

# Stability of the quantum supermembrane in a manifold with boundary

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## Abstract

We point out an effect which may stabilize a supersymmetric membrane moving on a manifold with boundary, and lead to a light-cone Hamiltonian with a discrete spectrum of eigenvalues. The analysis is carried out explicitly for a closed supermembrane in the regularized  $SU(N)$  matrix model version.

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1. In ref. [1] it was shown that the light-cone Hamiltonian of the eleven dimensional supermembrane theory [2] has a continuous spectrum. The proof was given for membranes moving in eleven-dimensional Minkowski space-time. It is important to investigate whether this instability persists if the manifold has a boundary. In particular, recent results in string dualities [3] indicate that the strong coupling limit of ten-dimensional heterotic string theory is described by an eleven-dimensional theory compactified on  $S^1/\mathbf{Z}_2 = I$  ( $I$  is the unit interval). It is plausible that the eleven-dimensional supermembrane theory on the orbifold  $\mathbf{R}^{10} \times I$  may be of relevance to this theory. Another view is to regard the membrane states just as solitonic objects which need not to be quantized in order to have a description of gravitons (see e.g. [3-5]). In either case, it would be desirable to identify possible stable membrane states, which are not protected topologically.

The relevance of the present results to M-theory on  $S^1/\mathbf{Z}_2$  is nevertheless unclear (for a review on M-theory, see ref. [6]). In particular, we do not know whether the boundary conditions for the membrane wave function that will be used here can be consistently implemented in this theory. In conventional quantum field theory, a boundary in space-time requires imposing (e.g. Dirichlet or Neumann) boundary conditions on the fields, to prevent momentum and quantum information from leaking out from the physical space. In particular, a wave function with quantum mechanical probabilistic interpretation must vanish at the boundary. This is the assumption that will be made here for the membrane wave function. In the orbifold  $S^1/\mathbf{Z}_2$ , however, the only restriction is that wave functions are  $\mathbf{Z}_2$ -invariant, modulo a suitable action on the internal quantum numbers. But they are generally not required to vanish at the fixed points; one is essentially dealing with a compact space, where the states which are not invariant under the action of the discrete group have been projected out from the Hilbert space. Thus the only direct implication of the present results to a supermembrane moving on  $\mathbf{R}^{10} \times S^1/\mathbf{Z}_2$  is perhaps the observation that its instabilities are all caused by modes whose associated wave functions do not vanish at the fixed points. In other words, the spectrum of the supermembrane is discrete upon the restriction to the Hilbert space of membrane states with nodes at the fixed points.

At long wavelengths, the dynamics of the supersymmetric membrane is dictated by the action of [2] (recent discussions on different aspects of membranes and five-branes can be found in refs. [7,8]). Classically, the presence of a boundary in the space may appear to have no influence in solving the instability problem, since the wave function could leak out to infinity along one of the remaining Minkowski spatial directions. Fortunately, this is not what happens in the quantum theory: we shall see that the boundary modifies the

asymptotic zero-point energy of oscillators which are transverse to the potentially dangerous direction, in such a way the resulting motion of the membrane modes are confined (analogue to the Casimir effect which stabilizes the bosonic membrane).

We start with the example which in ref. [1] was used to illustrate the instability of the supermembrane. The Hamiltonian is

$$H = \frac{1}{2} \begin{pmatrix} -\Delta + x^2 y^2 & x + iy \\ x - iy & -\Delta + x^2 y^2 \end{pmatrix} . \quad (1)$$

The quantum mechanical system we want to consider is governed by the Hamiltonian (1) and, in addition, we put an infinite barrier at  $xy = 0$ , the accessible space being defined by  $xy \geq 0$ . This breaks supersymmetry explicitly. (The breaking may also be regarded as “spontaneous”, if the condition  $\psi(0) = 0$  is interpreted as a physical restriction on the Hilbert space. It is also possible to define the system (1), including the infinite potential barrier, as a limit of a supersymmetric quantum system, i.e. with  $H = \frac{1}{2}\{Q, Q^\dagger\}$  ). Following [1], we consider the wave packet

$$\psi_t(x, y) = \chi(x - t)\varphi_0(x, y)\xi_F , \quad \xi_F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} , \quad (2)$$

which is designed so as the wave packet may escape along the potential valley at  $y = 0$ . Here  $\chi(x)$  is a smooth function with compact support, and the spinor  $\xi_F$  is chosen to give a maximal negative contribution to the energy of the wave packet, i.e.

$$\xi_F^T H \xi_F = H_B - \frac{1}{2}x , \quad H_B = -\frac{1}{2}\Delta + \frac{1}{2}x^2 y^2 .$$

We would like to study the motion of  $\chi$  in the ground state of  $\varphi_0(x, y)$ .  $\varphi_0(x, y)$  represents oscillations of the  $y$ -coordinate at fixed  $x$  about the bottom of the potential valley:

$$H_2 \varphi_0 = E_0 \varphi_0 , \quad H_2 \equiv -\frac{1}{2} \frac{\partial^2}{\partial y^2} + \frac{1}{2} x^2 y^2 , \quad (3)$$

where  $H_2$  is viewed as an operator on  $\mathcal{H}_y = L^2(\mathbf{R}, dy)$  . This is the equation of the harmonic oscillator. In the absence of a boundary, the ground state energy is  $E_0 = \frac{1}{2}|x|$ , which would just cancel against the fermionic contribution, and the motion for  $\chi$  would be unbounded. Because of the wall,  $\varphi_0$  must satisfy the boundary condition  $\varphi_0(y = 0) = 0$ . The wave-function of the ground state is

$$\varphi_0(x, y) = \frac{\sqrt{2}}{\pi^{1/4}} |x|^{3/4} y \exp(-\frac{1}{2}|x|y^2) , \quad (4)$$

with a zero-point energy  $E_0 = \frac{3}{2}|x|$ . Consequently, we have for  $\chi$

$$\lim_{t \rightarrow \infty} (\psi_t, H\psi_t) = \int dx \chi^*(x) \left( -\frac{1}{2} \partial_x^2 + |x| \right) \chi(x), \quad (5)$$

which implies that the motion is bounded, and that the spectrum of  $H$  is discrete (from eq. (5) it can be explicitly proven that  $\text{Tr } e^{-tH} < \infty$ , e.g. using the ‘‘sliced bread inequalities’’ [9]).

This mechanism relies on the uncertainty principle; the zero-point energy increases as the classically accessible region for the particle is squeezed. Clearly, a similar effect holds for more general boundaries, for example, a boundary of the form  $-1 \leq y \leq 1$  would also increase the zero-point energy of  $\varphi_0$ .

2. Let us consider a quantum system governed by the following Hamiltonian (analogue to eq. (3))

$$H_2 = -\frac{1}{2} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y^i} + \frac{1}{2} (z^T z)_{ij} y^i y^j, \quad i, j = 1, \dots, n, \quad (6)$$

where  $z$  is an  $n \times n$  antisymmetric real matrix. For  $n > 2$  the matrix  $z^T z$  will in general not be diagonal. By a suitable  $SO(n)$  rotation  $M$ ,  $u_i = M_{ij} y^j$ , we can take the Hamiltonian to the form

$$H_2 = -\frac{1}{2} \frac{\partial}{\partial u_i} \frac{\partial}{\partial u^i} + \frac{1}{2} w_i^2 u_i u^i. \quad (7)$$

First, let us investigate the effect of a wall at  $u_i = 0$ ,  $i = 1, \dots, n$ , on the zero-point energy of this system. The eigenvalue equation is decoupled, and the result is similar to the example of the previous section,

$$E_0 = \frac{3}{2} \sum_i w_i. \quad (8)$$

Next, suppose that we change the orientation of the wall, which we now place at  $y_i = 0$ . Finding the ground state energy of this system is a complicated problem, and perhaps not solvable by analytic methods. But there is a generic feature which can be stated as follows.

*Theorem 1.* Let a quantum mechanical system be described by the Hamiltonian (7) with a potential barrier  $y_i \geq 0$ ,  $i = 1, \dots, n$ , with  $y_i = M'_{ij} u^j$ . Then, for all  $M' \in SO(n)$  and all  $w_i \in \mathbf{R}$ ,  $w_i \neq 0$ , its ground state energy is greater than the ground state energy of the similar system without walls.

A simple proof is as follows. Locally, the eigenvalue equation for both systems is the same, the difference being that in one of them there is the additional requirement that the wave function vanishes on the boundary. From the standpoint of the system without walls,

these wave functions are just particular solutions with nodes at  $y_i = 0$ . By the oscillation theorem of quantum mechanics, the expectation value of the energy on any of these states (in particular, on the normal state of the system with the walls) must be greater than the ground-state energy. Intuitively, the presence of the wall, irrespective of the orientation, reduces the classically allowed region; by virtue of the uncertainty relation, the ground state energy must increase.<sup>1</sup>

The basic property that will be used in the next section is contained in the following theorem, which is a generalization of the former to the case when the boundary is imposed in terms of a Fourier-like transform.

*Theorem 2.* Same as theorem 1, but now the potential barrier is  $X(\sigma) \geq 0$ ,  $X(\sigma) = \sum_i u_i f_i(\sigma)$ , where the  $f_i(\sigma)$  represent a set of orthonormal functions on some space.

Consider the function  $X(\sigma) \equiv 0$ . Because of orthonormality, this corresponds to the choice  $u_i = 0$  for all  $i$ . Thus, at the point  $u_i = 0, i = 1, 2, \dots$ , the eigenfunctions of the system with walls must have a node. On the other hand, the system without walls is a collection of decoupled harmonic oscillators. The ground state is just the direct product of the ground states of individual oscillators, and it does not have any node. Therefore its energy must be lower, as stated above.

3. By expanding the coordinates in a complete orthonormal basis of functions  $Y^A(\sigma)$  on the membrane,  $\mathbf{X}(\sigma) = \sum_A \mathbf{X}^A Y_A(\sigma)$ , and similarly for the fermionic variables and the momenta, the light-cone gauge Hamiltonian takes the form [10]

$$H = \frac{1}{2} P_\mu^A P_{\mu A} + \frac{1}{4} f_{ABE} f_{CD}^E X_\mu^A X_\nu^B X_\mu^C X_\nu^D - \frac{1}{2} i f_{ABC} X_\mu^A \theta^B \gamma^\mu \theta^C, \quad (9)$$

where  $\mu, \nu = 1, 2, \dots, 9$  and  $f_{ABC}$  are the structure constants of the group of area-preserving diffeomorphisms of the parameter manifold. Here  $\theta_\alpha^A$ , with  $\alpha = 1, \dots, 16$ , are real  $SO(9)$  spinors. The classical instabilities occur along the Cartan directions, where the potential vanishes. For concreteness, we will consider the case of spherical or toroidal membranes, where the group can be regarded as  $SU(N)$ , with  $N \rightarrow \infty$ . The generalization to other

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<sup>1</sup> For certain configurations of  $w_i$ , the statement is perhaps rather obvious. If  $w_1$  is the lowest frequency, we write  $H_2 = H'_2 + H'$ ,  $H'_2 = -\frac{1}{2} \frac{\partial}{\partial u_i} \frac{\partial}{\partial u^i} + \frac{1}{2} w_1^2 u_i u^i$ ,  $H' = \frac{1}{2} \sum_{i=2}^n (w_i^2 - w_1^2) u_i^2$ . Since  $H' > 0$ , the ground-state energy of  $H_2$  satisfies  $E_0 > \frac{3}{2} n w_1$ , which is sufficient to demonstrate the theorem for those frequencies satisfying  $\frac{3}{2} n w_1 > \frac{1}{2} \sum_i w_i$ , or  $w_1 > (3n - 1)^{-1} \sum_{i=2}^n w_i$ . For other frequencies  $H'$  cannot be ignored, since  $(w_i^2 - w_1^2)$  will not be small for all  $i$ .

compact Lie groups should be straightforward. It is convenient to split off the coordinates  $X_\mu^A$  into the form  $X_\mu^A \rightarrow (Z_\mu^i, Y_\mu^I)$  where indices  $i, j, k = 1, \dots, N - 1$  correspond to the Cartan directions, and  $I, J = N, \dots, N^2 - 1$  label the remaining directions. Upon gauge fixing, where  $Y_9^I$  is removed, only a residual invariance under the Cartan subgroup remains, and the Hamiltonian takes the form [1]

$$H = H_1 + H_2 + H_3 + H_4 , \quad (10)$$

$$H_1 = -\frac{1}{2} \left( \frac{\partial}{\partial Z^k} \right)^2 - \frac{1}{2} \left( \frac{\partial}{\partial Z_a^k} \right)^2 , \quad H_2 = -\frac{1}{2} \left( \frac{\partial}{\partial Y_a^I} \right)^2 + \frac{1}{2} (z^T z)_{IJ} Y_a^I Y_a^J , \quad (11)$$

$$H_3 = -\frac{1}{2} i \theta^I (z_{IJ} \gamma_9 + z_{IJ}^a \gamma_a) \theta^J , \quad (12)$$

$$z_{IJ}^a = Z^{ak} f_{kJI} , \quad Z^k = Z_9^k , \quad a = 1, \dots, 8 .$$

$H_4$  is not important in the analysis of stability, and here it will not be considered (as in [1], all terms in  $H_4$  vanish in the asymptotic region). The eigenvalues of  $z_{IJ}$  are the (non-vanishing) roots of the Lie algebra of  $SU(N)$ , and can be expressed in terms of the eigenvalues  $i\lambda_m, i\lambda_m^a, \lambda_m \in \mathbf{R}, m = 1, \dots, N$  of  $Z$  and  $Z_a$ . The following relations will be useful:

$$\det z = \prod_{m < n}^N (\lambda_m - \lambda_n)^2 = \det \Omega , \quad \text{tr } \Omega = 2 \sum_{m < n}^N (\lambda_m - \lambda_n) , \quad \Omega \equiv \sqrt{z^T z} . \quad (13)$$

Now we consider the wave-packet which in  $\mathbf{R}^{11}$  causes a instability (representing a mode escaping along the potential valley). This has the form

$$\psi_t(Z, Z_a, Y_a^I) = \chi(Z - tV, Z_a) \phi_0(Z, Y_a^I) \xi_F(Z, Z_a) , \quad (14)$$

$$V_{mn} = i \left[ \frac{1}{2} (N + 1) - m \right] \delta_{mn} . \quad (15)$$

The fermionic variable has the ground state energy

$$H_3 \xi = -8 \sum_{m < n} \omega_{mn} \xi , \quad (16)$$

where

$$\omega_{mn} = \sqrt{\lambda_{mn}^2 + (\lambda_{mn}^a)^2} , \quad \lambda_{mn} = \lambda_m - \lambda_n . \quad (17)$$

As far as the fermionic ground state energy is concerned, the presence of a boundary, e.g. of the type  $Y_8^I \geq 0$ , is irrelevant ( $H_3$  does not even depend on  $Y_a^I$ ). The derivation of (16) is identical to the derivation of ref. [1], so it will not be reproduced here.

Now we determine the zero-point energy of  $H_2$  (see eq. (11)). Denote by  $U_a^I$  the basis in which  $H_2$  is diagonal, where  $U_a^I$  is related to the  $Y_a^I$  by an orthogonal transformation,

$$U_a^I = M_{IJ} Y_a^J .$$

Let us first investigate the effect of a “boundary” imposed in terms of  $U_8^I$ ,  $U_8^I \geq 0$  for all  $I$ . In this case the eigenstates of  $H_2$  can be explicitly constructed, being a straightforward extension of the quantum mechanical systems considered in the previous sections. The  $U_a$  with  $a = 1, \dots, 7$  will give a contribution to the zero-point energy proportional to  $\frac{7}{2} \text{tr } \Omega$ , whereas the contribution of  $U_8$  will be proportional to  $\frac{3}{2} \text{tr } \Omega$ . Explicitly,

$$H_2 \phi_0 = 5(\text{tr } \Omega) \phi_0 , \quad (18)$$

where  $\phi_0$  is given by (cf. eq. (4))

$$\phi_0(Z, Y_a^I) = \text{const.} (\det \Omega)^{5/2} \left( \prod_I U_8^I \right) \exp \left( -\frac{1}{2} \Omega_{IJ} Y_a^I Y_a^J \right) . \quad (19)$$

The potential for  $\chi$  arises from the contribution of  $H_2$  and  $H_3$  to the total energy:

$$(H_2 + H_3) \psi_t = 2 \sum_{m < n}^N (5|\lambda_{mn}| - 4\omega_{mn}) \psi_t . \quad (20)$$

For large  $t$  (see eqs. (14), (15)),

$$\lambda_{mn} = (n - m)t + O(1) , \quad (21)$$

and  $\omega_{mn} \rightarrow |\lambda_{mn}|$ . As a result we get

$$\lim_{t \rightarrow \infty} \|(H_2 + H_3) \psi_t\| = \text{tr } \Omega = \left( 2 \sum_{m < n}^N (n - m) \right) t .$$

Thus  $\chi$  cannot move off to infinity; the system can only execute a finite motion in the  $Z$  direction. The theorem 1 states that the same conclusion applies when the boundary is imposed in terms of the coordinate  $Y_8^I$ , i.e.  $Y_8^I \geq 0$ , related by an orthogonal rotation to the  $U_8^I$ . By theorem 2, the motion of  $\chi$  will also be finite when the boundary is imposed in terms of the physical coordinate  $X_8(\sigma)$ , i.e.  $X_8(\sigma) \geq 0$  for all  $\sigma$ .

Nevertheless, for this system with a single wall, the spectrum will be continuous in virtue of the fact that there is still a direction along which a mode can leak out to infinity.

This is the direction orthogonal to the wall, say, orthogonal to the hyperplane  $Y_8 = 0$ . Indeed, for large values of  $Z_8$ , the boundary has no effect in the oscillators transverse to this direction. To see this explicitly, we gauge away  $Y_8$  (instead of  $Y_9$ ) and consider a similar wave packet with  $\chi(Z_8 - tV)$ . Then  $H_2$  only involves coordinates which are unbounded, so that its ground state energy is  $4 \text{ tr } \Omega$ , which in the asymptotic region cancels against the fermionic contribution. To eliminate the possibility of infinite motion in this direction, we generalize the above discussion by adding an extra wall. In particular, systems with either an extra boundary component, such as  $0 \leq Y_8 \leq 1$ , or a boundary in another direction, e.g.  $Y_8, Y_7 \geq 0$ , do not have that possibility of leakage. The motion in all Cartan directions is finite and the spectrum of eigenvalues must therefore be discrete.

It may seem counterintuitive that the inclusion of a boundary in a single dimension can stabilize the supermembrane. What happens is that all the coordinates couple to the same matrix  $(z^T z)_{IJ}$ . A change of the zero-point energy of a single coordinate modifies the total zero-point energy of  $H_2$  by a numerical factor; it no longer cancels against the fermionic contribution and a confining potential for  $\chi$  is generated. The discussion can be formally generalized to the continuum, by using a framework recently introduced in ref. [11].

4. So far our discussion has only included spherical and toroidal membranes. A similar analysis applies for open membranes of cylindrical topology. Indeed, the basis functions are essentially equivalent to those of the torus [12],  $Y_K(\sigma) = e^{ik_1\sigma_1 + ik_2\sigma_2}$ ,  $K = (k_1, k_2)$ ,  $\sigma_{1,2} \in [0, 2\pi)$ . By making use of 't Hooft's twist matrices [13], it is possible to construct exactly  $N^2 - 1$  traceless independent matrices  $T_K$ , satisfying an algebra which approaches the Lie algebra of  $SU(N)$ , as  $N \rightarrow \infty$ .

An open supermembrane with boundary components living on the boundary of the manifold  $\mathbf{R}^{10} \times I$ , perhaps of relevance to the strong coupling limit of heterotic string theory, is likely to have a discrete spectrum as well (in the restricted Hilbert space of membrane wave functions which vanish at the space-time boundary). The only difference lies on the dynamics of the membrane boundary, represented by closed strings propagating in  $\mathbf{R}^{10}$ , but this dynamics should not spoil stability.

Another question concerns the dependence of the eigenvalue spectrum of the Hamiltonian on the membrane topology. The analysis using the regularized Hamiltonian does depend on the properties of the specific Lie group, such as roots and structure constants. However, it is believed that the Poisson algebra of functions on a manifold that is a regular



coadjoint orbit of some group  $G$  can always be approximated by  $SU(N)$ , with possible restrictions to some subalgebra (see e.g. refs. [12,14]). Some dependence of the spectrum on the topology is expected, given that different membrane topologies should in fact represent inequivalent quantum states. In the eleven dimensional theory on  $\mathbf{R}^{11}$  there is no coupling parameter that can suppress higher topologies. On the other hand, in the compactified theory, by combining the membrane tension  $T_3$  and the size  $R_{11}$  of the eleven dimension, it is possible to define an adimensional coupling parameter  $g^2 = T_3 R_{11}^3$  (which in the standard correspondence with heterotic superstring theory is in turn related to the dilaton field [3]). The way the D-2-brane is quantized in type IIA superstring theory suggests that a consistent quantization of the eleven dimensional membrane may require including states associated with higher topologies. Adding small handles should not considerably increase the mass, since the cost of energy is proportional to the membrane area. Just as would be the case for a Ramond-Ramond soliton of type IIA string theory, for  $g^2 = O(1)$  the low-energy excitations of the membrane should be generically constituted of both oscillation modes and tiny handles. In this case a more adequate formalism may rather involve a suitable quantization of the world-volume field theory [15]. While the D-brane picture is not justified in extrapolating from weak to strong string coupling, where the eleven dimension emerges, it is indicative in discriminating the relevant degrees of freedom in the various limits of the product  $T_3 R_{11}^3$  (see also ref. [8]).

5. A priori, possible unbroken supersymmetries do not imply a relation between the zero-point energies of  $H_2$  and  $H_3$ . Note that what is calculated here is just the zero-point energy of  $H_2 + H_3$  for the particular decomposition of the trial wave function (14) (in the quantum subsystem  $\mathcal{H} \equiv H_2 + H_3$ , with  $z_{IJ}$  fixed, the presence of a boundary breaks supersymmetry). Although our analysis does not therefore exclude the existence of a set of normalizable eigenvectors of  $H$  with zero eigenvalue (representing a massless multiplet in the supermembrane spectrum), there are several reasons to expect that the ground state in the stabilized theory will be massive. As in the  $D = 10$  superstring theory, in supermembrane theory the existence of a massless multiplet seems to be ascribed to a complete cancellation of the bosonic and fermionic ground state energies [16]. In the  $\mathbf{R}^{10} \times S^1/\mathbf{Z}_2$  orbifold compactification, half of the supersymmetries remain, but in the present context they could be broken because of the boundary conditions. Just from the zero-mode structure, a contingent unbroken supersymmetry, together with the discreteness of the spectrum, could be used to argue that there is a unique ground state, constituted

by the eleven dimensional supergravity multiplet [17]. Unfortunately, the general ground-state wave functional is not known, so it is not presently possible to determine whether the boundary condition is eliminating a square-integrable massless state. (The structure of the equation  $Q\Psi = 0$  actually suggests that there cannot be any normalizable massless state in supermembrane theory [10], but the problem is still open). For compactifications in more complicated topologies, say  $T^2/\mathbf{Z}_2$ , the generic situation is that there is no solution to a differential equation that is everywhere non-vanishing. From this point of view, demanding the wave functional to have nodes at the fixed points seems less restrictive. Clearly, further work is needed. In particular, it would be of interest to identify the membrane topologies which can lead to a non-vanishing Witten index, which seems to be computable in the regularized low  $N$  theory.

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