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# Exceptional SW Geometry from ALE Fibrations

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We show that the genus 34 Seiberg-Witten curve underlying  $N = 2$  Yang-Mills theory with gauge group  $E_6$  yields physically equivalent results to the manifold obtained by fibration of the  $E_6$  ALE singularity. This reconciles a puzzle raised by  $N = 2$  string duality.

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## 1. Introduction

There is now an extensive body of literature on the construction of the quantum effective actions on the Coulomb branch of  $N = 2$  supersymmetric Yang-Mills theories [1]. The work of [1] was extended to other gauge groups: for example, to  $G = SU(n)$  in [2,3], to  $G = SO(2n + 1)$  in [4], and to  $G = SO(2n)$  in [5]. In these approaches, the curves are given in terms of the appropriate simple singularities  $W_{ADE}$  [6], and are generically of the form

$$y^2 = W^2(x; u_j) - \mu^2, \quad (1.1)$$

with  $\mu = \Lambda^{h^\vee}$ , where  $h^\vee$  is the dual Coxeter number of  $G$ , and  $\Lambda$  is the quantum scale. The  $u_j, j = 1, \dots, \ell$  are the fundamental Casimir invariants (with degree increasing with the subscript  $j$ ) and  $\ell$  is the rank of  $G$ ; the top Casimir,  $u_\ell$ , has degree  $h^\vee$ . For example, for  $G = SU(n)$ , one has  $W_{A_{n-1}}(x; u_j) = x^n - \sum_{j=1}^{n-1} u_j x^{n-1-j}$ .

In a complementary, unifying approach based on integrable systems [7], a general scheme for obtaining Seiberg-Witten (SW) curves for all groups was presented in [8]. As explained in more detail below, these curves are of the form

$$\zeta + \frac{\mu^2}{\zeta} + P_{\mathcal{R}}(x; u_j) = 0, \quad (1.2)$$

where  $P_{\mathcal{R}}$  is a polynomial in  $x$  of order  $\dim(\mathcal{R})$ , where  $\mathcal{R}$  is some representation of  $G$ . For  $G = SU(n)$ , one can take  $P_{\mathcal{R}}(x; u_j) \equiv W_{A_{n-1}}(x; u_j)$  so that the curves (1.2) and (1.1) are manifestly the same, up to a simple reparametrization. This however cannot be done for other groups, since  $\dim(\mathcal{R})$  will in general<sup>1</sup> not match the degree of  $W_{ADE}$ , which is equal to  $h^\vee$ .

More recently, it was found in [9] how, via  $N = 2$  heterotic-type II string duality [10], local SW geometry can be derived from fibrations of ALE spaces: the relevant manifolds are described by

$$\zeta + \frac{\mu^2}{\zeta} + W_{ADE}^{ALE}(x_1, x_2, x_3; u_j) = 0, \quad (1.3)$$

where  $W_{ADE}^{ALE}$  is the (non-compact) ALE space of type  $ADE$ ; for  $G = SU(n)$ ,  $W_{A_{n-1}}^{ALE} \equiv W_{A_{n-1}}(x_1; u_j) + x_2^2 + x_3^2$ . Obviously, by trivially integrating out the quadratic pieces in  $x_2$  and  $x_3$  (which does not change the singularity type) this manifold is equivalent to the above SW curves. It also gives rise to the same periods [9].

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<sup>1</sup> However, this can easily be reconciled [5] for  $G = SO(2n)$ .

Actually, by simple reparametrization, (1.3) can be brought into the form (1.1) (but with  $W$  depending on more than one variable) for all gauge groups, and it was conjectured in [2] that this should describe the Seiberg-Witten effective action. String duality implies that this must indeed be true. However, for the groups  $G = E_n, n \geq 6$ ,  $x_2$  does not enter quadratically and thus cannot easily be integrated out (for example, one has  $W_{E_6}^{\text{ALE}} = x_1^3 + x_2^4 + x_3^2 + \dots$ ). This means that for  $E_n$ , the manifolds (1.3) have *a priori* no obvious relation to Riemann surfaces, and appear intrinsically as higher dimensional surfaces.

The question then immediately arises as to how the manifolds (1.3) are related to the curves (1.2) for exceptional gauge groups. It is very natural to believe that somehow the periods must be the same, but how this precisely works was not clear until now. It is the purpose of this letter to show that from the point of view of SW theory, the curves (1.2) and the manifolds (1.3) are indeed physically equivalent.

There is also a physics aspect to this. In fact, (1.3) represents (a local, non-compact piece of) a threefold, and the BPS states correspond to wrappings of type IIB 3-branes [11] around the local vanishing homology  $H_3$ . For  $G = SU(n)$ , it follows from the considerations in [9] that upon integrating out the quadratic terms in  $x_2$  and  $x_3$ , the wrapped 3-branes are *effectively* equivalent to wrapping 1-branes around the curve (1.2); the 1-branes, which are the left-over pieces of the 3-branes, are precisely the non-critical, (anti-)self dual strings of [12]. On physical grounds, one would expect this to be true also for the other, and in particular the exceptional, gauge groups, but this is at first sight not so obvious. By performing the integrals over  $x_2$  and  $x_3$  in  $W_{E_6}^{\text{ALE}}(x_1, x_2, x_3; u_j)$ , we will argue that indeed the 3-branes, when wrapped around the vanishing cycles of (1.3), are equivalent to 1-branes on the curves (1.2). We expect that a similar story holds for the other exceptional groups as well.

The potential importance of this type of considerations lies not in the generalization to other gauge groups of what has been known before, but ultimately in the generalization to other, non-field theory limits of type IIB strings. In a threefold, more general singularities than those describing YM theories typically do arise, and the question is under what conditions  $\alpha' \rightarrow 0$  physics can effectively be described in terms of SW-like curves (with anti-self-dual strings wrapping around them).

## 2. Seiberg–Witten Curves

In [8] a general scheme was presented for obtaining a family of Seiberg-Witten Riemann surfaces for an  $N = 2$  supersymmetric Yang-Mills theory with arbitrary gauge group,  $G$ . Specifically, given any representation  $\mathcal{R}$  of  $G$ , one considers the polynomial:

$$P_{\mathcal{R}}(x; u_j) = \det(x - \Phi_0) , \quad (2.1)$$

where  $\Phi_0$  is a generic adjoint Higgs v.e.v. written in the representation  $\mathcal{R}$ . One can diagonalize  $\Phi_0$  to some Cartan subalgebra element  $v \cdot H$ , where  $v$  is an  $\ell$ -dimensional vector. In the representation  $\mathcal{R}$ , the eigenvalues of  $\Phi_0$  are  $\lambda \cdot v$ , where  $\lambda$  are the weights of  $\mathcal{R}$ . Thus one can write:

$$P_{\mathcal{R}} = \prod_{\lambda} (x - \lambda \cdot v) . \quad (2.2)$$

The polynomial,  $P_{\mathcal{R}}(x; u_j)$ , naturally decomposes into factors corresponding to Weyl orbits of weights. If the representation  $\mathcal{R}$  is miniscule, then by definition, the weights of  $\mathcal{R}$  form exactly one such orbit. If the representation is not miniscule, then we will take  $P_{\mathcal{R}}$  to be any one of the Weyl orbit factors in the determinant (2.1). Let  $M$  be the degree of  $P_{\mathcal{R}}$ .

To get the Riemann surface for a simply laced group,  $G$ , one first defines a function  $\tilde{P}_{\mathcal{R}}(x, \zeta; u_j, \mu)$  via

$$\tilde{P}_{\mathcal{R}}(x, \zeta; u_j, \mu) \equiv P_{\mathcal{R}}(x; u_1, \dots, u_{\ell-1}, u_{\ell} + \zeta + \mu^2/\zeta) . \quad (2.3)$$

That is, one shifts the top Casimir by  $u_{\ell} \rightarrow u_{\ell} + \zeta + \mu^2/\zeta$ . The Riemann surface is then given by

$$\tilde{P}_{\mathcal{R}}(x, \zeta; u_j, \mu) = 0 \quad (2.4)$$

The canonical way to view this surface [13,14,8] is as an  $M$ -sheeted foliation by  $x(\zeta)$  over the base  $\zeta$ -sphere. The sheets are then in one-to-one correspondence with the weights of  $\mathcal{R}$  (or at least the Weyl orbit of weights that one has chosen). There are  $2(\ell + 1)$  branch points in the base, and they come in pairs related by the involution symmetry  $\zeta \rightarrow \mu^2/\zeta$ . The first  $\ell$  pairs of branch points can be labelled by a system of simple roots  $\{\alpha_j, j = 1, \dots, \ell\}$  of  $G$ , and the monodromy around the branch point is then given by the Weyl reflection  $r_{\alpha_j}$ . Above the  $\alpha_j$ -cut one then connects sheets in pairs according to whether the weights labelling the sheets are exchanged by the Weyl reflection  $r_{\alpha_j}$ . The  $(\ell + 1)$ -th pair of branch points is  $\{\zeta = 0, \zeta = \infty\}$  and each of these points has the Coxeter monodromy corresponding to the product of the fundamental Weyl reflections. The sheets are joined above  $\zeta = 0$  and  $\zeta = \infty$  according to the Coxeter orbits of the weight labels.

The genus of the Riemann surface is usually far larger than the rank,  $\ell$ , of  $G$ . Thus one needs to isolate a special sub-Jacobian, or, more precisely, a special Prym Variety, whose periods give the effective action of the  $U(1)^\ell$  on the Coulomb branch. This can be done using the underlying integrable system, and can be implemented directly in a number of ways [13,14,8]. The simplest is to first take the  $\ell$  cycles that surround the  $\alpha_j$ -cuts on the  $\zeta$ -sphere, and then take the inverse image under the projection of the Riemann surface to the  $\zeta$ -sphere. This gives  $\ell$   $A$ -cycles. The  $B$ -cycles are then obtained by finding the cycles that intersect the  $A$ -cycles in the proper manner. As we will see later, once one has the Seiberg-Witten differential  $\lambda_{SW}$ , the issue of the proper  $A$  and  $B$  cycles is essentially moot.

Because of the connection with integrable systems, the Seiberg-Witten differential takes the universal form:

$$\lambda_{SW} = -2x \frac{d\zeta}{\zeta} . \tag{2.5}$$

We now focus on the details of the curve for  $E_6$ . The simplest  $E_6$  curve is obtained from the (miniscule) 27-dimensional representation. Given a system of simple roots,  $\alpha_1, \dots, \alpha_6$ , one finds that for each root,  $\alpha_j$ , there are six<sup>2</sup> weights,  $\lambda_j^{(a)}$ , of the 27 such that  $\lambda_j^{(a)} \cdot \alpha_j = +1$ . Consequently,  $\lambda_j'^{(a)} \equiv \lambda_j^{(a)} - \alpha_j$  are weights of the 27 with  $\lambda_j'^{(a)} \cdot \alpha_j = -1$ . Thus above *each* of the six  $\alpha_j$ -cuts on the  $\zeta$ -sphere, the sheets of the foliation are connected in six pairs, making a total of 36 such interconnections. Under the action of the Coxeter element the weights form three orbits of order 12, 12 and 3 respectively. If one imagines assembling the surface by first making the connections at  $\zeta = 0$  and  $\zeta = \infty$ , one first gets three disjoint spheres (the Coxeter orbits) that must then be laced together by the 36 pieces of plumbing mentioned earlier. The result is a genus 34 surface.

This surface is explicitly given by

$$\frac{1}{2} x^3 \tau^2 - q_1 \tau + q_2 = 0 ; \quad \tau \equiv \zeta + \frac{\mu^2}{\zeta} + u_6 , \tag{2.6}$$

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<sup>2</sup> The fact that this is the rank of  $E_6$  is a coincidence.

where

$$\begin{aligned}
q_1 &= 270 x^{15} + 342 u_1 x^{13} + 162 u_1^2 x^{11} - 252 u_2 x^{10} + (26 u_1^3 + 18 u_3) x^9 \\
&\quad - 162 u_1 u_2 x^8 + (6 u_1 u_3 - 27 u_4) x^7 - (30 u_1^2 u_2 - 36 u_5) x^6 \\
&\quad + (27 u_2^2 - 9 u_1 u_4) x^5 - (3 u_2 u_3 - 6 u_1 u_5) x^4 - 3 u_1 u_2^2 x^3 \\
&\quad - 3 u_2 u_5 x - u_2^3 ; \\
q_2 &= \frac{1}{2x^3} (q_1^2 - p_1^2 p_2) ; \\
p_1 &= 78 x^{10} + 60 u_1 x^8 + 14 u_1^2 x^6 - 33 u_2 x^5 + 2 u_3 x^4 - 5 u_1 u_2 x^3 - u_4 x^2 \\
&\quad - u_5 x - u_2^2 ; \\
p_2 &= 12 x^{10} + 12 u_1 x^8 + 4u_1^2 x^6 - 12 u_2 x^5 + u_3 x^4 - 4 u_1 u_2 x^3 - 2 u_4 x^2 \\
&\quad + 4 u_5 x + u_2^2 .
\end{aligned} \tag{2.7}$$

Note that (2.6) is *not* a hyperelliptic curve, unlike the simplest curves for the  $A_n$  and  $D_n$  groups. Moreover, like the curves for  $D_n$ , the genus of the curve exceeds the rank of the gauge group, but unlike  $D_n$ , there is no obvious, elementary symmetry that picks out the cycles that yield the quantum effective action [5]. For the curve defined by (2.6) one has to use the methods outlined above and in [8] to determine the cycles of interest.

It is useful to note that since (2.6) is quadratic in  $\tau$ , one can solve it to obtain a more convenient presentation of the curve:

$$\tau = \zeta + \mu^2/\zeta + u_6 = \frac{1}{x^3} [ q_1 \pm p_1 \sqrt{p_2} ] , \tag{2.8}$$

where  $q_1, p_1$  and  $p_2$  are defined above.

This expression for  $\tau$  will be of importance in the next section, but it is also of interest for other reasons. One may view (2.8) as determining how each eigenvalue,  $x$ , of the v.e.v.,  $\Phi_0$ , changes as one varies the top Casimir,  $u_6$ , while holding the remaining Casimirs,  $u_1, \dots, u_5$  fixed. As a result, the explicit expression (2.8), which has degree 12, can be interpreted as a single variable (though non-polynomial) version of the Landau-Ginzburg potential for  $E_6$ , in the sense of [15]. It can probably be used to describe the coupling of  $E_6$  minimal matter to topological gravity directly, using the residue methods of [16,15]<sup>3</sup>.

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<sup>3</sup> Indeed, there was a long-standing question how the KdV type Lax operator, which is derived from the 27-dimensional representation of  $E_6$  via Drinfeld-Sokolov reduction and which thus has *a priori* degree 27, is related to the  $E_6$  simple singularity of degree 12. This simple singularity supposedly figures as the superpotential of a topological LG theory that describes the matter-gravity system [16].

Finally, we observe that one can also obtain (2.8) from decomposing Casimir invariants invariants of  $E_6$  into those of  $SO(10) \times U(1)$ . If one writes the  $u_j$  in terms of the five invariants of  $SO(10)$  and the  $U(1)$  eigenvalue  $x$ , and then inverts this relationship, one can easily extract an expression for  $u_6$  in terms of  $u_1, \dots, u_5$  and  $x$ . Thus one can easily derive (2.8) from the results of [17]. That this is equivalent to the procedure described above follows from the fact that the stabilizer of a weight space in the 27 of  $E_6$  is  $SO(10) \times U(1)$ .

### 3. The quantum effective action from string theory

It was argued in [10,18,9] that one can obtain the quantum effective action of a pure gauge theory from the IIB string compactified on a Calabi-Yau manifold that is degenerating to a nearly singular ALE fibration over a  $\mathbb{P}^1$  base. The Riemann surface can then be constructed<sup>4</sup> by using the monodromy data on the non trivial 2-cycles of the fiber to produce the monodromies of the Riemann sheets. The Seiberg-Witten differential is obtained by integrating the holomorphic (3, 0)-form,  $\Omega$ , of the Calabi-Yau manifold over the 2-cycles in the fiber, yielding a meromorphic 1-form on the Riemann surface [9]. This is easily verified for  $A_n$  gauge groups,<sup>5</sup> but it is far less obvious for the exceptional groups. Here we will describe, in detail, how it works for  $E_6$ .

The singular ALE fibration for  $E_6$  has the form:

$$\mathcal{P}(y_1, y_2, y_3; \zeta) \equiv W_{E_6}(y_1, y_2, y_3) + \nu (\zeta + \mu^2/\zeta) = 0, \quad (3.1)$$

where  $\zeta$  is the coordinate on the base  $\mathbb{P}^1$  and

$$W_{E_6}(y_1, y_2, y_3) = y_1^3 + y_2^4 + y_3^2 + w_1 y_1 y_2^2 + w_2 y_1 y_2 + w_3 y_2^2 + w_4 y_1 + w_5 y_2 + w_6. \quad (3.2)$$

The parameter  $\nu$  in (3.1) is a normalization constant that has been inserted for later convenience. The holomorphic (3, 0)-form in these local coordinates can be written:

$$\Omega = \left( \frac{d\zeta}{\zeta} \right) \wedge \frac{dy_1 \wedge dy_2}{y_3} = \left( \frac{d\zeta}{\zeta} \right) \wedge \frac{dy_1 \wedge dy_2}{\sqrt{\mathcal{P}(y_1, y_2, y_3 = 0; \zeta)}}. \quad (3.3)$$

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<sup>4</sup> Note that, strictly speaking, the SW curve itself is not geometrically embedded in the Calabi-Yau manifold.

<sup>5</sup> It is straightforward to extend the arguments of [9] to  $D_n$  gauge groups as well.

The appearance of  $d\zeta/\zeta$  in (3.3) follows from the Ricci flatness of the underlying Calabi-Yau and the Ricci flatness of the fiber.

We need to find the 2-cycles in the fiber space defined by  $W_{E_6} = \text{const}$ , and then we need to integrate (3.3) over these fibers. The key to doing this is to recall some beautiful facts of classical algebraic geometry [19]. One starts by recasting (3.2) as a cubic in  $\mathbb{P}^3$  [20]. That is, given

$$W(x_i) = x_3^2 x_4 + x_1^3 + 2i x_2^2 x_3 + w_1 x_1 x_2^2 + w_2 x_1 x_2 x_4 + w_3 x_2^2 x_3 + w_4 x_1 x_4^2 + w_5 x_2 x_4^2 + w_6 x_4^3, \quad (3.4)$$

in homogeneous coordinates,  $x_i$ , one can obtain (3.2) by going to the patch  $x_4 \neq 0$ , setting  $y_1 = x_1/x_4$ ,  $y_2 = x_2/x_4$  and  $y_3 = (x_3/x_4 + ix_2^2/x_4^2)$ . A cubic in  $\mathbb{P}^3$  can be thought of as the blow-up of six points in general position in  $\mathbb{P}^2$  [19]. The second homology,  $H_2$ , of this surface is seven dimensional, and consists of these six (non-intersecting) spheres, and the canonical class of the  $\mathbb{P}^2$ , which has intersection number 1 with each of the six blown up spheres. The six dimensional integer homology of the local  $E_6$  singularity consists on a six-dimensional subspace of  $H_2$ . This subspace is obtained by orthogonalizing with respect to a vector with coordinates  $(3, -1, -1, -1, -1, -1, -1)$ , and the result is a set of homology cycles that have the intersection matrix of  $E_6$  [19,20,13].

Thus far, we have seen very little that looks like the 27-sheeted Riemann surface described in the previous section. However, a celebrated fact about a cubic in  $\mathbb{P}^3$  is that it contains 27 lines. That is, there are 27 holomorphically embedded  $\mathbb{P}^1$ 's in this space. Each of these 27 lines is a non-trivial element of homology, and together they (over) span the (seven dimensional) homology. Thus, we can exhibit the homology by exhibiting these lines. Moreover, it is also well known that the monodromy group of the  $E_6$  singularity is the Weyl group of  $E_6$ , and that it acts on these lines as on the 27 of  $E_6$ <sup>6</sup>.

The explicit computation of the 27 lines for the generic  $E_6$  singularity was recently given by Minahan and Nemeschansky [21], and while the motivation of these authors was rather different from ours, our analysis will closely parallel theirs. Make a change of variables from  $(y_1, y_2)$  to  $(x, y)$  in (3.2), where  $y_1 = x y + \alpha(x)$ , and  $y_2 = y + \beta(x)$ . As yet,  $\alpha$  and  $\beta$  are arbitrary functions of  $x$ . With these changes of variable, the function  $\mathcal{P}$

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<sup>6</sup> As a representation of the Weyl group of  $E_6$ , the 27 is reducible:  $27 = 20 + 6 + 1$ . The singlet is the cycle with respect to which one orthogonalizes to get the cycles of the  $E_6$  singularity, and the 6 is the fundamental reflection representation (the Cartan subalgebra representation) on the compact homology basis of the  $E_6$  singularity.



in (3.1) is a quartic in  $y$ . Choosing  $\beta = -(x^3 + w_1x)/4$  cancels the  $y^3$  term in this quartic. One next chooses  $\alpha(x)$  so as to get rid of the linear term in  $y$ . This involves solving a quadratic equation for  $\alpha(x)$ , however, before doing this, we wish to reparametrize the versal deformation of the  $E_6$  singularity via:

$$\begin{aligned} w_1 &= \frac{1}{2} u_1 ; & w_2 &= -\frac{1}{4} u_2 ; & w_3 &= \frac{1}{96} (u_3 - u_1^3) ; \\ w_4 &= \frac{1}{96} (u_4 + \frac{1}{4}u_1 u_3 - \frac{1}{8}u_1^4) ; & w_5 &= -\frac{1}{48} (u_5 - \frac{1}{4}u_1^2u_2) ; \\ w_6 &= \frac{1}{3456}(u_6 + \frac{1}{16} u_1^6 - \frac{3}{16} u_1^3 u_3 + \frac{3}{32}u_3^2 - \frac{3}{4} u_1^2 u_4) . \end{aligned} \quad (3.5)$$

One then finds that

$$\alpha(x) = \frac{1}{48x} \left[ (2u_1 x^3 + u_1^2 x + 2u_2) \pm 2 \sqrt{p_2} \right] . \quad (3.6)$$

where  $p_2$  is given by (2.7).

The lines in cubic occur when one can analytically solve  $\mathcal{P} = 0$ . Since  $\mathcal{P}$  is quadratic in  $y_3$ , this happens precisely when the rest of  $\mathcal{P}$  is a perfect square. That is, we must find the points at which  $\mathcal{P}(y_1, y_2, y_3 = 0; \zeta)$  becomes a perfect square. With the changes of variable above, and the choices of  $\alpha(x)$  and  $\beta(x)$ , the function  $\mathcal{P}(x, y, y_3 = 0)$  takes the form of a quadratic in  $y^2$ . The lines are thus defined by the vanishing of the discriminant,  $\Delta$ , of this quadratic in  $y^2$ . A straightforward computation shows that:

$$\Delta = \frac{1}{864} \left[ \frac{q_1 \mp p_1 \sqrt{p_2}}{x^3} - u_6 \right] - 4 \nu (\zeta + \mu^2/\zeta) . \quad (3.7)$$

Taking  $\nu = \frac{1}{3456}$ , one thus finds that is discriminant vanishes precisely when  $x$  and  $\zeta$  satisfy the equation (2.6). We thus see that the Riemann surface is nothing other than a fibration, over the  $\zeta$ -sphere, of the locations of the 27 lines in the  $E_6$  singularity.

The lines are all defined by a relationship of the form:

$$y_3^2 = -(y^2 + a(x))^2 , \quad (3.8)$$

and hence, from the perspective of the singularity polynomial, these lines are all non-compact cycles in the closed homology of singularity relative to the boundary of the singularity [6]. Note that the coefficient of  $y^2$  in (3.8) is independent of  $x$ , and so all the lines meet at infinity. This means that the difference of any two lines defines a compact cycle of the singularity. It is over these compact cycles that we want to integrate  $\Omega$ .

With the change of variables above,  $\Omega$  now takes the form

$$\Omega = \left( \frac{d\zeta}{\zeta} \right) \wedge \frac{(y + \beta'(x) - x\alpha'(x)) dx \wedge dy}{\sqrt{y^4 + b_1(x, \zeta; u_j) y^2 + b_0(x, \zeta; u_j)}} \quad (3.9)$$

At generic values of  $x$  and  $\zeta$  there are two branch cuts for  $\Omega$  (related by  $y \rightarrow -y$ ) in the  $y$ -plane. These cuts simultaneously disappear at each of the 27 lines. As in [9], the “latitude circles” of the homology 2-cycles correspond to contours that surround the branch cuts, and these “latitude circles” vanish at the North and South poles of the 2-cycles defined by the vanishing of the branch cuts at the lines. In this way, a pair of lines defines a homology 2-cycle. The only difference here is that there are now two branch cuts (as opposed to one such cut for  $A_n$  and  $D_n$ ), and this introduces a minor subtlety.

Given two lines at which the cuts simultaneously vanish, there are two ways in which the branch points can behave. Suppose that near one line,  $L$ , the branch points appear in pairs  $\{y = \xi, y = \eta\}$ , and  $\{y = -\xi, y = -\eta\}$  with  $\xi \rightarrow \eta$  as one approaches  $L$ . Take the cuts,  $C_{\pm}$ , to run from  $\pm\xi$  to  $\pm\eta$  respectively. As one approaches another line,  $L'$ , either  $\xi \rightarrow \eta$  again, or  $\xi \rightarrow -\eta$ . For the former possibility, a contour surrounding either cut defines a non-trivial 2-cycle, and the two 2-cycles defined by the two cuts do not intersect. If, however, one has  $\xi \rightarrow -\eta$ , then the cuts  $C_+$  and  $C_-$  annihilate one another. Each cut then defines the half of a 2-cycle, and the annihilation of the cuts corresponds to gluing hemispheres of the 2-cycle together. These two possibilities are reflected in the fact that the difference of two different weights in the 27 of  $E_6$  can either have length-squared 4 or 2. For the former, the difference weights is the sum of two orthogonal roots of  $E_6$ , and the latter corresponds to a single root of  $E_6$ . It follows that if the lines  $L$  and  $L'$  give rise to a pair of non-intersecting 2-cycles, then there must be another intermediate line,  $L''$ , such that paths from  $L$  to  $L''$  and from  $L''$  to  $L'$  describe each of the spheres separately.

It should now be evident that the proper contour over which to integrate  $\Omega$  is one that surrounds both cuts, and that does not get pinched off as the cuts annihilate. This simple closed loop in the  $y$ -plane can then be deformed to large  $|y|$ , and the integral  $\oint \Omega dy$  becomes the elementary integral:  $\oint \frac{dy}{y} = 2\pi i$ . The integral over  $x$  is now trivial:  $2\pi i \int dx$  evaluated between the lines defining the poles and equatorial gluing of hemispheres of the 2-cycle in question. One thus finds that the integral of  $\Omega$  over the 2-cycle,  $L_{ij}$ , defined by the lines  $L_i$  and  $L_j$  is:

$$\int_{L_{ij}} \Omega = 2\pi i (x_j - x_i) \left( \frac{d\zeta}{\zeta} \right) = -\pi i (\lambda_{SW}|_{x=x_j} - \lambda_{SW}|_{x=x_i}), \quad (3.10)$$

where  $x_i$  and  $x_j$  are the locations of the lines. Thus we recover the difference of the Seiberg-Witten differential between the sheets of the Riemann surface, exactly as in [9].

Given that there are really only six compact 2-cycles in the fiber of the foliation, it follows that there are 12 compact 3-cycles (six  $A$ -cycles and six  $B$ -cycles) in the total

space of the foliation (3.1). Recalling the discussion of the branch cuts in the base, and the plumbing of the Riemann surface, we see that above each  $\alpha_j$ -cut, there are six copies of the  $\alpha_j$  2-cycle expanding from a branch point and collapsing back to a branch point. The result is the  $A$ -type 3-cycle  $\alpha_j$ . (The  $B$ -type cycles are made by going from one of these branch points out to  $\zeta = 0$  or  $\zeta = \infty$ .) Thus the redundancy of the genus 34 Riemann surface stems from the highly redundant description of a six-dimensional fiber homology in terms of 27 lines.

This establishes a useful result about the integrals of  $\lambda_{SW}$  on the Riemann surface (2.6): since there are only six  $A$ -type 3-cycles and six  $B$ -type 3-cycles, there can only be six independent integrals of  $\Omega$  over the sets of  $A$  and  $B$  cycles. This means that there can only be six independent integrals of  $\lambda_{SW}$  over the  $A$ -cycles of the Riemann surface, and six independent integrals over  $B$ -cycles, even though the genus of the surface is very large. Moreover, a basis of such integrals can be obtained by choosing *any* representative cycle in the surface that lies above each of the  $\alpha_j$  cuts.

#### 4. Comments

The lines on the surface in projective space, or the non-compact cycles of the local singularity type, have played a central role in defining the sheets of the Riemann surface. This was also true of the analysis in [9], though it was not explicitly stated there. The fact that the lines in the projective space span a vector space of dimension  $\ell + 1$  suggests that there may also be a further 2-cycle or 3-cycle in the Calabi-Yau manifold that is playing an interesting implicit role.

While we have treated the  $E_6$  theory in detail here, there is very probably a similar story for the other  $E_\ell$  groups. The corresponding singularities can all be realized as the blow-up of  $\ell$  points in general position ( $\ell \leq 8$ ) in a  $\mathbb{P}^2$  [13]. One can then presumably realize these surfaces in some weighted projective spaces, and then use the lines on these surfaces to relate the integration of  $\Omega$  over 3-cycles to the integration of  $\lambda_{SW}$  over cycles of Riemann surfaces, in analogy to what we did above.

In [21] the  $E_6$  singularity and the 27 lines were used to construct a candidate Seiberg-Witten curve for a superconformal theory with matter and  $E_6$  global symmetry. In string theory, global symmetries are generically gauged [22], and so one might expect some connection between the results here and those of [21] via some kind of flavour gauging. It turns out that we can come fairly close: there are two extreme degenerations of (2.6) at which the theory becomes superconformal: (i) the  $E_6$  Argyres-Douglas points [23,24], and (ii)

the point where the Argyres-Douglas points come together:  $\mu = 0$ , *i.e.* when  $\tau = \zeta + u_6$ . In the latter instance, the Riemann surface collapses to a genus 4 surface foliated over the  $x$ -sphere:

$$\left[ \frac{x^3 (\zeta + u_6) - q_1}{p_1} \right]^2 = p_2 . \quad (4.1)$$

The Seiberg-Witten differential, (2.5), is holomorphic on this surface except at infinity and at the zeroes of  $\zeta$ . At infinity the differential has a double pole, while at the zeroes of  $\zeta$  it has a simple pole with residue  $-2x$ . From (2.2), (2.3) and (2.4) one immediately sees that there are 27 such points on the base  $x$ -sphere, and the residues are simply  $-2\lambda \cdot v$ . If one views the six-dimensional vector  $v$  as the masses of some matter fields, then one has obtained a theory with matter with an  $E_6$  global symmetry. The most significant physical difference between this matter theory, and that of [21], is that this “matter theory” lacks an additional free parameter, called  $\rho$  in [21]. This parameter describes a Higgs v.e.v. on the Coulomb branch, but in our degeneration limit, the Coulomb branch has been collapsed to provide the masses.

In summary, we have made precise (for  $E_6$ ) something that was conjectured some while ago [2]: namely that the  $ADE$  singularity types must be the key ingredients in the construction of the quantum effective actions of the  $N = 2$  supersymmetric Yang-Mills theories with  $ADE$  gauge groups. As in [9] we have also established an important string theory result: the  $E_6$  Yang-Mills theory can be represented in terms of compactifying the six-dimensional self-dual string [12] on the genus 34 Riemann surface defined by (2.6).

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