Central Extension of Extended Supergravities in Diverse Dimensions

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Abstract

We generalize central–charge relations and differential identities of $N = 2$ Special Geometry to N extended supergravity in any dimension $4 \leq D < 10$, and p–extended objects.

We study the extremization of the ADM mass per unit of p–volume of BPS extended objects. Runaway solutions for a "dilaton" degree of freedom leading to a vanishing result are interpreted as BPS extremal states having vanishing Bekenstein–Hawking Entropy.

1   introduction

In recent time attempts to study non perturbative properties of gauge theories [1] and string theories [2], [3] have made an essential use of low energy effective lagrangians incorporating the global and local symmetries of the fundamental theories.

In this analysis BPS states play an important role [4, 5], especially in connection with enhancement of gauge symmetries [6, 7, 8] and more generally for phase transitions which may be signaled by some BPS state becoming massless at some point of the underlying moduli space.

Many of these phenomena can be studied, to some extent, by properties of the effective supergravity theories and their central extensions [9]–[11], which are the analogue of the non linear chiral lagrangians for QCD.

The BPS states often appear as solitonic solutions of the supergravity field equations in backgrounds preserving some of the supersymmetries depending on the degree of extremality of the solitonic state (see for instance [12]–[15]).

Recently a lot of information on black holes and black p-branes in diverse dimensions have been obtained using these methods [16]–[26].

For $N = 2$, $N = 4$ and $N = 8$ black holes in $D = 4, 5$ an almost complete analysis of their solitonic configurations has been given [27]–[33] and partial results for $D > 4$ theories, both for extremal and non extremal situations, are available [13]–[21], [23]–[27]. The underlying geometry of the moduli space plays a fundamental role in finding these solutions since the ADM mass or, more generally, the mass per unit of p–volume for p–extended objects depends on the asymptotic value of the moduli and some other physical quantities, such as the classical determination of the Bekenstein - Hawking entropy formula, are also related to properties of the moduli space [28, 29]. For instance, extremal black holes preserving one supersymmetry in $D = 4$ and 5 dimensions have an entropy formula obtained in a rather moduli independent way by minimizing the ADM mass in the moduli space [34].

These results heavily rely on properties which connect space-time supergravity with the underlying moduli space of the theory. For instance, properties of $N = 2$ extremal black holes at $D = 4$ and 5 depend merely on the underlying geometry of the moduli space.

These considerations lead to a simple understanding of no hair theorems and to the possibility of describing the physics of the black hole horizon in terms of an effective potential encoding the thermodynamical properties of the system [35].
In view of several non perturbative dualities between different kinds of theories, a given theory is truly specified by the dimension of space time in which it lives, the number of unbroken supersymmetries and the massless matter content.

The aim of the present work is to further extend these results generalizing some of these considerations to higher N supergravities in diverse dimensions. We study differential identities between different kinds of charges and establish sum rules which generalize the ones previously obtained for $N = 2$ theories. We also study the extremization of the ADM mass of several p–extended BPS states and draw some conclusions about the Bekenstein–Hawking entropy formula.

An expanded version of the present paper, focussed on the relation between central and matter charges in extended supergravities in any dimensions, will appear in a forth-

coming publication. In particular it will be discussed the relation between $N = 2$ Special Geometry and the existence of a flat symplectic connection in all higher N theories at $D = 4$ [36].

2 Differential identities and sum rules for central and matter charges

With the exception of $D = 4$, $N = 1, 2$ and $D = 5$, $N = 2$ all supergravity theories contain scalar fields whose kinetic Lagrangian is described by σ–models of the form G/H. Here G is a non compact group acting as an isometry group on the scalar manifold while H, the isotropy subgroup, is of the form:

$$H = H_{Aut} \otimes H_{matter} \tag{2.1}$$

H_{Aut} being the automorphism group of the supersymmetry algebra while H_{matter} is related to the matter multiplets. (Of course $H_{matter} = \mathbb{1}$ in all cases where supersymmetric matter doesn't exist, namely $N > 4$ in $D = 4, 5$ and in general in all maximally extended supergravities)

The coset manifolds G/H and the automorphism groups for various supergravity theories for any D and N can be found in the literature (see for instance the reference book [37]). As it is well known, the group G acts linearly on the $n = p+2$–forms field strengths $H^{\Lambda}_{a_1 \cdots a_n}$ corresponding to the various $p+1$–forms appearing in the gravitational and matter multiplets. Here and in the following the index Λ runs over the dimensions of some representation of the duality group G.

The true duality symmetry, acting on integral quantized electric and magnetics charges, is the restriction of the continuous group G to the integers [2].

All the properties of the given supergravity theories for fixed D and N are completely fixed in terms of the geometry of G/H namely in terms of the coset representatives L satisfying the relation

$$g L(\phi) = L(\phi') h^{-1}(g, \phi) \tag{2.2}$$

where $g \in G$, $h \in H$ and $\phi' = \phi'(\phi)$, ϕ being the coordinates of G/H. In particular, as explained in the following, the kinetic metric for the $p+2$ forms H^{Λ} is fixed in terms of L and the physical field strengths of the interacting theories are "dressed" with scalar fields in terms of the coset representatives.

This allows us to write down the central charges associated to the $p+1$– forms in the gravitational multiplet in a neat way in terms of the geometrical structures of the moduli space.

In an analogous way also the matter $p+1$–forms of the matter multiplets give rise to charges which, as we will see, are closely related to the central charges. Note that when $p > 1$ these central charges do not appear in the usual supersymmetry algebra, but in the extended version of it containing central generators $Z_{a_1 \cdots a_p}$ associated to p– dimensional extended objects ($a_1 \cdots a_p$ are a set of space–time antisymmetric Lorentz indices) [9, 10, 38, 11]

Our main goal is to write down the explicit form of the dressed charges and to find relations among them analogous to those worked out in $D = 4$, $N = 2$ case by means of the Special Geometry relations [39][6].

To any $p+2$–form H^Λ we may associate a magnetic charge ($D-p-4$–brane) and an electric (p–brane) charge given respectively by:

$$g^\Lambda = \int_{S^{p+2}} H^\Lambda \qquad e_\Lambda = \int_{S^{D-p-2}} \mathcal{G}_\Lambda \qquad (2.3)$$

where $\mathcal{G}_\Lambda = \frac{\partial \mathcal{L}}{\partial H^\Lambda}$.

These charges however are not the physical charges of the interacting theory; the latter ones can be computed by looking at the transformation laws of the fermion fields, where the physical field–strengths appear dressed with the scalar fields.

Let us first introduce the central charges: they are associated to the dressed $p+2$–forms H^Λ appearing in the supersymmetry transformation law of the gravitino 1-form. Quite generally we have, for any D and N:

$$\delta\psi_A = D\epsilon_A + \sum_i c_i L_{\Lambda_i AB}(\phi) H^{\Lambda_i}_{a_1 \cdots a_{n_i}} \Delta^{a a_1 \cdots a_n} \epsilon^B V_a + \cdots \qquad (2.4)$$

where:

$$\Delta_{a a_1 \cdots a_n} = \left(\Gamma_{a a_1 \cdots a_{n_i}} - \frac{n}{n-1}(D-n-1)\delta^a_{[a_1} \Gamma_{a_2 \cdots a_{n_i}]} \right). \qquad (2.5)$$

Here c_i are coefficients fixed by supersymmetry, V^a is the space–time vielbein, $A = 1, \cdots, N$ is the index acted on by the automorphism group, $\Gamma_{a_1 \cdots a_n}$ are γ–matrices in the appropriate dimensions, and the sum runs over all the $p+2$–forms appearing in the gravitational multiplet. Here and in the following the dots denote trilinear fermion terms. $L_{\Lambda AB}$ is given in terms of the coset representative matrix of G. Actually it coincides with a subset of the columns of this matrix except in $D=4$ ($N>1$) and the for maximally extended $D=6,8$ supergravities since in those cases we have the slight complication that the action of G on the $p+2 = D/2$–forms is realized through the embedding of G in $Sp(2n, \mathbb{R})$ or $O(n,n)$ groups.

Excluding for the moment these latter cases, $L_{\Lambda AB}$ is actually a set of columnes of the (inverse) coset representative L of G. Indeed, let us decompose the representative of G/H as follows:

$$L = (L^\Lambda{}_{AB}, L^\Lambda{}_I) \qquad L^{-1} = (L^{AB}{}_\Lambda, L^I{}_\Lambda) \qquad (2.6)$$

where the couple of indices AB transform as a symmetric tensor under H_{Aut} and I is an index in the fundamental representation of H_{matter} which in general is an orthogonal group (in absence of matter multiplets $L \equiv (L^\Lambda{}_{AB})$). Quite generally we have:

$$L_{AB\Lambda} L^{AB}{}_\Sigma - L_{I\Lambda} L^I{}_\Sigma = \mathcal{N}_{\Lambda\Sigma} \qquad (2.7)$$

where $\mathcal{N}$ defines the kinetic matrix of the $(p+2)$–forms H^Λ and the indices of H_{Aut} and H_{matter} (generally given by a pseudoorthogonal group) are raised and lowered with the appropriate metric in the given representation. For maximally extended supergravities $\mathcal{N}_{\Lambda\Sigma} = L_{AB\Lambda} L^{AB}{}_\Sigma$.

When G contains an orthogonal factor $O(m,n)$, what happens for matter coupled supergravities in $D=5,7,8,9$, where $G = O(10-D,n) \times O(1,1)$ and in all the matter coupled $D=6$ theories, the coset representatives of the orthogonal group satisfy:

$$L^t \eta L = \eta \quad \rightarrow \quad L_{A\Lambda} L^A{}_\Sigma + L_{I\Lambda} L^I{}_\Sigma = \eta_{\Lambda\Sigma} \qquad (2.8)$$

$$L^t L = \mathcal{N} \quad \rightarrow \quad L_{A\Lambda} L^A{}_\Sigma - L_{I\Lambda} L^I{}_\Sigma = \mathcal{N}_{\Lambda\Sigma} \qquad (2.9)$$

where $\eta = \begin{pmatrix} \mathbb{1}_{m \times m} & 0 \\ 0 & -\mathbb{1}_{n \times n} \end{pmatrix}$ is the $O(m,n)$ invariant metric and $A = 1, \cdots, m$; $I = 1, \cdots, n$. (In particular, setting the matter to zero, we have in these cases $\mathcal{N}_{\Lambda\Sigma} = \eta_{\Lambda\Sigma}$).

Note that both for matter coupled and maximally extended supergravities we have:

$$L_{\Lambda AB} = \mathcal{N}_{\Lambda\Sigma} L^{\Sigma}{}_{AB} \tag{2.10}$$

From equation (2.4) we see that the dressed graviphoton n_i–forms field strengths are:

$$T^{(i)}_{AB} = L_{\Lambda_i AB}(\phi) H^{\Lambda_i} \tag{2.11}$$

The magnetic central charges for BPS saturated $D - p - 4$–branes can be now defined (modulo numerical factors to be fixed in each theory) by integration of the dressed field strengths as follows:

$$Z^{(i)}_{(m)AB} = \int_{S^{p+2}} L_{\Lambda_i AB}(\phi) H^{\Lambda_i} = L_{\Lambda_i AB}(\phi_0) g^{\Lambda_i} \tag{2.12}$$

where ϕ_0 denote the $v.e.v.$ of the scalar fields, namely $\phi_0 = \phi(\infty)$ in a given background.

The corresponding electric central charges are:

$$Z^{(i)}_{(e)AB} = \int_{S^{D-p-2}} L_{\Lambda_i AB}(\phi) {}^{\star}H^{\Lambda_i} = \int_{S^{D-p-2}} \mathcal{N}_{\Lambda_i \Sigma_i} L^{\Lambda_i}{}_{AB}(\phi) {}^{\star}H^{\Sigma_i} = L^{\Lambda_i}{}_{AB}(\phi_0) e_{\Lambda_i} \tag{2.13}$$

These formulae make it explicit that $L^{\Lambda}{}_{AB}$ and $L_{\Lambda AB}$ are related by electric–magnetic duality via the kinetic matrix.

Note that the same field strengths (the graviphotons) which appear in the gravitino transformation laws are also present in the dilatino transformation laws in the following way:

$$\delta \chi_{ABC} = \cdots + \sum_i b_i L_{\Lambda_i AB}(\phi) H^{\Lambda_i}_{a_1 \cdots a_{n_i}} \Gamma^{a_1 \cdots a_{n_i}} \epsilon_C + \cdots \tag{2.14}$$

In an analogous way, when vector multiplets are present, the matter vector field strengths are dressed with the columns $L_{\Lambda I}$ of the coset element (2.6) and they appear in the transformation laws of the gaugino fields:

$$\delta \lambda^I_A = \Gamma^a P^I_{AB,i} \partial_a \phi^i \epsilon^B + c L_{\Lambda}{}^I(\phi) F^{\Lambda}_{ab} \Gamma^{ab} \epsilon_A + \cdots \tag{2.15}$$

where $P^I_{AB,i}$ is the vielbein of the coset manifold spanned by the scalar fields of the vector multiplets and c is a constant fixed by supersymmetry (in $D = 6$, $N = (2,0)$ and $N = (4,0)$ the 2–form $F^{\Lambda}_{ab} \Gamma^{ab}$ is replaced by the 3–form $F^{\Lambda}_{abc} \Gamma^{abc}$).

In the same way as for central charges, one finds the magnetic matter charges:

$$Z^I_{(m)A} = \int_{S^{p+2}} L_{\Lambda}{}^I F^{\Lambda} = L_{\Lambda}{}^I(\phi_0) g^{\Lambda} \tag{2.16}$$

while the electric matter charges are:

$$Z_{(e)I} = \int_{S^{D-p-2}} L_{\Lambda I}(\phi) {}^{\star}F^{\Lambda} = \int_{S^{D-p-2}} \mathcal{N}_{\Lambda\Sigma} L^{\Lambda}{}_I(\phi) {}^{\star}F^{\Sigma} = L^{\Lambda}{}_I(\phi_0) e_{\Lambda} \tag{2.17}$$

The important fact to note is that the central charges and matter charges satisfy relations and sum rules analogous to those derived in $D = 4$, $N = 2$ using Special Geometry

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techniques [39]. In the general case these sum rules are inherited from the properties of the coset manifolds G/H, namely from the differential and algebraic properties satisfied by the coset representatives L^Λ_Σ.

Indeed, for a general coset manifold we may introduce the left–invariant 1–form $\Omega = L^{-1}dL$ satisfying the relation (see for instance [41]):

$$d\Omega + \Omega \wedge \Omega = 0 \tag{2.18}$$

where

$$\Omega = \omega^i T_i + P^\alpha T_\alpha \tag{2.19}$$

T_i, T_α being the generators of G belonging respectively to the Lie subalgebra $\mathbb{H}$ and to the coset space algebra $\mathbb{K}$ in the decomposition

$$\mathbb{G} = \mathbb{H} + \mathbb{K} \tag{2.20}$$

$\mathbb{G}$ being the Lie algebra of G. Here ω^i is the $\mathbb{H}$ connection and P^α is the vielbein of G/H.

Since in all the cases we will consider G/H is a symmetric space ($[\mathbb{K}, \mathbb{K}] \subset \mathbb{H}$), $\omega^i C_i{}^{\alpha\beta}$ ($C_i{}^{\alpha\beta}$ being the structure constants of G) can be identified with the Riemannian spin connection of G/H.

Suppose now we have a matter coupled theory. Then, using the decomposition (2.20), from (2.18) and (2.19) we get:

$$dL^\Lambda{}_{AB} = L^\Lambda{}_{CD}\omega^{CD}{}_{AB} + L^\Lambda{}_I P^I_{AB} \tag{2.21}$$

where P^I_{AB} is the vielbein on G/H and $\omega^{CD}{}_{AB}$ is the $\mathbb{H}$–connection in the given representation. It follows:

$$D^{(H)}L^\Lambda{}_{AB} = L^\Lambda{}_I P^I_{AB} \tag{2.22}$$

where the derivative is covariant with respect to the $\mathbb{H}$–connection $\omega^{CD}{}_{AB}$. Using the definition of the magnetic dressed charges given in (2.12) we obtain:

$$D^{(H)}Z_{AB} = Z_I P^I_{AB} \tag{2.23}$$

This is a prototype of the formulae one can derive in the various cases for matter coupled supergravities. To illustrate one possible application of this kind of formulae let us suppose that in a given background preserving some number of supersymmetries $Z_I = 0$ as a consequence of $\delta\lambda^I_A = 0$. Then we find:

$$D^{(H)}Z_{AB} = 0 \rightarrow d(Z_{AB}Z^{AB}) = 0 \tag{2.24}$$

that is the square of the central charge reaches an extremum with respect to the *v.e.v.* of the moduli fields.

For the maximally extended supergravities there are no matter field–strengths and the previous differential relations become differential relations between central charges only. Indeed in this case the Maurer–Cartan equations become:

$$dL^\Lambda{}_{AB} = L^\Lambda{}_{CD}\Omega^{CD}{}_{AB} + L^\Lambda{}_{CD}P^{CD}{}_{AB} \tag{2.25}$$

where now AB runs over the same set of values as Λ. Therefore we get:

$$D^{(H)}L^\Lambda{}_{AB} = L^\Lambda{}_{CD}P^{CD}{}_{AB} \tag{2.26}$$

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that is:
$$D^{(H)}Z_{AB} = Z^{CD}P_{CDAB} \qquad (2.27)$$

This relation implies that the vanishing of a subset of central charges forces the vanishing of the covariant derivatives of some other subset. Typically, this happens in some supersymmetry preserving backgrounds where the requirement $\delta\chi_{ABC} = 0$ corresponds to the vanishing of just a subset of central charges.

Finally, from the coset representatives relations (2.7) (2.8) it is immediate to obtain sum rules for the central and matter charges which are the counterpart of those found in $N = 2$, $D = 4$ case using Special Geometry. Indeed, let us suppose e.g. that the group G is $G = O(10 - D, n) \times O(1, 1)$, as it happens in general for all the minimally extended supergravities in $7 \leq D \leq 9$, $D = 6$ type IIA and $D = 5$, $N = 2$. The coset representative is now a tensor product $L \rightarrow e^\sigma L$, where e^σ parametrizes the $O(1, 1)$ factor.

We have, from (2.8)
$$L^t \eta L = \eta \qquad (2.28)$$

where η is the invariant metric of $O(10 - D, n)$ and from (2.7)
$$e^{-2\sigma}(L^t L)_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma}. \qquad (2.29)$$

Using the decomposition (2.20) one finds:
$$Z_{AB}Z^{AB} + Z_I Z^I = g^\Lambda \eta_{\Lambda\Sigma} g^\Sigma e^{-2\sigma} \qquad (2.30)$$

$$Z_{AB}Z^{AB} - Z_I Z^I = g^\Lambda \mathcal{N}_{\Lambda\Sigma} g^\Sigma \qquad (2.31)$$

In more general cases analogous relations of the same kind can be derived.

Let us now see how these considerations modify in the case of extended objects which can be dyonic, i.e. for $p = (D - 4)/2$. Following Gaillard and Zumino [40], for p even (D multiple of 4) the duality group G must have a symplectic embedding in $Sp(2n, \mathbb{R})$; for p odd (D odd multiple of 2), the duality group is always $O(n, m)$ where n,m are respectively the number of self–dual and anti self–dual $p + 2$–forms.

In $D = 4$, $N > 2$ we may decompose the vector field–strengths in self–dual and anti self–dual parts:
$$\mathcal{F}^\mp = \frac{1}{2}(\mathcal{F} \mp i \,{}^\star\mathcal{F}) \qquad (2.32)$$

According to the Gaillard–Zumino construction, G acts on the vector $(\mathcal{F}^{-\Lambda}, \mathcal{G}_\Lambda^-)$ (or its complex conjugate) as a subgroup of $Sp(2n_v, \mathbb{R})$ (n_v is the number of vector fields) with duality transformations interchanging electric and magnetic field–strengths:
$$\mathcal{S}\begin{pmatrix} \mathcal{F}^{-\Lambda} \\ \mathcal{G}_\Lambda^- \end{pmatrix} = \begin{pmatrix} \mathcal{F}^{-\Lambda} \\ \mathcal{G}_\Lambda^- \end{pmatrix}' \qquad (2.33)$$

where:
$$\mathcal{G}_\Lambda^- = \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} \qquad (2.34)$$
$$\mathcal{G}_\Lambda^+ = \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Sigma} \qquad (2.35)$$

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \subset Sp(2n_v, \mathbb{R}) \qquad (2.36)$$

and $\mathcal{N}_{\Lambda\Sigma}$, is the matrix appearing in the kinetic part of the vector Lagrangian:

$$\mathcal{L}_{kin} = i\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}\mathcal{F}^{-\Sigma} + h.c. \tag{2.37}$$

Using a complex basis in the vector space of $Sp(2n_v)$, we may rewrite the symplectic matrix in the following way:

$$\mathcal{S} \to U = \begin{pmatrix} \phi_0 & \overline{\phi}_1 \\ \phi_1 & \overline{\phi}_0 \end{pmatrix} \tag{2.38}$$

where:

$$\phi_0 = \frac{1}{2}(A - iB) + \frac{i}{2}(C - iD) \tag{2.39}$$

$$\phi_1 = \frac{1}{2}(A - iB) - \frac{i}{2}(C - iD) \tag{2.40}$$

Defining:

$$f^\Lambda = -\frac{1}{\sqrt{2}}(\phi_0 + \phi_1) \tag{2.41}$$

$$h_\Lambda = -\frac{1}{\sqrt{2}}(\phi_0 - \phi_1). \tag{2.42}$$

f and h are coset representatives of G embedded in $Sp(2n_v, \mathbb{R})$ and can be constructed in terms of the L's. The kinetic matrix $\mathcal{N}$ turns out to be:

$$\mathcal{N} = hf^{-1} \tag{2.43}$$

and transforms projectively under duality rotations:

$$\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \tag{2.44}$$

The requirement $\mathcal{S} \in Sp(2n_v, \mathbb{R})$ implies:

$$\begin{cases} i(f^\dagger h - h^\dagger f) &= \mathbb{1} \\ (f^\dagger\overline{h} - h^\dagger\overline{f}) &= 0 \end{cases} \tag{2.45}$$

By using (2.43) we find that

$$(f^t)^{-1} = i(\mathcal{N} - \overline{\mathcal{N}})\overline{f} \tag{2.46}$$

As a consequence, in the transformation law of gravitino (2.4) and gaugino (2.15) we have to substitute

$$(L_{\Lambda AB}, L_{\Lambda I}) \to (f_{\Lambda AB}, f_{\Lambda I}) \tag{2.47}$$

In particular, the dressed graviphotons and matter vectors take the symplectic invariant form:

$$T_{AB}^- = f_{AB}^\Lambda(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} = f_{AB}^\Lambda\mathcal{G}_\Lambda^- - h_{\Lambda AB}\mathcal{F}^{-\Lambda} \tag{2.48}$$

$$T_I^- = f_I^\Lambda(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} = f_I^\Lambda\mathcal{G}_\Lambda^- - h_{\Lambda I}\mathcal{F}^{-\Lambda} \tag{2.49}$$

The corresponding central and matter charges become:

$$Z_{AB} = f_{AB}^\Lambda e_\Lambda - h_{\Lambda AB}g^\Lambda \tag{2.50}$$

$$Z_I = f_I^\Lambda e_\Lambda - h_{\Lambda I}g^\Lambda \tag{2.51}$$

We see that the presence of dyons in $D = 4$ is related to the symplectic embedding. Also in this case one can obtain differential relations and a sum rule among the charges. The sum rule has the following form:

$$Z_{AB}\overline{Z}_{AB} + Z_I \overline{Z}_I = -\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P \tag{2.52}$$

where $\mathcal{M}(\mathcal{N})$ and P are:

$$\mathcal{M} = \begin{pmatrix} \mathbb{1} & 0 \\ -Re\mathcal{N} & \mathbb{1} \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & Im\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & -Re\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \tag{2.53}$$

$$P = \begin{pmatrix} g^\Lambda \\ e_\Lambda \end{pmatrix} \tag{2.54}$$

Furthermore the Maurer–Cartan equations (2.22) for the coset representatives of G/H imply analogous Maurer–Cartan equations for the embedding coset representatives (f, h):

$$\nabla^{(\mathbf{H})}(f, h) = (\overline{f}, \overline{h}) P^{(\mathbf{K})} \tag{2.55}$$

The differential relations among central and matter charges and their sum rules can then be found in a way analogous to that shown before for the odd dimensional cases.

For $D = 8$, $N = 2$ the situation is exactly similar to the $D = 4$ case, where the 2–forms field–strengths are to be understood as 4–forms. In the case at hand the 4–form in the gravitational multiplet and its dual are a doublet under the duality group $Sl(2, \mathbb{R})$.

Finally in $D = 6$ the 3–form field strengths H^Λ which appear in the gravitational and/or tensor multiplet have a definite self–duality

$$H^{\pm \Lambda} = \frac{1}{2}(H^\Lambda \pm^\star H^\Lambda) \tag{2.56}$$

In this case the duality group is of the form $G = O(m, n)$. Except for the left–right symmetric cases $N = (4, 4)$ and $N = (2, 2)$, the number of self–dual tensors $H^{+\Lambda_1}$ in the gravitational multiplet, $\Lambda_1 = 1, \cdots m$ and antiself–dual tensors $H^{-\Lambda_2}$ in the matter multiplet, $\Lambda_2 = 1, \cdots n$ are different in general and G acts in its fundamental representation on $(H^{+\Lambda_1}, H^{-\Lambda_2})$ so that no embedding is required.

The procedure to find the charges and their relations is thus completely analogous to the odd dimensional case, that is

$$Z_{AB} = L_{\Lambda AB} g^\Lambda \tag{2.57}$$

However, due to the relation:
$$\mathcal{N}_{\Lambda\Sigma}{}^\star H^\Lambda = \eta_{\Lambda\Sigma} H^\Sigma, \tag{2.58}$$

where η and $\mathcal{N}$ are defined in terms of the coset representatives of $\frac{O(n,m)}{O(m) \times O(n)}$ as in (2.8), (2.7), we have no distinction among electric and magnetic charges. Indeed

$$e_\Lambda = \int \mathcal{N}_{\Lambda\Sigma}{}^\star H^\Lambda = \int \eta_{\Lambda\Sigma} H^\Sigma = \eta_{\Lambda\Sigma} g^\Sigma \tag{2.59}$$

For the maximally extended case, we have an equal number (5) of self–dual and anti self–dual field strengths and therefore a Lagrangian exists. The group $G = O(5, 5)$ rotates

among themselves H^+ and H^- in the representation $\underline{10}$. The analogous of the Gaillard–Zumino construction in this case would define an $O(5,5)$ embedding of $O(5)$ rotating among themselves $H^+, \mathcal{G}^+$ or $H^-, \mathcal{G}^-$ where

$$\mathcal{G}^\pm = \mathcal{N}_\pm H^\pm \tag{2.60}$$

where $\mathcal{N}_- = -(\mathcal{N}_+)^t$ is the kinetic metric of the tensors in the Lagrangian. In this case we obtain a formula analogous to (2.50) which is however invariant under $O(5,5)$ instead than $Usp(n,n)$:

$$Z_{\pm AB} = f_\pm^\Lambda e_\Lambda + h_{\Lambda\pm} g^\Lambda \tag{2.61}$$

3   Considerations on maximally and non maximally extended supergravities

We now consider some properties and differences of maximally extended theories versus non maximal ones. $D = 4, 5$ maximally extended theories ($N = 8$) with solutions preserving one supersymmetry have been studied in ref. [31, 34] and will not be discussed further. The Bekenstein–Hawking entropy is expressed in terms of the quartic and cubic invariant [42] of $E_{7(7)}$ [31] and $E_{6(6)}$ [34] respectively. We give here a simple proof of why this is the case. The matrix L defining the coset representative transforms under $G_L \times G_R$ as $L \to g_L L g_R^{-1}$, The central charge matrix (with its complex conjugate) is a vector $Z = LP$ under G_R where $P = (e, g)$ is a vector under G_L. therefore any G_R–invariant I constructed out of Z is independent of L: $I(Z) = I(LP) = I(P)$.

We now consider maximal supergravities for $D > 5$; for $5 \le D \le 7$ we have only p=0,1–branes (with their duals $p' = 0, 1, 2, 3$) while for $D = 8, 9$ also $p = 2$–branes occur (together with their duals $p' = 2, 3, 4, 5$) Here we report as an example how the coset representatives spell out in $D = 9$. The duality group is, in this case, $Sl(2, \mathbb{R}) \times O(1, 1)$ while the coset manifold is [37]:

$$G/H = \frac{Sl(2, \mathbb{R})}{O(2)} \times O(1, 1) \tag{3.62}$$

and the field content and group assignements are given in Table 1, where A, B, C are $O(2)$ vector indices, $L^\Lambda{}_{AB}$ is the coset representative of $\frac{Sl(2,\mathbb{R})}{O(2)}$ symmetric and traceless in A, B, e^σ parametrizes $O(1,1)$, $\Lambda = 1, 2$ are indices of $Sl(2, \mathbb{R})$ in the defining representation, χ_{ABC} is completely symmetric and can be decomposed as

$$\chi_{ABC} = \overset{\circ}{\chi}_{ABC} + \delta_{(AB}\chi_{C)} \tag{3.63}$$

From the analysis of the fermions transformation laws we get the following magnetic central charges:

$$Z^{(4)} = e^{-\sigma} g \qquad g = \int H^{(4)} \tag{3.64}$$

$$Z^{(3)}_{AB} = e^{-\sigma} L_{\Lambda AB} g^\Lambda \qquad g^\Lambda = \int H^{(3)\Lambda} \tag{3.65}$$

$$Z^{(2)}_{(AB)} = L_{\Lambda AB} m^\Lambda \qquad m^\Lambda = \int F^{(2)\Lambda} \tag{3.66}$$

$$Z^{(2)} = e^{-\sigma} m \qquad m = \int F^{(2)} \tag{3.67}$$

Table 1: *transformation properties of fields in D=9, N=2*

D=9, N=2	V_μ^a	$C_{\mu\nu\rho}$	$B_{\mu\nu}^\Lambda$	A_μ^Λ	A_μ	L^Λ_{AB}	e^σ	ψ_μ^A	χ^{ABC}
$Sl(2,\mathbb{R})$	1	1	2	2	1	2	1	1	1
$O(2)$	1	1	1	1	1	2	1	2	2 + 2
$O(1,1)$	0	1	1	0	1	0	1	0	0

where the superscript in the field–strengths denotes their order as forms.

Using now the Maurer–Cartan equations for the coset representative $e^{-\sigma}L_{\Lambda AB}$ we find:

$$\nabla^{O(2)}(e^{-\sigma}L_{\Lambda AB}) = e^{-\sigma}(L_{\Lambda AC}P_{CB} - d\sigma L_{\Lambda AB}) \qquad (3.68)$$

Therefore:

$$\partial_\sigma \begin{pmatrix} Z^{(4)} \\ Z^{(3)}_{AB} \\ Z^{(2)} \end{pmatrix} = - \begin{pmatrix} Z^{(4)} \\ Z^{(3)}_{AB} \\ Z^{(2)} \end{pmatrix} \qquad (3.69)$$

$$\nabla_i \begin{pmatrix} Z^{(3)}_{AB} \\ Z^{(2)}_{AB} \end{pmatrix} = \begin{pmatrix} Z^{(3)}_{AC} \\ Z^{(2)}_{AC} \end{pmatrix} P_{CB,i} \qquad (3.70)$$

From eqs. (3.69), (3.70) we see that the extremization of the central charges of the singlet 0–brane, the doublet of 1–branes, and the singlet 2–brane occur at zero value of the central charge. This corresponds to a minimum with runaway behaviour of the $O(1,1)$ dilatonic field at $\sigma = \infty$. We can conclude in this case that for this p–extended object there is necessarily a zero of the area–entropy formula. Similar conclusions do not follow immediately for maximally extended theories at $D < 9$ where the duality group does not contain such a $O(1,1)$ factor.

If we consider instead all non maximal theories with 16 supersymmetries (which reduce to $N = 4$ in 4 and 5 dimensions) they all have a $O(1,1)$ factor in the duality group (with the exception of chiral $(4,0)$ theory in $D = 6$, which does not have vectors altogether). The vectors are in the fundamental of the $O(10 - D, n)$ T–duality group and are also charged under the S–duality group $O(1,1)$ which reduces to $\mathbb{Z}_2$ when it is restricted to the integers. Therefore, by the same argument as before, one can prove that the extremal 0–branes have only the runaway solution $\sigma = \infty$ as an extremum. As a consequence all 0–branes in this theories have vanishing area–entropy formula at the extremum.

We incidentally note that the $D = 5$ case is exceptional in this respect due to the fact that in top of the $10 - D + n$ vectors having the same $O(1,1)$ charges there is an extra vector singlet (dual to $B_{\mu\nu}$) with different $O(1,1)$ charge, whose virtue is precisely to stabilize the $O(1,1)$ mode and to give a finite non zero extremized ADM mass.

We can rephrase the previous results in a more group–theoretical setting: since the extremized ADM mass is G invariant, it may only depend on G invariant quantities constructed with charged representations of G. As G contains a $O(1,1)$ factor, an invariant is possible only if the charges carry different $O(1,1)$ quantum numbers. This happens only in $D = 5$, $N = 4$, but not for $D > 5$. Indeed, in $D = 5$, the vectors are in the reducible representation $(-2,1) \oplus (1, 5+n)$ of $O(1,1) \times O(5,n)$, where the $O(1,1)$ quantum number denotes the scaling properties of fields under $\sigma \to \sigma + c$.

This argument can actually be extended to more general cases, where the $O(1,1)$ factor is not present, by using group–theoretical arguments on the G–representatives of the charges. In fact, if the G–representation of the charges does not admit an invariant, either the extremum doesn't exist or the extremized mass can not depend on the charges. Inspection of the representation content shows that this is indeed the case for any p–branes in $D > 6$.

An exception is $D = 6$, for which invariants exist for $p = 1$ BPS states (for $(2,0)$, $(4,0)$ chiral theories and for $(4,4)$ maximal theory). However in the latter case the $p = 0$ BPS states (black–holes) are in the spinor representation of $O(5,5)$ with no quadratic invariant. Hence there are not black–holes with finite entropy at $D = 6$, at least if we assume that they can be obtained by decompactification of $D = 5$ massless black–holes corresponding to the chain $E_6 \to O(1,1) \times O(5,5)$.

Note that in $D = 4$ the invariant was possible because the G group is $Sl(2,\mathbb{Z}) \times O(6,n;\mathbb{Z})$ with invariant $T_{\Lambda\Sigma}T^{\Lambda\Sigma}$ where $T_{\Lambda\Sigma}$ is the $Sl(2,\mathbb{R})$ invariant skew tensor

$$T_{\Lambda\Sigma} = P_\Lambda Q_\Sigma - P_\Sigma Q_\Lambda \qquad (3.71)$$

4   Conclusions

In this note we have generalized previous results on central charges and their properties to generic extended supergravities in D dimensions and to generic BPS states describing extremal p–branes.

We have assumed that the area per unit of p–brane volume is proportional to the BPS mass per unit of p–brane volume under the condition that a solution with one residual supersymmetry can be found [43, 44, 45, 46]. Under this condition we have been able to prove that in all theories with 16 supersymmetries there are no extremal 0–branes with finite horizon area for $D > 5$. Similar results are also obtained for the $D \geq 7$ maximally extended theories (with 32 supersymmetries). If, on the other hand, solutions with finite entropy will be found, this will imply that one of our hypotesis has been evaded. BPS saturated 0–branes and 1–branes and their duals are expected to have finite area–entropy formula for $D = 4, 5, 6$ respectively.

Finally it is worth noticing that our framework can also be applied to the study of phase-transitions corresponding to a vanishing central charge at some point of the moduli space. For BPS saturated p–branes this correspond to tensionless extended objects with infinitely many point–particles becoming light [49, 50, 16, 51].

Acknowledgements

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