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Anisotropic domain walls

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Abstract

We find an anisotropic, non-supersymmetric generalization of the extreme supersymmetric domain walls of simple non-dilatonic supergravity theory. As opposed to the isotropic non- and ultra-extreme domain walls, the anisotropic non-extreme wall has the *same* spatial topology as the extreme wall. The solution has naked singularities which vanish in the extreme limit. Since the Hawking temperature on the two sides is different, the generic solution is unstable to Hawking decay.

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1 Introduction

Domain walls [1] are surfaces interpolating between regions with different expectation values of some matter field(s). Such objects are interesting for a variety of reasons. Whenever the vacuum manifold has a non-trivial homotopy group $\pi_0(\mathcal{M})$, domain walls can exist as topological defects [2]. Therefore, the possibility of domain wall formation in the early universe—as a result of spontaneous symmetry breaking in unified gauge theories—has attracted much interest. But domain walls, and more generally solitons, are of interest from a purely theoretical perspective. Within string theory, it has recently been recognized that Bogomol'nyi–Prasad–Sommerfield saturated states could play an important rôle in its non-perturbative dynamics.

Over the last few years domain walls have been studied within four-dimensional $N = 1$ supergravity theory (see Ref. [3] for a review). After discovery of the "ordinary" supersymmetric supergravity domain walls [4, 5], their global space–time structure has been analyzed [6, 7], and the relation to the corresponding non-supersymmetric domain wall bubbles has been clarified [8, 9] (see Refs. [10, 11, 12] for generalizations to the dilatonic case). The isotropic vacuum domain walls can be classified according to the value of their surface density σ, compared to the energy-densities of the vacua outside the wall [8, 9]. The three kinds of isotropic walls are the static planar *extreme walls* with $\sigma = \sigma_{\text{ext}}$, the *non-extreme* two-centred bubbles with $\sigma > \sigma_{\text{ext}}$, and the *ultra-extreme* vacuum decay bubbles with $\sigma < \sigma_{\text{ext}}$.

In this paper we consider the *anisotropic* case, where the various components of the metric tensor has a different functional dependence of the distance z from the wall. We shall restrict the analysis to space–times with a line element of the form[1]

$$ds^2 = A_{(t)}^2 dt^2 - A_{(x)}^2 dx^2 - A_{(y)}^2 dy^2 - dz^2 \tag{1}$$

where $A_{(i)}$ $(i \in \{t, x, y\})$ all are functions of z, and where we have used the gravitational gauge (coordinate) freedom to normalize g_{zz} to -1. Throughout this paper, indices in parentheses, such as those in the metric above, *shall not* be subject to the Einstein summation convention.

2 The supersymmetric solution

Consider the bosonic piece of an $N = 1$ supersymmetric theory with one chiral matter superfield $\mathcal{T}$ in $3 + 1$ space–time dimensions:

$$\mathcal{L} = -\frac{1}{2}R + K_{T\overline{T}}\partial^\mu \overline{T}\partial_\mu T - V(T, \overline{T}) \tag{2a}$$

[1] We use units so that $\kappa \equiv 8\pi G = c = 1$.

where $K(T,\overline{T})$ is the Kähler potential,

$$V(T,\overline{T}) = \mathrm{e}^{K}\left(K^{T\overline{T}}|D_T W|^2 - 3|W|^2\right), \tag{2b}$$

is the scalar potential, and

$$D_T W \equiv \mathrm{e}^{-K}\left[\partial_T(\mathrm{e}^{K}W)\right] = W_T + W K_T \tag{2c}$$

is the Kähler covariant derivative acting on the superpotential W.

In a supersymmetric vacuum $D_T W = 0$, and thus the effective cosmological constant in such a vacuum is

$$\Lambda_{\mathrm{susy}} = -3\,\mathrm{e}^{K}|W|^2 \equiv -3\alpha^2.$$

Hence, it is non-positive in this theory.

Let us now review the supersymmetric anti-de Sitter (AdS)–Minkowski wall already discussed in Refs. [6, 9, 13]. In that case the metric (1) takes the form

$$ds^2 = A^2(dt^2 - dx^2 - dy^2) - dz^2 \tag{3}$$

(cf. the Appendix). The behaviour of the function $A(z)$ some distance from the wall, which is placed at $z = 0$, is given by

$$A(z) = \begin{cases} \mathrm{e}^{\alpha z} & \text{for } z < 0\text{: the AdS vacuum} \\ 1 & \text{for } z > 0\text{: the Minkowski vacuum.} \end{cases} \tag{4}$$

The fact that $A(z)$ vanish as $z \to -\infty$ suggests that the line element is geodesically incomplete on the AdS side of the wall. Further investigation shows that null geodesics leave the AdS side with finite affine parameter. Since $A(z)$ is a function of z only, the (2+1)-dimensional space–times with constant z (the slices parallel to the wall) are simply Minkowski space. Therefore, the interesting directions for possible coordinate extensions are (t,z). A Penrose conformal diagram for the compactified (t,z) coordinates is shown in Fig. 1.

3 Anisotropic vacuum solutions

In the thin wall approximation the energy–momentum tensor of the wall-forming matter field(s) is approximated by a cosmological constant outside the wall where the (nearly constant) potential term is dominating and a delta-function singularity in the wall surface where the kinetic term is dominating.

Let us first look at the gravitational field off the wall. Using the natural orthonormal frame, and the definitions

$$H_{(i)} \equiv \frac{d\ln A_{(i)}(z)}{dz}, \tag{5}$$

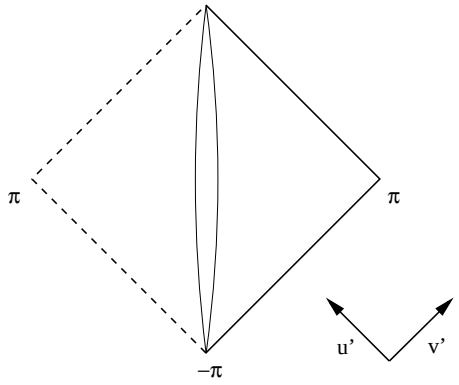

Figure 1: Penrose conformal diagram of one diamond formed by compactifying the coordinates (t, z). The wall is the lens-shaped region splitting the diamond in half. To the right of the wall is the Minkowski region. Solid lines symbolize geodesically completeness. To the left is the AdS region, where dashed lines indicates the need for coordinate extensions.

the Einstein tensor for the metric (1) is

$$G^0{}_0 = -H_{(x)}^2 - H_{(x)}H_{(y)} - H_{(y)}^2 - H'_{(x)} - H'_{(y)} \tag{6a}$$
$$G^1{}_1 = -H_{(t)}^2 - H_{(t)}H_{(y)} - H_{(y)}^2 - H'_{(t)} - H'_{(y)} \tag{6b}$$
$$G^2{}_2 = -H_{(t)}^2 - H_{(t)}H_{(x)} - H_{(x)}^2 - H'_{(t)} - H'_{(x)} \tag{6c}$$
$$G^3{}_3 = -H_{(t)}H_{(x)} - H_{(t)}H_{(y)} - H_{(x)}H_{(y)} \tag{6d}$$

where a prime means the derivative with respect to z.

We now solve the Einstein equations with the stress-energy tensor of a vacuum with a non-positive energy density

$$G^\mu{}_\nu = \delta^\mu{}_\nu \Lambda,$$

where Λ is a negative cosmological constant which we shall parametrize by $\Lambda = -3\alpha^2$.

By combining the Einstein equations in various ways, we get

$$H'_{(i)} + 3HH_{(i)} = 3\alpha^2 \tag{7a}$$
$$H' + 3H^2 = 3\alpha^2. \tag{7b}$$

Here $H = \frac{1}{3}\sum_i H_{(i)}$. Integrating Eq. (??b), we get

$$H = \alpha \frac{(\xi+\alpha)^2\,\mathrm{e}^{6\alpha z} + \xi^2}{(\xi+\alpha)^2\,\mathrm{e}^{6\alpha z} - \xi^2} \tag{8}$$

where ξ is an integration constant.

Given H, the solution for H_i is easily found to be[2]

$$H_{(i)} = H + c_{(i)}h \tag{9}$$

where

$$h = \frac{2\alpha\xi(\alpha+\xi)\,\mathrm{e}^{3\alpha z}}{(\alpha+\xi)^2\,\mathrm{e}^{6\alpha z} - \xi^2}. \tag{10}$$

Note that under the transformation

$$\xi \to \xi' = -\alpha\xi/(\alpha+2\xi), \tag{11}$$

H is invariant and h simply changes sign.

Moreover, both H and h are invariant under the transformations

$$\alpha \to \alpha' = -\alpha \quad \text{and} \quad \xi \to \xi' = \xi + \alpha. \tag{12}$$

The space–time geometry and the surface energy of the wall are therefore left unchanged under this transformation. It is thus sufficient to study the case $\alpha > 0$.

The constants satisfy

$$\sum_i c_{(i)} = 0 \quad \text{and} \quad \sum_i c_{(i)}^2 = 6, \tag{13}$$

which represents (in three dimensions) a plane going through origo with normal vector making equal angles with all three axes, and a spherical shell with radius $\sqrt{6}$ centred in origo respectively. The allowed values for the constants therefore lie on the circle where the plane cuts through the sphere, and it is easy to verify that $|c_{(i)}| \leq 2$. We note that ξ may be interpreted as an anisotropy-parameter because one gets the isotropic solution by letting $\xi \to 0$ or $\xi \to -\alpha$.

By taking the limit $\alpha \to 0$, we obtain the Kasner [15] type solution, where $H_{(i)}$ is defined by taking

$$H_{\alpha=0} = h_{\alpha=0} = \frac{\xi}{1+3\xi z} \tag{14}$$

in Eq. (9). Now we get the isotropic Minkowski solution by letting $\xi \to 0$.

4 Domain wall solutions

We shall now match the above vacuum solutions by means of an infinitely thin domain wall junction. To this end we employ the Israel formalism [16] for singular hypersurfaces and thin shells.

[2]The solution is related to a corresponding generalization [14] of the Kasner [15] cosmological solution by a complex coordinate transformation, see Ref. [12] for applications of such transformations to the dilatonic domain walls.

4.1 Israel matching

We place the wall at $z = 0$, and define the spacelike unit normal vector to its surface by

$$\boldsymbol{n} \equiv \frac{\partial}{\partial z} \quad \text{and} \quad \boldsymbol{n} \cdot \boldsymbol{n} = n^\mu n_\mu \equiv -1. \tag{15}$$

The extrinsic curvature of the wall is a three-dimensional tensor whose components are defined by the covariant derivative of this unit normal:

$$K^i{}_j \equiv -n^i{}_{;j}. \tag{16}$$

For our choice of coordinates we have

$$K_{ij} = -\zeta \frac{1}{2} g_{ij,z},$$

where $\zeta = \pm 1$ is a sign factor depending on the direction of the unit normal. For a kink-like matter source $\zeta|_- = \zeta|_+ = 1$, so in the sequel we drop this sign factor. Hence, we get

$$K^i{}_j = -\delta^i{}_j H_{(i)}. \tag{17}$$

The stress-energy tensor for the wall is given by

$$S^i{}_j = -(\gamma^i{}_j - \delta^i{}_j \gamma^k{}_k),$$

where $\gamma^i{}_j \equiv \lim_{\epsilon \to 0}[K^i{}_j(z = +\epsilon) - K^i{}_j(z = -\epsilon)]$. Using Eq. (17) and inserting the solution (9), we get

$$S^i{}_j = -\delta^i{}_j \left[c_{(i)} h + 2H \right]_{z=-\epsilon}^{z=+\epsilon}. \tag{18}$$

The square brackets stand for the *difference* taken at the points indicated with the super- and subscript on the closing bracket. Now, for a domain wall we must have a surface energy density $\sigma \equiv S^t{}_t$ and a tension $\tau \equiv S^x{}_x = S^y{}_y = \sigma$, *i.e.*, a boost invariant energy–momentum tensor on the world volume. Together with Eqs. (13) this implies that

$$c_{(i)}\big|_- = \lambda \, c_{(i)}\big|_+ \quad \text{and} \quad h|_- = \lambda \, h|_+ \,, \tag{19}$$

where $\lambda = \pm 1$.

4.2 Wall between two vacua with $\Lambda = 0$

In this case the vacuum solutions on both sides of the wall are given by Eqs. (14). The domain wall equation of state $\sigma = \tau$, together with the constraints on the $c_{(i)}$-constants gives the following expression when we choose $\Lambda = 0$ on both sides of the wall:

$$\sigma = 4\xi_- \tag{20}$$

where the parameters are related as

$$\xi_- = -\xi_+ \quad \text{and} \quad c_{(i)}\big|_- = -\, c_{(i)}\big|_+ \tag{21}$$

or $\lambda = -1$. (With $\lambda = 1$ and $\xi_- = \xi_+$ there is no wall at all.) Thus, in order to have a domain wall with a positive energy density, we must have $\xi_- > 0$. We note that $H > 0$ for $z < 0$ and $H < 0$ for $z > 0$. Hence, the average scale factor is decreasing away from the wall on both sides. This is a non-extreme solution for which the extreme limit is trivial Minkowski space–time. It was discovered by Tomita [17].

4.3 Walls between $\Lambda < 0$ and $\Lambda = 0$ vacua

Let $\Lambda < 0$ for $z < 0$ and $\Lambda = 0$ for $z > 0$. Then

$$\sigma = 2\alpha_- + 4\xi_-, \tag{22}$$

where again $\lambda = -1$ relates the values of h and $c_{(i)}$ on each side of the wall as in Eq. (19). Selecting $\lambda = 1$ adds nothing new due to the invariance of H and h under the transformation given in Eq. (11). For the wall to have a positive energy density, $\alpha_- > -2\xi_-$. Now the anisotropy parameter on the side with vanishing cosmological constant is related to the parameters on the other side by

$$\xi_+ = \frac{-2\xi_-(\alpha_- + \xi_-)}{\alpha_- + 2\xi_-}. \tag{23}$$

In the isotropic limit $\xi \to 0$, we recover the extreme anti-de Sitter–Minkowski wall [4, 5, 6].

4.3.1 Non-extreme solution

If $\alpha_-, \xi_- > 0$, then the solution is a kink-like non-extreme solution with $\sigma > \sigma_{\rm ext} = 2\alpha$. It is smoothly related to the extreme solution in the limit $\xi \to 0$.

4.3.2 Ultra-extreme solution

If $\alpha_- > 0$ and $-\alpha_-/2 < \xi_- < 0$, then the solution is a kink-like ultra-extreme solution with $\sigma < \sigma_{\rm ext} = 2\alpha$. The average scale factor is monotoneously increasing from the $\Lambda < 0$ vacuum through the wall and into the $\Lambda = 0$ vacuum. This solution is smoothly related to the extreme solution in the limit $\xi \to 0$.

Physically this solution would correspond to a planar vacuum decay wall, but since the Euclidean action would be infinite in this case, only O(4) symmetric bubbles are expected to be realized by vacuum tunnelling. We therefore regard this solution as unphysical.

4.4 Space–time structure and Hawking temperature

In Fig. 2 we present a Penrose conformal diagram for the compactified (t, z) coordinates of the space–time discussed in Sect. 4.3.1. On both sides of the wall there are naked singularities.

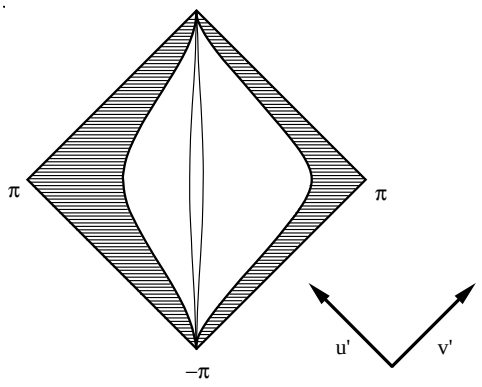

Figure 2: Penrose conformal diagram formed by compactifying the coordinates (t, z). Compare this diagram with Fig. 1. The wall is again the lens-shaped region splitting the diamond in half. To the right of the wall is the Kasner region, and to the left is the AdS region. The shaded area on both sides represents the forbidden regions beyond the singularities. They are *not* part of the classical space–time.

In general, the naked singularities have infinite Hawking temperatures. Here we calculate the temperature (equivalently, the surface gravity) at a wall-induced singularity. It is given by

$$T = -\frac{z_{\rm sng}}{|z_{\rm sng}|}\frac{\hbar}{2\pi k}\lim_{z\to z_{\rm sng}} A'_{(t)}, \tag{24}$$

where k is Boltzmann's constant, and

$$z_{\rm sng} = \begin{cases} \frac{1}{6\alpha}\ln\frac{\xi^2}{(\alpha+\xi)^2} & \text{if } \alpha \neq 0 \\ -\frac{1}{3\xi} & \text{if } \alpha = 0 \end{cases} \tag{25}$$

is the value of z at the singularity. Inserting for $A'_{(t)}$ and rearranging, we get a dimensionless and finite representation of the temperature:

$$T\frac{2\pi k}{\hbar\xi L} = -\frac{z_{\rm sng}}{|z_{\rm sng}|}(1+c_{(t)})\left[\frac{2\xi}{\alpha+2\xi}\right]^{(1-c_{(t)})/3}\left[\frac{\alpha+\xi}{\xi}\right]^{1/3}, \tag{26}$$

where

$$L \equiv \lim_{z \to z_{\rm sng}} \begin{cases} \left[\frac{(\alpha+\xi)\, e^{3\alpha z} - \xi}{\alpha}\right]^{(c_{(t)}-2)/3} & \text{if } \alpha \neq 0 \\ (1+3\xi z)^{(c_{(t)}-2)/3} & \text{if } \alpha = 0 \end{cases} \tag{27}$$

with the appropriate $z_{\rm sng}$ as defined in Eq. (25). It is understood that $\xi_\pm$ and $c_{(t)}\big|_\pm$ are used in place of ξ and $c_{(t)}$ in the above expressions.

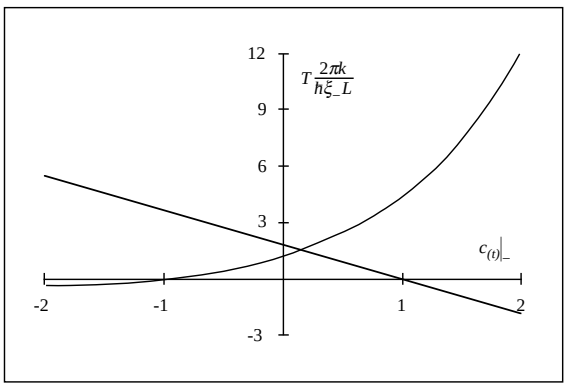

Figure 3: Plot of $T\frac{2\pi k}{\hbar \xi_- L}$ versus $c_{(t)}\Big|_-$ for the case discussed in Sect. 4.3.1 with $\xi_- = \alpha_-/10$. The curved graph represents the temperature on the $\Lambda < 0$, $z < 0$ side of the wall, while the straight graph yields the temperature on the $\Lambda = 0$, $z > 0$ side of the wall. Bear in mind that these graphs must be multiplied with a factor L, which is generally diverging, in order to obtain the temperature. Note that $|c_{(t)}| \leq 1$ in order to have a non-negative temperature everywhere.

At the singularities L diverges for all $c_{(t)}$ except $c_{(t)} = 2$, for which it equals 1. A plot of $T\frac{2\pi k}{\hbar \xi_- L}$ versus $c_{(t)}\Big|_-$ for the case discussed in Sect. 4.3.1 is provided in Fig. 3.

In order to have non-negative temperatures on both sides of the wall $|c_{(t)}| \leq 1$. However, in general the Hawking temperatures are different on the two sides, and therefore we expect the anisotropic domain wall solutions to be unstable to Hawking decay.

5 Conclusion

Only the planar, static and isotropic domain wall has a Killing spinor. Gravitational anisotropy therefore breaks supersymmetry and the corresponding non-extreme topological defects have a larger surface energy density.

The ultra-extreme walls are planar vacuum decay walls with a smaller energy-density. Due to their infinite Euclidean action, we consider these planar ultra-extreme solutions to be unphysical.

The anisotropic domain walls generally have a non-vanishing temperature gradient. These solutions are therefore unstable to Hawking decay.

A Killing spinor implies isotropy

The supercovariant derivative acting on the Majorana 4-spinor ϵ is given by

$$\widehat{\nabla}_\rho\epsilon = \left[2\nabla_\rho + i\,\mathrm{e}^{K/2}(WP_\mathrm{R} + \overline{W}P_\mathrm{L})\gamma_\rho - \mathrm{Im}(K_T\partial_\rho T)\gamma^5\right]\epsilon.$$

With a static and anisotropic line element of the planar form (1), we get the following explicit form of the supercovariant derivative acting on the spinor

$$\begin{aligned}
\widehat{\nabla}_t\epsilon &= \left[2\partial_t + \partial_z A_t\gamma^0\gamma^3 + iA_t\gamma^0(WP_\mathrm{L} + \overline{W}P_\mathrm{R})\,\mathrm{e}^{K/2}\right]\epsilon \\
\widehat{\nabla}_x\epsilon &= \left[2\partial_x + \partial_z A_x\gamma^1\gamma^3 - iA_x\gamma^1(WP_\mathrm{L} + \overline{W}P_\mathrm{R})\,\mathrm{e}^{K/2}\right]\epsilon \\
\widehat{\nabla}_y\epsilon &= \left[2\partial_y + \partial_z A_y\gamma^2\gamma^3 - iA_y\gamma^2(WP_\mathrm{L} + \overline{W}P_\mathrm{R})\,\mathrm{e}^{K/2}\right]\epsilon \\
\widehat{\nabla}_z\epsilon &= \left[2\partial_z - i\gamma^3(WP_\mathrm{L} + \overline{W}P_\mathrm{R})\,\mathrm{e}^{K/2} - \gamma^5\mathrm{Im}(K_T\partial_z T)\right]\epsilon.
\end{aligned}$$

The Killing spinor is one which satisfies the equation $\widehat{\nabla}_\rho\epsilon = 0$. Using the Weyl basis

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

where σ^i are the Pauli matrices, the Majorana spinor $\epsilon = \epsilon^c$ is

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_2^* \\ -\epsilon_1^* \end{pmatrix}. \tag{28}$$

Additionally, there is a constraint on the supersymmetric parameter $\epsilon_1 = \mathrm{e}^{i\Theta}\epsilon_2^*$. A Killing spinor therefore calls for the following equations to be satisfied

$$\partial_z \ln A_{(i)} = H_{(i)} = iW\,\mathrm{e}^{K/2}\,\mathrm{e}^{-i\Theta} \tag{29}$$

which implies $H_{(i)} = H$, satisfied only in the isotropic limit $\xi \to 0$.

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