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Hard Thermal Loops in a Magnetic Field and the Chiral Anomaly

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Abstract

The fermionic dispersion relation in the presence of a background magnetic field and a high temperature QED plasma is calculated exactly in the external field, using the Hard Thermal Loop effective action. As the field strength increases there is a smooth transition from the weak-field ($qB \ll q^2T^2$) thermal dispersion relations to the vacuum Landau levels when the background field is much stronger than any thermal effects ($qB \gg q^2T^2$). The self-energy at finite field strength acquires an imaginary part. The spectral width becomes important for critical field strengths ($qB \sim q^2T^2$), necessitating the use of the full spectral function. It is shown that the spectral function satisfies the usual condition of normalization and causality. Using the exact spectral function I also show that the production of chirality in an external electromagnetic field at high temperature is unaffected by the presence of the thermal masses of the fermions.

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1 Introduction

During the last few years an increasing understanding of high temperature gauge theories has been achieved through the consistent resummation of leading diagrams called Hard Thermal Loops (HTL) [1, 2]. The HTL resummed theory has been used to improve IR divergences in a number of processes, the most famous one being thermal gluon scattering and the related gluon damping problem [3]. The HTL effective action has been constructed in several different ways, and it yields a gauge-invariant extension of the electric screening mass for non-static fields. Since strong background fields are likely to have been present in the early Universe it is of interest to look for solutions to the effective HTL equations of motion in such backgrounds. Most of the literature so far has been concentrated on the small fluctuations around the trivial background with zero field. It has been known for a long time that static magnetic fields are not screened by the HTL resummation and thus a constant magnetic field is also a solution to the resummed effective equation of motion. In this paper I explore the fermionic excitations around such a constant field at very high temperature. One interesting point is that this can be done exactly, i.e. to all orders in the external field. At the end we recover two well-known limits. On the one hand there is the zero-field limit where the thermal dispersion relations are well known [4, 5]. On the other hand, when the field is very strong compared with the temperature, thermal corrections become less important and in that limit we recover the zero-temperature Landau levels. For intermediate field strengths the self-energy acquires a non-negligible imaginary part due to synchrotron radiation and scattering with the heat bath, even above the light-cone. It is, therefore, necessary to study the full spectral function and not only an on-shell relation for the real part. It turns out that in the lowest Landau level the spectral width is rather large, when the field strength gets comparable with the thermal mass squared ($qB \sim \mathcal{M}^2$), and the quasi-particle picture is not reliable.

Since there is a mass gap in the thermal fermionic spectrum even for a chirally invariant theory, it is not immediately obvious how the chiral anomaly mechanism works. The standard level crossing picture is not applicable since no levels ever cross the Dirac surface. I show, using the full spectral function in a background of electric and magnetic fields, how the spectral weight associated with particles and antiparticles can move continuously between the positive and negative energy solution without crossing the Dirac surface, and through this mechanism satisfy the anomaly equation.

In section 2, I describe the method of diagonalizing the HTL effective action in a background magnetic field, and compare the exact result with an approximate weak field formula in section 3. The spectral function is calculated in section 4. The main issue of the paper, namely the anomaly mechanism at finite temperature, is treated in section 5 for the case of a free field in 1+1 dimension and in the HTL approximation in 3+1 dimension. Some properties such as normalization and causality are discussed in an appendix.

2 Dispersion relations from Hard Thermal Loops

The HTL effective action for QED can be written as [6]

$$\mathcal{L}_{\mathrm{HTL}} = -\frac{1}{4}F^2 + \frac{3}{4}\mathcal{M}_\gamma^2 F_{\mu\alpha}\left\langle \frac{u^\alpha u^\beta}{(\partial u)^2}\right\rangle F_\beta{}^\mu$$
$$+\overline{\Psi}(\not{\Pi} - m)\Psi - \mathcal{M}_e^2 \overline{\Psi}\gamma_\mu \left\langle \frac{u^\mu}{u\cdot\Pi}\right\rangle \Psi \quad , \tag{2.1}$$

where $\Pi_\mu = i\partial_\mu - qA_\mu$ and the average $\langle\cdot\rangle$ is defined by

$$\langle f(u_0, \boldsymbol{u})\rangle = \int \frac{d\Omega}{4\pi} f(u_0, \boldsymbol{u}) \quad , \tag{2.2}$$

where $u_0 = 1$ and $\boldsymbol{u}$ is a spatial unit vector. The thermal mass of the photon $\mathcal{M}_\gamma^2$ is given by $q^2T^2/9$ and for the electron we have $\mathcal{M}_e^2 = q^2T^2/8$. The equation of motion for Ψ that follows from Eq. (2.1) is

$$\left[\not{\Pi} - m - \mathcal{M}_e^2\gamma_\mu \left\langle \frac{u^\mu}{u\cdot\Pi}\right\rangle\right]\Psi = 0 \quad . \tag{2.3}$$

Equation (2.3) is a non-local and non-linear differential equation, which is, in general, very difficult to solve. What makes this equation much less tractable than the thermal Dirac equation, in the absence of an external electromagnetic field, is that the average over $\boldsymbol{u}$ is difficult to perform explicitly since $[\Pi_\mu, \Pi_\nu] = -ieF_{\mu\nu} \neq 0$, i.e. not all components of Π_μ can be diagonalized simultaneously. We shall in this section only deal with an external magnetic field and fix it to be in the z-direction. The solutions to Eq. (2.3) in vacuum ($\mathcal{M}_e = 0$) are given by the standard Landau levels. Since the spatial symmetries of the system are unchanged by the thermal heat bath, we expect the eigenfunctions to have the same spatial form as at zero temperature. In fact, after performing the u-integral in Eq. (2.3) the result can only be a function of the invariants $\Pi_\perp^2$, p_0^2 and p_z^2, and the γ-structure has to be proportional to $\gamma\Pi_\perp$, $\gamma_0 p_0$ and $\gamma_z p_z$.[1]

We shall therefore compute the matrix elements

$$\langle\Phi_{\kappa'}|\left\langle\frac{u^\mu}{u\cdot\Pi}\right\rangle|\Phi_\kappa\rangle \tag{2.4}$$

between the vacuum eigenstates. To be specific we use the gauge $A_\mu = (0,0,-Bx,0)$. Then the eigenstates are given by

$$\langle x|\Phi_\kappa\rangle = \exp[i(-p_0 t + p_y y + p_z z)] I_{n;p_y}(x) \quad , \tag{2.5}$$

$$\begin{aligned} I_{n;p_y}(x) &= \left(\frac{|qB|}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}|qB|\left(x-\frac{p_y}{qB}\right)^2\right] \\ &\quad \times \frac{1}{\sqrt{n!}} H_n\left[\sqrt{2|qB|}\left(x-\frac{p_y}{qB}\right)\right] \quad , \end{aligned} \tag{2.6}$$

where $\kappa = \{p_0, n, p_y, p_z\}$ and $H_n[x]$ are Hermite polynomials defined by

$$H_n[x] = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \quad . \tag{2.7}$$

These states form a complete set of functions in four dimensions when the energy is off shell. In the chiral representation and with $qB > 0$, suitable spinors can be formed from Φ_κ as $\Psi_\kappa = \mathrm{diag}[\Phi_\kappa, \Phi_{\kappa-1}, \Phi_\kappa, \Phi_{\kappa-1}]\chi_\kappa$ where χ_κ is a space-time-independent spinor, which can be determined from the Dirac equation. The vacuum Dirac operator in Eq. (2.3) gives by construction an eigenvalue when acting on Ψ_κ, but it is more difficult to determine the action of the thermal part since Φ_κ cannot be an eigenfunction to $u\cdot\Pi$ for all u. One way of calculating the matrix element in Eq. (2.4) is to find a basis such that $v\cdot\Pi|v_p\rangle = v\cdot p|v_p\rangle$ and insert a unit operator $\int[d^4p]|v_p\rangle\langle v_p|$ into Eq. (2.4). (We use the notation $\int[d^np] = \int d^np/(2\pi)^n$.) The

[1] We use the notation $a\cdot b_\perp = a_x b_x + a_y b_y$ for any two four-vectors a and b. In our convention three-vectors such as $\boldsymbol{p} = (p_x, p_y, p_z)$ and $\boldsymbol{\gamma} = (\gamma_x, \gamma_y, \gamma_z)$ are the contravariant components of the corresponding four-vector and thus have Lorentz indices $i = 1,2,3$ upstairs, i.e. $p_x = p^1$ etc. We use the Minkowski metric $\mathrm{diag}(+,-,-,-)$ so that $p_i = -p^i$ and $\gamma_i = -\gamma^i$ for $i = 1,2,3$.

unit operator is, of course, independent of v, so in particular we can choose $v = u$ and change the order of integrations between p and u. In the gauge we use, an eigenvector to $v \cdot \Pi$ is given by

$$\langle x|v_p\rangle = \exp\left[-ip_0 t + ip_z z + ip_y y + i\left(p_x x + \frac{qBv_y}{v_x}\frac{x^2}{2}\right)\right] \quad . \tag{2.8}$$

After computing the matrix elements in Eq. (2.4) we find indeed that they are diagonal in κ for u_0 and u_z, and have a mixing with the first subdiagonals for u_x and u_y. We define $\langle u_{0,z,\pm}\rangle$ by

$$\langle\Phi_{\kappa'}|\left\langle\frac{u_{0,z}}{u\cdot\Pi}\right\rangle|\Phi_\kappa\rangle = (2\pi)^3\delta_{\kappa',\kappa}\langle u_{0,z}\rangle_\kappa \quad , \tag{2.9}$$

$$\langle\Phi_{\kappa'}|\left\langle\frac{u_x \pm iu_y}{u\cdot\Pi}\right\rangle|\Phi_\kappa\rangle = (2\pi)^3\delta_{\kappa',\kappa\mp 1}\langle u_\pm\rangle_\kappa \quad , \tag{2.10}$$

and $\kappa \mp 1 = \{p_0, n \mp 1, p_y, p_z\}$. These are exactly the components that occur naturally when we include the γ-matrices in the chiral representation. The explicit calculation of $\langle u_{0,z,\pm}\rangle$ is a bit lengthy but straightforward and is done by performing the integrals over x, x', p and u in

$$\langle\Phi_{\kappa'}|\left\langle\frac{u_\mu}{u\cdot\Pi}\right\rangle|\Phi_\kappa\rangle = \int dx\, dx'\, dp\, \frac{d\Omega}{4\pi}\langle\Phi_{\kappa'}|x'\rangle\langle x'|u_p\rangle\frac{u_\mu}{u\cdot p}\langle u_p|x\rangle\langle x|\Phi_\kappa\rangle \quad . \tag{2.11}$$

The result reads

$$\begin{aligned}\langle u_0\rangle_\kappa &= \frac{1}{n!\sqrt{2\pi}}\int_{-\infty}^{\infty} ds\, H_n^2(s)e^{-s^2/2} \\ &\quad\times\left\{\frac{p_z}{2p^2}\ln\frac{p_0+p_z}{p_0-p_z} + \frac{p_0 s\sqrt{2qB}}{2p^2\sqrt{p_0^2-p^2}}\arctan\frac{s\sqrt{2qB}}{2\sqrt{p_0^2-p^2}}\right\} \quad ,\end{aligned} \tag{2.12}$$

$$\begin{aligned}\langle u_z\rangle_\kappa &= \frac{1}{n!\sqrt{2\pi}}\int_{-\infty}^{\infty} ds\, H_n^2(s)e^{-s^2/2} \\ &\quad\times\left\{-\frac{p_z}{p^2} + \frac{p_0(2p_z^2 - qBs^2)}{4p^4}\ln\frac{p_0+p_z}{p_0-p_z}\right. \\ &\quad\left. + \frac{p_z(2p_0^2-p^2)}{2p^4}\frac{s\sqrt{2qB}}{\sqrt{p_0^2-p^2}}\arctan\frac{s\sqrt{2qB}}{2\sqrt{p_0^2-p^2}}\right\} \quad ,\end{aligned} \tag{2.13}$$

$$
\begin{aligned}
\langle u_+\rangle_\kappa &= \frac{i}{\sqrt{2\pi n!(n-1)!}} \int_{-\infty}^{\infty} ds\, H_n(s) H_{n-1}(s) e^{-s^2/2} \\
&\quad \times \left\{ \frac{s\sqrt{2qB}}{2p^2} - \frac{p_0 s p_z \sqrt{2qB}}{2p^4} \ln \frac{p_0+p_z}{p_0-p_z} \right. \\
&\quad \left. + \frac{2p_z^2(p_0^2-p^2) - p_0^2 qBs^2}{2p^4\sqrt{p_0^2-p^2}} \arctan \frac{s\sqrt{2qB}}{2\sqrt{p_0^2-p^2}} \right\} \quad , \qquad (2.14) \\
\langle u_-\rangle_\kappa &= -\langle u_+\rangle_{\kappa+1} \quad , \qquad (2.15)
\end{aligned}
$$

where $p^2 = p_z^2 + qBs^2/2$. With these definitions the Dirac equation effectively reduces to a 4×4 matrix in the spinor indices, since the other quantum numbers have been diagonalized. In the massless limit ($m=0$) the left- and right-handed parts factorize and the Dirac equation takes the form

$$
\left[\not{\Pi} - \mathcal{M}_e^2 \gamma_\mu \left\langle \frac{u^\mu}{u\cdot\Pi} \right\rangle \right] \chi \equiv \begin{pmatrix} 0 & D_L(\kappa) \\ D_R(\kappa) & 0 \end{pmatrix} \begin{pmatrix} \chi_R(\kappa) \\ \chi_L(\kappa) \end{pmatrix} = 0 \quad , \qquad (2.16)
$$

where

$$
\begin{aligned}
D_R(\kappa) &= \begin{pmatrix} -p_0 + p_z + \mathcal{M}_e^2(\langle u_0\rangle_\kappa - \langle u_z\rangle_\kappa) & i\sqrt{2qBn} - \mathcal{M}_e^2\langle u_-\rangle_{\kappa-1} \\ -i\sqrt{2qBn} - \mathcal{M}_e^2\langle u_+\rangle_\kappa & -p_0 - p_z + \mathcal{M}_e^2(\langle u_0\rangle_{\kappa-1} + \langle u_z\rangle_{\kappa-1}) \end{pmatrix} \quad , \\
D_L(\kappa) &= \begin{pmatrix} -p_0 - p_z + \mathcal{M}_e^2(\langle u_0\rangle_\kappa + \langle u_z\rangle_\kappa) & -i\sqrt{2qBn} + \mathcal{M}_e^2\langle u_-\rangle_{\kappa-1} \\ i\sqrt{2qBn} + \mathcal{M}_e^2\langle u_+\rangle_\kappa & -p_0 + p_z + \mathcal{M}_e^2(\langle u_0\rangle_{\kappa-1} - \langle u_z\rangle_{\kappa-1}) \end{pmatrix} \quad ,
\end{aligned}
$$
$$(2.17)$$

In the lowest Landau level ($n=0$) Eq. (2.16) reduces to a 2×2 matrix, since only one orientation of the magnetic moment is possible. It is easy to take the determinant of Eq. (2.17) to find the dispersion relations, which for the right-handed component are

$$
\begin{aligned}
n \geq 1: \quad & \Big(p_0 - p_z - \mathcal{M}_e^2(\langle u_0\rangle_\kappa - \langle u_z\rangle_\kappa)\Big) \\
& \times\Big(p_0 + p_z - \mathcal{M}_e^2(\langle u_0\rangle_{\kappa-1} + \langle u_z\rangle_{\kappa-1})\Big) \\
& - \Big(\sqrt{2qBn} - i\mathcal{M}_e^2\langle u_+\rangle_\kappa\Big)^2 = 0 \quad , \qquad (2.18) \\
n = 0: \quad & p_0 - p_z - \mathcal{M}_e^2(\langle u_0\rangle_\kappa - \langle u_z\rangle_\kappa) = 0 \quad .
\end{aligned}
$$

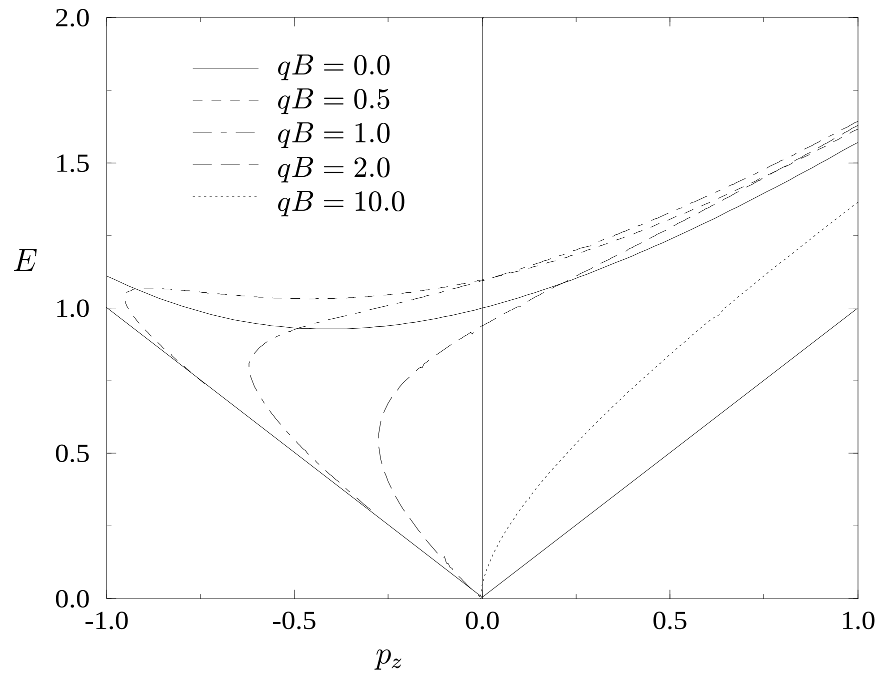

Figure 1: *Dispersion relations for the right-handed branch in the lowest Landau level ($n = 0$), neglecting the imaginary part. As the B-field increases thermal effects become less important and the dispersion relation approaches the light cone, which is indicated by solid lines. All dimensionful parameters are given in units of the thermal mass $\mathcal{M}_e$.*

These relations are only meaningful for stable propagating quasi-particles with well-defined relations between momentum and energy, i.e. when the imaginary parts are negligible. In general there are imaginary parts in the functions $\langle u_{0,z,\pm}\rangle_\kappa$, which are discussed in Section 4. It is anyway useful to first solve Eq. (2.18), ignoring for the moment the imaginary part, since the zeros of the real part indicate where the spectral functions are peaked, at least when the imaginary part is small enough. This can conveniently be done numerically as all the integrals in Eqs. (2.12) to (2.15) are well convergent. The dispersion relations for several field strengths in the lowest Landau level are shown in Fig. 1.

In the lowest Landau level a right-handed particle (positive chirality, $\chi = R = +1$) with q and B positive, can only propagate in the positive z-direction since the magnetic moment, and thus the spin, has to point along the field. Positive chirality implies positive helicity for particles and thus positive p_z. The left-hand side of Fig. 1 has to violate one of these sign arguments. The solution is that hole excitations have opposite chirality–helicity relation and can thus propagate for negative p_z. As the field increases the hole branch develops a new sub-branch and disappears continuously for large enough fields. The new branch must not be taken too seriously, since it only appears when the imaginary part is non-negligible and then only the full spectral function is meaningful. The particle branch approaches the light cone, i.e. the vacuum dispersion relation, in a smooth way as the field strength increases. This is physically very reasonable since, for very strong field strengths, the thermal effects should disappear. Once again it should be emphasized that the above analysis is based only on the real part of the self-energy and it can only serve as a guiding line to describe what kind of modes propagate in the plasma. For a complete description, which is necessary for $qB \simeq \mathcal{M}_e^2$, where the imaginary part is comparable with the real part, the full spectral function has to be used, as we do in section 4.

The HTL effective action is derived under the condition that the temperature is much larger then the momentum. Here, the magnetic field enters only through the covariant momentum and should thus satisfy the condition $\Pi^2 \sim qB \ll T^2$. On the other hand, already when $qB \gg \mathcal{M}_e^2 \sim q^2T^2$ (which is the only scale where T enters in the HTL approximation) the thermal corrections start to get small compared with the tree-level part. Thus, for small coupling the HTL corrections become small before they are invalid.

3 Comparison with an approximate formula

In a direct one-loop calculation of the fermionic self-energy in a magnetic field [7] one is naturally led to an approximation where the full B-dependence is kept only where it is added linearly to the momentum squared. In other places it enters only to $\mathcal{O}(B^2)$ (see [7] for details). Since the result from this approximation is surprisingly simple, it is worth commenting on its relation with the exact solution. From [7] we find the Dirac equation

$$
\begin{aligned}
&[\not{\Pi} - m - \hat{\Sigma}(p_0, p_z, \mathbf{\Pi}_\perp)]\Psi = \\
&\left[s(p_0, \mathbf{\Pi}^2)\gamma_0 p_0 - r(p_0, \mathbf{\Pi}^2)\gamma_z p_z - r(p_0, (\mathbf{\Pi}\cdot\boldsymbol{\gamma})^2)\not{\Pi}_\perp - m\right]\Psi = 0 \quad , \qquad (3.1)
\end{aligned}
$$

where $\mathbf{\Pi}^2 = \Pi_\perp^2 + p_z^2$. The functions $s(p_0, \mathbf{\Pi}^2)$ and $r(p_0, \mathbf{\Pi}^2)$ are derived from the HTL effective action without background field and they are given by:

$$p_0 s(p_0, \mathbf{\Pi}^2) = p_0 - \mathcal{M}_e^2 \left\langle \frac{u_0}{u\cdot p} \right\rangle \Big|_{p\to\Pi} = p_0 - \frac{\mathcal{M}_e^2}{2\,|\mathbf{\Pi}|} \ln\left|\frac{p_0 + |\mathbf{\Pi}|}{p_0 - |\mathbf{\Pi}|}\right| , \tag{3.2}$$

$$p_z r(p_0, \mathbf{\Pi}^2) = p_z - \mathcal{M}_e^2 \left\langle \frac{u_z}{u\cdot p} \right\rangle \Big|_{p\to\Pi} = p_z + \frac{p_z \mathcal{M}_e^2}{\mathbf{\Pi}^2} \left(1 - \frac{p_0}{2\,|\mathbf{\Pi}|} \ln\left|\frac{p_0 + |\mathbf{\Pi}|}{p_0 - |\mathbf{\Pi}|}\right|\right) . \tag{3.3}$$

It is *almost* possible to guess the expression in Eq. (3.1) from the standard expression for the HTL Dirac equation [4, 5]. The usual momentum p_μ should be replaced with the gauge-invariant momentum Π_μ, but there is an ambiguity in replacing p^2 by Π^2 or by $\not\Pi\not\Pi$. The correct way follows from the calculations in [7].

The difference between Eq. (3.1) and the exact formula is related to the order of doing the average over u and replacing p by Π. Comparing the exact expression for $\langle u_0 \rangle$ with Eq. (3.2), before doing the u-integration, we would like to specify under which circumstances the approximate equality

$$\frac{1}{n!\sqrt{2\pi}} \int_{-\infty}^{\infty} ds\, H_n^2(s) e^{-s^2/2} \frac{d\Omega}{4\pi} \frac{u_0}{u_0 p_0 - u_z p_z - u_\perp \sqrt{\frac{qB}{2}}\, s}$$
$$\simeq \int \frac{d\Omega}{4\pi} \frac{u_0}{u_0 p_0 - u_z p_z - u\cdot p_\perp}\Big|_{p_\perp \to \Pi_\perp} \tag{3.4}$$

is valid. The first term comes from the exact expression and the second from the approximate formula Eq. (3.2). In a standard coordinate system with $\boldsymbol{u} = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$, so that $u_\perp = \sin\theta$, we see that the difference lies in the integral over the azimuthal angle ϕ. In the exact formula the integral over ϕ is replaced by a more complicated integral over s, involving the exact external states, i.e. the Landau levels. The two expressions in Eq. (3.4) do not coincide, except in some particular limits. Expanding both sides of Eq. (3.4) formally in powers of qB and $\Pi_\perp^2$, we are led to comparing the integrals

$$\frac{1}{n!\sqrt{2\pi}} \int_{-\infty}^{\infty} ds\, H_n^2(s) e^{-s^2/2} \left(u_\perp \sqrt{\frac{qB}{2}}\, s\right)^{2k} = (u_\perp^2 qB)^k \frac{(2k)!}{2^{2k}} \sum_{l=0}^{\min(n,k)} \frac{2^l n!}{(n-l)!(k-l)!(l!)^2} ,$$

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \left(\boldsymbol{u}_\perp \cdot p_\perp\right)^{2k}\Big|_{p_\perp \to \Pi_\perp} \quad = \quad (u_\perp^2 qB)^k \frac{(2k)!}{2^{2k}(k!)^2}(2n+1)^k \quad , \tag{3.5}$$

where we used the fact that $\Pi_\perp^2 = qB(2n+1)$ when acting on a Landau level. First we notice that for $k = 1$ the two integrals coincide. Then, we find that the leading terms in the limit of large n, for fixed k, also coincide. We can thus expect that the approximative formula Eq. (3.1) is useful both for weak fields and for very high Landau levels. It should, however, be noticed that the expansion converges badly close to the light cone, and that it eventually breaks down for hole excitation of high momentum [7].

In many applications it is only the dispersion relation for small momenta that is important. Using Eq. (3.1) in the limit $m = 0$, we can easily obtain an approximate formula for the dispersion relation in the lowest Landau level around $p_z = 0$ in the presence of a weak magnetic field and $\chi = 1$

$$E(p_z) \simeq \mathcal{M}_e \left(1 + \frac{qB}{6\mathcal{M}_e^2}\right) + \frac{p_z}{3}\left(1 - \frac{7qB}{15\mathcal{M}_e^2}\right) \quad . \tag{3.6}$$

For the right-handed branch of the lowest Landau level in a weak magnetic field ($qB = 0.2\,\mathcal{M}_e^2$), the dispersion relation following from Eq. (3.1) is shown in Fig. 2, where it is also compared with the exact solution of Eq. (2.18).

4 Spectral function

The dispersion relation was solved in section 2, ignoring the imaginary parts of the self-energy. This is only a good approximation for small magnetic fields, where the imaginary parts are small. Since we have the exact expression for the self-energy it is not too difficult to study the complete spectral function and to see how good the quasi-particle picture is. The spectral function can be defined from the representation of the retarded and advanced propagator as

$$S(E \pm i\epsilon, \boldsymbol{p}) = \int_{-\infty}^{\infty} dE' \frac{\mathcal{A}(E', \boldsymbol{p})}{E - E' \pm i\epsilon} \quad . \tag{4.1}$$

In the real-time formalism of thermal field theory the spectral representation goes through in much the same way as at zero temperature, the only essential difference being doubling the

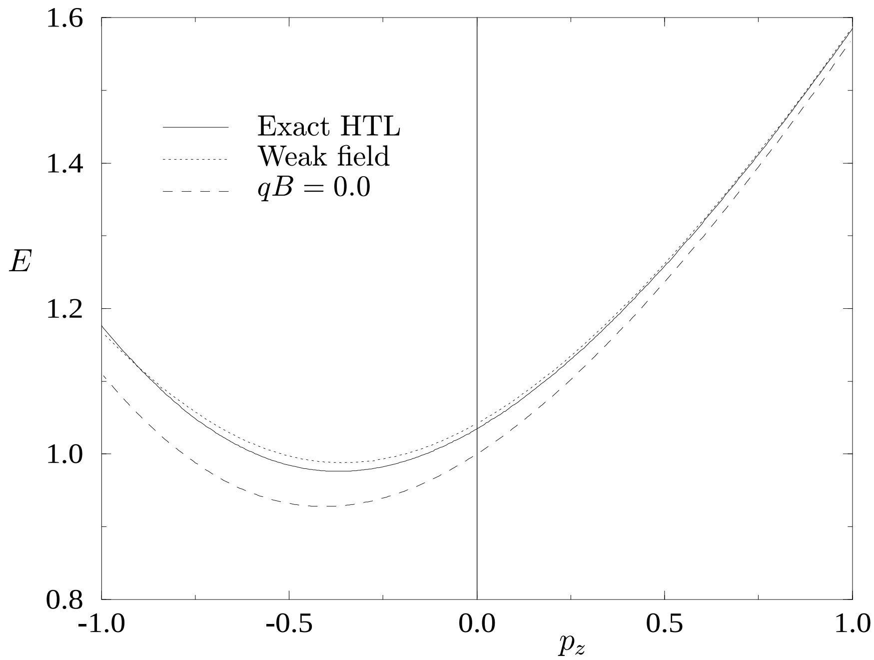

Figure 2: *Comparison of the dispersion relation from the HTL effective action and the weak field approximation in the lowest Landau level for $qB = 0.2\mathcal{M}_e^2$. All dimensionful parameters are given in units of the thermal mass $\mathcal{M}_e$.*

degrees of freedom (for a recent review see [8]). The full thermal propagator takes the form [8]

$$S^{(ab)}(E,\boldsymbol{p}) = \int_{-\infty}^{\infty} dE' \mathcal{A}(E',\boldsymbol{p})\sigma_z \mathcal{B}^{-1}(E') \begin{pmatrix} \frac{1}{E-E'+i\epsilon} & 0 \\ 0 & \frac{1}{E-E'-i\epsilon} \end{pmatrix} \mathcal{B}(E') \quad , \tag{4.2}$$

where σ_z is a Pauli spin matrix and $\mathcal{B}(E)$ can be chosen to be

$$\mathcal{B}(E) = \begin{pmatrix} (e^{-\beta(E-\mu)}+1)^{-1} & (e^{\beta(E-\mu)}+1)^{-1} \\ 1 & 1 \end{pmatrix} \quad . \tag{4.3}$$

For a free Dirac fermion we have

$$\mathcal{A}(E,\boldsymbol{p}) = (\gamma_0 E - \boldsymbol{\gamma}\boldsymbol{p} + m)\mathrm{sign}(E)\delta(E^2 - \boldsymbol{p}^2 - m^2) \quad , \tag{4.4}$$

but in general $\mathcal{A}(E,\boldsymbol{p})$ can have both a δ-function part for the quasi-particles and a continuous part. The HTL fermion propagator without any external B-field is given by

$$S(E,\boldsymbol{p}) = \frac{s(E,\boldsymbol{p})\gamma_0 E - r(E,\boldsymbol{p})\boldsymbol{\gamma p} + m}{s(E,\boldsymbol{p})^2 E^2 - r(E,\boldsymbol{p})^2 \boldsymbol{p}^2 - m^2} \quad , \tag{4.5}$$

where the functions $s(E,\boldsymbol{p})$ and $r(E,\boldsymbol{p})$ are defined in Eqs. (3.2) and (3.3). For $E > |\boldsymbol{p}|$ the only imaginary part comes from the analytic continuation using $\pm i\epsilon$ and the contribution to the spectral function become δ-functions at the solutions of the dispersion relation. Below the light cone, i.e. when $E < |\boldsymbol{p}|$, there is a finite imaginary part emerging from the logarithms in Eqs. (3.2) and (3.3) giving a continuous contribution to $\mathcal{A}(E,\boldsymbol{p})$. In the appendix it is shown that $S(E,\boldsymbol{p})$ fulfils the general requirements of normalization and causality.

4.1 Spectral function in the presence of a B-field

Since the self-energy in the presence of the B-field does not have any singular points away from the real axis, and since the HTL corrections are negligible for large complex E, we expect that the propagator still has the correct analyticity properties and that the normalization and causality properties, discussed in appendix A, are satisfied. We have checked the sum rule in Eq. (A.3) by direct numerical calculations and it is indeed satisfied. The analytic continuation $E \to E \pm i\epsilon$ is more complicated in the presence of the background field, but it can be summarized by the formula:

$$\begin{aligned}
&\frac{1}{\sqrt{E^2-p^2}} \arctan \frac{s\sqrt{2qB}}{2\sqrt{E^2-p^2}} \to \\
&\theta(E^2-p^2)\frac{1}{\sqrt{E^2-p^2}} \arctan \frac{s\sqrt{2qB}}{2\sqrt{E^2-p^2}} \\
&+\theta(p^2-E^2)\Bigg\{\theta(p_z^2-E^2)\frac{1}{2\sqrt{p^2-E^2}} \ln\left|\frac{2\sqrt{p^2-E^2}-s\sqrt{2qB}}{2\sqrt{p^2-E^2}+s\sqrt{2qB}}\right| \\
&\qquad +\theta(E^2-p_z^2)\left[\frac{\mathrm{sign}(s)}{2\sqrt{p^2-E^2}} \ln\left|\frac{|s|\sqrt{2qB}-2\sqrt{p^2-E^2}}{|s|\sqrt{2qB}+2\sqrt{p^2-E^2}}\right|\right. \\
&\qquad\qquad \left.+\frac{i\pi \mathrm{sign}(s)\mathrm{sign}(E)}{2\sqrt{p^2-E^2}}\right]\Bigg\} \quad ,
\end{aligned} \tag{4.6}$$

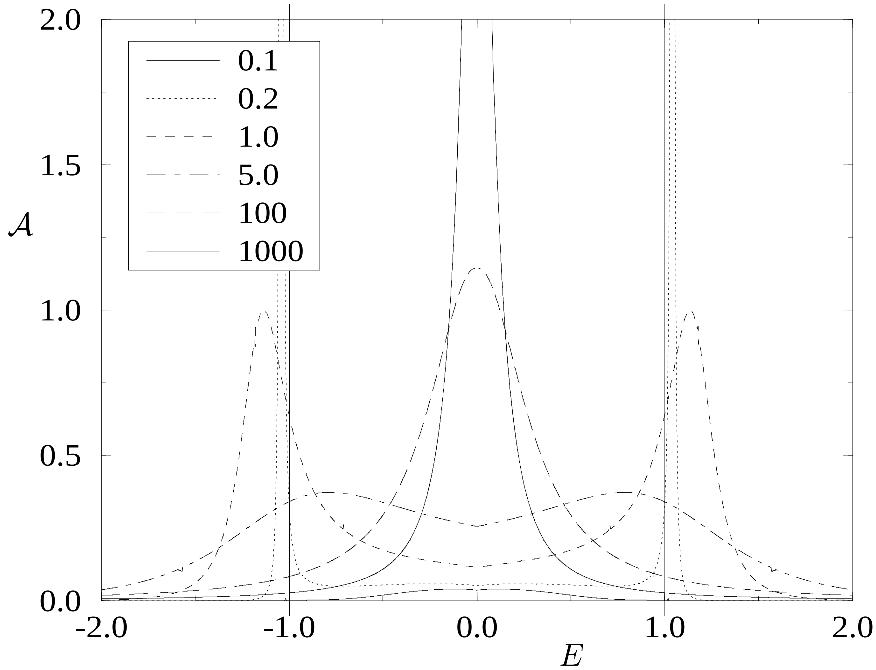

Figure 3: *Spectral function for various magnetic field strengths at the momentum* $p_z = 0.0$ *in the lowest Landau level. For weak fields there are two equal peaks around* $E \simeq \pm 1.0$ *(indicated by the vertical solid lines), corresponding to the thermal mass of the particle and anti-hole solutions. As the field increases the width of the peaks increases and the positions are shifted to a slightly higher value. For intermediate fields the spectral function is very wide and eventually it gets more concentrated around* $E =$ 0.0, *which is the position it should have without thermal correction. All dimensionful parameters are given in units of the thermal mass* $\mathcal{M}_e$.

for $E \to E - i\epsilon$.

With this explicit expression it is straightforward to calculate the spectral function numerically. We have done so for $p_z = 0$ and $p_z = 0.5$ for the right-handed branch in the lowest Landau level; the result for the term proportional to γ_0, namely

$$
\begin{aligned}
\mathcal{A}^R_{\rm LLL}(E, p_z) &= \mathrm{tr}\left[\frac{1}{2}(1+\gamma_5)\gamma_0 \mathcal{A}(E, p_z, n=0)\right] \\
&= \frac{1}{2\pi i}\Big(S^R_{\rm LLL}(E - i\epsilon, p_z) - S^R_{\rm LLL}(E + i\epsilon, p_z)\Big) \ , \\
S^R_{\rm LLL}(E, p_z) &= \frac{1}{p_0 - p_z - \mathcal{M}_e^2(\langle u_0\rangle - \langle u_z\rangle)} \ ,
\end{aligned}
\tag{4.7}
$$

is presented in Figs. 3 and 4. These figures should be compared with the solution of the dispersion relation from the real part in section 2 and correspond to two vertical cuts in Fig. 1 at $p_z = 0$ and $p_z = 0.5$. The tendencies are the same. For zero momentum (Fig. 3) there is no distinction between particles and holes, and the two δ-function peaks correspond to positive and negative energy solutions. As the B-field is increased, there is a broadening in the width and the positions of the peaks are shifted towards the zero temperature value, which at $p_z = 0$ is a peak at $E = 0$. At non-zero momentum ($p_z = 0.5$, Fig. 4) the two peaks at $B = 0$ correspond to a particle solution at $E \simeq 1.1\,\mathcal{M}_e$ and an anti-hole solution at $E \simeq -\mathcal{M}_e$. In addition there is a continuous part in the interval $-p_z < E < p_z$. Also in this case the peaks get broader as B increases and eventually there is only one wide peak around $p_z = 0.5$ for very strong fields. In vacuum there is an imaginary part of the self-energy, describing the decay to a lower energy level due to synchrotron radiation, only for the higher Landau levels, but at finite temperature even the particle in the lowest Landau level can scatter with the surrounding plasma and this is what causes the imaginary part.

It should be noticed here that, as already discussed at the end of section 2, the HTL approximation is only valid for momenta smaller than the temperature. For larger momenta, or stronger fields, the tree level contribution dominates the real part in the Dirac equation, but since there is no imaginary part at tree-level it comes entirely from the HTL term, which is then not reliable (though small). In the very strong field limit ($qB \gg T^2$), only the lowest Landau level is occupied and the imaginary part comes from annihilations with the antiparticles [7]. This term is linear in temperature and does not appear in the HTL approximation.

5 The chiral anomaly

The classical action for massless fermions is invariant under chiral transformations, but the corresponding chiral current is not conserved on the quantum level due to the chiral anomaly [9]. The divergence of the chiral current in 3+1 dimensions is given by

$$\partial_\mu j_5^\mu = \partial_\mu \overline{\Psi} \gamma^\mu \gamma_5 \Psi = \frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \quad . \tag{5.1}$$

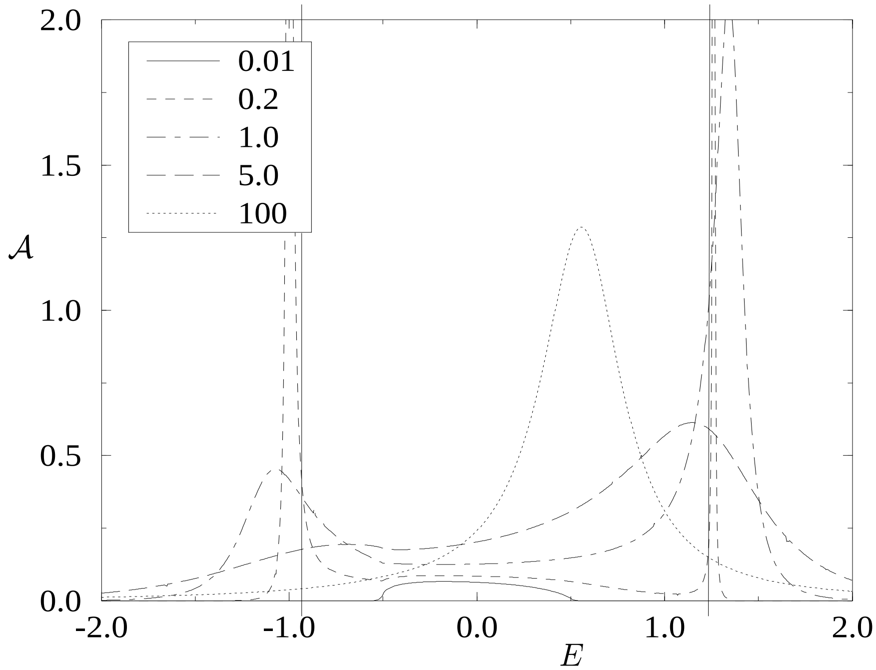

Figure 4: *Spectral function for various magnetic field strengths at the momentum $p_z = 0.5$ in the lowest Landau level. For very weak fields there are two δ-functions at the positions of the particle and anti-hole poles (indicated by the vertical solid lines), and a continuous part in the interval $[-0.5, 0.5]$ which is below the light cone. When the field increases the anti-hole peak around $E \simeq -1.0$ disappears faster than the particle peak at $E \simeq 1.3$. At $qB = 5.0\mathcal{M}_e^2$ there is only a very wide peak left of the particle around $E \simeq 1.0$, which for increasing field is slowly shifted towards $E = 0.5$ while also getting narrower.*

Finite temperature effects do not break chirality and, as a classical action, the HTL effective action is still chirally invariant. Since the anomalous term in Eq. (5.1) originates from the UV-divergent part of the propagator, it is not expected to change at finite temperature and Eq. (5.1) is expected to hold. This has also been verified explicitly by several authors [10, 11].

In vacuum there is a clear physical picture, related to the IR properties of the fields, of how chirality is created by moving particles from the Dirac sea up to positive energy by switching on an external electric field [12, 13]. This picture works because the massless dispersion relation crosses the Dirac surface and particle–antiparticle pairs can be created continuously. By adding a chirality-breaking mass term the dispersion relation no longer crosses the Dirac surface and, in fact, no chirality is produced if the external gauge field varies adiabatically. New particle pairs are again created [13] when the variation of the gauge field is rapid with respect to the mass of the fermion.

At very high temperature, it is well known that even massless chiral fermions pick up an effective thermal mass, in the sense that the energy of the propagating modes does not go to zero for vanishing momentum. From this IR picture it is not obvious how the anomaly equation (Eq. (5.1)) can be fulfilled at finite temperature. What happens, as I show below, is that as a quasi-particle of one chirality moves along the dispersion relation from a particle-like excitation to a hole-like one, it looses its spectral weight, while a quasi-particle of opposite chirality gains the same amount of spectral weight. In this way the spectral weight is shifted between chiralities (or between states above and below the Dirac surface) without any dispersion curve actually crossing the surface.

Most discussions of the anomaly at zero temperature are performed using a language of single-particle excitations. Even though there are stable quasi-particles in the leading HTL effective action without any external field, this is far from being the whole picture. There is some danger in treating the branches of the dispersion relation as ordinary particles. Each of the branches does not have a full spectral weight, and not even the sum of the spectral weight for the particle and the hole adds up to 1. With non-zero external field there are also imaginary parts, which cannot be accounted for within a quasi-particle picture. We shall therefore carry out the calculations entirely in terms of Green's functions and without reference to single-particle states. The HTL effective action is, after all, only a way of writing a set of Green's functions.

5.1 The 1+1 D anomaly

In order to see what are the essential parts of the thermal anomaly equation, we shall first briefly repeat the standard calculations of the anomaly in 1+1 dimensions using both an operator and a Green's functions language. We shall follow ref. [13] very closely in the operator formalism. The chiral anomaly at finite temperature for a non-interacting theory has been studied by several authors [10], in particular in 1+1 dimensions. While Smilga [11] gave a physical interpretation of the thermal effect in terms of scattering with particles in the heat bath, I take a slightly different approach to reach similar conclusions. Once we have understood the mechanism in 1+1 dimensions it should be easier to see that the same holds true in 3+1 dimensions.

There is an intrinsic consistency problem in treating a time-dependent gauge field, i.e. an electric field, and an equilibrium ensemble at the same time. Starting from an equilibrium ensemble it will not remain in equilibrium if we switch on an electric field, unless we consider an adiabatic limit where the ensemble has time to readjust itself to equilibrium much faster than the variation of the gauge field. Even though Eq. (5.1) is true at the operator level, and thus true for any expectation value of the equation, it does not determine the time-integrated form $\int^t dt' \mathrm{tr}\left[\rho(t')\overline{\Psi}(x)\gamma_0\gamma_5\Psi(x)\right]$ for an explicitly time-dependent density matrix $\rho(t)$. The fields in Eq. (5.1) are in the Heisenberg picture and if the density matrix has no explicit time dependence it enters only as an initial condition. In an adiabatic limit one could effectively take into account interactions with the heat bath by using a density matrix, which at all times corresponds to thermal equilibrium. This would then be an explicitly time-dependent density matrix and the time-integrated anomaly equation may not fulfil the standard anomaly equation. Such a replacement with an effective density matrix is to some extent arbitrary and depends on which physical situation is imagined. It is possible to consider that only energy is equilibrated by scattering processes, and that chirality is conserved in each process, and thus not equilibrated. Or, one can consider the system to be in contact with a heat reservoir with which it can also exchange chirality.

Another possible situation is when we neglect interactions between particles altogether and follow the exact time evolution of the non-interacting plasma, after its initial condition is given. This is the situation we shall consider in detail.

The chiral charge has to be defined using a gauge-invariant point splitting regularization in the spatial z-direction:

$$\langle Q_5^\gamma \rangle = \int d\xi_z\, d\eta_z \delta_\gamma(\xi_z - \eta_z) \langle \overline{\Psi}(\xi_z, t)\gamma_0\gamma_5\Psi(\eta_z, t)\rangle \exp\left[ie\int_{\eta_z}^{\xi_z} A_z(\xi_z', t) d\xi_z'\right] \quad , \tag{5.2}$$

where

$$\delta_\gamma(\xi) = \frac{\exp[-\frac{\xi^2}{2\gamma}]}{\sqrt{2\pi\gamma}} \quad . \tag{5.3}$$

Including a chirality-breaking Dirac mass the anomaly equation in 1+1 dimensions, integrated over space and over time from 0 to τ, reads

$$\lim_{\gamma\to 0}\langle Q_5^\gamma(\tau)\rangle = \frac{e}{2\pi}\int_0^\tau dt\int dx\,\varepsilon_{\mu\nu}F^{\mu\nu} + 2im\int_0^\tau dt\langle\overline{Q}_5^\gamma(t)\rangle \quad , \tag{5.4}$$

where $\langle\overline{Q}_5^\gamma\rangle$ is defined as $\langle Q_5^\gamma\rangle$ in Eq. (5.2) but with $\gamma_0\gamma_5$ replaced by γ_5. With a field operator

$$\Psi(t,\xi_z) = \int [dp_z] e^{ip_z\xi_z}[u_{p_z}(t)b_{p_z} + v_{-p_z}(t)d^\dagger_{-p_z}] \quad , \tag{5.5}$$

where the notation is taken from [13], the chiral charge in the massless limit can be computed as

$$\langle Q_5^\gamma(\tau)\rangle = -\frac{L}{2\pi}\int dp_z\, e^{-\frac{\gamma}{2}(p_z-eA_z(\tau))^2}\Big(1-\langle b^\dagger_{p_z}b_{p_z}\rangle - \langle d^\dagger_{-p_z}d_{-p_z}\rangle\Big)\Big[\theta(p_z)-\theta(-p_z)\Big] \quad . \tag{5.6}$$

The initial expectation values of the number of particles $\langle b^\dagger_{p_z}b_{p_z}\rangle$ and antiparticles $\langle d^\dagger_{-p_z}d_{-p_z}\rangle$ depend on which physical situation we consider, but in any case they should go rapidly to zero for large $|p_z|$. In thermal equilibrium, with zero chemical potential, we would for instance have $\langle b^\dagger_{p_z}b_{p_z}\rangle = \langle d^\dagger_{-p_z}d_{-p_z}\rangle = (\exp[\beta|p_z|]+1)^{-1}$. It is thus only in the vacuum part that the point splitting is needed. It follows that only the vacuum part can depend on $A_z(t)$ and the chirality production is, therefore, independent of the initial thermal condition. If, on the other hand, we consider a situation where the particles relax rapidly to thermal equilibrium, so that the distribution of particles with quantum number p_z is determined by the energy of the states after switching on the A_z-field, then we should rather use $\langle b^\dagger_{p_z}b_{p_z}\rangle = (\exp[\beta E_{p_z}]+1)^{-1}$, where $E_{p_z} = |p_z| - \text{sign}(p_z)eA_z(t)$. In this case also the thermal part of Eq. (5.6) depends on $A_z(t)$, not directly through the anomaly, but from the interaction with the heat bath that we assume to be present in order to maintain thermal equilibrium.

Since in the (3+1)-dimensional case we want to avoid the use of single particle states as in Eq. (5.5), we shall now see how the above calculation can be performed using Green's functions. The field expectation values can be related to the time-ordered Feynman Green's function via

$$\langle\overline{\Psi}_a(\xi)\Psi_b(\eta)\rangle = -i\, S_F(\eta,\xi)_{ba}\big|_{\xi_0>\eta_0} \quad . \tag{5.7}$$

In a time-dependent background field, where energy is not conserved, it is not obvious what the correct ϵ-prescription for a time-ordered Green's function should be. In this simple non-interacting case we can, however, compute everything explicitly and, starting with the vacuum part, we find

$$\langle \mathbf{T}\Psi(\xi)\overline{\Psi}(\eta)\rangle^0 = \int [d^2p] \exp\left[-ip_0\xi_0 + ip_z\xi_z + i\gamma_5 \int^{\xi_0} dt A_z(t)\right] \frac{i}{\not{p}} \exp\left[ip_0\eta_0 - ip_z\eta_z + i\gamma_5 \int^{\eta_0} dt A_z(t)\right] \quad , \tag{5.8}$$

where the usual Feynman prescription $p_0 \to (1+i\epsilon)p_0$ should be used in $\frac{i}{\not{p}}$. All complications from the external A_z field is thus taken into account in the phases of the diagonalizing wave functions, and the Feynman prescription is unchanged. The significance of p_0 depends on the representation of the wave functions we use to diagonalize the propagator, and here p_0 is the initial energy of a particle in a state labelled by p_z before the A_z field is turned on. The actual energy of that state then varies as $p_0 - \chi A_z(t)$, where χ is the chirality of the state.

With this form of the propagator we obtain the chiral charge from

$$\langle \overline{\Psi}(\xi)\gamma_0\gamma_5\Psi(\eta)\rangle^0 = -i\int [d^2p] e^{ip(\xi-\eta)} \left(\frac{e^{-i\int_{\eta_0}^{\xi_0} eA_z(t)dt}}{p_0 - p_z + i\epsilon p_0} - \frac{e^{i\int_{\eta_0}^{\xi_0} eA_z(t)dt}}{p_0 + p_z + i\epsilon p_0} \right)\Bigg|_{\xi_0>\eta_0} \quad . \tag{5.9}$$

The condition $\xi_0 > \eta_0$ tells us that the p_0 contour must be closed in the upper half-plane. The poles in Eq. (5.9) give two θ-functions in p_z in the standard manner. The rest of the calculation can be found in [13] and the result is

$$\langle Q_5^\gamma(\tau)\rangle^0 = -\frac{L}{2\pi}\int dp_z e^{-\frac{\gamma}{2}(p_z - eA_z)^2}[\theta(p_z) - \theta(-p_z)] \overset{\gamma\to 0}{\to} -\frac{L}{2\pi}2eA_z(\tau) \quad , \tag{5.10}$$

in accordance with Eq. (5.6). From this exercise we learn that in the massless case, where we can find an explicit basis diagonalizing the propagator, the mathematical mechanism that gives us the correct anomaly is that the poles in p_0 cross the real axes when $p_z = 0$.

In the equilibrium real-time finite temperature formalism the free propagator can be written as in Eq. (4.2), but since we compute a one-point function we only need the 11-part:

$$iS_F^\beta(p) = iS_F^0(p) - f_F(p_0)\left(iS_F^0(p) - iS_F^{0*}(p)\right) \quad , \tag{5.11}$$

where $f_F(p_0)$ is the thermal distribution function. The problem we have at hand is not one of an equilibrium, but as we saw in Eq. (5.8) the time dependence can be entirely absorbed in the phases of the wave functions. Since in this basis p_0 has the meaning of the energy of the initial state, the thermal version of Eq. (5.9) is obtained by the substitution

$$\frac{i}{p_0 - \chi p_z + i\epsilon p_0} \to 2\pi \mathrm{sign}(p_0) f_F(p_0) \mathcal{A}^\chi(p_0, p_z) = 2\pi \mathrm{sign}(p_0) f_F(p_0) \delta(p_0 - \chi p_z) \quad , \tag{5.12}$$

where $f_F(p_0)$ is the initial particle distribution. With this propagator the thermal contribution to the anomaly is given by

$$\langle Q_5^\gamma(\tau)\rangle^\beta = \frac{L}{2\pi}\int dp_z e^{-\frac{\gamma}{2}(p_z - eA_z)^2}\int_0^\infty dp_0 \Big(f_F(p_0) + f_F(-p_0)\Big)\Big[\mathcal{A}^R(p_0,p_z) - \mathcal{A}^L(p_0,p_z)\Big] \quad , \tag{5.13}$$

which agrees with the thermal part of Eq. (5.6). Since the spectral functions are rapidly convergent in p_z for fixed p_0 the point splitting γ can be sent to zero before doing the integrations. There is no $A_z(t)$ dependence left, which shows again that there is no thermal correction to the anomaly.

5.2 The anomaly in 3+1 dimensions at high temperature

The HTL effective action is chirally invariant even though there is a mass gap in the dispersion relation. We, therefore, expect that the chirality produced in vacuum cannot be undone by chirality-conserving interaction with the thermal heat bath. From a mathematical point of view we can argue that since the anomaly equation is true at the operator level it must remain true in whatever average we take, including a thermal average. We shall see that this is correct by explicitly calculating the chirality production in 3+1 dimensions within the HTL approximation. The only thing we need is an explicit expression for the propagator. The fermionic part of the HTL effective action is simply related to the inverse of the propagator by

$$\mathcal{L}^{\mathrm{f}}_{\mathrm{HTL}} = \overline{\Psi}(x) S^{-1}(x,y)\Psi(y) \quad . \tag{5.14}$$

In order to write down the propagator itself we have fixed the boundary conditions when inverting the kernel of Eq. (5.14). At finite temperature the standard inversion gives Eq. (5.11), but with

$$iS_F^0(x,y) = \langle \mathbf{T}[\Psi(x)\overline{\Psi}(y)]\rangle = \left\langle x \left| \frac{i}{\not{\Pi} - m - \mathcal{M}_e^2 \gamma_\mu \left\langle \frac{u^\mu}{u\cdot\Pi}\right\rangle} \right| y \right\rangle \quad . \tag{5.15}$$

The ϵ-prescription for time ordering can be obtained by comparing the present calculation with the explicit calculation in section 5.1. We shall here use similar, explictly time-dependent, wave functions to diagonalize the propagator.

The (3+1)-dimensional anomaly equation at finite temperature, in a classical background field, is given by

$$\langle\partial_\mu\overline{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\rangle = \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \quad . \tag{5.16}$$

We shall compute the left-hand side of Eq. (5.16) from the propagator in Eq. (5.15) in a background consisting of orthogonal E and B fields. Scattering with external thermal particles, described by the thermal part of Eq. (5.11), is discussed at the end. In the massless vacuum case we saw that the essential mechanism of generating the correct anomaly was that the poles of the propagator crossed the real axis when p_z changed sign. This does not happen at finite temperature due to thermal masses.

First, we need to diagonalize the HTL Dirac equation in the presence of a B-field in the z-direction and a parallel E-field. We choose the gauge $A_\mu = (0,0,0,A_3(\xi_0))$ for the electric part of the background field. The diagonalization can be done in the same way as in section 2 but instead of Eq. (2.8) we use the basis

$$\langle\xi|v_p\rangle = \exp\left[-ip_0\xi_0 + ie\frac{v_z}{v_0}\int^{\xi_0}A_z(t)dt + ip_z\xi_z + ip_y\xi_y + i\left(p_x\xi_x + \frac{qBv_y}{v_x}\frac{\xi_x^2}{2}\right)\right] \quad , \tag{5.17}$$

but the eigenvalue of $v\cdot\Pi$ remains $v\cdot p$. In this way the propagator can be calculated exactly, even for non-adiabatic electric background fields, just as in the massless case at zero temperature. The prescription for the time-ordered Green's function is again $p_0 \to (1+i\epsilon)p_0$, in this basis. We write the relevant trace of the propagator as

$$\mathrm{tr}\, S_F(y,x)\gamma_0\gamma_5 = \sum_\kappa\sum_{i,j,a=1}^{4}\langle y,a|\kappa,i\rangle[S_F(\kappa)\gamma_0\gamma_5]_{ij}\langle\kappa,j|x,a\rangle \quad , \tag{5.18}$$

and use the basis

$$\begin{aligned}\Phi_a^{(i)}(\xi;p_0,n,p_y,p_z) &= \langle\xi,a|\kappa,i\rangle\\ &= e^{-i(p_0\xi_0-p_y\xi_y-p_z\xi_z)}\mathrm{diag}[I_{n,p_y}(\xi_x),I_{n-1,p_y}(\xi_x),I_{n,p_y}(\xi_x),I_{n-1,p_y}(\xi_x)]_{ab}u_b^{(i)} \quad ,\end{aligned} \tag{5.19}$$

where $u_b^{(i)}$ is a set of 4-spinor base vectors, which can be taken to be $u_b^{(i)} = \delta_{ib}$. For $n = 0$ there are only two states, $u^{(1)}$ and $u^{(3)}$, the others being identically zero. With the Dirac operator in Eq. (2.16), diagonalized in the spatial quantum numbers, we obtain for the higher Landau levels ($n > 0$):

$$\mathrm{tr}\, S_F(\kappa)\gamma_0\gamma_5 = \mathrm{tr}\, D_R^{-1}(\kappa) - \mathrm{tr}\, D_L^{-1}(\kappa) \quad . \tag{5.20}$$

In the lowest Landau level the matrices $D_{R,L}$ are not invertible, since there are only two states in total. Its explicit form then is ($\kappa_0 = \{n = 0, p_y, p_z\}$):

$$\mathrm{tr}\, S_F(\kappa_0)\gamma_0\gamma_5 = \int_{-\infty}^{\infty} \frac{\mathcal{A}^R_{\mathrm{LLL}}(E, p_z) - \mathcal{A}^L_{\mathrm{LLL}}(E, p_z)}{p_0 - E + i\epsilon p_0} \quad , \tag{5.21}$$

with $\mathcal{A}^{R,L}_{\mathrm{LLL}}(E, p_z)$ given by

$$\left.\frac{1}{p_0 - p_z - \mathcal{M}_e^2(\langle u_0\rangle_{\kappa_0} - \langle u_z\rangle_{\kappa_0})}\right|_{p_0 \to p_0 + i\epsilon p_0} = \int_{-\infty}^{\infty} dE \frac{\mathcal{A}^R_{\mathrm{LLL}}(E, p_z)}{p_0 - E + i\epsilon p_0} \quad , \tag{5.22}$$

and $\mathcal{A}^L_{\mathrm{LLL}}(E, p_z) = \mathcal{A}^R_{\mathrm{LLL}}(E, -p_z)$. We shall start by computing the contribution to $\langle Q_5\rangle$ from the lowest Landau level. This is, in fact, the only part that contributes, as we shall see later. Taking the point splitting only in the z-direction we find

$$\langle Q_5\rangle_{\mathrm{LLL}} = -iV \int d\xi_z \delta_\gamma(\xi_z) \int [dp_0][dp_y][dp_z] \langle 0|\kappa_0\rangle \mathrm{tr}\, S_F(\kappa_0)\gamma_0\gamma_5 \langle \kappa_0|\xi\rangle e^{ieA_z\xi_z} \quad . \tag{5.23}$$

Using

$$\int [dp_y] I_{n,p_y^2}(\xi_x) = \frac{eB}{2\pi} \quad , \tag{5.24}$$

and the fact that the p_0-contour should be closed in the upper half-plane, the produced chiral charge reduces to

$$\langle Q_5\rangle_{\mathrm{LLL}} = -\frac{VeB}{4\pi^2} \int dp_z e^{-\frac{\gamma}{2}(p_z - eA_z)^2} \left[W(p_z) - W(-p_z)\right] \quad . \tag{5.25}$$

This equation has a clear resemblance with Eq. (5.10). The function $W(p_z)$ is defined by

$$W(p_z) = \int_0^{\infty} dE \mathcal{A}^R_{\mathrm{LLL}}(E, p_z) \quad . \tag{5.26}$$

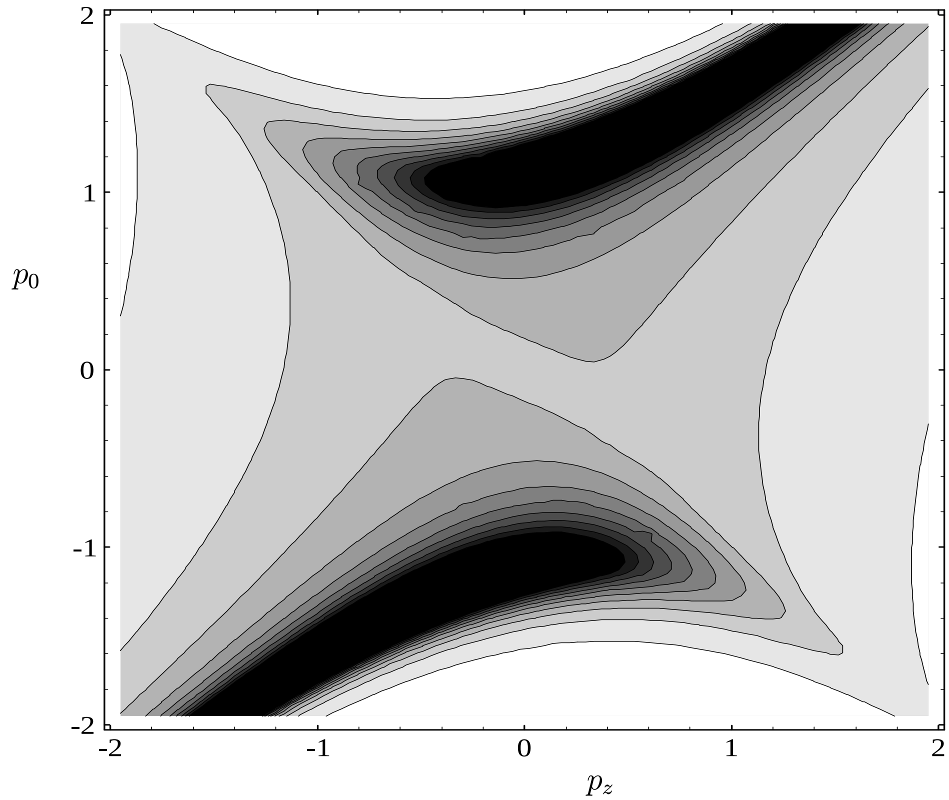

Figure 5: *A density plot of the spectral function $\mathcal{A}^R_{\rm LLL}(p_0, p_z)$ for $qB = \mathcal{M}_e^2$. Even though there is no crossing of the Dirac surface, the spectral density goes continuously between positive and negative energy states when p_z decreases.*

It is the spectral weight for the right-handed positive energy solution in the lowest Landau level. For very large $|p_z|$ there are no collective excitations, such as holes, but only the standard particle solution. With our convention that the B-field points in the positive z-direction, we find that for $p_z > 0$, $\mathcal{A}^R_{\rm LLL}(E, p_z)$ is concentrated at a positive energy particle solution at $E = |p_z|$, while for $p_z < 0$ it is peaked on a negative energy antiparticle solution at $E = -|p_z|$ (see Fig. 5). Thus, we have

$$\lim_{p_z \to -\infty} W(p_z) = 0 \ , \quad \lim_{p_z \to \infty} W(p_z) = 1 \ , \tag{5.27}$$

and all derivatives of $W(p_z)$ vanish for large $|p_z|$. It can then be shown that

$$\langle Q_5(t) \rangle_{\rm LLL} = -\frac{VeB}{2\pi^2} eA_z = \int d^3x \int^t dt' \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \ . \tag{5.28}$$

We note that this agrees with the anomaly in Eq. (5.16) for the particular background field configuration that we have chosen. When it comes to the higher Landau levels it turns out that the two terms in Eq. (5.20) are separately well convergent for large p_z, as opposed to Eq. (5.21), which needs the point splitting in order to be well defined. We can, therefore, change $p_z \to -p_z$ in $D_L(\kappa)$, after which the sum cancels when $\gamma \to 0$.

Scattering with particles in the thermal heat bath is taken into account in the same way as in 1+1 dimensions. The thermal part is UV-convergent and, exactly as in Eq. (5.13), it has no time dependence.

6 Conclusions

The main computational part of this paper is the diagonalization of the fermionic part of the Hard Thermal Loop effective action in the presence of a constant background magnetic field. This makes it possible to write down the explicit expression for the spectral function of fermions and to see how it depends on the magnetic field strength. We find that, starting from weak fields, the spectral weight moves from the standard particle and hole solutions at high temperature over to the vacuum Landau levels for very strong fields.

It has been recognized in the literature that it is difficult to reconcile the standard picture of level crossing as a mechanism for anomalies, with the thermal masses of fermions at high temperature [11]. Using the exact spectral function, I have shown in this paper how the spectral weight can move continuously between chiralities, when a background electric field is switched on, without the dispersion relation ever crossing the Dirac surface. The difference from the vacuum is that the spectral weight on a dispersion curve varies continuously between zero and one in the HTL approximation, while it is always exactly one in vacuum. The conclusion is that the anomaly equation remains valid at high temperature, even after taking interactions into account. It is however possible to obtain different production rates by coupling the system to an external heat reservoir, which means effectively using an explicitly time-dependent density matrix. The chirality production in that case would then depend on the exact experimental setup, and I have not discussed this possibility in any detail.

Even though the main problems formulated in this paper have also been solved here, there are some related issues that still call for solutions. One problem is to extend the analysis of the dispersion relation to non-Abelian gauge bosons, but this turns out to be far more complicated due to self-interaction. Another problem of a certain interest is to see how this anomaly mechanism fits into the language of index theorems, which has shown to be useful for the anomalies at zero temperature.

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A Normalization and causality

The thermal expectation value of the canonical anticommutation relation for the fermionic fields

$$\langle\{\Psi(x), \Psi^\dagger(y)\}\rangle \equiv C(x-y) \tag{A.1}$$

should vanish for space-like $x-y$ and should be equal to a δ-function in $\boldsymbol{x}-\boldsymbol{y}$ when $x_0 = y_0$. These are basic requirements of the fundamental fields which we do not expect to be violated by the HTL approximation. In terms of the spectral function the normalization condition, derived from the equal-time commutator, becomes

$$\int_{-\infty}^{\infty} dE \mathcal{A}(E,\boldsymbol{p}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\Big(S(E-i\epsilon,\boldsymbol{p}) - S(E+i\epsilon,\boldsymbol{p})\Big) = \gamma_0 \quad . \tag{A.2}$$

There are two ways of showing the validity of Eq. (A.2) for the propagator in Eq. (4.5), analytically and numerically. First we present the analytic proof. Consider the contribution from the advanced and the retarded propagators separately. For a given $\boldsymbol{p}$ they are analytic in the lower and upper half-plane, respectively [14], and the integration contour can be deformed to suitable arcs at infinity. The integrals along those arcs do not vanish, but the HTL corrections go to zero. The expression for a free fermion can then be used and it trivially satisfies Eq. (A.2). In this way it is clear that the only important ingredients for the spectral sum rule to be fulfilled is the analyticity away from the real E-axis, and that the correction goes away for large complex E. It is only the γ_0-part of the propagator in Eq. (4.5) that contributes to Eq. (A.2) since the other parts of Eq. (4.5) decay too fast on the arcs at infinity, and they are also antisymmetric in E. It is, therefore, common to write the normalization condition as

$$\int_{-\infty}^{\infty} dE \frac{1}{4} \mathrm{tr}\,[\gamma_0 \mathcal{A}(E, \boldsymbol{p})] = 1 \quad . \tag{A.3}$$

The other way to check Eq. (A.3) is by a direct numerical calculation. The residues at the poles above the light cone ($E > |p|$) have been computed by several authors [5, 15] and it is well known that they do not add up to 1 for $|\boldsymbol{p}| > 0$. It is straightforward to calculate the integral in Eq. (A.3) below the light cone, and it turns out to make up for the missing part, as expected.

The causality condition, i.e. that $C_F(x)$ vanishes for space-like x, has been discussed in the HTL approximation in [16], where

$$C(x) = \int \frac{dE d^3\boldsymbol{p}}{(2\pi)^3} \exp[-i(Et - \boldsymbol{p}\boldsymbol{x})] \mathcal{A}(E, \boldsymbol{p}) \tag{A.4}$$

was calculated numerically for gauge bosons. I shall here give an analytic demonstration that the commutator indeed vanishes outside the light cone, starting with the part of Eq. (A.4) proportional to γ_0, $C_0(x) \equiv \mathrm{tr}\,\frac{1}{4}\gamma_0 C(x)$. Assuming that $\mathrm{tr}\,[\gamma_0 \mathcal{A}(E, \boldsymbol{p})]$ only depends on $p = |\boldsymbol{p}|$, we can perform the angular integral and obtain

$$\begin{aligned} C_0(x) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dE \int_0^{\infty} dp\, p^2 e^{-iEt} \frac{e^{ip|x|} - e^{-ip|x|}}{ip|\boldsymbol{x}|} \mathrm{tr}\,[\frac{1}{4}\gamma_0 \mathcal{A}(E, p)] \\ &= -\frac{1}{4\pi^2 |\boldsymbol{x}|} \frac{d}{d|\boldsymbol{x}|} \int_{-\infty}^{\infty} dE\, dp\, e^{-iEt + ip|x|} \mathrm{tr}\,[\frac{1}{4}\gamma_0 \mathcal{A}(E, p)] \quad . \end{aligned} \tag{A.5}$$

In order to more easily see the analytic structure, we change the variables to u ($= E + p$) and v ($= E - p$):

$$\begin{aligned} C_0(x) &= -\frac{1}{8\pi^2 |\boldsymbol{x}|} \frac{d}{d|\boldsymbol{x}|} \int_{-\infty}^{\infty} du\, dv \exp\left[-\frac{i}{2} u(t - |\boldsymbol{x}|) - \frac{i}{2} v(t + |\boldsymbol{x}|)\right] \\ &\quad \times \frac{1}{2\pi i} \mathrm{tr}\,\left[\frac{1}{4}\gamma_0 \Big(S(u - i\epsilon, v - i\epsilon) - S(u + i\epsilon, v + i\epsilon)\Big)\right] \quad . \end{aligned} \tag{A.6}$$

Let us first see how it works for a free massless scalar, where

$$S(E - i\epsilon, p) = \frac{1}{E^2 - p^2 - iE\epsilon} = \frac{1}{u - \frac{i}{2}\epsilon} \frac{1}{v - \frac{i}{2}\epsilon} \quad . \tag{A.7}$$

The integrals over u and v factorize and we can use

$$\int_{-\infty}^{\infty} du \frac{e^{i\alpha u}}{u - i\epsilon} = 2\pi i \theta(\alpha) \quad , \tag{A.8}$$

to show that[2]

$$\begin{aligned} C_0(x) &= -\frac{i}{4\pi|\boldsymbol{x}|}\frac{d}{d|\boldsymbol{x}|}\left[\theta(-t+|\boldsymbol{x}|)\theta(-t-|\boldsymbol{x}|) - \theta(t-|\boldsymbol{x}|)\theta(t+|\boldsymbol{x}|)\right] \\ &= -\frac{i}{4\pi|\boldsymbol{x}|}[\delta(t-|\boldsymbol{x}|) - \delta(t+|\boldsymbol{x}|)] = -\frac{i}{2\pi}\mathrm{sign}(t)\delta(t^2 - |\boldsymbol{x}|^2) \quad . \end{aligned} \tag{A.9}$$

From this follows also the canonical commutation relation for scalar fields:

$$[\phi(t,\boldsymbol{x}), \partial_t \phi(t,\boldsymbol{y})] = \partial_{y_0}\, C_0(x-y)\big|_{y_0 \to x_0} = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \quad . \tag{A.10}$$

The basic reason why the commutator vanishes outside the light cone is the occurrence of θ-functions coming from Eq. (A.8) and this follows from the property of analyticity in the correct half-plane. Since $S_0(u - i\epsilon, v - i\epsilon)$ is analytic in the lower half-plane for both u and v, and it vanishes fast enough for large arguments, the integration contours can be closed in the lower half-plane for positive $t - |\boldsymbol{x}|$ or $t + |\boldsymbol{x}|$, respectively. We conclude that for any $S(u - i\epsilon, v - i\epsilon)$ with the correct analyticity properties we have, using $\theta(\alpha)\theta(\beta) = \theta(\alpha\beta)\theta(\beta)$,

$$\int_{-\infty}^{\infty} du\, dv \exp\left[-\frac{i}{2}u(t-|\boldsymbol{x}|) - \frac{i}{2}v(t+|\boldsymbol{x}|)\right] S(u - i\epsilon, v - i\epsilon) = \theta(t^2 - |\boldsymbol{x}|^2) F(t, |\boldsymbol{x}|) \quad , \tag{A.11}$$

for some function $F(t, |\boldsymbol{x}|)$, at least for non-zero $t^2 - |\boldsymbol{x}|^2$. There can be other singularities right on the light cone. A similar argument applies to $S_0(u + i\epsilon, v + i\epsilon)$, leading to another factor $\theta(t^2 - |\boldsymbol{x}|^2)$.

Let us return to the HTL propagator and see if it has the correct analyticity properties. It is given by Eq. (4.5), and in terms of u and v we have, for $E \to E - i\epsilon$:

$$\begin{aligned} s(u - i\epsilon, v - i\epsilon) &= 1 - \frac{2\mathcal{M}_e^2}{u^2 - v^2} \ln\left(\frac{u - i\epsilon}{v - i\epsilon}\right) \quad , \\ r(u - i\epsilon, v - i\epsilon) &= 1 + \frac{4\mathcal{M}_e^2}{(u-v)^2}\left(1 - \frac{u+v}{2(u-v)} \ln\left(\frac{u - i\epsilon}{v - i\epsilon}\right)\right) \quad . \end{aligned} \tag{A.12}$$

[2]Note that there is a sign error in the revised version of ref. [16].

The cuts from the logarithm start slightly above the real axis and the branch cuts remain in the upper half-plane. There is a potential singularity at $u = v$, but it can be shown that both $s(u, v)$ and $r(u, v)$ have power series expansion around that point. Thus, none of these functions have any non-analyticity in the lower half-plane. Then we only have to verify that the denominator in Eq. (4.5) does not have any pole in the lower half-plane. This should be done for each of u and v keeping the other one real, and it is not very difficult to do this numerically. With a fine enough grid one can demonstrate that there are no singularities away from the real axis. When deriving Eq. (A.5), we assumed that the spectral function only depended on $|\boldsymbol{p}|$. The HTL spectral function also has a term proportional to $\boldsymbol{\gamma p}$, but it can be rewritten as $i\boldsymbol{\gamma} \cdot \nabla$ acting on a rotationally invariant function, so that it does not really affect the above reasoning.

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