surrounding electron condensate.

I wish to address myself to a general audience without detailed knowledge of quantized Hall fluids, and will begin with a review of the long distance physics.

2 Laughlin's theory of the long distance physics

Most of our understanding of the fractionally quantized Hall effect is based on a highly original trial wave function for the ground state proposed by Laughlin [1]:

$$\Psi_m(z_1, z_2, ..., z_N) = \prod_{i<j}^{N} (z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}eB|z_i|^2} . \quad (1)$$

This wave function describes a circular droplet of an incompressible electron fluid in a strong perpendicular magnetic field B. The fact that all the electrons live in the lowest Landau level constrains the wave function to an analytic function in the complex particle positions $z = x+iy$ times a Gaussian; the Jastrow factor $\prod(z_i - z_j)$ raised to an odd integer power m very effectively suppresses unwanted configurations in which electrons come close to each other.

The Landau level filling fraction is defined as

$$\nu \equiv \frac{\partial N}{\partial N_\Phi}, \tag{2}$$

where N is the number of electrons and N_Φ the number of Dirac flux quanta through the liquid. The latter is equal to the number of zeros of the wave function $\Psi(z_1, z_2, ..., z_N)$ seen by an individual test electron with coordinate z_1 while all the other electron coordinates $z_2, ..., z_N$ are held at fixed positions. For the Laughlin state (1) above, such a test electron will see m zeros at the positions of each other electron, and no additional zeros elsewhere. This implies $\nu = 1/m$.

The elementary excitations, quasiholes and quasielectrons, correspond to additional zeros which are not attached to electrons, or of deficits of zeros in given regions, respectively. Laughlin's explicit trial wave function for the quasi-hole is given by

$$\Psi_m^\eta\,(z_1, z_2, ..., z_N) = \prod_{i=1}^{N} (z_i - \eta)\ \Psi_m(z_1, z_2, ..., z_N)\,. \tag{3}$$

It is immediately obvious that m quasiholes at the same point η amount to a true hole in the liquid, which has charge $+e$; the convention here is $e > 0$. The quasihole charge is therefore e/m. There is a similar trial wave function for the quasielectron, which involves derivatives in the z_i's.

The experimentally observed plateaus in the Hall resistivity around filling fractions 1/3 and 1/5 are due to localization by impurities of all the quasiparticles present to account for the excess density, which then cease to contribute to the transport properties.

The trial wave function (1) is actually a rather good approximation to the exact ground state of two dimensional electrons with Coulomb interactions in the lowest Landau level; at $\nu = \frac{1}{3}$, a numerical comparison for 6 electrons on a sphere yields [2]

$$\langle\, \Psi_{m=3} \mid \Psi_{\text{exact}} \,\rangle \;=\; 0.9964\,. \tag{4}$$

The reason for this remarkable agreement, or more generally for the success of Laughlin's

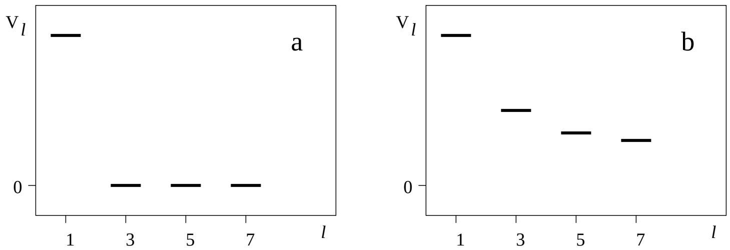

Figure 1: Pseudopotentials for (a) which the Laughlin 1/3 state is exact (b) a Coulomb interaction projected onto the lowest Landau level.

theory, is that it captures the correct long distance physics. The essential physics contained in the trial wave function (1)—in fact the only physics except for the magnetic field—is that the electrons become *superfermions* for $m = 3, 5, ...$etc. The notion of superfermions makes sense in two space dimensions only. It means that the phase picked up by the wave function when one electron encircles another is not 2π, as Fermi statistics requires it, but an odd multiple $2\pi m$, which is consistent with Fermi statistics as well. The fractional quantum numbers of the quasiparticles, for example, are a direct consequence of the superfermions.

Before closing this review, I would like to point out a technical detail [3] which will ease the exposition in the following chapter. In the lowest Landau level, any two-body potential can be parameterized by a discrete set of pseudopotentials V_l, which denote the energy cost of having relative angular momentum l between two particles. The Laughlin 1/3 state is the exact ground state of a model Hamiltonian where only the pseudopotential $V_1 > 0$ while all the other $V_l = 0$ for $l = 3, 5, ...$etc., as shown in Figure 1a. The reason for this is simply that the superfermions have—it follows directly from their definition—no amplitude to be in a state of relative angular momentum $l = 1$.

3 Quantum Hall quarks and the short distance physics

Now imagine we adiabatically deform the set of pseudopotentials shown in Figure 1a into the corresponding set for Coulomb interactions shown in Figure 1b. Then the ground state will evolve from a Laughlin 1/3 state into the exact Coulomb ground state at $\nu = \frac{1}{3}$. We know from the overlap (4) that the state cannot change very much, and from the correctness of Laughlin's theory that the long distance physics cannot change at all—the changes must occur at short distances. The superfermions must evolve into *approximate superfermions*, that is, particles which look like superfermions from far away, yet are different from the exact superfermions contained in Laughlin's trial wave function.

To elucidate this notion, consider once more the zeros of the wave function as seen by an individual test electron z_1 while all the other electron coordinates $z_2, ..., z_N$ are fixed. The exact

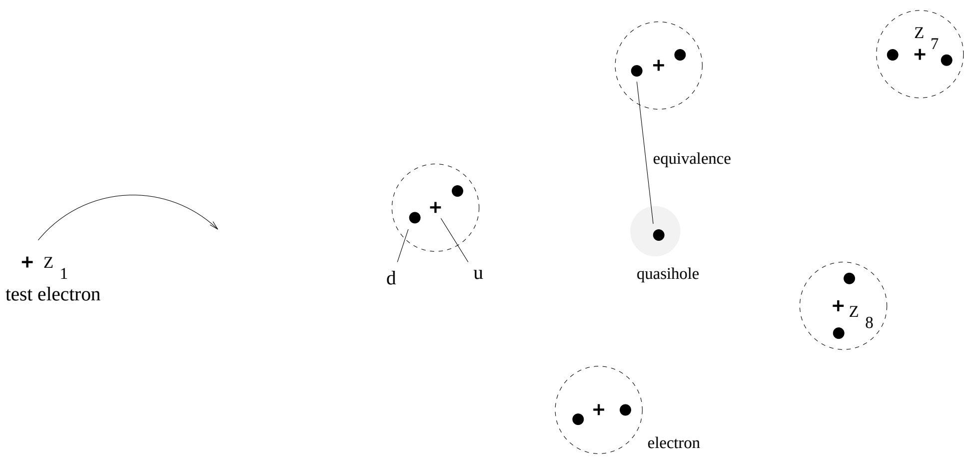

Figure 2: Zeros of $\Psi_{\text{Coul.}}(z_1, z_2, ..., z_N)$ as seen by an individual test electron z_1. The zeros denoted by crosses stem from a Jastrow factor and coincide with the electron positions $z_2, ..., z_N$, while those denoted by dots are in general very complicated functions of all the electron coordinates in the vicinity. Also shown is an isolated zero not associated with any electron, which corresponds to a quasihole excitation.

Coulomb ground state is of the general form

$$\Psi_{\text{Coul.}}(z_1, z_2, ..., z_N) = \prod_{i<j}^{N} (z_i - z_j)\; P(z_1, z_2, ..., z_N) \prod_{i=1}^{N} e^{-\frac{1}{4}eB|z_i|^2}\,. \tag{5}$$

The Jastrow factor must be present since $\Psi_{\text{Coul.}}(z_1, z_2, ..., z_N)$ is antisymmetric; $P(z_1, z_2, ..., z_N)$ is, in general, a complicated symmetric polynomial. A cartoon of the zeros of $\Psi_{\text{Coul.}}$ in a given region, as seen by a test electron from far away, is shown in Figure 2. There are three zeros associated with each electron: one of them (denoted by a cross) stems from the Jastrow factor in (5) and coincides with the electron coordinate z_i; the other two (denoted by dots) stem from the polynomial $P(z_1, z_2, ..., z_N)$ and are, in general, very complicated meromorphic functions of all the electron coordinates in a range which depends on the range of the interaction potential.[a] In the limit of the minimally short ranged potential shown in Figure 1a, the positions of these two zeros depend only on the coordinate of the electron they are associated with—in fact, they coincide with this coordinate: $P(z_1, z_2, ..., z_N)$ becomes a Jastrow factor squared, and the general ground state (5) the Laughlin 1/3 state.

The reason the test electron must be far away is that the positions of the zeros associated with each electron depend on the position of the other electrons nearby. If we were to pick an electron nearby as a test electron, the zeros seen in this region would be those seen by another test electron from far away if the test electron nearby would not exist. The positions of the zeros would therefore depend on which of the electrons nearby we were to pick as a test electron. If we, however, choose an electron far away as the test electron, the positions of the zeros in the region nearby will not depend on our choice and an interpretation of the zeros as particle coordinates, as I will advocate below, is conceivable.

[a]The corresponding observation for mixing with higher Landau levels has been made by Halperin [4]; my considerations here, however, only involve the lowest Landau level.

This brings me to the heart of the matter. In order to provide a *local* description of the short distance physics of fractionally quantized Hall fluids, I propose to view the zeros associated with the electrons as particles. The electron effectively decomposes into three smaller constituent particles,[b]

$$e^- \rightarrow udd \tag{6}$$

where the u and d particles, or quantum Hall quarks, are the zeros due to the Jastrow factor and the polynomial $P(z_1, ..., z_N)$, respectively, as shown in Figure 2. The d particles are equivalent to quasiholes, in the sense that a quasihole is nothing but an isolated d. The charge of the d must therefore be equal to the charge of the quasihole, which we know to be $+1/3$. Since the vacuum or ground state is neutral on a level on which the quasihole assumes this charge, the total charge of the udd composite must be zero, which implies that the charge of the u is $-2/3$.

Most of the remainder of this talk is devoted to motivating and elucidating this idea. To begin with, I will use the hierarchy of quantized Hall fluids [3,5,6] to establish an interpretation of the quasiparticles in quantized Hall fluids as particles.

4 Particle interpretation of quasiparticles and the hierarchy

The quasiparticle excitations of quantized Hall liquids, quasielectrons and quasiholes, were originally conceived as vortices [1], and are adequately interpreted as such when a plateau in the Hall resistivity results from their localization by disorder. There are situations, however, where an alternative interpretation as quantum mechanical particles is not only possible, but inevitable. The hierarchy of quantized Hall states provides us with an example: the quasiparticles *themselves* condense into a Laughlin-Jastrow type fluid, and it is necessary to assign a wave function to them in order to describe this condensation.[c] More precisely, we write an $[m, +p]$ state, that is a p daughter state of quasihole excitations of an m parent state, as [7,8]

$$\begin{aligned}
\Psi_{[m,+p]}(z_1, ..., z_N) &= \int D[\xi, \bar{\xi}]\ \Phi_m(\bar{\xi}_1, ..., \bar{\xi}_{N_1}) \times \\
&\times \prod_{k<l}^{N_1} (\xi_k - \xi_l)^{\frac{1}{m}} \prod_{k=1}^{N_1} e^{-\frac{1}{4m}eB|\xi_k|^2} \prod_{k=1}^{N_1} \prod_{i=1}^{N} (z_i - \xi_k) \times \\
&\times \Psi_m(z_1, ..., z_N)
\end{aligned} \tag{7}$$

with the quasiparticle wave function

$$\Phi_m(\bar{\xi}_1, ..., \bar{\xi}_{N_1}) = \prod_{k<l}^{N_1} (\bar{\xi}_k - \bar{\xi}_l)^{p+\frac{1}{m}} \prod_{k=1}^{N_1} e^{-\frac{1}{4m}eB|\xi_k|^2} \tag{8}$$

and $N_1 = N/p$. The first two factors in the second line of (7) serve to normalize the quasiparticle Hilbert space. The fact that we have to integrate over the quasiparticle coordinates to obtain

[b] The idea illustrated here for $\nu = 1/3$ generalizes in an obvious way to $\nu = 1/m$.

[c] An alternative approach to hierarchical Hall states in terms of "composite fermions" has been proposed [9]; there is good reason to believe, however, that the explicit wave functions motivated by this notion are, in fact, hierarchy wave functions in disguise [8].

a wave function for electrons is entirely consistent with their nature as quantum mechanical particles, as quantum mechanical degrees of freedom always have to be integrated out with a wave function as a measure whenever we wish to calculate a meassureable quantity (e.g. a transition probability).

The explicit trial wave function (7), and its cousin for the $[m,-p]$ state in which quasi-electrons rather than quasiholes condense, are excellent approximations to the exact Coulomb ground states; at $\nu=\frac{2}{5}$, the overlap for 6 electrons on a sphere is [d]

$$\langle\, \Psi_{[3,-2]} \mid \Psi_{\text{exact}} \,\rangle \;=\; 0.9995\,, \tag{9}$$

a number which compares favorably even with the Laughlin 1/m states.

It is perhaps worth noting that we may assign a wave function for an isolated quasiparticle as well. Specifically, we may write

$$\begin{aligned}\Psi_m^\eta(z_1,z_2,...,z_N) \;&=\; \int D(\xi,\bar{\xi})\;\Phi^\eta(\bar{\xi})\,e^{-\frac{1}{4m}eB|\xi|^2}\,\prod_{i=1}^{N}(z_i-\xi)\;\times \\ &\times\;\;\Psi_m(z_1,z_2,...,z_N)\end{aligned} \tag{10}$$

with a wave function for a charged particle localized in a magnetic field as the quasihole wave function:

$$\Phi^\eta(\bar{\xi}) = e^{-\frac{1}{4m}eB|\xi-\eta|^2}\;e^{+\frac{1}{4m}eB(\bar{\xi}\eta-\xi\bar{\eta})}\,. \tag{11}$$

This wave function is, after performing the integration, equivalent to (3); the only virtue it bears is that it suggests the possibility of interpreting the quasihole as a quantum mechanical particle.

The particle nature of the quasiparticles leads us to the question of their origin, to the question of where new particles of fractional charge may come from. The answer is the obvious one, and this is precisely why it is so hard to swallow: The charges of the quasiparticles are parts of electron charges, and *the quasiparticles themselves are parts of electrons.* In order for quasiparticle excitations to exist, the vacuum or ground state must contain them already in a confined phase—the vacuum must be a phase in which pieces of electrons bind together to form electrons. [e]

5 Induced dynamics and scattering in Monte Carlo time

Particle physicists usually establish the existence of new particles by observing them as resonances in scattering experiments. This is not possible for quantum Hall quarks, once because the kinetic energy of all the particles involved is quenched due to Landau level quantization,

[d] Strictly speaking, I have only been able to compute this number without the long distance normalization factor $|\xi_k-\xi_l|^{2/m}$ in (7) and (8) or with a factor $|\xi_k-\xi_l|^2$ instead; the results are 0.999546 and 0.999543, respectively [8].

[e] The possibility of an interpretation along these lines has been suggested by Wilczek. [10]

and a concept of time does consequently not exist, but even more profoundly so because we invoke quantum Hall quarks to describe the vacuum, which trivially excludes the possibility of scattering experiments.

Fortunately, there is a way around these problems. While we do not have a concept of real time, we can perform a Monte Carlo simulation and monitor scattering events as particle configurations evolve in Monte Carlo time. Let me briefly review the technique: a Monte Carlo simulation is a numerical method to approximate an integral over many variables with a probability ρ as a measure. Instead of integrating over the variables directly, we interpret them as dynamical variables, and let them evolve in Monte Carlo time. This concept of time is discrete; at each step we randomly pick one of the variables, and define a new configuration by randomly choosing a new value for this variable according to a certain distribution, which is usually taken as a Gaussian centered at the present value. Finally, we randomly decide whether to update the configuration or not according to probabilities proportional to the measure ρ for the new and for the present configuration, respectively. The desired integral is obtained by averaging the integrand (not including the measure) over a long span in Monte Carlo time; the approximation becomes exact as this span tends to infinity.

In our case, the Monte Carlo variables are the electron coordinates z_i, and the measure ρ is the probability $|\Psi_{\text{Coul.}}(z_1, ..., z_N)|^2$. A snapshot of a typical Monte Carlo configuration including all the zeros or quantum Hall quarks, is shown in Figure 3a. Only the electron coordinates, or u quantum Hall quarks, are truly dynamical variables in Monte Carlo time; the dynamics of the remaining zeros, or d quantum Hall quarks, is induced through the surrounding electron condensate. This, however, does not emerge from Figure 3a, nor does it ever manifest itself as we follow the evolution of this configuration on a continuous time scale—that is, a time scale on which all the variables evolve simultaneously.

Let us now look at a particular scattering event, as shown in Figure 3. In this event, two electrons scatter off each other, and interchange one of their constituent particles: two electron coordinates happen to come very close to each other, and remain unchanged for a number of Monte Carlo steps, while the configuration of the additional zeros associated with them evolves with the surrounding electron liquid; this configuration will, in general, have changed significantly by the time the two electrons separate again. Thus there is a finite amplitude for zeros to get interchanged—the zeros are *indistinguishable* when interpreted as particles, and scatter into each other as identical particles do in quantum mechanics.

This thought experiment does not only motivate the possibility of quantum Hall quarks, but in my opinion also shows the necessity for invoking them: for what other framework could describe objects which scatter into each other like identical particles if not the framework of identical particles? Moreover, it nicely illustrates the underlying reason why it is possible for these fictitious or induced degrees of freedom to become particles: induced and fundamental degrees of freedom are *locally* equivalent, in the sense that no local experiment, and in particular no scattering experiment, is capable of resolving the difference. [f]This is precisely the reason why

[f]Note that this ambiguity in interpretation, the ambiguity between fictitious and fundamental degrees of freedom, exists for scattering experiments performed on hadrons as well.

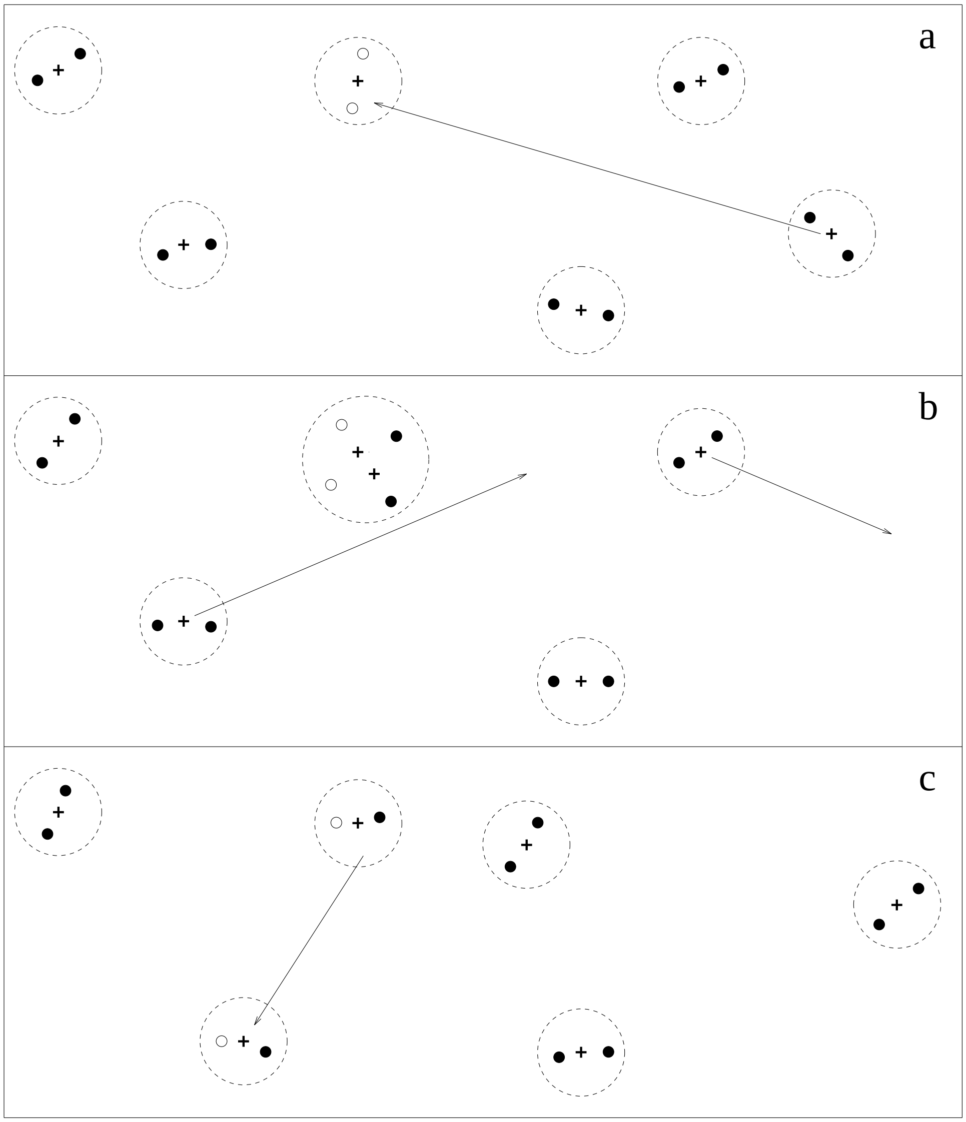

Figure 3: Electron–electron scattering in Monte Carlo time for an exact Coulomb ground state at $\nu = 1/3$: a) two electron coordinates (or u quantum Hall quarks) happen to come very close to each other. b) the surrounding electron configuration evolves in Monte Carlo time, and with it the configuration of the zeros (or d quantum Hall quarks) of the two electrons close to each other. c) the two electrons separate again, having interchanged one of their constituent particles.

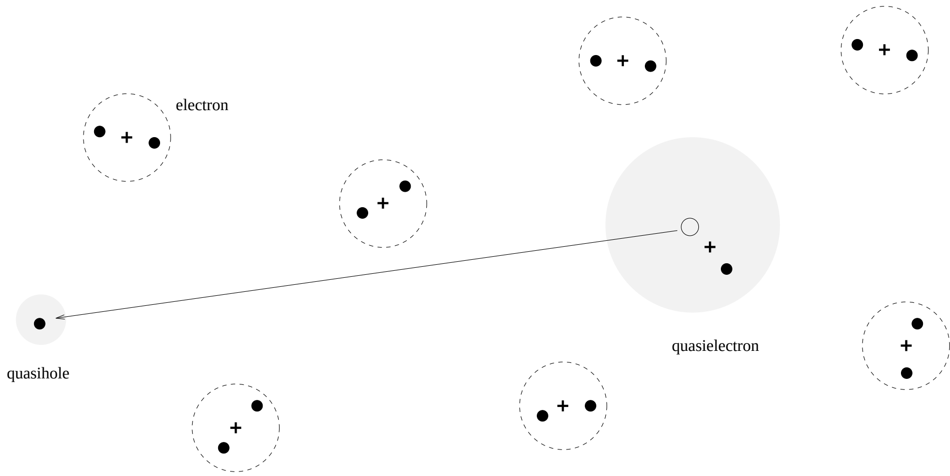

Figure 4: Quasihole-quasielectron pair created by removing a d-particle from the vacuum in a given region and placing it somewhere else.

it is perfectly reasonable to invoke quantum Hall quarks in order to provide a *local* description of fractionally quantized Hall fluids at short distances.

6 Quantum Hall quarks and quasiparticles

As already mentioned in chapter 3, the d particle is equivalent to a quasihole excitation, in the sense that a quasihole is nothing but a d in isolation. To see this, we just need to perform another Monte Carlo experiment with an exact quasihole for Coulomb interactions at some location η, and we will find that the position of the zero associated with the quasihole does not exactly coincide with the position η, but rather depends on all the electron coordinates in the vicinity, as indicated in Figure 2. Moreover, we will find that this zero has a finite amplitude to get interchanged with other zeros or d particle in the liquid as electrons scatter off the quasihole in Monte Carlo time.

This thought experiment is very instructive, since it illustrates the precise sense in which the exact quasihole for realistic interaction potentials differs from Laughlin's trial wave function (3). This difference has troubled me for many years, as it is known from numerical studies [2] that the quasihole expands in size as the interactions get softer, while the Landau level quantization for the quasihole (10,11) seems to indicate that Laughlin's trial wave function is the *only* possible form with the correct quantum numbers if we assume that the zero is distinguishable in the sense of having no amplitude to get interchanged with another zero.

The equivalence of confined and isolated zeros can also be deduced from the fact that a quasihole-quasielectron pair is created by removing a zero from the vacuum in a certain region and placing it into another region, as illustrated in Figure 4. This observation immediately suggests a possible application for a quantitative version of this theory. It would enable one to calculate quasiparticle energies from the local field configurations for the quasihole and the quasielectron, and therewith calculate them for the first time: for all the previous evaluations of these energies, which numerically compare finite systems with and without excitations, amount strictly speaking to measurements.

7 Delusion of composite fermions

I would like to pause briefly now to rectify a misconception, which is important because it has spread so widely over the literature. This misconception is the notion of "composite fermions", defined [9] as "an electron bound to a flux tube" or "the composite of an electron and an even number of flux quanta". It is the main point of this lecture to illustrate that the electron itself becomes a composite particle, a bound state of a u with an even number of d particles, which implies that nothing is bound to electrons.

Let me explain once more why this must be so. An isolated flux tube or quasihole is a particle, is a piece of an electron. This flux tube or zero in the wave function is equivalent to the other flux tubes or zeros in the condensate, to those associated with the electrons. Therefore the interpretation for the isolated and confined flux tube must be the same, and both are particles, both are pieces of electrons. It clearly does not make sense to think of composites which are formed by binding pieces of electrons to electrons; the correct interpretation is that pieces of electrons bind to other pieces of electrons, and that the electron itself becomes the composite particle.

8 Conclusions

Most of what I have explained in this lecture concerns the ground state or vacuum of fractionally quantized Hall fluids. The concept of quantum Hall quarks has some applications to quasiparticle excitations, as we have seen in chapter 6, but is not nearly as important in descriping them as Laughlin's theory of the long distance physics, while excitations are all that matters to experiments performed on quantum Hall systems. The real significance of the analysis presented here lies in the general message we can learn from it, and the potential relevance of this message to other systems, in particular to the vacuum of our universe, the ground state which supports all the elementary particles known to us as excitations.

This general message is that the particles we see or detect as excitations above a certain vacuum may not be contained in this vacuum on a fundamental level, and might possibly be pieces of larger particles invisible to us. The degrees of freedom we perceive as fundamental may in fact be fictitious or induced, and fractional quantum numbers—but in particular the fractional charges of quarks in quantum chromodynamics—may arise through a mechanism related to the one responsible for quantum Hall quarks. If we specifically imagine an observer who lives in a quantized Hall fluid and consists of quasiparticles, this observer would never see electrons, but only fictitious particles of fractional charge, and would naturally be inclined to accept those as fundamental. Note in particular that scattering experiments, both the ones performed by this observer as well as the ones performed by us in particle accelerators, are incapable of resolving the ambiguity between induced and fundamental degrees of freedom.

In this spirit, I wish to express my hope that condensed matter and high energy physics may never cease to exchange inspiration.

Acknowledgments

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