HUB-EP-96/47

hep-th/9608154

PERTURBATIVE COUPLINGS AND MODULAR FORMS IN $N = 2$ STRING MODELS WITH A WILSON LINE

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ABSTRACT

We consider a class of four parameter $D = 4, N = 2$ string models, namely heterotic strings compactified on $K_3 \times T_2$ together with their dual type II partners on Calabi-Yau three-folds. With the help of generalized modular forms (such as Siegel and Jacobi forms), we compute the perturbative prepotential and the perturbative Wilsonian gravitational coupling F_1 for each of the models in this class. We check heterotic/type II duality for one of the models by relating the modular forms in the heterotic description to the known instanton numbers in the type II description. We comment on the relation of our results to recent proposals for closely related models.

August 1996

CERN-TH/96-227

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1 Introduction

Recently, accumulating evidence for the existence of various types of strong–weak coupling duality symmetries was gathered, such as S-duality of the four-dimensional $N=4$ heterotic string [1, 2, 3] and string-string dualities between the heterotic and type II strings [4, 5, 6]. The string-string duality between four-dimensional strings with $N=2$ space-time supersymmetry [6] is of particular interest, since $N=2$ strings exhibit a very rich non-perturbative structure which, in the point particle limit, contains [7] the non-perturbative effects of rigid $N=2$ gauge theories [8]. Furthermore, the $N=2$ strings are "half way" in between the well controlled $N=4$ models and the phenomenologically interesting, but much less understood $N=1$ string-string dualities [9].

The $N=2$ string-string duality between heterotic strings on $K3 \times T_2$ and corresponding type II strings on a suitably chosen Calabi–Yau three-fold has been successfully tested [6],[10]–[15] for models with a small number of vector multiplets. Most of these tests were based on the comparison of lower order gauge and gravitational couplings [16, 17, 18] of the perturbative heterotic string with the corresponding couplings of the dual type II string in some corner of the Calabi–Yau moduli space. One key point in establishing the string-string duality between heterotic and type II $N=2$ strings is the appearance [19] of certain modular functions in the low-energy effective action of these theories.

To be more specific, the discussion so far was essentially limited to models with number of massless Abelian vector multiplets $N_V = 3$ and $N_V = 4$. For the rank four case, $N_V = 4$, one is dealing with the heterotic S-field, with two T_2 moduli T and U plus the graviphoton. The perturbative heterotic vector multiplet couplings are given in terms of modular functions of the perturbative T-duality group $SO(2,2;\mathbf{Z})$. Due to the required embedding of this T-duality group into the $N=2$ symplectic transformations it follows [16, 17] that the heterotic one-loop prepotential must obey well-defined transformation rules under this group. In addition it was shown in [18] that the one-loop prepotential can be expressed in terms of the coefficients of the q expansion of certain modular forms. This heterotic S-T-U model is supposed [6] to be dual to the type II string compactified on the Calabi–Yau space $P_{1,1,2,8,12}(24)$ with $h_{1,1}=3$, $h_{2,1}=243$. In fact, it was shown for this example that the perturbative heterotic prepotential and the function F_1 (which specifies the non-minimal gravitational interactions involving the square of the Riemann tensor) agree with the corresponding type II functions in the limit where one specific Kähler class modulus of the underlying Calabi–Yau space becomes large. A set of interesting relations between certain topological Calabi–Yau data (rational and elliptic instanton numbers) and various modular forms has emerged when performing these tests [18, 15].

It is clearly an interesting problem to extend this kind of discussion to $N = 2$ string models with a larger number of vector multiplets, $N_V > 4$. It is the purpose of this paper to compute the heterotic one-loop couplings as well as to discuss the heterotic/type II string-string duality for these type of $N = 2$ string models, where we will concentrate on the particular case $N_V = 5$. Whereas the heterotic moduli T and U are related to the compactification from six to four dimensions on T_2, the additional vector fields originate from the ten-dimensional gauge group $E_8 \times E_8$ which survive after the compactification on $K3$. Usually the corresponding complex moduli are called Wilson lines; in case of $N_V = 5$ we denote the single Wilson line vector multiplet by V. The corresponding class of theories is called S-T-U-V models.

The classical moduli space as well as the classical T-duality transformations for heterotic string compactifications with Wilson line moduli were derived in [20, 21, 22]. For p non-vanishing Wilson lines the classical moduli space is locally given by the coset $\frac{SO(2,2+p)}{SO(2)\times SO(2+p)}$, and the T-duality group is given by $SO(2, 2 + p, \mathbf{Z})$. Together with the dilaton S-field moduli space one therefore deals at the classical level with the special Kähler spaces $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+p)}{SO(2)\times SO(2+p)}$, and the corresponding classical $N = 2$ prepotential can be easily constructed [23, 24, 16]. At the heterotic one-loop level the effective action is given in terms of automorphic functions of the duality group $SO(2, 2 + p, \mathbf{Z})$, which are functions of T, U and the Wilson line moduli [25, 22]. One generically encounters singularities at those points in the moduli space where certain perturbative BPS states become massless. (The automorphic functions can be constructed as infinite sum over the perturbative BPS spectrum.)

In this context it is important to realize that one encounters a very special situation in the presence of a single Wilson line V only, i.e. $N_V = 5$. In this case, as it was observed in [22], the classical T-duality group $SO(2, 3, \mathbf{Z})$ is isomorphic to $Sp(4, \mathbf{Z})$, which has a standard action on the Siegel upper half plane $\mathcal{H}_2$. The corresponding automorphic functions of $Sp(4, \mathbf{Z})$ are just given by the Siegel modular forms, which are directly associated to genus two Riemann surfaces. In the limit of vanishing Wilson line, $V \to 0$, the genus two Riemann surface degenerates into the product of two T_2, and the Siegel modular forms approach the $SO(2, 2, \mathbf{Z})$ modular functions of the S-T-U model in this limit.

In our paper we will show how the heterotic one-loop prepotential and the gravitational F_1-function for a class of $N = 2$ models with $N_V = 5$ can be constructed in terms of Siegel modular forms, Jacobi forms and ordinary (functions of τ only) modular functions. The models we are investigating are characterized by the embedding of the $SU(2)$ instanton numbers into the heterotic gauge group $E_8^{(1)} \times E_8^{(2)}$. We discuss the corresponding dual type II Calabi-Yau compactifications with $h_{1,1} = 4$ and find in this way the relation between the relevant modular forms and the rational Calabi-Yau instanton numbers. This relation will be shown to be satisfied for a particular example based on the Calabi-Yau space $P_{1,1,2,6,10}(20)$, recently discussed in [26].

Our paper is organized as follows. In the next section we define the class of models, that we will be investigating in the following, together with their massless spectrum. The models are discussed from the heterotic as well as from the dual type II point of view. In particular we discuss the points in the classical moduli space where extra states become massless. The various enhancement loci are given in terms of Humbert surfaces in the classical moduli space and are related to specific Siegel modular form such as $\mathcal{C}_{30}(T, U, V)$ and $\mathcal{C}_5(T, U, V)$. In section three we present the construction of the supersymmetric index for the $N = 2$ models with one Wilson line. In 3.1. we first review the computation [18] of the supersymmetric index of the S-T-U model. This construction can be nicely extended to the case $N_V = 5$ by a well defined "hatting" procedure of Jacobi functions, which describes the transition of going from Jacobi forms to ordinary modular forms. The physical interpretation of the hatting procedure is just the gauge symmetry breaking $SU(2) \to U(1)$ by turning on the Wilson line V. In section four we use the results of the previous chapter to write down the heterotic one-loop prepotential as a power series expansion in terms of hatted Jacobi functions. Comparing with the corresponding type II prepotential we relate the Calabi-Yau instanton numbers to the coefficients of the heterotic power series expansion. Using the known rational instanton numbers for the dual Calabi-Yau $P_{1,1,2,6,10}(20)$ we show that this relation holds for this specific example. In section five we compute the one-loop heterotic function F_1 in terms of the Siegel forms $\mathcal{C}_{30}$ and $\mathcal{C}_5$. A summary concludes the main body of the paper. In appendix A we review some interesting properties of Siegel and Jacobi modular forms. We also provide more details of the hatting procedure and its relation to theta functions and lattices, which is used to construct the supersymmetric index and the heterotic one-loop prepotential in the presence of a Wilson line V. In appendix B we show in some detail the computation of an integral which is needed for the computation of F_1.

During the process of finishing our calculations and writing up our results, some related work appeared in [27]. In [27] a four parameter model based on the Calabi-Yau $P_{2,2,3,3,10}(20)$ is discussed. We will make further comments on [27] in our paper. It is worth noting that recently the Siegel modular forms proved to be relevant for the computation of the non-perturbative elliptic genus of four-dimensional $N = 4$ strings [28].

2 $N = 2$ four parameter string models

In the following, we will discuss a class of heterotic 4 parameter $N = 2$ models, obtained by compactifying the $E_8 \times E_8$ string on $K3 \times T_2$. The four moduli comprise the dilaton S, the two toroidal moduli T and U as well as a Wilson line V. We will refer to these models as S-T-U-V models. Any of the S-T-U-V models in the class we will consider here has a dual type IIA description. Two such duals type II models have been recently discussed in the literature. The first one [26] consists of a type IIA compactification on the Calabi–Yau three-fold $P_{1,1,2,6,10}(20)$ with $h_{1,1} = 4$, $h_{2,1} = 190$ and consequently Euler number $\chi = -372$. This model has a Higgs transition [26] to the well known type IIA compactification on $P_{1,1,2,8,12}(24)$ with $h_{1,1} = 3$, $h_{2,1} = 243$ and $\chi = -480$, the so-called S-T-U model [6]. The next 4 parameter model, discussed by Kawai in [29, 27], is based on the Calabi–Yau $P_{2,2,3,3,10}(20)$ with $h_{1,1} = 4$, $h_{2,1} = 70$ and $\chi = -132$. Finally we will discuss two 4 parameter models with $h_{1,1} = 4$, $h_{2,1} = 214$, $\chi = -420$ and $h_{1,1} = 4$, $h_{2,1} = 202$, $\chi = -396$ respectively; the corresponding Calabi-Yau spaces were discussed in [30, 31]. Any of the S-T-U-V models considered here can be truncated to the 3 parameter S-T-U model upon setting $V \to 0$. Note that this is a truncation as far as the vector moduli sector is concerned; in the hyper moduli space one has to move to a generic point in the course of the Higgs transition [26].

The perturbative heterotic $N = 2$ models we will consider in the following will be constructed as follows. Following [6, 30, 32], we start with a compactification of the heterotic $E_8^{(1)} \times E_8^{(2)}$ string on $K3$ with $SU(2)$ bundles with instanton numbers $(d_1, d_2) = (12 - n, 12 + n)$. For $0 \leq n \leq 8$, the gauge group is $E_7^{(1)} \times E_7^{(2)}$, and the spectrum of massless hypermultiplets follows from the index theorem [33, 6] as

$$\frac{1}{2}(8-n)(\mathbf{56},\mathbf{1}) + \frac{1}{2}(8+n)(\mathbf{1},\mathbf{56}) + 62(\mathbf{1},\mathbf{1}). \tag{2.1}$$

For the standard embedding, $n = 12$, the gauge group is $E_8^{(1)} \times E_7^{(2)}$ with massless hypermultiplets

$$10(\mathbf{1},\mathbf{56}) + 65(\mathbf{1},\mathbf{1}). \tag{2.2}$$

These gauge groups can be further broken by giving vevs to the charged hypermultiplets. Specifically, $E_7^{(2)}$ can be completely broken through the chain

$$E_7 \to E_6 \to SO(10) \to SU(5) \to SU(4) \to SU(3) \to SU(2) \to SU(1), \tag{2.3}$$

where $SU(1)$ denotes the trivial group consisting of the identity only. In the following, we will concentrate on the cases where we break $E_7^{(2)}$ either completely or down to $SU(2)$.

On the other hand, $E_7^{(1)}$ can be perturbatively broken only to some terminal group $G_0^{(1)}$ that depends on n (see [30] for details); e.g. for $n=4$ this group is given by $G_0^{(1)} = SO(8)$. For $n=8$ it is $G_0^{(1)} = E_8$. It is only for $n=0,1,2$ that $E_7^{(1)}$ can be completely broken. Finally, when compactifying to four dimensions on T_2, three additional vector fields arise, namely the fields S, T and U.

Let us first discuss in slightly more detail the class of models where $E_7^{(2)}$ is completely broken. We will call, as it will be plausible in the following, these models the "S-T-U" class of models. In the dual type II description the corresponding Calabi-Yau spaces are given by elliptic fibrations over the Hirzebruch surface F_n. (For $n=2$ the corresponding Calabi-Yau is given by $P_{1,1,2,8,12}(24)$.) The models with $n=0,1,2$ all contain $N_V = h_{1,1}+1 = 4$ Abelian vector multiplets, the fields S, T, U plus the graviphoton, and in addition $N_H = h_{2,1}+1 = 244$ neutral hypermultiplets. In fact, at the heterotic perturbative level all three models are the same; the models with even $n=0,2$ are even identical at the non-perturbative level.

For $n>2$ both N_V and N_H increase (see the chain in the first column of table A.1 in [30]). However, suppose that $G_0^{(1)}$ could be completely broken and that $\dim(G_0^{(1)})$ hypermultiplets could be made massive by some mechanism, such that the spectrum would be given by $N_V=4$, $N_H=244$ for all n. Then it is natural to conjecture that all models are perturbatively equivalent; moreover we conjecture that the models with even respectively odd n are non-perturbatively equivalent.

Now let us come to the models with unbroken $SU(2)^{(2)}$. The corresponding Hodge numbers are given in the second column of table A.1 in [30]. The universal vector fields are now given by S, T, U and V, where the Wilson line V is in the Cartan subalgebra of $SU(2)^{(2)}$. The commutant of $SU(2)^{(2)}$ in $E_7^{(2)}$ is $SO(12)^{(2)}$. Then, it follows from the index theorem that the charged spectrum consists of $\frac{1}{2}(8-n)$ $\mathbf{56}$ of $E_7^{(1)}$, as well as of $\frac{1}{2}(8+n)$ $\mathbf{32}$ of $SO(12)^{(2)}$ plus 62 gauge neutral moduli.

As for the S-T-U models, it is only possible to perturbatively higgs the $E_7^{(1)} \times SO(12)^{(2)}$ completely for $n=0,1,2$. Thus, these heterotic models will have a massless spectrum comprising $N_V=5$ vector multiplets, S, T, U, V plus the graviphoton, as well as

$$N_H = (\frac{1}{2}(8+n)32 - 66 + \frac{1}{2}(8-n)56 - 133 + 62 = 12d_1 + 71 = 215 - 12n \qquad (2.4)$$

neutral hyper multiplets. Note that, unlike for the S-T-U models with $N_H = 244$, the number of hypermultiplets now depends on n. Furthermore, as we will discuss, for the four parameter models also the vector multiplet couplings are sensitive to n already at the perturbative level.

In the dual type IIA description, based on compactifications on 4 parameter Calabi–Yau three-folds X_n, the Euler numbers are $\chi(X_n) = 2(h_{1,1} - h_{2,1}) = 24n - 420$ and Hodge numbers are given by $h_{1,1} = N_V - 1 = 4$, $h_{2,1} = N_H - 1 = 214 - 12n$. The $n = 2$ Calabi–Yau three-fold X_2, for instance, is given by the space $P_{1,1,2,6,10}(20)$ of [26]. The Calabi–Yau spaces X_0 and X_1 and are given in [30, 31].

For $n > 2$ $E_7^{(1)}$ can only be higgsed to $G_0^{(1)}$ in a perturbative way and hence $N_V > 5$. However, suppose again for the moment that $G_0^{(1)}$ can be completely broken by some mechanism, and that $\dim(G_0^{(1)})$ massless hypermultiplets could disappear. Then $N_V = 5$ and the number of massless hypermultiplets is given by eq.(2.4). This would imply that on the dual type IIA side there exist Calabi–Yau spaces X_n with $\chi(X_n) = 24n - 420$ for $0 \leq n \leq 8$ and $n = 12$. In fact, for $n = 12$ a candidate Calabi–Yau really exist, namely the $n = 12$ Calabi–Yau space X_{12} is given by the space $P_{2,2,3,3,10}(20)$ of [29, 27]. Note that X_{12} and the $n = 2$ space $X_2 = P_{1,1,2,6,10}(20)$ both directly show the same K3 fibre $P_{1,1,3,5}(10)$. Futhermore, this also holds for X_0 and X_1 [30].

In summary, we will focus our proceeding discussion on the cases $n = 0, 1, 2, 12$ where the Hodge numbers of the corresponding Calabi-Yau space are summarized in the following table.

-	X_0	X_1	X_2	X_{12}
$-\chi$	420	396	372	132
$h^{2,1}$	214	202	190	70

At the transition point $V = 0$, the $U(1)$ associated with the Wilson line modulus V becomes enhanced to an $SU(2)$. Let $N_V' = 2$ and N_H' denote the number of additional vector and hyper multiplets becoming massless at this transition point. Then

$$\frac{1}{2}(N_H' - N_V') = 6n + 15 \quad . \tag{2.5}$$

This will prove to be a useful relation later on. It follows from the fact that the Euler number of the Calabi–Yau space $\chi(X_n)$ and of the S-T-U models ($\chi = -480$) differ by $2(N_H' - N_V') = \chi(X_n) + 480$.

In addition to the $V = 0$ locus of gauge symmetry enhancement, there are also the enhancement loci (such as $T = U$), associated with the toroidal moduli T and U, already known from the S-T-U model. All these loci correspond to surfaces/lines of gauge symmetry enhancement in the heterotic perturbative moduli space $\mathcal{H}_2 = \frac{SO(3,2)}{SO(3)\times SO(2)}$ and have a common description as follows.

Consider the Narain lattice $\Gamma = \Lambda \oplus U(-1)$ of signature $(3,2)$, where $U(-1)$ denotes the hyperbolic plane $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and where $\Lambda = U(-1) \oplus <2>= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ in a basis which we will denote by (f_2, f_{-2}, f_3); we will use the coordinate $z = iTf_{-2} + iUf_2 - iVf_3$ in $\Lambda \otimes \mathbf{C}$. Note here that the perturbative moduli space $\frac{SO(3,2)}{SO(3)\times SO(2)}$, which is a hermitian symmetric space, has a representation as a bounded domain of type IV, that is, as a connected component of $\mathcal{D} = \{[\omega] \in \mathbf{P}(\Gamma \otimes \mathbf{C}) | \omega^2 = 0,\ \omega \cdot \bar{\omega} > 0\} = \Lambda \otimes \mathbf{R} + iC(\Lambda) \subset \Lambda \otimes \mathbf{C}$, where $C(\Lambda) = \{x \in \Lambda \otimes \mathbf{R} | x^2 < 0\}$ (this last condition ensures again that $2\mathrm{Re}T\,\mathrm{Re}U - 2(\mathrm{Re}V)^2 > 0$; the connected component can then be realised as $\mathcal{D}^+ = \Lambda \otimes \mathbf{R} + iC^+(\Lambda)$,where $C^+(\Lambda)$ denotes the future light cone component of $C(\Lambda)$). Now in the basis $\varepsilon_1 = f_{-2} - f_2, \varepsilon_2 = f_3, \varepsilon_3 = f_2 - f_3$, Λ is equivalent to the intersection matrix $A_{1,0} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}$ associated to the Siegel modular form $\mathcal{C}_{35}$ of [34]. To each element ε_i, which squares to 2, is associated the Weyl reflection $s_i : x \to x - (x \cdot \varepsilon_i)\varepsilon_i$. The fixed loci of these Weyl reflections give the enhancement loci [25]. As these reflection planes are given by planes orthogonal to the elements ε_i, this gives rise to the following loci: the orthogonality conditions $(a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3)\varepsilon_i = 0$ yield $c = 2a$, $b = c$ and $a = 2(c-b)$. Since a, b and c are related to T, U and V by $a = iT$, $b = iT + iU - iV$ and $c = iT + iU$, as can be seen by comparing $a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 = a(f_{-2} - f_2) + bf_3 + c(f_2 - f_3) = af_{-2} + (c-a)f_2 + (b-c)f_3$ with $z = iTf_{-2} + iUf_2 - iVf_3$, the above orthogonality conditions result in the enhancement loci $T = U$,$V = 0$ and $T - 2V = 0$. Note that these are the conditions for enhancement loci related to $\mathcal{C}_{35} = \mathcal{C}_{30} \cdot \mathcal{C}_5$ (cf. appendix A). Also note that the locus $T - 2V = 0$ locus goes over into the locus $T - U = 0$ under the target space duality transformation [20] $T \to T + U + 2V, U \to U, V \to V + U$. Thus, the enhancement lines of the S-T-U model have become the Humbert surfaces H_4 and H_1 (cf. the discussion about rational quadratic divisors given in ch. 5 of [18] ($s = 1$) as well as in [34]).

Furthermore, [2] for the $K3$-fibre $P_{1,1,3,5}(10)$ of X_n, one finds that (cf. [26] for $n = 2$) in the

[2]Note that, since the heterotic perturbative gauge group is reflected, on the dual type IIA side, in the

basis j_1, j_3, j_4 (where we denote the intersections of the CY divisors with the $K3$ (J_2) by small letters) the intersection form is given by $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 0 & 2 \\ 4 & 2 & 6 \end{pmatrix}$, which is equivalent (over $\mathbf{Z}$) to $-\Lambda$ under the base change $f_2 = j_1 - j_3$, $f_{-2} = j_3$ and $f_3 = 2j_1 - j_4$. The enhancement loci will become the conditions $t_3 = 0$ resp. $t_4 = 0$ for the Kähler moduli on the type II side (cf. section 4).

3 The supersymmetric index

It was shown in [38, 39] that threshold corrections in $N = 2$ heterotic string compactifications can be written in terms of the supersymmetric index

$$\frac{1}{\eta^2} Tr_R F(-1)^F q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \quad . \tag{3.1}$$

This quantity is, as shown in [18], also related to the computation of the perturbative heterotic $N = 2$ prepotential. In the next subsection we will first review the computation of the index (3.1) for an S-T-U model. In the following subsection, we will then discuss its computation in an S-T-U-V model.

3.1 The S-T-U models

For the S-T-U model with instanton number embedding $(d_1, d_2) = (0, 24)$, the supersymmetric index (3.1) was calculated in [18] and found to be equal to

$$\frac{1}{\eta^2} Tr_R F(-1)^F q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} = -2i Z_{2,2} \frac{E_4 E_6}{\Delta} \quad , \tag{3.2}$$

where $Z_{2,2}$ denotes the sum over the Narain lattice $\Gamma_{2,2}$, $Z_{2,2} = \sum_{p \in \Gamma^{2,2}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}$, and where $\frac{E_4 E_6}{\Delta} = \sum_{n \geq -1} \tilde{c}_{STU}(n) q^n$. Here the subscript on the trace indicates the Ramond sector as right-moving boundary condition; F denotes the right-moving fermion number, $F = F_R$. Let us recall how this expression came about. First, one can reduce (3.1) to $\frac{1}{\eta^2} Tr_R (-1)^F q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24}$, where the contributions are weighted with $\pm 2\pi i$ depending on whether a BPS hyper or vector multiplet contributes. The expression resulting from

(monodromy invariant part of the) Picard group of the generic $K3$ fibre of the Calabi-Yau [35, 36], the discussion presented here agrees precisely with the one of [37] concerning the zero divisor of the period map for the (mirror of the) $K3$. $\mathcal{D}^+$ can be matched with the domain of the period map $\Phi(z)$.

the evaluation of the trace consists of the product of three terms, namely of $Z_{2,2}/\eta^4$, of the partition function for the first $E_8^{(1)}$ in the bosonic formulation (leading to the contribution E_4/η^8) and of the elliptic genus for the second $E_8^{(2)}$ containing the gauge connection on $K3$.

This last quantity decomposes now additively (taking into account the appropriate weightings) into contributions from the following sectors, namely: 1) the (NS,R) sector, which we will also denote by (NS^+,R), 2) the "twisted" sector (NS^-,R), where a factor $(-1)^{F_L}$ is inserted in the trace (this contribution is weighted with (-1)) and 3) the (R,R) sector, which we will also denote by (R^+,R). Since we are using the fermionic representation for $E_8^{(2)}$, we decompose the fermionic $D_8^{(2)} \subset E_8^{(2)}$, so that each of these contributions splits again multiplicatively into a free $D_6^{(2)}$ part and into a D_2 part, to be called $D_2^{(2)}K3$, containing the gauge connection A_1 which describes the corresponding gauge bundle on $K3$. The corresponding contributions are summarized in the following table, where we also indicate the connection to the generic elliptic genus $Z(\tau,z) = Tr_{R,R} y^{F_L}(-1)^{F_L+F_R} q^{L_o-c/24}\bar{q}^{\tilde{L}_o-\tilde{c}/24} = 6\frac{\theta_2^2\theta_4^2}{\eta^4}\frac{\theta_3^2(\tau,z)}{\eta^2} - 2\frac{\theta_4^4-\theta_2^4}{\eta^4}\frac{\theta_1^2(\tau,z)}{\eta^2}$, where $y = \mathbf{e}[z] = \exp 2\pi i z$ (cf. [40, 41]).

Tr	D_6	$K3D_2$
(NS^+,R)	$\frac{\theta_3^6}{\eta^6}$	$-2\frac{\theta_4^4-\theta_2^4}{\eta^4}\frac{\theta_3^2}{\eta^2} = q^{\frac{1}{4}}Z(\tau,\frac{\tau+1}{2})$
(NS^-,R)	$\frac{\theta_4^6}{\eta^6}$	$2\frac{\theta_2^4+\theta_3^4}{\eta^4}\frac{\theta_4^2}{\eta^2} = q^{\frac{1}{4}}Z(\tau,\frac{\tau}{2})$
(R^+,R)	$\frac{\theta_2^6}{\eta^6}$	$2\frac{\theta_3^4+\theta_4^4}{\eta^4}\frac{\theta_2^2}{\eta^2} = Z(\tau,\frac{1}{2})$
(R^-,R)	$\frac{\theta_1^6}{\eta^6} = 0$	$6\frac{\theta_2^2\theta_3^2\theta_4^2}{\eta^4\cdot\eta^2} = 24 = Z(\tau,0)$

Now recall that E_4 and E_6 have the following θ-function decomposition

$$\begin{aligned} 2E_4 &= \theta_2^6\cdot\theta_2^2 + \theta_3^6\cdot\theta_3^2 + \theta_4^6\cdot\theta_4^2 \\ 2E_6 &= -\theta_2^6(\theta_3^4+\theta_4^4)\cdot\theta_2^2 + \theta_3^6(\theta_4^4-\theta_2^4)\cdot\theta_3^2 + \theta_4^6(\theta_2^4+\theta_3^4)\cdot\theta_4^2\ ; \end{aligned} \tag{3.3}$$

the θ_i^2 contributions ($i=2,3,4$) are due to the $SO(4)$ piece in the fermionic decomposition of $E_8 \supset SO(12)\times SO(4)$. Hence the sum of the three non-vanishing terms in the table precisely leads to (3.2).

On the other hand, in the case of a general (d_1,d_2) embedding (using now a fermionic representation for both E_8's), one first has to decompose the $D_2^{(1)}K3D_2^{(2)}$ part into

$D_2^{(1)}K3 \times D_2^{(2),free} + D_2^{(1),free} \times K3D_2^{(2)}$, where the factors in each summand are now in different, and hence commuting, E_8's. Furthermore, since the rudimentary $K3$ gauge bundles are structurally completely the same as before, the amount of contribution realised by them can - by comparison with the "complete" $K3$ bundle considered above - be read off from the R^- sector. Note that $Z(\tau, 0)$ is the Witten index, which gives the Euler number of $K3$ resp. the second Chern class of the relevant vector bundle.
This results in a contribution proportional to

$$\frac{1}{\Delta}(\frac{d_1}{24}E_6 \cdot E_4 + E_4 \cdot \frac{d_2}{24}E_6) = \frac{1}{\Delta}E_4 E_6 = \sum_{n \geq -1} c_{STU}(n) q^n \quad , \tag{3.4}$$

so that the result is independent of the particular instanton embedding.

3.2 The S-T-U-V models

In the presence of a Wilson line, which we will take to lay in the second $E_8^{(2)}$, the symmetry between the two E_8's is broken and thus, contrary to the 3 parameter case, the prepotential will already depend perturbatively on the type (d_1, d_2) of the instanton embedding (we take $d_2 \geq d_1$).
The supersymmetric index (3.1) will now have the form

$$\frac{1}{\eta^2} Tr_R F(-1)^F q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} = -2i Z_{3,2}(\tau, \bar{\tau}) F(\tau) \quad , \tag{3.5}$$

where $Z_{3,2}$ denotes the sum over the Narain lattice $\Gamma_{3,2}$, $Z_{3,2} = \sum_{p \in \Gamma^{3,2}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}$. The presence of the Wilson line in $E_8^{(2)}$ has the following effect on the θ_i^2 pieces appearing in the decomposition (3.3) of E_4 and E_6

$$\begin{aligned} 2\widehat{E_{4,1}(\tau, z)} &= \theta_2^6 \cdot \widehat{\theta_2^2(\tau, z)} + \theta_3^6 \cdot \widehat{\theta_3^2(\tau, z)} + \theta_4^6 \cdot \widehat{\theta_4^2(\tau, z)} \quad , \\ 2\widehat{E_{6,1}(\tau, z)} &= -\theta_2^6(\theta_3^4 + \theta_4^4) \cdot \widehat{\theta_2^2(\tau, z)} + \theta_3^6(\theta_4^4 - \theta_2^4) \cdot \widehat{\theta_3^2(\tau, z)} + \theta_4^6(\theta_2^4 + \theta_3^4) \cdot \widehat{\theta_4^2(\tau, z)} \quad , \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \widehat{\theta_1^2(\tau, z)} &= \theta_2(2\tau) - \theta_3(2\tau) \quad , \\ \widehat{\theta_2^2(\tau, z)} &= \theta_2(2\tau) + \theta_3(2\tau) \quad , \\ \widehat{\theta_3^2(\tau, z)} &= \theta_3(2\tau) + \theta_2(2\tau) \quad , \\ \widehat{\theta_4^2(\tau, z)} &= \theta_3(2\tau) - \theta_2(2\tau) \end{aligned} \tag{3.7}$$

are the two $SU(2)$ characters of the surviving A_1 when written in the boundary condition picture instead of the usual conjugacy class picture. We refer to appendix A.4 and A.5 for description and interpretation of the hatting procedure.

The replacement $E_4 \to \widehat{E_{4,1}}$, in particular, amounts to replacing the E_8 partition function $P_{E_8} = P_{E_7^{(0)}} \cdot P_{A_1^{(0)}} + P_{E_7^{(1)}} \cdot P_{A_1^{(1)}}$ with $P_{E_7^{(0)}} + P_{E_7^{(1)}}$. This precisely describes the breaking of the $E_8^{(2)}$ to $E_7^{(2)} \times U(1)$ when turning on a Wilson line.

Thus, the effect of turning on a Wilson line can be described as follows. Introducing

$$A_n(\tau) = \frac{1}{\Delta}\left(\frac{d_1}{24}E_6 \cdot \widehat{E_{4,1}} + E_4 \cdot \frac{d_2}{24}\widehat{E_{6,1}}\right) \quad , \tag{3.8}$$

it follows that turning on a Wilson line results in the replacement

$$\begin{aligned} Z_{2,2} \quad &\to \quad Z_{3,2} \ , \\ \frac{1}{\Delta}\left(\frac{d_1}{24}E_6 \cdot E_4 + E_4 \cdot \frac{d_2}{24}E_6\right) \quad &\to \quad F(\tau) = A_n \ . \end{aligned} \tag{3.9}$$

(The first few expansion coefficients of A_0, A_1, A_2 and A_{12} are listed in the second table in appendix A.6.) The product $Z_{3,2}A_n$ transforms covariantly under modular transformations, since $F(\tau)$ has weight $-2\frac{1}{2}$. (Recall that $\frac{E_4 E_6}{\Delta}$ has weight -2.)

The occurence of modular forms $F(\tau)$ of half-integral weight is naturally understood by realising that the present case (of $s = 1$ Wilson lines turned on) interpolates between the $s = 0$ and $s = 8$ cases of [18], where the relevant modular forms E_4E_6/Δ ($s = 0$) and E_6/Δ ($s = 8$) are of weight -2 and -6, respectively.

4 The perturbative prepotential for the S-T-U-V models

In this section we discuss the relation between the type II and the heterotic prepotentials for the S-T-U-V models, that is between rational instanton numbers on the type II side and Siegel modular forms on the heterotic side. The appearance of Siegel modular forms in the context of threshold corrections in the presence of Wilson lines was first pointed out in [22].

As discussed in the previous section, the supersymmetric index is given in terms of

$$F(\tau) = A_n = \sum_{N \in \mathbf{Z}, \mathbf{Z}+\frac{3}{4}} c_n(4N)q^N \quad . \tag{4.1}$$

As explained in appendix A, the modular function $A_n(\tau)$ is in one-to-one correspondence with the index-one Jacobi form with the same expansion coefficients $c_n(k, b) = c_n(4k - b^2)$: $A_n(\tau) = \widehat{A_n(\tau, z)}$, $A_n(\tau, z) = \frac{1}{\Delta(\tau)}\left(\frac{d_1}{24}E_6(\tau) \cdot E_{4,1}(\tau, z) + E_4(\tau) \cdot \frac{d_2}{24}E_{6,1}(\tau, z)\right) = \sum_{k,b} c_n(4k - b^2)q^k r^b$.[3]

[3] $A_n(\tau, z)$ can be eventually seen as the order s expansion coefficient of a Siegel modular form $F_n(T, U, V)$, again with identical expansion coefficients. Specifically, the index-one Jacobi form $A_n(\tau, z)$ is the order s expansion coefficient of the Siegel form $\frac{1}{132}(-\mathcal{E}_4\mathcal{E}_6 + (31 \cdot 12^3 - 11 \cdot 12^2 n)\mathcal{C}_{10})$.

The expansion coefficients $c_n(4N)$ of $F(\tau)$ govern the perturbative, i.e. 1-loop, corrections to the heterotic prepotential $F_0^{\rm het}$ [18]. For the class of S-T-U-V models considered here, the perturbative heterotic prepotential is given by

$$F_0^{\rm het} = -S(TU-V^2)+p_n(T,U,V)-\frac{1}{4\pi^3}\sum_{\substack{k,l,b\in\mathbf{Z}\\(k,l,b)>0}} c_n(4kl-b^2)Li_3(\mathbf{e}[kiT+liU+biV]), \tag{4.2}$$

where $\mathbf{e}[x]=\exp 2\pi i x$. The first term $-S(TU-V^2)$ is the tree-level prepotential of the special Kähler space $\frac{SO(3,2)}{SO(3)\times SO(2)}$; $p_n(T,U,V)$ denotes the one-loop cubic polynomial which depends on the particular instanton embedding n. The condition $(k,l,b)>0$ means that: either $k>0, l,b\in\mathbf{Z}$ or $k=0, l>0, b\in\mathbf{Z}$ or $k=l=0, b<0$ (cf. [18]). It is shown in appendix B how the worldsheet expansion coefficients $c_n(4N)$ turn into the target-space coefficients $c_n(4kl-b^2)$ appearing in the prepotential.

Next, consider truncating an S-T-U-V model to the S-T-U model by setting $V=0$. Then, the sum over b in (4.2) yields independently from n the coefficients of the 3 parameter model,

$$c_{STU}(kl)=\sum_b c_n(4kl-b^2)\ , \tag{4.3}$$

as it can be checked by explicit comparison. Therefore the prepotential (4.2) truncates correctly to the prepotential for the S-T-U model.

The (Wilsonian) Abelian gauge threshold functions are related (see [16] for details) to the second derivatives of the one-loop prepotential $h(T,U,V)=p_n(T,U,V)-\frac{1}{4\pi^3}\sum_{(k,l,b)>0}c_n(4kl-b^2)Li_3(\mathbf{e}[kiT+liU+biV])$. At the loci of enhanced non-Abelian gauge symmetries some of the Abelian gauge couplings will exhibit logarithmic singularities due to the additional massless states. First consider $\partial_T\partial_U h$. At the line $T=U$ one $U(1)$ is extended to $SU(2)$ without additional massless hypermultiplets. It can be easily checked that, as $T\to U$,

$$\partial_T\partial_U h=-\frac{1}{\pi}\log(T-U)\ , \tag{4.4}$$

as it should. The Siegel modular form which vanishes on the $T=U$ locus and has modular weight 0 is given by $\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5}$. It can be shown that, as $V\to 0$,

$$\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5}\to (j(T)-j(U))^2\ , \tag{4.5}$$

up to a normalization constant. Hence one deduces that

$$\partial_T\partial_U h=-\frac{1}{2\pi}\log\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5}+regular. \tag{4.6}$$

On the other hand, at the locus $V=0$, a different $U(1)$ gets enhanced to $SU(2)^{(2)}$, and at the same time N'_H hyper multiplets, being doublets of $SU(2)^{(2)}$, become massless. Using eq.(2.5), $N'_V=2$ and that $c_n(-1)=-N'_H, c_n(-4)=N'_V$, it can be checked that, as $V\to 0$,

$$-\frac{1}{4}\partial_V^2 h=\frac{3}{2\pi}(2+n)\log V=-\frac{1}{\pi}(1-\frac{1}{8}N'_H)\log V \quad . \tag{4.7}$$

Observe that the factor $(1-\frac{1}{8}N'_H)$ is precisely given by the $N=2$ $SU(2)$ gauge β-function coefficient with $N'_H/2$ hypermultiplets in the fundamental representation of $SU(2)$. The Siegel modular form which vanishes on the $V=0$ locus and has modular weight 0 is given by $\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}}$. It can be shown that, as $V\to 0$,

$$\mathcal{C}_5\to V\,(\Delta(T)\Delta(U))^{\frac{1}{2}} \quad , \tag{4.8}$$

So we now conclude that

$$-\frac{1}{4}\partial_V^2 h=\frac{3}{4\pi}(2+n)\log\left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}}\right)^2+regular. \tag{4.9}$$

Let us now compare the heterotic models with the corresponding type II models on the Calabi–Yau spaces X_n. The cubic parts of the type II prepotentials of X_0, X_1 and X_2 are given in [26, 31] and can be written in an universal, n-dependent function as follows:

$$\begin{aligned} F^{II}_{\text{cubic}} &= t_2(t_1^2+t_1t_3+4t_1t_4+2t_3t_4+3t_4^2) \\ &+ \frac{4}{3}t_1^3+8t_1^2t_4+\frac{n}{2}t_1t_3^2+(1+\frac{n}{2})t_1^2t_3+2(n+2)t_1t_3t_4 \\ &+ nt_3^2t_4+(14-n)t_1t_4^2+(4+n)t_3t_4^2+(8-n)t_4^3. \end{aligned} \tag{4.10}$$

We believe that this expression is also valid for X_n with $n>2$, in particular also for the Calabi–Yau model X_{12}. Note that for $t_4=0$, F^{II}_{cubic} precisely reduces to the cubic prepotential of the S-T-U models [42, 31]. In order to match (4.10) with the cubic part of the heterotic prepotential given in (4.2), we will perform the following identification of type II and heterotic moduli (which differs from the one given in [26])

$$\begin{aligned} t_1 &= U-2V, \qquad t_2=S-\frac{n}{2}T-(1-\frac{n}{2})U, \\ t_3 &= T-U, \qquad t_4=V \quad , \end{aligned} \tag{4.11}$$

which is valid in the chamber $T>U>2V$. Then, (4.10) turns into

$$F^{II}_{\text{cubic}}=-F^{\text{het}}_{\text{cubic}}=S(TU-V^2)+\frac{1}{3}U^3+(\frac{4}{3}+n)V^3-(1+\frac{n}{2})UV^2-\frac{n}{2}TV^2 \quad . \tag{4.12}$$

Note that using the heterotic moduli the prepotential is independent of n in the limit $V=0$.

Next, let us consider the contributions of the world sheet instantons to the type II prepotential of a 4 parameter model. Generically, they are given by

$$F^{II}_{\rm inst} = -\frac{1}{(2\pi)^3}\sum_{d_1,\ldots,d_4} n^r_{d_1,\ldots,d_4} Li_3(\prod_{i=1}^{4} q^{d_i}) \ . \tag{4.13}$$

The $n^r_{d_1,d_2,d_3,d_4}$ denote the rational instanton numbers. The heterotic weak coupling limit $S\to\infty$ corresponds to the large Kähler class limit $t_2\to\infty$. In this limit, only the instanton numbers with $d_2=0$ contribute in the above sum. Using the identification $kT+lU+bV=d_1t_1+d_3t_3+d_4t_4$, it follows that (independently of n)

$$\begin{aligned} k &= d_3 \ , \\ l &= d_1 - d_3 \ , \\ b &= d_4 - 2d_1 \ . \end{aligned} \tag{4.14}$$

Then, (4.13) turns into

$$F^{II}_{\rm inst} = -\frac{1}{(2\pi)^3}\sum_{k,l,b} n^r_{k,l,b} Li_3(e^{-2\pi(kT+lU+bV)}) \ . \tag{4.15}$$

Comparison with (4.2) shows that the rational instanton numbers have to satisfy the following constraint

$$n^r_{k,l,b} = n^r(4kl-b^2) \tag{4.16}$$

as well as

$$n^r_{k,l,b} = -2c_n(4kl-b^2) \ . \tag{4.17}$$

Note that the constraint (4.16) is non-trivial. We conjecture that an analogous constraint has to hold for an arbitrary number of Wilson lines after the proper identification of T and U. Also note that $c_n(0)=\chi(X_n)$ and racall that $c_n(-1)=-N'_H, c_n(-4)=N'_V$.

For concreteness, let us now check above relations for the 4 parameter model of [26], which has a dual type II description based on the Calabi–Yau space $X_2=P_{1,1,2,6,10}(20)$. Using the instanton numbers given in [26][4], it can be checked that both (4.16) and (4.17) for c_2 indeed hold, as can be seen from the second table in appendix A.6 and the table given below.

[4]We are grateful to B. Andreas and P. Mayr for providing us the higher instanton numbers which are not given in [26].

d_1	d_3	d_4	k	l	b	$N = 4kl - b^2$	$n_{d_1,0,d_3,d_4}$
0	0	3	0	0	3	-9	0
0	1	0	1	-1	0	-4	-2
0	0	2	0	0	2	-4	-2
1	0	0	0	1	-2	-4	-2
1	0	4	0	1	2	-4	-2
0	0	1	0	0	1	-1	56
1	0	3	0	1	1	-1	56
1	0	1	0	1	-1	-1	56
1	0	2	0	1	0	0	372
2	1	3	1	1	-1	3	53952
2	1	4	1	1	0	4	174240

The truncation to the three parameter Calabi–Yau model is made by setting $V = 0$. The instanton numbers $n^r_{k,l}$ of the S-T-U model are then given by [26]

$$n^r_{k,l} = \sum_b n^r(4kl - b^2) \quad , \tag{4.18}$$

where the summation range over b is finite. For example, $n^r_{1,0} = -2 + 56 + 372 + 56 - 2 = 480$ [26].

5 The heterotic perturbative Wilsonian gravitational coupling F_1

5.1 BPS orbits

An important role in the computation of the Wilsonian gravitational coupling F_1 is played by BPS states [43, 44, 18, 14],

$$F_1 \propto \log \mathcal{M} \quad , \tag{5.1}$$

where $\mathcal{M}$ denotes the moduli-dependent holomorphic mass of an $N=2$ BPS state. For the S-T-U-V models under consideration, the tree-level mass $\mathcal{M}$ is given by [25, 29, 45]

$$\mathcal{M} = m_2 - im_1U + in_1T + n_2(-UT + V^2) + ibV \quad . \tag{5.2}$$

Here, $l = (n_1, m_1, n_2, m_2, b)$ denotes the set of integral quantum numbers carried by the BPS state. The level matching condition for a BPS state reads

$$2(p_L^2 - p_R^2) = 4n^T m + b^2 \quad . \tag{5.3}$$

Of special relevance to the computation of perturbative corrections to F_1 are those BPS states, whose tree-level mass vanishes at certain surfaces/lines in the perturbative moduli space $\mathcal{H}_2 = \frac{SO(3,2)}{SO(3)\times SO(2)}$. Note that the condition $\mathcal{M}=0$ is the condition (see appendix A.1) for a rational quadratic divisor

$$\mathrm{H}_l = \{\begin{pmatrix} iT & iV \\ iV & iU \end{pmatrix} \in \mathcal{H}_2 | m_2 - im_1U + in_1T + n_2(-UT + V^2) + ibV = 0\} \tag{5.4}$$

of discriminant

$$\mathrm{D}(l) = 2(p_L^2 - p_R^2) = 4m_1n_1 + 4n_2m_2 + b^2 \quad . \tag{5.5}$$

Consider, for instance, BPS states becoming massless at the surface $V=0$, the so-called Humbert surface H_1 (cf. appendix A.1). They lay on the orbit $\mathrm{D}(l)=1$, that is, on the orbit $n^Tm = 0, b^2 = 1$. On the other hand, BPS states becoming massless at $T=U$, the Humbert surface H_4, lay on the orbit $\mathrm{D}(l)=4$, that is, they carry quantum numbers satisfying $n^Tm = 1, b^2 = 0$ [25].

5.2 The coupling F_1 in the S-T-U model

The perturbative Wilsonian gravitational coupling for the S-T-U model is given by[5] (in the chamber $T > U$)

$$F_1 = 24S_{\text{inv}} - \frac{b_{\text{grav}}}{\pi} \log \eta(T)\eta(U) + \frac{2}{\pi} \log(j(T) - j(U)) \quad . \tag{5.6}$$

Using that [16]

$$\begin{aligned} S_{\text{inv}} &= \tilde{S} + \frac{1}{8}L \ , \\ \tilde{S} &= S - \frac{1}{2}\partial_T\partial_U h \quad , \quad L = -\frac{4}{\pi}\log(j(T) - j(U)) \ , \end{aligned} \tag{5.7}$$

[5] The dilaton is defined to be $S = 4\pi/g^2 - i\theta/2\pi$.

it follows that F_1 can be rewritten as

$$F_1 = 24\tilde{S} - \frac{1}{\pi}\Big[10\log(j(T)-j(U)) + b_{\text{grav}}\log\eta(T)\eta(U)\Big] \ . \tag{5.8}$$

The perturbative gravitational coupling is related to the perturbative Wilsonian coupling by

$$\frac{1}{g^2_{\text{grav}}} = \Re F_1 + \frac{b_{\text{grav}}}{4\pi}K = 12(S+\bar{S}+V_{GS}) + \Delta_{\text{grav}} \ . \tag{5.9}$$

This relates the Wilsonian gravitational coupling F_1 to the supersymmetric index, that is to $\Delta_{\text{grav}} = -\frac{2}{4\pi}\tilde{I}_{2,2}$ [14], where [18]

$$\tilde{I}_{2,2} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\Big[Z_{2,2}\frac{E_4E_6}{\eta^{24}}(E_2 - \frac{3}{\pi\tau_2}) - \tilde{c}_1(0)\Big] \ . \tag{5.10}$$

It follows from (5.9) that

$$\begin{aligned} F_1 &= 24S - \frac{2}{\pi}\sum_{r>0}\tilde{c}_1(-\frac{r^2}{2})Li_1 \\ &= 24\tilde{S} - \frac{1}{\pi}\Big[10\log(j(T)-j(U)) + b_{\text{grav}}\log\eta(T)\eta(U)\Big] \ , \end{aligned} \tag{5.11}$$

where the coefficients $\tilde{c}_1$ are given by [18]

$$\frac{E_2E_4E_6}{\Delta} = \sum \tilde{c}_1(n)q^n \ , \quad \Delta = \eta^{24} \ . \tag{5.12}$$

Here, we have ignored the issue of ambiguities in (5.11) linear in T and in U.

5.3 The coupling F_1 in the S-T-U-V models

The classical moduli space of a heterotic S-T-U-V model is locally given by the Siegel upper half plane $\mathcal{H}_2 = \frac{SO(3,2)}{SO(3)\times SO(2)}$. Because of target space duality invariance, one has to consider modular forms on $\mathcal{H}_2$, i.e. Siegel modular forms (cf. appendix A).

The Siegel modular form which vanishes on the $T=U$ locus and has modular weight 0 is given by $\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5}$. It can be shown that, as $V \to 0$,

$$\frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} \to (j(T)-j(U))^2 \ , \tag{5.13}$$

up to a normalization constant. On the other hand, the Siegel modular form which vanishes on the $V=0$ locus and has modular weight 0 is given by $\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}}$. It can be shown that, as $V \to 0$,

$$\mathcal{C}_5 \to V\,(\Delta(T)\Delta(U))^{\frac{1}{2}} \ , \tag{5.14}$$

up to a proportionality constant. Finally, the Siegel form $\mathcal{C}_{12}$ generalises $\Delta(T)\Delta(U)$, that is

$$\mathcal{C}_{12} \rightarrow \Delta(T)\Delta(U) \tag{5.15}$$

as $V \rightarrow 0$.

Then, in analogy to (5.6), the perturbative Wilsonian gravitational coupling for an S-T-U-V model is now given by (in the chamber $T > U$)

$$F_1 = 24 S_{\rm inv} - \frac{b_{\rm grav}}{24\pi} \log \mathcal{C}_{12} + \frac{1}{\pi} \log \frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} - \frac{1}{2\pi}(N'_H - N'_V) \log \left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}} \right)^2 . \tag{5.16}$$

Here, N'_V and N'_H denote the vector and the hyper multiplets which become massless at the $V = 0$ locus. Since at $V = 0$ there is a gauge symmetry restoration $U(1) \rightarrow SU(2)$, we have $N'_V = 2$.

The invariant dilaton $S_{\rm inv}$ is given by [16]

$$\begin{aligned} S_{\rm inv} &= \tilde{S} + \frac{1}{10} L \ , \\ \tilde{S} &= S - \frac{4}{10}(\partial_T \partial_U - \frac{1}{4}\partial_V^2) h \ , \end{aligned} \tag{5.17}$$

where the role of the quantity L is to render $S_{\rm inv}$ free of singularities. Using eqs.(4.6) and (4.9), it follows that

$$\tilde{S} = S + \frac{1}{5\pi} \log \frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} - \frac{3}{10\pi}(2+n) \log \left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}} \right)^2 + regular \tag{5.18}$$

and, hence,

$$L = -\frac{2}{\pi} \log \frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} + \frac{3}{\pi}(2+n) \log \left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}} \right)^2 . \tag{5.19}$$

It follows that the Wilsonian gravitational coupling (5.16) can be rewritten into

$$\begin{aligned} F_1 &= 24 \tilde{S} - \frac{1}{\pi} \Big[\frac{19}{5} \log \frac{\mathcal{C}_{30}^2}{\mathcal{C}_{12}^5} + \frac{b_{\rm grav}}{24} \log \mathcal{C}_{12} \\ &+ \left(-\frac{72}{10}(2+n) + \frac{1}{2}(N'_H - N'_V) \right) \log \left(\frac{\mathcal{C}_5}{\mathcal{C}_{12}^{5/12}} \right)^2 \Big] . \end{aligned} \tag{5.20}$$

Now recall from (2.5) that $N'_H - N'_V = 12n + 30$. Inserting this into (5.20) yields

$$F_1 = 24 \tilde{S} - \frac{1}{\pi} \Big[\frac{19}{5} \log \mathcal{C}_{30}^2 + \frac{3}{5}(1 - 2n) \log \mathcal{C}_5^2 \Big] . \tag{5.21}$$

Note that the $\log \mathcal{C}_{12}$ terms have completely canceled out!

Now consider the perturbative gravitational coupling, which is again related to the perturbative Wilsonian coupling by

$$\frac{1}{g^2_{\rm grav}} = \Re F_1 + \frac{b_{\rm grav}}{4\pi}K = 12(S+\bar S+V_{GS})+\Delta_{\rm grav} \ , \tag{5.22}$$

where this time $\Delta_{\rm grav} = -\frac{2}{4\pi}\tilde I_{3,2}$ with

$$\tilde I_{3,2} = \int_{\mathcal F}\frac{d^2\tau}{\tau_2}\Big[Z_{3,2}A_n\Big(E_2-\frac{3}{\pi\tau_2}\Big)-d_n(0)\Big] \ . \tag{5.23}$$

Here, we have introduced

$$B_n(\tau) = E_2A_n = \frac{12-n}{24}\frac{E_2E_6\widehat{E_{4,1}}}{\Delta}+\frac{12+n}{24}\frac{E_2E_4\widehat{E_{6,1}}}{\Delta} = \sum_{N\in\mathbf{Z}\, or\, \mathbf{Z}+\frac34} d_n(4N)q^N \ . \tag{5.24}$$

The world-sheet integral (5.23) can be evaluated using the techniques of [38, 18, 29, 27, 45]. A more detailed discussion can be found in appendix B. Then we find from (5.22) that

$$\begin{aligned} F_1 &= 24S - \frac{2}{\pi}\sum_{(k,l,b)>0} d_n(4kl-b^2)Li_1 \\ &= 24\tilde S - \frac{1}{\pi}\Big[\frac{19}{5}\log\mathcal{C}_{30}^2+\frac35(1-2n)\log\mathcal{C}_5^2\Big] \ . \end{aligned} \tag{5.25}$$

Here, we have again ignored the issue of ambiguities linear in T, U and V. Equation (5.25) gives a highly non-trivial consistency check on (4.2) and on (5.23). Namely, it yields, using the product expansions for $\mathcal{C}_5$ and $\mathcal{C}_{30}$ given in [34] (cf. appendix A.3),

$$d_n(N) = -\frac65 Nc_n(N) - \frac{19}{5}f_2'(N) - \frac35(1-2n)f(N) \ , \tag{5.26}$$

where $N = 4kl-b^2 \in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3$. As a matter of fact, (5.26) is equivalent to the following set of non-trivial relations

$$\begin{aligned} d_n^{(1)}(N) &= -\frac65 Nc_n^{(1)}(N) - \frac{19}{5}f_2'(N) - \frac35 f(N) \ , \\ d_n^{(2)}(N) &= -\frac65 Nc_n^{(2)}(N) - \frac15 f(N) \ , \end{aligned} \tag{5.27}$$

where we have decomposed $A_n(4\tau)$ and $B_n(4\tau)$ into

$$\begin{aligned} A_n(4\tau) &= \sum_{N\in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3} c_n^{(1)}(N)q^N - 6n \sum_{N\in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3} c_n^{(2)}(N)_n q^N \ , \\ B_n(4\tau) &= \sum_{N\in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3} d_n^{(1)}(N)q^N - 6n \sum_{N\in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3} d_n^{(2)}(N)q^N \ . \end{aligned} \tag{5.28}$$

In order to show that (5.27) really holds, consider introducing [27]

$$\hat{Z} = \frac{1}{72} \frac{(E_4^2 \widehat{E_{4,1}} - E_6 \widehat{E_{6,1}})}{\Delta} \ ,$$

$$J_C = \frac{2 E_6 \widehat{E_{6,1}}}{\Delta} + 81 \hat{Z} \ , \tag{5.29}$$

as well as

$$\tilde{Z}(\tau) = \hat{Z}(4\tau) = 2 \sum_{N \in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3} f(N) q^N \ , \tag{5.30}$$

$$\tilde{J}_C(\tau) = J_C(4\tau) = \sum_{N \in 4\mathbf{Z}, 4\mathbf{Z}+3} c_J(N) q^N = 2q^{-4} - 14 q^{-1} + 65664 q^3 + 262440 q^4 + \cdots \ .$$

Then, it can be verified that

$$f_2'(N) = \frac{1}{2} c_J(N) + 6 f(N) \ . \tag{5.31}$$

One also has [27]

$$\Theta_q E_m = \frac{m}{12} (E_2 E_m - E_{m+2}) \ , \ m = 4, 6$$

$$\Theta_q \hat{E}_{m,1} = \frac{2m-1}{24} \left(E_2 \hat{E}_{m,1} - \hat{E}_{m+2,1} \right) \ , \ m = 4, 6$$

$$\Theta_q \tilde{E}_{m,1} = \frac{2m-1}{6} \left(\tilde{E}_2 \tilde{E}_{m,1} - \tilde{E}_{m+2,1} \right) \ , \ m = 4, 6 \tag{5.32}$$

where

$$\tilde{E}_2(\tau) = E_2(4\tau) \ ,$$

$$\tilde{E}_{m,1}(\tau) = \hat{E}_{m,1}(4\tau) \ , \tag{5.33}$$

and where $\Theta_q = q \frac{d}{dq}$. Then, using (5.31) as well as (5.32), it can be shown that (5.27) indeed holds.

6 Conclusions

In this paper we have computed the perturbative threshold corrections, i.e. the one-loop prepotential and the one-loop gravitational coupling F_1, for $D = 4$, $N = 2$ heterotic string models compactified on $K3 \times T_2$ as a function of the toroidal moduli T, U and the single Wilson line V. The considered chain of models with generic Abelian gauge group $U(1)^5$ is characterized by the embedding of the $SU(2)$ instanton numbers $(d_1, d_2) = (12-n, 12+n)$ into $E_8^{(1)} \times E_8^{(2)}$. At special points in the classical moduli space $\frac{SO(3,2)}{SO(3) \times SO(2)} / \Gamma$, where

$\Gamma = SO(3, 2, \mathbf{Z})$ is the classical T-duality group, the Abelian gauge group $U(1)^5$ can be enhanced. The enhancement loci correspond to the Humbert surfaces in the classical moduli space. The one-loop prepotential and the function F_1 can be expressed in terms of a set of very beautiful modular functions, namely the Siegel and Jacobi modular forms. The construction of the supersymmetric index as a power series in the parameter $q = e^{2\pi i\tau}$ involves a so-called hatting procedure, which describes the transition of going from Jacobi forms to ordinary modular functions. The physical interpretation of the hatting procedure is just the turning on of the Wilson line modulus V. If follows that the one-loop prepotential is given in terms of the same expansion coefficients as the supersymmetric index.

For the S-T-U-V class of heterotic string models the spectrum (the number of massless hyper multiplets) and the perturbative threshold corrections explicitly depend on the particular instanton embedding, parametrized by the integer n. This situation is in contrast to the three parameter S-T-U class of models, where the spectrum and the perturbative couplings do not depend on n. In this case the models with $n = 0, 2$ are even equivalent at the non-perturbative level. A priori, four-parameter models with gauge group $U(1)^5$ are obtained for the cases $n = 0, 1, 2$ only. In perturbation theory, $E_8^{(1)}$ can only be broken to some group $G_0^{(1)}$ for $n > 2$. However, we believe that our results also remain valid if there were a mechanism to get rid of the gauge group $G_0^{(1)}$ as well as of $\dim(G_0^{(1)})$ hypermultiplets (leaving $215 - n$ massless hyper multiplets). In fact, for $n = 12$ our results perfectly agree with the recent results of [27].

Besides the heterotic construction and the heterotic perturbative couplings, we also discussed the corresponding dual type II string models on Calabi–Yau three-folds X_n with Hodge number $h_{1,1} = 4$ and Euler number $\chi = 24n - 420$. For $n = 0, 1, 2$ these Calabi–Yau spaces are known and can be explicitly constructed. For $n = 2$ there is a Higgs transition [26] to the three parameter Calabi–Yau $P_{1,1,2,8,12}(24)$; the possibility of this Higgs transition reflects itself in a consistent truncation $V \to 0$ of the S-T-U-V vector couplings to the corresponding couplings in the S-T-U models. If the "complete" gauge symmetry breaking to $U(1)^5$ on the heterotic side could be realized for $n > 2$, it would predict the existence of new Calabi–Yau spaces X_n. Since the truncation $V \to 0$ to the perturbative couplings of three-parameter model consistently works for all n, we conjecture that all Calabi–Yau spaces X_n, if existent, allow for a Higgs transition to the three parameter Calabi–Yau spaces. Specifically for n even, the relevant three parameter Calabi–Yau space should be based on the elliptic fibration over the Hirzebruch surface F_2 (or F_0), whereas for n odd the three parameter Calabi–Yau should be given by the elliptic fibration over F_1. The possibility of having Calabi–Yau spaces X_n with $n > 2$ is in fact supported by the known existence of the $n = 12$ Calabi–Yau $P_{2,2,3,3,10}(20)$ [27].

Clearly, it would be very interesting to extend these results to models with a larger number of Wilson lines. Finally, it would be very interesting to see if there is any relation between the perturbative $N = 2$ couplings, considered here, and the non-perturbative $N = 4$ supersymmetric index of [28], where the Siegel modular forms also play a prominent role.

Acknowlegement

We would like to thank E. Derrick, T. Mohaupt and S. Theisen for discussions. We would also like to thank B. Andreas for participation at an earlier stage of this work, as well as P. Berglund and especially P. Mayr for fruitful discussions on the model of [26].

A Modular forms

A.1 On Siegel modular forms

Here we review some properties of Siegel modular forms. A more detailed account can be found in [46].

The classical moduli space of a heterotic S-T-U-V model is locally given by the Siegel upper half plane $\mathcal{H}_2 = \frac{SO(3,2)}{SO(3)\times SO(2)}$ (note the exceptional isomorphism $SO(5) = B_2 = C_2 = Sp(4)$, here in a noncompact formulation). The standard action of $Sp(4, Z)$ on an element τ of the Siegel upper half plane $\mathcal{H}_2$ is given by

$$M \to M \cdot \tau = (a\tau + b)(c\tau + d)^{-1} \quad , \tag{A.1}$$

where

$$\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} = \begin{pmatrix} iT & iV \\ iV & iU \end{pmatrix} \quad , \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = Sp(4, Z) \quad , \tag{A.2}$$

and where $\det \mathrm{Im}\tau = \mathrm{Re}T\mathrm{Re}U - (\mathrm{Re}V)^2 > 0$. Note that a, b, c and d denote 2×2 matrices. A Siegel modular form F of even weight k transforms as

$$F(M \cdot \tau) = det(c\tau + d)^k F(\tau) \tag{A.3}$$

for every $M \in G = Sp(4, Z)$, whereas a modular form of odd weight k transforms as

$$F(M \cdot \tau) = \varepsilon(M) det(c\tau + d)^k F(\tau) \quad . \tag{A.4}$$

Here $\varepsilon : G \to G/G(2) = S_6 \to \{\pm 1\}$ is the sign of the permutation in S_6. $G(2)$ denotes the principal congruence subgroup of level 2.

The Eisenstein series are given by

$$\mathcal{E}_k = \sum det(c\tau + d)^{-k} \quad . \tag{A.5}$$

Now, recall that the usual modular forms of $Sl(2, \mathbf{Z})$ are generated by the (normalized) Eisenstein series E_4 and E_6. These are related to the two modular forms E_{12} and Δ of weight 12 by

$$\begin{aligned} aE_4^3 + bE_6^2 &= (a+b)E_{12} \ , \\ E_4^3 - E_6^2 &= \alpha\Delta \ , \end{aligned} \tag{A.6}$$

where $\Delta = \eta^{24}$ is the cusp form, and where $a = (3 \cdot 7)^2, b = 2 \cdot 5^3, c = a + b = 691, \alpha = 2^6 \cdot 3^3 = 1728$.

Similarly, the ring of Siegel modular forms is generated by the (algebraic independent) Eisenstein series $\mathcal{E}_4, \mathcal{E}_6, \mathcal{E}_{10}, \mathcal{E}_{12}$ and by one further cusp form of odd weight $\mathcal{C}_{35}$, whose square can again be expressed in terms of the even generators. Alternatively, instead of using $\mathcal{E}_{10}$ and $\mathcal{E}_{12}$, one can also use the cusp forms $\mathcal{C}_{10}$ and $\mathcal{C}_{12}$.

A Siegel cusp form is defined as follows. Since a modular form f is invariant under the translation group $U = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G\}$, where the integer valued 2×2- matrix b is symmetric, it has a Fourier expansion $F = \sum_M a(M) e^{2\pi i tr M\tau}$. Here, the summation extends over all symmetric half-integral 2×2-matrices (that is, over symmetric matrices which have integer valued diagonal entries and half-integer valued off-diagonal entries). The Fourier coefficient $a(M)$ depends only on the class of M under conjugation by $Sl(2, \mathbf{Z})$, and it is zero unless M is positive semidefinite.

Now, consider the Siegel operator Φ which, to every Siegel modular form F with Fourier coefficients $a(M)$, associates the ordinary $SL(2, \mathbf{Z})$ modular form ΦF with Fourier coefficients $a(n) = a(\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix})$. This yields a surjective homomorphism of graded rings of modular forms. The forms in the kernel are the cusp forms. Thus, identities between ordinary modular forms lead to Siegel cusp forms, as follows:

$$\begin{aligned} E_4 E_6 = E_{10} \quad &\to \quad \mathcal{E}_4 \mathcal{E}_6 - \mathcal{E}_{10} =: p\,\mathcal{C}_{10} \ , \\ aE_4^3 + bE_6^2 = cE_{12} \quad &\to \quad a\mathcal{E}_4^3 + b\mathcal{E}_6^2 - c\mathcal{E}_{12} =: \alpha^2 \frac{ab}{c} \mathcal{C}_{12} \ , \end{aligned} \tag{A.7}$$

where p denotes a normalisation constant given by $p = \frac{2^{10}\cdot 3^5\cdot 5^2\cdot 7\cdot 53}{43867}$. We will drop this normalisation constant in the following, for notational simplicity.

Next, consider restricting the Siegel modular forms to the diagonal $D = \{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \}$ (corresponding to the embedding $\frac{SO(2,2)}{SO(2)\times SO(2)} \to \frac{SO(3,2)}{SO(3)\times SO(2)}$). Then, interestingly,

$$\mathcal{E}_k \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = E_k(\tau_1) E_k(\tau_2) \ . \tag{A.8}$$

Specifically

$$\begin{aligned} \mathcal{E}_4 &\to E_4(\tau_1)E_4(\tau_2) \ , \\ \mathcal{E}_6 &\to E_6(\tau_1)E_6(\tau_2) \ , \\ \mathcal{C}_{10} &\to 0 \ , \\ \mathcal{C}_{12} &\to \Delta(\tau_1)\Delta(\tau_2) \ . \end{aligned} \tag{A.9}$$

More precisely, one finds that, up to a normalisation constant, $\mathcal{C}_{10} \to \tau_3^2 \Delta(\tau_1)\Delta(\tau_2)$ as $\tau_3 \to 0$.

Now, consider the behaviour on D of the odd generator $\mathcal{C}_{35}$. Since $\mathcal{C}_{35}$ is a more complicated object, one first reexpresses its square in terms of the other, even generators. Namely, by using the results in [46], one finds that

$$\begin{aligned} \alpha^2 \mathcal{C}_{35}^2 = \frac{1}{3^3}\mathcal{C}_{10} \ \ [\ \ & 2^{24}\cdot 3^{15}\mathcal{C}_{12}^5 \\ & -2^{13}\cdot 3^9 \mathcal{C}_{12}^4(\mathcal{E}_4^3 + \mathcal{E}_6^2) \\ + \ \ & 3^3\mathcal{C}_{12}^3(\mathcal{E}_4^6 - 2\mathcal{E}_4^3\mathcal{E}_6^2 - 2^{14}\cdot 3^5\mathcal{E}_4^2\mathcal{E}_6\mathcal{C}_{10} \\ & \quad -2^{23}3^9 5^2\mathcal{E}_4\mathcal{C}_{10}^2 + \mathcal{E}_6^4) \\ & +2^{11}\cdot 3^6\mathcal{C}_{12}^2\mathcal{C}_{10}(37\mathcal{E}_4^4 + 5\cdot 7\mathcal{E}_4\mathcal{E}_6^2 - 2^{12}3^3 5^3\mathcal{E}_6\mathcal{C}_{10}) \\ + \ \ & 3^2\mathcal{C}_{12}\mathcal{C}_{10}^2(-\mathcal{E}_4^7 + 2\mathcal{E}_4^4\mathcal{E}_6^2 + 2^{11}3^3\cdot 5\cdot 19\mathcal{E}_4^3\mathcal{E}_6\mathcal{C}_{10} \\ & \quad +2^{20}3^6 5^3\cdot 11\mathcal{E}_4^2\mathcal{C}_{10}^2 - \mathcal{E}_4\mathcal{E}_6^4 + 2^4 3^3 5^2\mathcal{E}_6^3\mathcal{C}_{10}) \\ + \ \ & 2\cdot \mathcal{C}_{10}^3(-\mathcal{E}_4^4\mathcal{E}_6 - 2^{11}3^4\mathcal{E}_4^5\mathcal{C}_{10} + 2\mathcal{E}_4^3\mathcal{E}_6^3 \\ & \quad +2^{11}3^4 5^2\mathcal{E}_4^2\mathcal{E}_6^2\mathcal{C}_{10} + 2^{20}3^7 5^4\mathcal{E}_4\mathcal{E}_6\mathcal{C}_{10}^2 - \mathcal{E}_6^5 \\ & \quad +2^{31}3^9 5^5\mathcal{C}_{10}^3)] \ . \end{aligned} \tag{A.10}$$

Thus, on the diagonal D, $\mathcal{C}_{35} = 0$ as well as

$$\begin{aligned}
\alpha^2 \frac{\mathcal{C}_{35}^2}{\mathcal{C}_{10}} &= \alpha^2[\alpha^2 \mathcal{C}_{12}^5 - 2\mathcal{C}_{12}^4(\mathcal{E}_4^3 + \mathcal{E}_6^2) + \frac{1}{\alpha^2}\mathcal{C}_{12}^3(\mathcal{E}_4^6 - 2\mathcal{E}_4^3\mathcal{E}_6^2 + \mathcal{E}_6^4)] \\
&= \mathcal{C}_{12}^3[\alpha^4\mathcal{C}_{12}^2 - 2\alpha^2\mathcal{C}_{12}(\mathcal{E}_4^3 + \mathcal{E}_6^2) + (\mathcal{E}_4^3 - \mathcal{E}_6^2)^2] \\
&= \mathcal{C}_{12}^5 \frac{(\alpha^2\mathcal{C}_{12} - (\mathcal{E}_4^3 - \mathcal{E}_6^2))^2 - 4\alpha^2\mathcal{C}_{12}\mathcal{E}_6^2}{\mathcal{C}_{12}^2} \\
&= \mathcal{C}_{12}^5(j(\tau_1) - j(\tau_2))^2 = (\eta^2(\tau_1)\eta^2(\tau_2))^{60}(j(\tau_1) - j(\tau_2))^2 \ , \qquad \text{(A.11)}
\end{aligned}$$

where $j(\tau) = E_4^3/\Delta$. Then, using $\mathcal{C}_5$ and $\mathcal{C}_{30}$, which are related to the forms already defined by $\mathcal{C}_{10} = \mathcal{C}_5^2$ and $\mathcal{C}_{35} = \mathcal{C}_{30}\mathcal{C}_5$, respectively, it follows that

$$\alpha^2\mathcal{C}_{30}^2 \to \Delta^5(\tau_1)\Delta^5(\tau_2)(j(\tau_1) - j(\tau_2))^2 \qquad \text{(A.12)}$$

on the diagonal D.

A rational quadratic divisor of $\mathcal{H}_2$ is, by definition [34], the set

$$\mathrm{H}_l = \{\begin{pmatrix} iT & iV \\ iV & iU \end{pmatrix} \in \mathcal{H}_2 | in_1T + im_1U + ibV + n_2(-TU + V^2) + m_2 = 0\} \ , \qquad \text{(A.13)}$$

where $l = (n_1, m_1, b, n_2, m_2) \in \mathbf{Z}^5$ is a primitive (i.e. with the greatest commom divisor equals 1) integral vector. The number $\mathrm{D}(l) = b^2 - 4m_1n_1 + 4n_2m_2$ is called the discriminant of H_l. This divisor determines the Humbert surface H_{D} in the Siegel three-fold $Sp_4(\mathbf{Z}) \setminus \mathcal{H}_2$. The Humbert surface H_{D} is (the image in $Sp_4(\mathbf{Z}) \setminus \mathcal{H}_2$ of) the union of all H_l of discriminant $\mathrm{D}(l)$. Each Humbert surface H_{D} can be represented by a linear relation in T, U and V. For instance, the divisor of $\mathcal{C}_5$ is the diagonal $\mathrm{H}_1 = \{Z = \begin{pmatrix} iT & 0 \\ 0 & iU \end{pmatrix} \in Sp_4(\mathbf{Z}) \setminus \mathcal{H}_2\}$. Similarly, the divisor of the Siegel modular form $\mathcal{C}_{30}$ is the surface $\mathrm{H}_4 = \{Z = \begin{pmatrix} iT & iV \\ iV & iU \end{pmatrix} \in Sp_4(\mathbf{Z}) \setminus \mathcal{H}_2 | T = U\}$. The divisor of the Siegel modular form $\mathcal{C}_{35}$, on the other hand, is the sum (with multiplicity 1) of the surfaces H_1 and H_4.

A.2 On Jacobi forms

A Siegel modular form $F(T, U, V)$ of weight k has a Fourier expansion with respect to its variable iU

$$F(T, U, V) = \sum_{m=0}^{\infty} \phi_{k,m}(T, V)s^m \ , \qquad \text{(A.14)}$$

where $s = \mathbf{e}[iU], \mathbf{e}[x] = \exp 2\pi i x$. Each of the $\phi_{k,m}(T,V)$ is a Jacobi form of weight k and index m [47]. That is, for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbf{Z})$ and $\lambda, \mu \in \mathbf{Z}$

$$
\begin{aligned}
\phi_{k,m}\left(\frac{aT-ib}{icT+d}, \frac{V}{icT+d}\right) &= (icT+d)^k e^{2\pi i m \frac{c(iV)^2}{icT+d}} \phi(T,V) \ , \\
\phi_{k,m}(T, V+\lambda T+\mu) &= e^{-2\pi i m(\lambda^2 iT + 2\lambda iV)} \phi_{k,m}(T,V) \ .
\end{aligned} \tag{A.15}
$$

A Jacobi form $\phi_{k,m}(T,V)$ of index m has in turn an expansion

$$
\phi(T,V) = \sum_{n\geq 0} \sum_{l \epsilon \mathbf{Z}} c(n,l) q^n r^l \ , \tag{A.16}
$$

where $q = \mathbf{e}[iT], r = \mathbf{e}[iV]$. Of special relevance are the Jacobi forms $\phi_{k,1}$ of index 1. The summation in l extends in the usual case, and for the generators introduced above, over $4n - l^2 \geq 0$; for the forms divided by Δ, $4n - l^2 \geq -1$ or -4, depending on whether the form is a cusp form or not. Furthermore

$$
c(n,l) = c(4n - l^2) \ . \tag{A.17}
$$

Consider, for instance, the Eisenstein series, which have the expansion

$$
\mathcal{E}_k(T,U,V) = E_k(T) - \frac{2k}{B_k} E_{k,1}(T,V)\, s + \mathcal{O}(s^2) \ . \tag{A.18}
$$

Here, the B_k denote the Bernoulli numbers. Thus, for instance,

$$
\begin{aligned}
\mathcal{E}_4 &= E_4 + 240 E_{4,1} s + \cdots \ , \\
\mathcal{E}_6 &= E_6 - 504 E_{6,1} s + \cdots \ .
\end{aligned} \tag{A.19}
$$

The Jacobi forms $E_{4,1}(T,V)$ and $E_{6,1}(T,V)$ of index 1 have the expansion (the expansion coefficients are listed in the first table of appendix A.6)

$$
\begin{aligned}
E_{4,1} &= 1 + (r^2 + 56r + 126 + 56r{-}1 + r^{-2})q \\
&+ (126r^2 + 576r + 756 + 576r^{-1} + 126r^{-2})q^2 + \cdots \ , \\
E_{6,1} &= 1 + (r^2 - 88r - 330 - 88r^{-1} + r^{-2})q \\
&+ (-330r^2 - 4224r - 7524 - 4224r^{-1} - 330r^{-2})q^2 + \cdots \ .
\end{aligned} \tag{A.20}
$$

Note that $E_{k,1} \to E_k$ as $V \to 0$.

Similarly, the cusp forms $\mathcal{C}_{10}(T,U,V)$ and $\mathcal{C}_{12}(T,U,V)$ have the expansion

$$\begin{aligned} \mathcal{C}_{10}(T,U,V) &= \phi_{10,1}(T,V)s + \mathcal{O}(s^2) \ , \\ \mathcal{C}_{12}(T,U,V) &= \Delta(T) + \frac{1}{12}\phi_{12,1}(T,V)s + \mathcal{O}(s^2) \ , \end{aligned} \tag{A.21}$$

where

$$\begin{aligned} \phi_{10,1} &= \frac{1}{144}(E_6E_{4,1} - E_4E_{6,1}) \to \quad 0 \ , \\ \phi_{12,1} &= \frac{1}{144}(E_4^2E_{4,1} - E_6E_{6,1}) \to 12\Delta \ . \end{aligned} \tag{A.22}$$

Here, we have indicated the behaviour under the truncation $V \to 0$. The Jacobi forms $\phi_{10,1}$ and $\phi_{12,1}$ of index 1 have the following expansion (the expansion coefficients are listed in the first table in appendix A.6)

$$\begin{aligned} \phi_{10,1} &= (r - 2 + r^{-1})q + (-2r^2 - 16r + 36 - 16r^{-1} - 2r^{-2})q^2 + \cdots \ , \\ \phi_{12,1} &= (r + 10 + r^{-1})q + (10r^2 - 88r - 132 - 88r^{-1} + 10r^{-2})q^2 + \cdots \ . \end{aligned} \tag{A.23}$$

A.3 Product expansions

The Siegel modular forms $\mathcal{C}_5$ and $\mathcal{C}_{30} = \mathcal{C}_{35}/\mathcal{C}_5$ have the following product expansion [34]

$$\begin{aligned} \mathcal{C}_5 &= (qrs)^{1/2} \prod_{\substack{n,m,l\in\mathbf{Z} \\ (n,m,l)>0}} (1 - q^n r^l s^m)^{f(4nm-l^2)} \ , \\ \mathcal{C}_{30} &= (q^3rs^3)^{1/2}(q - s) \prod_{\substack{n,m,l\in\mathbf{Z} \\ (n,m,l)>0}} (1 - q^n r^l s^m)^{f_2'(4nm-l^2)} \ , \end{aligned} \tag{A.24}$$

where the condition $(n,m,l) > 0$ means that $n \geq 0, m \geq 0$ and either $l \in \mathbf{Z}$ if $n+m > 0$, or $l < 0$ if $n = m = 0$. The coefficients $f(4nm - l^2)$ and $f_2'(4nm - l^2)$, which are listed in the first table in appendix A.6, are defined as follows [34]. Consider the expansion of

$$\phi_{0,1} := \frac{\phi_{12,1}}{\Delta(T)} = \sum_{n\geq 0} \sum_{l\epsilon\mathbf{Z}} f(n,l) q^n r^l \ , \tag{A.25}$$

where the sum over l is restricted to $4n - l^2 \geq -1$. Then, $f(N) = f(n,l)$ if $N = 4n - l^2 \geq -1$, and $f(N) = 0$ otherwise. The coefficients $f_2'(N)$ are then given by $f_2'(N) = 8f(4N) + (2\left(\frac{-N}{2}\right) - 3)f(N) + f(\frac{N}{4})$. Here, $(\frac{D}{2}) = 1, -1, 0$ depending on whether $D \equiv 1 \bmod 8, 5 \bmod 8, 0 \bmod 2$.

Using the product expansions (A.24), we can perform a check on the expansion (A.21) of $\mathcal{C}_{10} = qrs\prod(1-q^n r^l s^m)^{2f}$. Namely, consider the term in $\mathcal{C}_{10}$ with $n = m = 0, l = -1$. It gives rise to $qsr(1-r^{-1})^2 = qs(r-2+r^{-1})$, which indeed matches the q-term of $\phi_{10,1}$.

Similarly, we can perform a check on (A.12). Setting $r = 1$ in (A.24), we see that the $m = 0$-terms have $f_2'(0) = 60$, and thus they match $\Delta^{5/2}(T) = \eta^{60}$ occuring in $\mathcal{C}_{30} \propto \Delta^{5/2}(T)\Delta^{5/2}(U)(j(T)-j(U))$. The sum over l for the terms with $m = n = 1$, on the other hand, yields $f_2'(4) + 2(f_2'(3) + f_2'(0)) = 196884$, which matches the q-term in the expansion of $j - 744 = q^{-1} + 196884q + \cdots$.

A.4 Theta functions and Jacobi forms

The standard Jacobi theta functions are defined as follows ($z = iV$)

$$
\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} r^{n-\frac{1}{2}} \ , \\
\theta_2(\tau,z) &= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2} r^{n-\frac{1}{2}} \ , \\
\theta_3(\tau,z) &= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}n^2} r^n \ , \\
\theta_4(\tau,z) &= \sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}n^2} r^n \ .
\end{aligned}
\tag{A.26}
$$

It is useful to introduce

$$
\begin{aligned}
\theta_{0,1}(\tau,z) &= \theta_3(2\tau,z) = \sum_{n\in\mathbf{Z}} q^{n^2} r^n \ , \\
\theta_{1,1}(\tau,z) &= \theta_2(2\tau,z) = \sum_{n\nu\mathbf{Z}} q^{(n-\frac{1}{2})^2} r^{n-\frac{1}{2}}
\end{aligned}
\tag{A.27}
$$

as well as

$$
\begin{aligned}
\theta_{ev}(\tau,z) &= \theta_{0,1}(\tau,2z) = \sum_{n\equiv 0(2)} q^{n^2/4} r^n \ , \\
\theta_{odd}(\tau,z) &= \theta_{1,1}(\tau,2z) = \sum_{n\equiv 1(2)} q^{(n-\frac{1}{2})^2} r^{n-\frac{1}{2}} \ .
\end{aligned}
\tag{A.28}
$$

Next, consider setting $z = 0$. The $\theta_i(\tau,0)$ will be simply denoted by θ_i, whereas the $\theta_i(2\tau,0)$ will be denoted by $\theta_i(2\cdot)$ ($i = 1,\ldots 4$). It is well known that $\theta_1 = 0$ and that $\theta_3^4 = \theta_2^4 + \theta_4^4$ as well as $\theta_2\theta_3\theta_4 = 2\eta^3$. Also

$$
\begin{aligned}
E_4 &= \frac{1}{2}\left(\theta_2^8 + \theta_3^8 + \theta_4^8\right) \ , \\
E_6 &= \frac{1}{2}\left((\theta_2^4 + \theta_3^4)(\theta_3^4 + \theta_4^4)(\theta_4^4 - \theta_2^4)\right) \\
&= \frac{1}{2}\left(-\theta_2^6(\theta_3^4 + \theta_4^4)\theta_2^2 + \theta_3^6(\theta_4^4 - \theta_2^4)\theta_3^2 + \theta_4^6(\theta_2^4 + \theta_3^4)\theta_4^2\right) \ .
\end{aligned}
\tag{A.29}
$$

Additional useful identities are given by

$$\begin{aligned}
2\theta_2(2\cdot)\theta_3(2\cdot) &= \theta_2^2 \ , \\
\theta_2^2(2\cdot)+\theta_3^2(2\cdot) &= \theta_3^2 \ , \\
\theta_3^2(2\cdot)-\theta_2^2(2\cdot) &= \theta_4^2 \ , \\
2\theta_2^2(2\cdot) &= \theta_3^2-\theta_4^2 \ , \\
2\theta_3^2(2\cdot) &= \theta_3^2+\theta_4^2 \ , \\
\theta_4^2(2\cdot) &= \theta_3\theta_4 \ .
\end{aligned} \tag{A.30}$$

Now consider Jacobi forms $f(\tau,z)=\sum_{\substack{n\geq 0\\ l\in\mathbf{Z}}} c(4n-l^2)q^n r^l$ of weight k and index 1. The following examples provide useful identities between Jacobi forms of index 1 and Jacobi theta functions

$$\begin{aligned}
\phi_{10,1} &= -\eta^{18}\theta_1^2(\tau,z) \ , \\
\phi_{12,1} &= 12\eta^{24}\frac{\theta_3^2(\tau,z)}{\theta_3^2}+(\theta_4^4-\theta_2^4)[-\eta^{18}\theta_1^2(\tau,z)]
\end{aligned} \tag{A.31}$$

as well as

$$E_{4,1} = \frac{1}{2}\left(\theta_2^6\theta_2^2(\tau,z)+\theta_3^6\theta_3^2(\tau,z)+\theta_4^6\theta_4^2(\tau,z)\right) \ , \tag{A.32}$$

$$E_{6,1} = \frac{1}{2}\left(-\theta_2^6(\theta_3^4+\theta_4^4)\,\theta_2^2(\tau,z)+\theta_3^6(\theta_4^4-\theta_2^4)\,\theta_3^2(\tau,z)+\theta_4^6(\theta_2^4+\theta_3^4)\,\theta_4^2(\tau,z)\right).$$

A Jacobi form of index 1 has the following decomposition [47, 29, 27]

$$f(\tau,z)=f_{ev}(\tau)\theta_{ev}(\tau,z)+f_{odd}(\tau)\theta_{odd}(\tau,z) \ , \tag{A.33}$$

where

$$\begin{aligned}
f_{ev} &= \sum_{N\equiv 0(4)} c(N)q^{N/4} \ , \\
f_{odd} &= \sum_{N\equiv -1(4)} c(N)q^{N/4} \ .
\end{aligned} \tag{A.34}$$

Consider, for instance, $E_{4,1}$. It has the decomposition [27]

$$\begin{aligned}
E_{4,1\,ev} &= \theta_3^7(2\cdot)+7\theta_3^3(2\cdot)\theta_2^4(2\cdot) \ , \\
E_{4,1\,odd} &= \theta_2^7(2\cdot)+7\theta_2^3(2\cdot)\theta_3^4(2\cdot) \ .
\end{aligned} \tag{A.35}$$

Furthermore one has (with $\theta_{ev} \equiv \theta_{ev}(\tau,z)$ and $\theta_{odd} \equiv \theta_{odd}(\tau,z)$)

$$\begin{aligned}
\theta_1^2(\tau,z) &= \theta_2(2\cdot)\theta_{ev} - \theta_3(2\cdot)\theta_{odd} \ , \\
\theta_2^2(\tau,z) &= \theta_2(2\cdot)\theta_{ev} + \theta_3(2\cdot)\theta_{odd} \ , \\
\theta_3^2(\tau,z) &= \theta_3(2\cdot)\theta_{ev} + \theta_2(2\cdot)\theta_{odd} \ , \\
\theta_4^2(\tau,z) &= \theta_3(2\cdot)\theta_{ev} - \theta_2(2\cdot)\theta_{odd} \ .
\end{aligned} \tag{A.36}$$

Next, consider the elliptic genus $Z(\tau,z)$ of $K3$, which is a Jacobi form of weight 0 and index 1, given by [41]

$$Z(\tau,z) = 2\frac{\phi_{12,1}}{\Delta} = 24\frac{\theta_3^2(\tau,z)}{\theta_3^2} - 2\frac{\theta_4^4-\theta_2^4}{\eta^4}\frac{\theta_1^2(\tau,z)}{\eta^2} \ . \tag{A.37}$$

It has the decomposition

$$\begin{aligned}
Z_{ev} &= 24\frac{\theta_3(2\cdot)}{\theta_3^2} - 2\frac{\theta_4^4-\theta_2^4}{\eta^4}\frac{\theta_2(2\cdot)}{\eta^2} = 20 + 216q + 1616q^2 + \cdots \ , \\
Z_{odd} &= 24\frac{\theta_2(2\cdot)}{\theta_3^2} + 2\frac{\theta_4^4-\theta_2^4}{\eta^4}\frac{\theta_3(2\cdot)}{\eta^2} = 2q^{-\frac{1}{4}} - 128q^{\frac{3}{4}} - 1026q^{\frac{7}{4}} + \cdots \ .
\end{aligned} \tag{A.38}$$

Now we introduce the hatted modular function $\widehat{f(\tau,z)}$ as

$$\widehat{f(\tau,z)} = f_{ev}(\tau) + f_{odd}(\tau) \ . \tag{A.39}$$

Hence the hatted modular function corresponds in an one-to-one way to the index 1 Jacobi form. In particular, the Jacobi form $f(\tau,z)$ and its hatted relative $\widehat{f(\tau,z)}$ possess identical power series expansion coefficients $c(N)$:

$$f(\tau,z) = \sum_{n,l} c(4n-l^2)q^n r^l, \qquad \widehat{f(\tau,z)} = \sum_{N\in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3} c(N)q^{N/4} \ . \tag{A.40}$$

Note that an ordinary modular form (that is a form not having any z-dependence), if occuring as a multiplicative factor in front of a proper Jacobi form, is left untouched by the hatting procedure (A.39). Thus, for instance,

$$\begin{aligned}
\widehat{E_{4,1}} &= \frac{1}{2}\left(\theta_2^6\,\widehat{\theta_2^2(\tau,z)} + \theta_3^6\,\widehat{\theta_3^2(\tau,z)} + \theta_4^6\,\widehat{\theta_4^2(\tau,z)}\right) \qquad \text{(A.41)} \\
&= \frac{1}{2}\left(\theta_2^6[\theta_2(2\cdot)+\theta_3(2\cdot)] + \theta_3^6[\theta_2(2\cdot)+\theta_3(2\cdot)] + \theta_4^6[\theta_3(2\cdot)-\theta_2(2\cdot)]\right) \ , \\
\widehat{E_{6,1}} &= \frac{1}{2}\left(-\theta_2^6(\theta_3^4+\theta_4^4)\,\widehat{\theta_2^2(\tau,z)} + \theta_3^6(\theta_4^4-\theta_2^4)\,\widehat{\theta_3^2(\tau,z)} + \theta_4^6(\theta_2^4+\theta_3^4)\,\widehat{\theta_4^2(\tau,z)}\right) \ ,
\end{aligned}$$

and similarly

$$\hat{Z} = Z_{ev} + Z_{odd} = 24\frac{\theta_2(2\cdot)+\theta_3(2\cdot)}{\theta_3^2} - 2\frac{(\theta_4^4-\theta_2^4)}{\eta^4}\frac{(\theta_2(2\cdot)-\theta_3(2\cdot))}{\eta^2} \ . \tag{A.42}$$

Furthermore, consider introducing

$$\tilde{f} = \hat{f}(4\cdot) = \sum_{N \in 4\mathbf{Z}\, or\, 4\mathbf{Z}+3} c(N) q^N \quad . \tag{A.43}$$

Note that $\tilde{f}$ is the $\Gamma_0(4)$ modular form of half-integral weight $k - 1/2$ associated to a Jacobi form of weight k and index 1 [47].

A.5 Lie algebra lattices and Jacobi forms

The relation between Lie algebra lattice sums (see e.g.[48, 49]) and Jacobi forms will be established in three steps. We start by reviewing the well known relationship between the Lie algebra lattice E_8 and the Eisenstein series E_4. Then we go on showing the relation between the Lie algebra lattice E_7 and the Jacobi Eisenstein series $E_{4,1}$. Finally, we will relate the processes of splitting off an A_1 and the hatting procedure. This will explain the relation between turning on a Wilson line and the hatting procedure.

First the relation between the Eisenstein series E_4 and the partition function of the E_8 lattice $\Lambda = \{x \in \mathbf{Z}^8 \cup \pi + \mathbf{Z}^8 | (x,\pi) \in \mathbf{Z}\}$ is well known ($\pi = (1/2, \cdots, 1/2) \in \mathbf{Z}^8$) and reads

$$E_4 = \sum_{x \in \Lambda} q^{\frac{1}{2}x^2} = \frac{1}{2}(\theta_2^8 + \theta_3^8 + \theta_4^8) \quad . \tag{A.44}$$

Because of the lattice relation $\Lambda_{E_8} = \Lambda_{D_8^{(0)}} + \Lambda_{D_8^{(S)}}$, this also shows that the fermionically computed partition function $P_{D_8^{(0)}} + P_{D_8^{(S)}}$ of E_8 is identical to the bosonically computed one, if one recalls the relation between the bosonic conjugacy class picture and the fermionic boundary condition picture

$$\begin{aligned} P_{D_n^{(0)}} &= \frac{\theta_3^n + \theta_4^n}{2} = \frac{NS^+ + NS^-}{2} \quad , \\ P_{D_n^{(V)}} &= \frac{\theta_3^n - \theta_4^n}{2} = \frac{NS^+ - NS^-}{2} \quad , \\ P_{D_n^{(S/C)}} &= \frac{\theta_2^n}{2} = \frac{R^+}{2} \quad . \end{aligned} \tag{A.45}$$

Now consider the Jacobi form $E_{4,1}(\tau, z) = \sum c(4n - l^2) q^n r^l$. Since the expression $\sum_{x \in \Lambda} q^{\frac{1}{2}x^2} r^{(x,\pi)}$ has the correct weights (and truncation), and since the space in question is one–dimensional, this represents $E_{4,1}$. If one considers the $l = 0$ resp. $l = 1$ sector, one finds $\sum_{(x,\pi)=0} q^{\frac{1}{2}(x,x)} = \sum_n c(4n) q^n = \sum_{N \equiv 0(4)} c(N) q^{N/4}$ resp. $\sum_{(x,\pi)=1} q^{\frac{1}{2}(x,x)} = \sum_n c(4n-1) q^n = \sum_{N \equiv -1(4)} c(N) q^{\frac{N+1}{4}}$, i.e. $E_{4,1ev} = \sum_{(x,\pi)=0} q^{\frac{1}{2}x^2}$ and

$E_{4,1odd} = q^{-1/4}\sum_{(x,\pi)=1} q^{\frac{1}{2}x^2} = \sum_{\substack{x\in -\frac{\pi}{2}+\Lambda \\ (x,\pi)=0}} q^{\frac{1}{2}x^2}$. Thus,

$$\begin{aligned} E_{4,1\,ev} = \sum_{(x,\pi)=0}^{E_8} q^{\frac{1}{2}x^2} &= \sum_{x\in(0)}^{E_7} q^{\frac{1}{2}x^2} = P_{E_7^{(0)}} \quad , \\ E_{4,1\,odd} = \sum_{\substack{x\in-\frac{\pi}{2}+\Lambda \\ (x,\pi)=0}}^{E_8} q^{\frac{1}{2}x^2} &= \sum_{x\in(1)}^{E_7} q^{\frac{1}{2}x^2} = P_{E_7^{(1)}} \quad , \end{aligned} \tag{A.46}$$

where the lattice sums $P_{E_7^{(i)}} = \sum_{x\in(i)} q^{\frac{1}{2}x^2}$ run over vectors within the conjugacy class (i). Besides this lattice theoretic argument, this can also be checked explicitely

$$\begin{aligned} E_{4,1\,ev} &= \theta_3^3(2\cdot)(\theta_3^4(2\cdot)+7\theta_2^4(2\cdot)) = \theta_3(2\cdot)\theta_3^2(2\cdot)(\theta_4^4(2\cdot)+8\theta_2^4(2\cdot)) \\ &= \theta_3(2\cdot)\frac{\theta_3^2+\theta_4^2}{2}[\theta_3^2\theta_4^2+2(\theta_3^2-\theta_4^2)^2] = \theta_3(2\cdot)[\theta_3^6+\theta_4^6-\frac{\theta_3^2\theta_4^2}{2}(\theta_3^2+\theta_4^2)] \\ &= \theta_3(2\cdot)\frac{1}{2}[\theta_3^6+\theta_4^6]+\theta_2(2\cdot)\frac{1}{2}\theta_2^6 = P_{E_7^{(0)}} \quad ; \end{aligned} \tag{A.47}$$

similarly $E_{4,1\,odd} = P_{E_7^{(1)}}$.

The last relation in (A.47) follows by noting the following lattice decomposition of $P_{E_7^{(0)}}$: $P_{E_7^{(0)}} = P_{D_6^{(0)}}\cdot P_{A_1^{(0)}} + P_{D_6^{(S)}}\cdot P_{A_1^{(1)}}$. Here one uses the following lattice sums for A_1, which has the root lattice $\Lambda_{A_1}^{(0)} = \sqrt{2}\mathbf{Z}$ and two conjugacy classes:

$$\begin{aligned} P_{A_1^{(0)}} &= \sum_{x\in(0)}^{A_1} q^{\frac{1}{2}x^2} = \sum_{n\in\mathbf{Z}} q^{n^2} = \theta_3(2\cdot) \quad , \\ P_{A_1^{(1)}} &= \sum_{x\in(1)}^{A_1} q^{\frac{1}{2}x^2} = \sum_{n\in\mathbf{Z}} q^{(n-1/2)^2} = \theta_2(2\cdot) \quad . \end{aligned} \tag{A.48}$$

Thus we get that

$$\begin{aligned} 2\widehat{E_{4,1}} &= \theta_2^6[\theta_2(2\cdot)+\theta_3(2\cdot)] + \theta_3^6[\theta_2(2\cdot)+\theta_3(2\cdot)] + \theta_4^6[\theta_3(2\cdot)-\theta_2(2\cdot)] \\ &= \theta_2^6\cdot\widehat{\theta_2^2(\tau,z)} + \theta_3^6\cdot\widehat{\theta_3^2(\tau,z)} + \theta_4^6\cdot\widehat{\theta_4^2(\tau,z)} \\ &= 2(P_{E_7^{(0)}}+P_{E_7^{(1)}}) \quad , \end{aligned} \tag{A.49}$$

which also holds, as is easily seen, in the dehatted version. Now we understand that the breaking of E_8 to E_7 by turning on a Wilson line, i.e. the splitting off of an A_1^{Wilson}, precisely corresponds to the replacement of E_4 by the hatted modular function $\widehat{E_{4,1}}$.

On the other hand, note that the truncation $V\to 0$

$$E_{4,1}(\tau,0) = E_4 = (E_{4,1})_{ev}\theta_3(2\cdot) + (E_{4,1})_{odd}\theta_2(2\cdot) \tag{A.50}$$

reflects the decomposition of $E_8 \supset E_7 \times A_1$

$$P_{E_8} = P_{E_7^{(0)}} \cdot P_{A_1^{(0)}} + P_{E_7^{(1)}} \cdot P_{A_1^{(1)}} \quad . \tag{A.51}$$

Let us again demonstate the hatting procedure by considering the Wilson line breaking of $D_2 = A_1 \times A_1^{\text{Wilson}}$ to A_1. The lattice decomposition of D_2 under $A_1 \times A_1$ has the form

$$\begin{aligned} P_{D_2^{(0)}} &= \frac{\theta_3^2 + \theta_4^2}{2} = P_{A_1^{(0)}} \cdot P_{A_1^{(0)}} = \theta_3(2\cdot)^2, \\ P_{D_2^{(V)}} &= \frac{\theta_3^2 - \theta_4^2}{2} = P_{A_1^{(1)}} \cdot P_{A_1^{(1)}} = \theta_2(2\cdot)^2, \\ P_{D_2^{(S,C)}} &= \frac{\theta_2^2}{2} = P_{A_1^{(0)}} \cdot P_{A_1^{(1)}} = \theta_2(2\cdot)\theta_3(2\cdot). \end{aligned} \tag{A.52}$$

Thus the corresponding hatted Jacobi forms become

$$\begin{aligned} \frac{\theta_3^2\widehat{(\tau,z)} + \theta_4^2\widehat{(\tau,z)}}{2} &= P_{A_1^{(0)}} = \theta_3(2\cdot), \\ \frac{\theta_3^2\widehat{(\tau,z)} - \theta_4^2\widehat{(\tau,z)}}{2} &= P_{A_1^{(1)}} = \theta_2(2\cdot), \\ \frac{\theta_2^2\widehat{(\tau,z)}}{2} &= \frac{1}{2}(P_{A_1^{(0)}} + P_{A_1^{(1)}}) = \frac{1}{2}(\theta_2(2\cdot) + \theta_3(2\cdot)). \end{aligned} \tag{A.53}$$

Finally, going back from the conjugacy class picture to the boundary condition picture one has

$$NS^{\pm}_{A_1} = P_{A_1^{(0)}} \pm P_{A_1^{(1)}} = \theta_3(2\cdot) \pm \theta_2(2\cdot) = \theta^2_{3/4}\widehat{(\tau,z)} \quad , \tag{A.54}$$

$$R^{+}_{A_1} = P_{A_1^{(0)}} + P_{A_1^{(1)}} = \theta_3(2\cdot) + \theta_2(2\cdot) = \theta^2_{2}\widehat{(\tau,z)} \quad . \tag{A.55}$$

A.6 Tables

This table displays some expansion coefficients of the Jacobi forms $E_{4,1}$, $E_{6,1}$, $\phi_{10,1}$, $\phi_{12,1}$ and of the Siegel forms $\mathcal{C}_5$, $\mathcal{C}_{30}$.

N	$e_{4,1}(N)$	$e_{6,1}(N)$	$c_{10,1}(N)$	$c_{12,1}(N)$	$f(N)$	$f_2'(N)$
-4	-	-	-	-	-	1
-1	-	-	-	-	1	-1
0	1	1	0	0	10	60
3	56	-88	1	1	-64	32448
4	126	-330	-2	10	108	131868
7	576	-4224	-16	-88	-513	***
8	756	-7524	36	-132	808	***
11	1512	-30600	99	1275	-2752	***
12	2072	-46552	-272	736	4016	***
15	4032	-130944	-240	-8040	-11775	***
16	4158	-169290	1056	-2880	16524	***
19	5544	-355080	-253	24035	***	***
20	7560	-464904	-1800	13080	***	***

In the following table some expansion coefficients of $\frac{E_{4,1}E_6}{\Delta}$, $\frac{E_4 E_{6,1}}{\Delta}$ and of A_n (see eq.(3.8)) for $n = 0, 1, 2, 12$ are listed.

N	$E_{4,1}E_6/\Delta$	$E_4E_{6,1}/\Delta$	$2A_0$	$2A_1$	$2A_2$	$2A_{12}$
-4	1	1	2	2	2	2
-1	56	-88	-32	-44	-56	-176
0	-354	-66	-420	-396	-372	-132
3	-26304	-27456	-52760	-53356	-53952	-54912
4	-88128	-86400	-174528	-174384	-174240	-172800

B The world sheet integral $\tilde{I}_{3,2}$

Consider the integral

$$\tilde{I}_{3,2} \;=\; \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\Big[Z_{3,2}F(\tau)\Big(E_2 - \frac{3}{\pi\tau_2}\Big) - d_n(0)\Big] \;, \tag{B.1}$$

where

$$\begin{aligned} F(\tau) &= A_n = \sum_{N\in\mathbf{Z},\mathbf{Z}+\frac{3}{4}} c_n(4N)q^N \;, \\ B_n(\tau) &= A_n E_2 = \sum_{N\in\mathbf{Z},\mathbf{Z}+\frac{3}{4}} d_n(4N)q^N \;. \end{aligned} \tag{B.2}$$

$\mathcal{F}$ denotes the fundamental domain for $SL(2,\mathbf{Z})$.

The calculation of (B.1) involves three contributions [38, 18, 29, 27, 45], that is $\tilde{I}_{3,2} = \mathcal{I}_0 + \mathcal{I}_{nd} + \mathcal{I}_{deg}$. In this appendix, we will evalute $\mathcal{I}_{nd}$ by closely following the procedure described in [38, 18, 29, 27, 45]. We will work in the chamber $T_2 > U_2 > 2V_2$. The other two contributions can be evaluated along similar lines.

Recall that

$$Z_{3,2}(\tau,\bar{\tau}) = \sum_{p\in\Gamma^{3,2}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2} = \sum_{m_1,m_2,n_1,n_2,b} q^{\frac{1}{2}(p_L^2-p_R^2)}q^{\frac{1}{2}p_R^2}\bar{q}^{\frac{1}{2}p_R^2} \;, \tag{B.3}$$

where

$$\begin{aligned} p_R^2 &= \frac{|m_2 + m_1 U + n_1 T + n_2(TU - V^2) + bV|^2}{2Y} \;, \\ \frac{1}{2}(p_L^2 - p_R^2) &= \frac{1}{4}b^2 - m_1 n_1 + m_2 n_2 \;, \\ Y &= T_2U_2 - V_2^2 > 0 \;. \end{aligned} \tag{B.4}$$

Performing a Poisson resummation on m_1 and m_2 yields [18, 45]

$$\sum_{m_1,m_2} q^{\frac{1}{2}p_R^2}\bar{q}^{\frac{1}{2}p_R^2} = \sum_{k_1,k_2} \frac{Y}{U_2\tau_2} q^{\frac{b^2}{4}} e^{\mathcal{G}} \;, \tag{B.5}$$

where

$$\begin{aligned} \mathcal{G} &= -\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T \det A + \frac{\pi b}{U_2}\left(V\tilde{\mathcal{A}} - \bar{V}\mathcal{A}\right) \\ &- \frac{\pi n_2}{U_2}\left(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}\right) + \frac{2\pi i V_2^2}{U_2^2}(n_1 + n_2\bar{U})\mathcal{A} \;. \end{aligned} \tag{B.6}$$

Here,

$$
\begin{aligned}
A &= \begin{pmatrix} n_1 & -k_1 \\ & \\ n_2 & k_2 \end{pmatrix} \quad , \\
\mathcal{A} &= (1,U)A(\tau,1)^T = -k_1 + n_1\tau + k_2 U + n_2\tau U \quad , \\
\tilde{\mathcal{A}} &= (1,\bar{U})A(\tau,1)^T = -k_1 + n_1\tau + k_2\bar{U} + n_2\tau\bar{U} \quad .
\end{aligned}
\tag{B.7}
$$

The contribution $\mathcal{I}_{nd}$ is obtained by restriction to non-denerate matrices A (that is, matrices with non-zero determinant) of the form [38, 18]

$$
A = \begin{pmatrix} n_1 & -k_1 \\ & \\ 0 & k_2 \end{pmatrix} \equiv \begin{pmatrix} k & j \\ & \\ 0 & p \end{pmatrix} \quad , \; p \neq 0 \, , \; k > j \geq 0 \quad . \tag{B.8}
$$

Then [38, 18]

$$
\mathcal{I}_{nd} = 2\frac{Y}{U_2} \sum_{b\in\mathbf{Z}} \sum_{\substack{p\in\mathbf{Z}\\ p\neq 0}} \sum_{k>0} \sum_{j=0}^{k-1} \int_{-\infty}^{\infty} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^2} q^{\frac{b^2}{4}} e^{\mathcal{G}} F(\tau) \Big(E_2 - \frac{3}{\pi\tau_2}\Big) \quad , \tag{B.9}
$$

where

$$
\begin{aligned}
\mathcal{G} &= -2\pi i T k p - \frac{\pi Y}{\tau_2 U_2^2}(k\tau_2 + pU_2)^2 - 2\pi b k \frac{V_2}{U_2}\tau_2 \\
&+ 2\pi i \frac{b}{U_2}(jV_2 - pV_1U_2 + pU_1V_2) \\
&+ 2\pi i \frac{V_2^2}{U_2^2} k(j + ik\tau_2 + pU) \\
&- \frac{\pi Y}{\tau_2 U_2^2} k^2 (\tau_1 + \frac{j + pU_1}{k})^2 + 2\pi i \frac{V_2}{U_2} b k \tau_1 + 2\pi i \frac{V_2^2}{U_2^2} k^2 \tau_1 \quad .
\end{aligned}
\tag{B.10}
$$

The integral over τ_1 is gaussian and yields

$$
\int_{-\infty}^{\infty} d\tau_1 e^{-\frac{\pi Y}{\tau_2 U_2^2}k^2(\tau_1 + \frac{j+pU_1}{k})^2 + 2\pi i \tau_1 \tilde{N}} = \frac{U_2}{k}\sqrt{\frac{\tau_2}{Y}} e^{-\frac{\pi\tau_2 U_2^2}{Yk^2}\tilde{N}^2 - 2\pi i \frac{j+pU_1}{k}\tilde{N}} \quad , \tag{B.11}
$$

where

$$
\tilde{N} = N + \frac{b^2}{4} + bk\frac{V_2}{U_2} + k^2\frac{V_2^2}{U_2^2} \quad . \tag{B.12}
$$

Then, $\mathcal{I}_{nd}$ turns into

$$
\begin{aligned}
\mathcal{I}_{nd} &= 2\sqrt{Y} \sum_{N\in\mathbf{Z},\mathbf{Z}+\frac{3}{4}} \sum_{b\in\mathbf{Z}} \sum_{\substack{p\in\mathbf{Z}\\ p\neq 0}} \sum_{k>0} \frac{1}{k} \sum_{j=0}^{k-1} \\
&\int_0^\infty \frac{d\tau_2}{\sqrt{\tau_2^3}} e^{\mathcal{G}'} e^{-2\pi\tau_2(N+\frac{b^2}{4})} e^{-\frac{\pi\tau_2 U_2^2}{Yk^2}\tilde{N}^2 - 2\pi i \frac{j+pU_1}{k}\tilde{N}} \Big(d_n(4N) - \frac{3c_n(4N)}{\pi\tau_2}\Big) \quad ,
\end{aligned}
\tag{B.13}
$$

where

$$\mathcal{G}' = -2\pi i T k p - \frac{\pi Y}{\tau_2 U_2^2}(k\tau_2 + pU_2)^2 - 2\pi b k \frac{V_2}{U_2}\tau_2$$
$$+ 2\pi i \frac{b}{U_2}(jV_2 - pV_1U_2 + pU_1V_2) + 2\pi i \frac{V_2^2}{U_2^2} k(j + ik\tau_2 + pU) \ . \qquad \text{(B.14)}$$

Next, consider summing over j. Then

$$\sum_{j=0}^{k-1} e^{-2\pi i \frac{j}{k}\tilde{N} + 2\pi i \frac{V_2}{U_2} jb + 2\pi i \frac{V_2^2}{U_2^2} kj} = \sum_{j=0}^{k-1} e^{-2\pi i \frac{j}{k}(N + \frac{b^2}{4})} = \begin{cases} k \ \text{ if } \frac{N+\frac{b^2}{4}}{k} = l \in \mathbf{Z} \ . \\ 0 \ \text{ otherwise } . \end{cases} \qquad \text{(B.15)}$$

Note that setting $N = kl - \frac{b^2}{4}$ is consistent with $N \in \mathbf{Z}, \mathbf{Z} + \frac{3}{4}$. It follows that

$$\mathcal{I}_{nd} = 2\sqrt{Y} \sum_{l\in\mathbf{Z}} \sum_{b\in\mathbf{Z}} \sum_{\substack{p\in\mathbf{Z} \\ p\neq 0}} \sum_{k>0} \qquad \text{(B.16)}$$
$$\int_0^\infty \frac{d\tau_2}{\sqrt{\tau_2^3}} e^{\mathcal{G}''} e^{-2\pi\tau_2 k l} e^{-\frac{\pi\tau_2 U_2^2}{Yk^2}\tilde{N}^2 - 2\pi i \frac{j+pU_1}{k}\tilde{N}} \Big(d_n(4kl - b^2) - \frac{3c_n(4kl - b^2)}{\pi\tau_2}\Big),$$

where now $\tilde{N} = k(l + b\frac{V_2}{U_2} + k\frac{V_2^2}{U_2^2})$, and where

$$\mathcal{G}'' = -2\pi i T k p - \frac{\pi T_2}{\tau_2 U_2}(k\tau_2 + pU_2)^2 - 2\pi b k \frac{V_2}{U_2}\tau_2 - \pi k^2 \frac{V_2^2}{U_2^2}\tau_2 + \frac{\pi V_2^2}{\tau_2}p^2$$
$$+ 2\pi i \frac{bp}{U_2}(-V_1U_2 + U_1V_2) + 2\pi i \frac{V_2^2}{U_2^2} U_1 p k \ . \qquad \text{(B.17)}$$

Next, rewrite the sum over $p \neq 0$ as

$$\mathcal{I}_{nd} = 2\sqrt{Y} \sum_{l\in\mathbf{Z}} \sum_{b\in\mathbf{Z}} \sum_{p>0} \sum_{k>0}$$
$$\Big(e^{2\pi i T k p + 2\pi i U_1 p l + 2\pi i V_1 p b} + e^{-2\pi i \bar{T} k p - 2\pi i U_1 p l - 2\pi i V_1 p b}\Big) e^{2\pi k p T_2}$$
$$\int_0^\infty \frac{d\tau_2}{\sqrt{\tau_2^3}} e^{-A\tau_2} e^{-\frac{B}{\tau_2}} \Big(d_n(4kl - b^2) - \frac{3c_n(4kl - b^2)}{\pi\tau_2}\Big), \qquad \text{(B.18)}$$

where

$$A = \pi(2kl + 2bk\frac{V_2}{U_2} + \frac{U_2^2}{Yk^2}\tilde{N}^2 + k^2\frac{T_2}{U_2} + k^2\frac{V_2^2}{U_2^2}) = \frac{\pi}{Y}(kT_2 + lU_2 + bV_2)^2 \ ,$$
$$B = \pi p^2 Y \ . \qquad \text{(B.19)}$$

Then, by using the following integral representations for the Bessel functions $K_{\frac{1}{2}}$ and $K_{\frac{3}{2}}$ (for $A > 0$, $B > 0$)

$$\int_0^\infty \frac{d\tau_2}{\sqrt{\tau_2^3}} e^{-A\tau_2} e^{-\frac{B}{\tau_2}} = \sqrt{\frac{\pi}{B}} e^{-2\sqrt{AB}} \ ,$$
$$\int_0^\infty \frac{d\tau_2}{\sqrt{\tau_2^5}} e^{-A\tau_2} e^{-\frac{B}{\tau_2}} = \frac{\sqrt{\pi}}{B} e^{-2\sqrt{AB}} \left(\sqrt{A} + \frac{1}{2\sqrt{B}}\right) \ , \qquad \text{(B.20)}$$

it follows that

$$
\begin{aligned}
\mathcal{I}_{nd} &= 2\sum_{l\in\mathbf{Z}}\sum_{b\in\mathbf{Z}}\sum_{p>0}\sum_{k>0}\left(e^{2\pi i p r\odot y}+e^{-2\pi i p r\odot y}\right) \\
&\left[\frac{d_n(4kl-b^2)}{p}-\frac{3c_n(4kl-b^2)}{\pi Y}\left(\frac{|kT_2+lU_2+bV_2|}{p^2}+\frac{1}{2\pi p^3}\right)\right], \qquad \text{(B.21)}
\end{aligned}
$$

where

$$
r\odot y = kT_1+lU_1+bV_1+i|kT_2+lU_2+bV_2| \ . \qquad \text{(B.22)}
$$

Note that, in the chamber $T_2 > U_2 > 2V_2$, $|kT_2+lU_2+bV_2| = kT_2+lU_2+bV_2$ and, hence, $r\odot y = kT+lU+bV$. This is due to the fact that the coefficients $c_n(4kl-b^2)$ and $d_n(4kl-b^2)$ vanish unless $4kl-b^2\geq -4$.

Then, summing over p yields

$$
\begin{aligned}
\mathcal{I}_{nd} &= 4\Re\Bigg(\sum_{l\in\mathbf{Z}}\sum_{b\in\mathbf{Z}}\sum_{k>0}\Big[d_n(4kl-b^2)Li_1(e^{2\pi i(kT+lU+bV)}) \\
&- \frac{3}{\pi Y}c_n(4kl-b^2)\mathcal{P}(e^{2\pi i(kT+lU+bV)})\Big]\Bigg), \qquad \text{(B.23)}
\end{aligned}
$$

where we introduced [18]

$$
\mathcal{P}(e^{2\pi i(kT+lU+bV)}) = (kT_2+lU_2+bV_2)Li_2(e^{2\pi i(kT+lU+bV)})+\frac{1}{2\pi}Li_3(e^{2\pi i(kT+lU+bV)}) \qquad \text{(B.24)}
$$

The term proportional to $\frac{1}{Y}c_n\mathcal{P}$ contributes to the Green–Schwarz term [18], whereas the term proportional to d_nLi_1 contributes to F_1.

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