Nonlinear Bunch Motion in an Accelerator with Reactive Impedance

E.Shaposhnikova, CERN, Geneva, Switzerland

1 INTRODUCTION

Analysis of bunch behaviour under the influence of space charge or inductive impedance in circular machines, besides numerical simulations, is usually based on two approaches. The first, dealing with potential well distortion, uses stationary solutions. Secondly, stability analysis applies perturbation theory, small deviations from stationary solutions being considered. In general there are no regular methods to describe the bunch motion in a self-consistent way. However, the elliptic distribution function plays a special role in that solutions can sometimes be obtained in analytic form, [1]-[3].

A self-consistent solution of the Vlasov equation for bunch transport with space charge was found for an elliptic distribution in [4]. Numerical integration of the envelope equation derived there is widely used to analyse the behaviour of space charge dominated bunches, [5]-[7].

Below we present closed form analytic solutions defining single particle and bunch motion with time in a machine with reactive impedance, also found for an elliptic distribution function, [8]. With RF off the system of equations parallels that describing the motion of a body under gravitational force with integrals of motion similar to Kepler's laws. This model gives different types of solutions. Defocusing induced voltage makes debunching faster. A focusing voltage slows debunching, but above some critical intensity leads to bunch shape oscillations. These results were used for impedance measurements in the CERN SPS, [9]. With RF on the amplitude and frequency of coherent oscillations are calculated for a bunch far from equilibrium.

2 MAIN EQUATIONS

We consider the motion of an intense bunch, short compared with the RF period, after injection at $t = 0$ into a machine with a reactive impedance ($\mathrm{ImZ}/n = const$). The initial distribution function is chosen to be

$$F = \mathcal{F}_0\,(1 - H_0/H_m)^{1/2}, \quad H_0 < H_m = \dot{\theta}_m^2, \qquad (1)$$

with parabolic line density $\lambda(\theta_0) = \lambda_{p0}\,(1 - \theta_0^2/\theta_m^2)$. Here θ is an azimuthal coordinate measured from the center of the bunch and $\dot{\theta} = d\theta/dt$. ($\theta_m$, $\dot{\theta}_m$) are the maximum values of (θ_0, $\dot{\theta}_0$), values at $t = 0$. We assume that the initial distribution function of the injected bunch is the function of the Hamiltonian H_0 of the injector. For short bunches $H_0 = \dot{\theta}_0^2 + \Omega^2\theta_0^2$, where $\Omega = \dot{\theta}_m/\theta_m = 2\eta/\tau_0(\Delta p_m/p_s)$, $\tau_0 = 2\theta_m/\omega_0$ is the length in seconds of the injected bunch, $\pm\Delta p_m/p_s$ is the initial maximum relative momentum spread in the bunch, $\eta = 1/\gamma_t^2 - 1/\gamma^2$ and ω_0 is the revolution frequency.

The equation governing the particle motion has the form

$$\frac{d^2\theta}{dt^2} + \omega_{s0}^2\theta + \frac{\epsilon\theta_m^2}{2\lambda_{p0}}\frac{\partial\lambda(\theta,t)}{\partial\theta} = 0, \qquad (2)$$

where ω_{s0} is the linear synchrotron frequency in the low intensity case, and intensity effects for the bunch with N particles are represented by the parameter $\epsilon = \mathrm{sgn}(\eta \mathrm{Im}Z)\,\Omega_\epsilon^2 = (6\eta e^2 N)/(\pi\beta c p_s \tau_0^3)\,\mathrm{ImZ}/n$.

We will show below that for the particle with initial coordinates $(\theta_0, \dot{\theta}_0)$ solutions of (2) can be found in the form

$$\theta(t) = \theta_0\, y_1(t) + \dot{\theta}_0\, y_2(t), \qquad (3)$$

$$\dot{\theta}(t) = \theta_0\, \dot{y}_1(t) + \dot{\theta}_0\, \dot{y}_2(t), \qquad (4)$$

where y_1 and y_2 are unknown functions of time with initial conditions: $y_1(0) = 1, y_2(0) = 0, \dot{y}_1(0) = 0, \dot{y}_2(0) = 1$. We suppose for the moment that the Wronskian of this system is constant, then from initial conditions $W = y_1\dot{y}_2 - \dot{y}_1 y_2 = 1$.

To obtain the distribution function at moment t we express $(\theta_0, \dot{\theta}_0)$ as functions of $(\theta, \dot{\theta})$ and time and substitute them into the initial distribution function (1). Integration over $\dot{\theta}$ gives the line density $\lambda(\theta, t)$ and (2) becomes

$$\frac{d^2\theta}{dt^2} + \left[\omega_{s0}^2 - \frac{\epsilon}{r^3(t)}\right]\theta = 0, \qquad (5)$$

where $r = r(t) = (y_1^2 + \Omega^2 y_2^2)^{1/2}$. As one would expect for an inductive impedance $(\mathrm{ImZ} > 0)$ the induced voltage defocuses $(\epsilon > 0)$ above transition and focuses $(\epsilon < 0)$ below transition. The requirement that (5) should be valid for arbitrary values of (θ_0,$\dot{\theta}_0$) gives a system of equations for y_1 and y_2. In new variables (r, ξ) defined by $y_1 = r\cos\xi$ and $y_2 = r\sin\xi/\Omega$, it can be written as:

$$r\ddot{\xi} + 2\dot{r}\dot{\xi} = 0, \qquad (6)$$

$$\ddot{r} - r\dot{\xi}^2 + \omega_{s0}^2 r - \frac{\epsilon}{r^2} = 0. \qquad (7)$$

This system has first integrals of motion:

$$\dot{r}^2 + \frac{C_2^2}{r^2} + \omega_{s0}^2 r^2 + \frac{2\epsilon}{r} = C_1, \qquad (8)$$

$$r^2\dot{\xi} = C_2. \qquad (9)$$

For $\omega_{s0} = 0$ this system of equations is also known to describe the motion of a body in the $(y_1, \Omega y_2)$ plane under the influence of gravitation with an attractive force for $\epsilon < 0$ and repulsive for $\epsilon > 0$. The first expression describes conservation of energy in the system and the second corresponds to the law of areas (second law of Kepler). From initial conditions we get $C_1 = \Omega^2 + \omega_{s0}^2 + 2\epsilon$ and $C_2 = \Omega$. As one can see the Wronskian $W = r^2\dot{\xi}/\Omega = 1$ satisfies our initial assumption.

The functions $y_1(t)$ and $y_2(t)$ also define a phase space distribution which is a time dependent solution of the non-linear Vlasov equation. Note that equation (7) (with $\dot{\xi}$ replaced by Ω/r^2) was obtained as an envelope equation in [4]. However its further analysis, to the best of our knowledge, was restricted to applying perturbation theory in the stationary case or to numerical integration.

The integrals of motion found above allow us to describe bunch motion by analysis of the equation

$$\frac{\dot{r}^2}{2} + U(r) = 0, \tag{10}$$

which can be considered as the equation of motion of some particle with the coordinate r in the potential

$$U(r) = \frac{\Omega^2(1-r)[1+ar-sr^2(1+r)]}{2r^2}, \tag{11}$$

where $a = 1 + 2\epsilon/\Omega^2$ and $s = \omega_{s0}^2/\Omega^2$. Here $r(t)$ is a positive defined function with the initial condition $r(0) = 1$, giving the variation with time of bunch length $\tau(t) = \tau_0 r(t)$ or of peak line density $\lambda_p(t) = \lambda_{p0}/r(t)$.

Solutions of equations (8)-(9) can be written as

$$\Omega t = \pm \int_1^r \frac{rdr}{\sqrt{\rho(r)}}, \quad \xi = \pm \int_1^r \frac{dr}{r\sqrt{\rho(r)}}, \tag{12}$$

where $\rho(r) = (r-1)[1+ar-sr^2(1+r)]$.

Bunch shape variation is fully presented by the function $r(t)$ while for single particle motion described by expressions (3) and (4) we need to know the phase $\xi(t)$ as well.

3 ANALYSIS OF SOLUTIONS

The character of the solutions depends on the relative values of Ω, ω_{s0} and Ω_ϵ. In Figs.1,2 we show some examples of potential $U(r)$ and the solutions for different situations analysed below.

Let us start first with the **RF off** (s=0) case. If the function $r(t)$ is known, the phase ξ can be found from expression

$$r[(\Omega^2+\epsilon)\cos\xi - \epsilon] = \Omega^2. \tag{13}$$

$\boldsymbol{a = 1}$. For low intensity ($\epsilon = 0$) the solution is

$$r(t) = (1+\Omega^2 t^2)^{1/2}. \tag{14}$$

When $\epsilon \neq 0$, there are two main types of solutions which correspond to infinite and finite (periodic) motion.

$\boldsymbol{a > 0}$. Motion in this case is only infinite, which means continuous debunching ($r \to \infty$ with $t \to \infty$):

$$\Omega t = \frac{\sqrt{\rho(r)}}{a} + \frac{a-1}{2a^{3/2}} \ln \frac{|2\sqrt{a\rho(r)} + 2ar + 1 - a|}{1+a}.$$

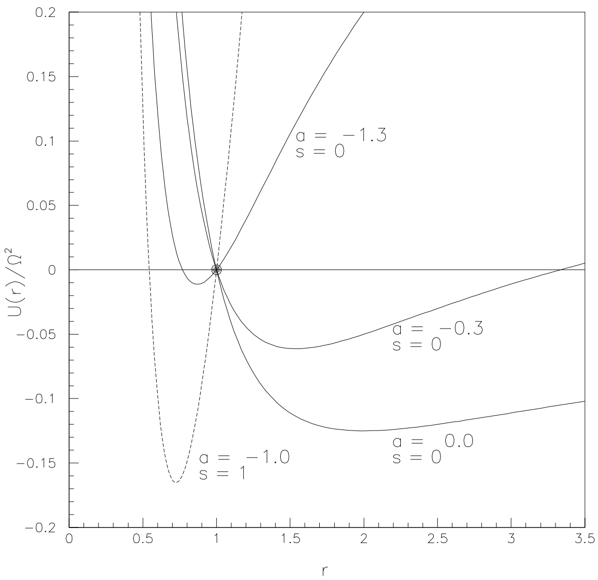

Figure 1: Effective potential $U(r)$ for different types of induced voltage with RF off ($s = 0$) and on ($s = 1$).

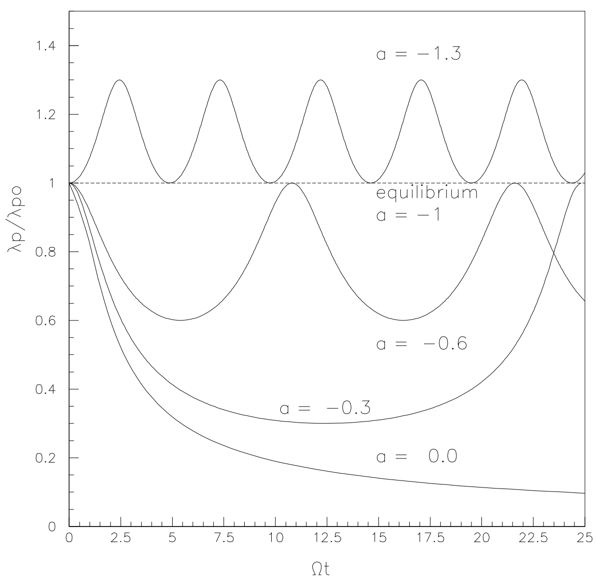

Figure 2: Variation of normalised peak line density $1/r$ for different types of induced voltage with RF off ($s = 0$).

$\boldsymbol{a = 0}$. This is a point of bifurcation where the character of the solution is changing. From (12) we have

$$\Omega t = 2(r-1)^{1/2}(r+2)/3. \tag{15}$$

$\boldsymbol{a < 0}$. The potential $U(r)$ has the form of a potential well and the solutions describe bunch shape oscillations with time. This is possible only for a focusing type of induced voltage ($\eta \mathrm{Im Z} < 0$) when $\Omega_\epsilon^2 > \Omega^2/2$.

For $-\mathbf{1} < \boldsymbol{a} < \mathbf{0}$ the solution has the form

$$\Omega t = \frac{\sqrt{\rho(r)}}{a} + \frac{1-a}{2a|a|^{1/2}}\left[\arcsin\frac{2ar+1-a}{|1+a|} - \frac{\pi}{2}\right]. \quad (16)$$

Oscillations begin with the bunch length increasing, so that $1 \leq r \leq 1/|a|$.

The period of the bunch shape oscillations is

$$T = \frac{\pi}{\Omega}\frac{(1-a)}{|a|^{3/2}} = \frac{2\pi\Omega_\epsilon^2}{(2\Omega_\epsilon^2-\Omega^2)^{3/2}}. \quad (17)$$

A bunch with an intensity such that $a = 0$ ($\Omega_\epsilon^2/\Omega^2 = 1/2$) has an infinitely large oscillation period and continuously debunches. The period and amplitude of the oscillations decrease with growing $|a|$. At $a = -1$, period $T = (2\pi)/\Omega$ but the oscillation amplitude is zero.

$\boldsymbol{a}= -\mathbf{1}$, ($\Omega_\epsilon^2 = \Omega^2$). This is the equilibrium situation when the initial bunch is matched to the induced voltage and is in the minimum of the potential well $U(r)$ with solution $r = 1$, not changing with time.

$\boldsymbol{a} < -\mathbf{1}$. This is the high intensity case with $\Omega_\epsilon^2 > \Omega^2$. Oscillations now start with the bunch initially shortening ($1/|a| \leq r \leq 1$). The solution has a form similar to (16) with a period defined by (17).

Now let us consider the case with **RF on.**

Without intensity effects ($\epsilon = 0$) the solution

$$r(t) = \left[\frac{s+1}{2s} + \frac{s-1}{2s}\cos(2\omega_{s0}t)\right]^{1/2} \quad (18)$$

describes pure quadrupole oscillations of the mismatched bunch ($s \neq 1$) with frequency $2\omega_{s0}$ and $r(t) = \tau/\tau_0$ varing between $r_0 = 1$ and $r_a = 1/\sqrt{s}$. This "low intensity" solution (the same as (14) in the RF off case) is independent of the initial distribution, if only it is a function of the Hamiltonian H_0 of linear synchrotron motion.

For the high intensity the expressions (12) can be written in an analytic form using elliptic integrals of the first and third kind, [8]. The solutions $r(t)$ have a shape similar to the low intensity case (18), however the amplitude and frequency of the oscillations depend on bunch intensity.

The positive roots of the equation $U(r) = 0$ are fixed points which define maximum and minimum values of the bunch length during the oscillation. One of the roots is $r = 1$. Another solution, r_a, can be presented in the implicit form

$$\frac{\epsilon}{\Omega^2} = \frac{(1+r_a)(sr_a^2-1)}{2r_a}. \quad (19)$$

Depending on the value of RF voltage and intensity, r_a can be less or more than the initial value 1, which means that the bunch shortens or lengthens during the oscillation.

The equilibrium state of the bunch is defined by the minimum of the potential well: $U'(r_{eq}) = 0$. The characteristics of this particular point in the solution are well known, [1]-[3]. The value r_{eq} can be obtained from equation

$$\frac{\epsilon}{\Omega^2} = \frac{sr_{eq}^4 - 1}{r_{eq}}. \quad (20)$$

The injected bunch is "matched" to the external plus induced voltage when $\Omega^2 = \omega_{s0}^2 - \epsilon$. Otherwise it will perfom oscillations around r_{eq} with amplitude r_a defined by (19).

The analytic form for the period of the bunch shape oscillations contains complete elliptic integrals of the second and third kind. The calculated coherent frequency $\omega_c = 2\pi/T_c$ as a function of intensity is shown in Fig.3. For the focusing type of induced voltage ($\epsilon < 0$) ω_c changes very rapidly, while for $\epsilon > 0$ it has a flat minimum. For $s = 1$ the frequency shift doesn't exceed 10% of the low intensity value with $\omega_c = 1.86\omega_{s0}$ at $\Omega_\epsilon = \omega_{s0}$ (when induced voltage equals external voltage). This is close to the value $\omega_c = 1.84\omega_{s0}$ measured in [7] and considered there as "a mystery".

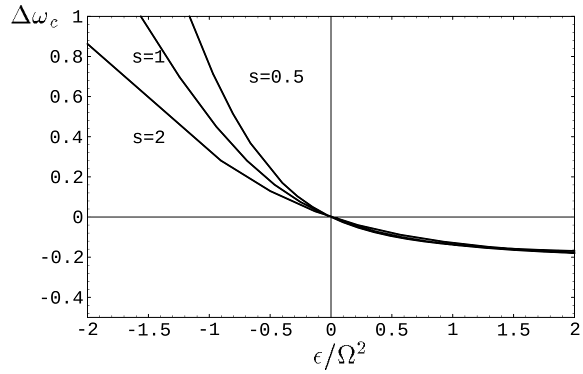

Figure 3: Coherent frequency of bunch shape oscillalations $\Delta\omega_c = (\omega_c - 2\omega_{s0})/\omega_{s0}$ as a function of intensity parameter ϵ/Ω^2 for different values of RF voltage (s).

In our model particles have no synchrotron frequency spread. From equation (5) the so called "incoherent" synchrotron frequency is $\omega_s^2 = \omega_{s0}^2 - \epsilon/r^3(t)$. This incoherent frequency is modulated by the coherent oscillation frequency ω_c with depth $\epsilon(1 - 1/r_a^3)$.

4 ACKNOWLEDGEMENTS

The author would like to thank T.Linnecar for many useful discussions of this work and D.Boussard, A.Hofmann and B.Zotter for reading the manuscript and valuable comments.

5 REFERENCES

[1] A.N.Lebedev and E.A.Zhilkov, NIM 45, 238 (1966).

[2] S.Hansen et al., IEEE Trans. Nucl. Sci., Vol.NS-22, No.3, 1381 (1975).

[3] A.Hofmann, F.Pedersen, IEEE Trans. Nucl. Sci., Vol. NS-26, No.3, 3526 (1979).

[4] D.Neuffer, PA, V.11, p.23 (1980).

[5] G.Kalisch et al., Proc. 1992 EPAC, Berlin, p.780.

[6] D.X.Wang et al., Proc. 1993 PAC, Washington, p.3627.

[7] T.J.P.Ellison et al., Proc. 1993 PAC, Washington, p.3536.

[8] E.Shaposhnikova, CERN-SL-95-78, 95-121 (RF) (1995).

[9] T.Linnecar, E.Shaposhnikova, SL-Note 95-82 (RF), (1995) and these proceedings.