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HIGHER-ORDER GRAVITATIONAL COUPLINGS AND MODULAR FORMS IN $N = 2, D = 4$ HETEROTIC STRING COMPACTIFICATIONS

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ABSTRACT

The restrictions of target–space duality are imposed at the perturbative level on the holomorphic Wilsonian couplings that encode certain higher-order gravitational interactions in $N = 2, D = 4$ heterotic string compactifications. A crucial role is played by non-holomorphic corrections. The requirement of symplectic covariance and an associated symplectic anomaly equation play an important role in determining their form. For models which also admit a type-II description, this equation coincides with the holomorphic anomaly equation for type-II compactifications in the limit that a specific Kähler-class modulus grows large. We explicitly evaluate some of the higher-order couplings for a toroidal compactification with two moduli T and U, and we express them in terms of modular forms.

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1 Introduction

Recently, substantial progress has been achieved in establishing various types of strong–weak coupling duality symmetries in superstring theories, such as S-duality of the four-dimensional $N = 4$ heterotic string [1, 2, 3] and string–string dualities between the heterotic and type-II strings [4, 5, 6]. In fact, it now seems that many of the known superstring theories are related to each other either by non-perturbative strong–weak coupling duality and/or by perturbative target–space duality. It seems that all these different string dualities can be understood in a unified manner in the new framework of either M-theory [5, 7] or F-theory [8].

One interesting case of string–string duality is the duality between heterotic and type-II strings with $N = 2$ supersymmetry in four space–time dimensions. Such theories exhibit a non-perturbative structure which, in the point–particle limit, contain [9] the non-perturbative effects of rigid $N = 2$ supersymmetric field theories studied in [10]. A large class of $N = 2$ heterotic vacua in four dimensions can be obtained by the compactification of the ten-dimensional heterotic string on $K_3 \times T_2$. The corresponding $N = 2$ type-II vacua are constructed by compactifying the ten-dimensional type-II string on suitably chosen Calabi–Yau three-folds. This string–string duality has been tested in several models with a small number of vector supermultiplets, and it has successfully passed various non-trivial explicit checks [6],[11]–[16]. Most of these tests were based on the comparison of lower-order gauge and gravitational couplings [17, 18, 19] of the perturbative heterotic string with the corresponding couplings of the dual type-II string in some corner of the Calabi–Yau moduli space. That is, it was shown that the perturbative heterotic prepotential $\mathcal{F}^{(0)}$ and the function $\mathcal{F}^{(1)}$ (which specifies the non-minimal gravitational interactions involving the square of the Riemann tensor) agree with the corresponding type-II functions in the limit where one specific Kähler-class modulus of the underlying Calabi–Yau space becomes large. A set of interesting relations between certain topological Calabi–Yau data (such as intersection numbers, rational and elliptic instanton numbers) and various modular forms has emerged when performing these tests [20, 16].

In order to show that a given $N = 2$ string model has two dual descriptions, both a heterotic and a type-II description, it is important to also verify whether string–string duality holds for higher-order (nonminimal) gravitational couplings. A particularly interesting subset of such couplings is based on chiral $N = 2$ densities which involve the square of the Riemann tensor multiplied by powers of the graviphoton field strength. These invariants take the form $\mathcal{F}^{(g)}\, R^2\, T^{2(g-1)}$, where T is related to the $N = 2$ gravipho-

ton field strength in a way we will specify later, and $\mathcal{F}^{(g)}$ is a function which depends on the vector moduli. For type-II string compactifications, it was shown in [21] that these higher-order couplings satisfy a holomorphic anomaly equation derived for the topological genus-g partition functions of twisted Calabi–Yau sigma models [22]. If a given type-II model is to have a dual heterotic description, then the heterotic higher-order gravitational couplings should satisfy similar holomorphic anomaly equations. For the case of a particular model with gauge group $U(1)^3$, the so called S-T model, it was shown in [23] that some of the heterotic higher-order couplings indeed satisfy the anomaly equations of [22], at least at the perturbative level.

On the other hand, target–space duality symmetry is a manifest symmetry at weak string coupling in heterotic string compactifications. Hence, this symmetry should be encoded in the perturbative heterotic prepotential as well as in all the gravitational couplings $\mathcal{F}^{(g)}$. These target–space duality transformations constitute a subgroup of the $N = 2$ symplectic reparametrizations. However, the (holomorphic) Wilsonian couplings $\mathcal{F}^{(g)}$ do not correspond directly to physical quantities and therefore are not themselves invariant under target–space duality transformations. In fact, it turns out that the holomorphic couplings do not transform as functions (or rather, sections) under symplectic reparametrizations and non-holomorphic terms are necessary in order to obtain quantities that do transform in a covariant form [24]. For these quantities one can then easily formulate the requirement of target–space duality invariance, which, as it turns out, can be translated into certain complicated restrictions on the original holomorphic functions. A particular proposal for the minimal non-holomorphic corrections required for symplectic covariance, was presented in the second paper of [24], where it should be noted that this construction does not exclude the possibility of additional non-holomorphic terms as long as they constitute an independent symplectic function. The result turns out to satisfy a certain holomorphic anomaly equation, which henceforth will be referred to as the 'symplectic' anomaly equation in order to distinguish it from the anomaly equation of [22].

As alluded to above, holomorphic anomaly equations can be derived in the context of topological field theories. They can also be understood in a space–time context as resulting from the propagation of massless modes. For those heterotic $N = 2$ models admitting a type-II description, we can make use of string–string duality and consider the anomaly equation of [22] in the limit where one of the type-II Kähler-class moduli is taken to be large so as to make contact with the perturbative heterotic description. Interestingly, in this limit the anomaly equation of [22] coincides with the symplectic anomaly equation of [24]. We further demonstrate that, in the heterotic weak-coupling limit, this anomaly

equation is consistent with target–space duality transformations. In doing so, one has to take into account that, at the one-loop level, the dilaton field is no longer invariant under target–space duality transformations and neither is the so-called Green–Schwarz term (describing the mixing of the dilaton field with the moduli), which also appears in the anomaly equation. We show that these two effects compensate each other, and by reformulating everything in terms of the invariant dilaton field and the invariant Green–Schwarz term [17], the results become manifestly covariant under target–space duality transformations.

In order to elucidate the above observations, we will consider the so-called S-T-U model in detail. This is a heterotic rank-4 model, that is, a model with gauge group $U(1)^4$, which is believed to have a dual type-II interpretation [6]. We will solve the relevant anomaly equation in the heterotic weak-coupling limit for the higher-order gravitational couplings $\mathcal{F}^{(2)\,\mathrm{cov}}$ and $\mathcal{F}^{(3)\,\mathrm{cov}}$. We will show that the results for the covariant non-holomorphic couplings $\mathcal{F}^{(2)\,\mathrm{cov}}$ and $\mathcal{F}^{(3)\,\mathrm{cov}}$ can be cast in a form that is explicitly covariant under target–space duality transformations by expressing them in terms of modular forms. In general, when solving for the non-holomorphic couplings $\mathcal{F}^{(g)\,\mathrm{cov}}$, one encounters holomorphic ambiguities [22], which cannot be fixed unless further inputs are provided. Interestingly, for the S-T-U model, $\mathcal{F}^{(2)\,\mathrm{cov}}$ is free from these holomorphic ambiguities, because target–space duality covariance and the knowledge of the leading holomorphic singularities, that are associated with known gauge–symmetry enhancement points/lines, fixes its structure completely. An unambiguous determination of $\mathcal{F}^{(2)\,\mathrm{cov}}$ provides one with information of genus-2 instanton numbers for the corresponding dual Calabi–Yau model $WP_{1,1,2,8,12}(24)$. This is an example of the utility of the second-quantized mirror map [25]. If the higher $\mathcal{F}^{(g)\,\mathrm{cov}}$ could also be determined unambiguously, then one would in principle be able to obtain information about the higher-genus instanton numbers on the dual type-II side, about which not much is known.

This paper is organized as follows. In section 2 we will discuss the projective transformation properties of the higher-order couplings $\mathcal{F}^{(g)}$. In section 3 we discuss their behaviour under symplectic reparametrizations and extend the discussion of the symplectic anomaly equation given in [24]. We will also provide the explicit solutions to the symplectic anomaly equation for a specific class of $N = 2$ effective field theories based on a cubic prepotential. In section 4 we test string–string duality by showing that the symplectic and the holomorphic anomaly equations agree with each other in the heterotic weak-coupling limit. We then solve the holomorphic anomaly equations for the heterotic higher-order couplings $\mathcal{F}^{(2)}$ and $\mathcal{F}^{(3)}$ in the S-T-U model. When solving the anomaly equation for $\mathcal{F}^{(2)}$, we will properly take into account the Green–Schwarz term which

arises as a dilaton–moduli mixing term in the one-loop prepotential. We present our conclusions in section 5. We refer to the various appendices for additional information on some of the more technical aspects of our calculations.

2 The holomorphic sections $\mathcal{F}^{(g)}(z)$

Consider an $N = 2$ supersymmetric effective field theory based on vector supermultiplets with generic couplings to supergravity, both of the 'minimal' and the 'nonminimal' type. The latter incorporate interactions of vector multiplets with the square of the Riemann tensor and are based on the Weyl multiplet: an $N = 2$ (reduced) chiral superfield W_{ab}^{ij}, which comprises the covariant quantities of the conformal supergravity sector.[2] This superfield is antisymmetric in both types of indices and anti-selfdual as a Lorentz tensor. Besides the graviton and gravitino field strenghts, the covariant quantities consist of the field strengths of the gauge fields associated with the chiral $SU(2) \times U(1)$ automorphism group of the $N = 2$ supersymmetry algebra, and an auxiliary spinor, scalar and tensor field. The anti-selfdual part of the latter, denoted by T_{ab}^{ij}, equals the lowest-θ component of W_{ab}^{ij}. The Riemann tensor resides at the θ^2-level, modified by terms proportional to $T_{ab\,ij}\,T_{cd}^{ij}$, as well as the $SU(2) \times U(1)$ field strengths. The highest-θ term contains second derivatives of $T_{ab\,ij}$ [26].

It is instructive to compare the Weyl multiplet to the vector supermultiplet. The covariant quantities of the latter constitute a reduced (scalar) chiral field (i.e., the superfield strength), whose lowest-θ component is the complex scalar of the vector multiplet.[3] We denote these scalars by X^I, where the index I labels the various vector multiplets ($I = 0, 1, \ldots, n$). Supergravity couplings of these multiplets depend sensitively on their assignment under dilatations and $U(1)$ transformations. As it turns out T_{ab}^{ij} and the scalar fields X^I share the same dilatational and $U(1)$ weights, equal to $+1$ and -1, respectively. Obviously, their complex conjugates, the selfdual tensor $T_{ab\,ij}$ and the scalars $\bar{X}^I$, carry opposite $U(1)$ weights.

From the Weyl multiplet one constructs a scalar chiral multiplet of weight 2, by taking its square $W^2 \equiv (W_{ab}^{ij}\varepsilon_{ij})^2$. Owing to its tensorial structure no other independent products of W can appear in a chiral scalar density. The general 'nonminimal' coupling of vector multiplets and supergravity is now encoded in a holomorphic function of the X^I and W^2. A consistent coupling to supergravity requires this function to be homogeneous of second

[2]The reduction of the Weyl multiplet is implemented by the following (linearized) constraint, Im $\left(\bar{D}_i \sigma^{ab} D_j\, W_{ab}^{ij}\right) = 0$, where i, j are chiral $SU(2)$ indices and a, b denote Lorentz indices [26].

[3]For the superfield strength Φ the reduction is effected by the superspace constraint Im $\left(\bar{D}_i D_j\, \Phi\right) = 0$.

degree [27],

$$F(\lambda X, \lambda^2 W^2) = \lambda^2 F(X, W^2)\,. \tag{2.1}$$

One may expand F in powers of W^2 and write

$$F(X, W^2) = \sum_{g=0}^{\infty} F^{(g)}(X)\,[W^2]^g\,. \tag{2.2}$$

The coefficient functions $F^{(g)}$ are holomorphic homogeneous functions of the X^I of degree $2-2g$. In this paper we intend to study the implication of target space duality invariance for these quantities in certain string compactifications.

The function $F^{(0)}$, which is thus homogeneous of second degree, determines the self-interactions of the vector supermultiplets with 'minimal' coupling to supergravity. Henceforth we drop the superscript (0) and simply write $F(X)$; to avoid confusion we will be careful and indicate the arguments X and W^2 whenever referring to the full function $F(X, W^2)$. Initially the action takes a form that is invariant under local dilatations. As a result of this, the coefficient of the Ricci scalar contains a field-dependent factor proportional to $i(\bar{X}^I\,F_I - X^I\,\bar{F}_I)$.[4] Without loss of generality we can apply a local dilatation to set this coefficient equal to the Planck mass such as to obtain the Einstein–Hilbert Lagrangian. Hereby the scalar fields will be restricted to an n-dimensional complex hypersurface. It is then convenient to parametrize the scalars in terms of holomorphic sections $X^I(z)$ depending on n complex coordinates z^A, which describe the hypersurface projectively. In terms of these sections the X^I read

$$X^I = m_{\rm Planck}\, e^{\frac{1}{2}K(z,\bar{z})}\, X^I(z)\,. \tag{2.3}$$

In order to distinguish the sections $X^I(z)$ from the original quantities X^I, we will always explicitly indicate their z-dependence. The overall factor $\exp[\frac{1}{2}K]$ is chosen such that $i(\bar{X}^I\,F_I - X^I\,\bar{F}_I) = m^2_{\rm Planck}$. With this requirement $K(z,\bar{z})$ equals

$$K(z,\bar{z}) = -\log\left(i\bar{X}^I(\bar{z})\,F_I(X(z)) - iX^I(z)\,\bar{F}_I(\bar{X}(\bar{z}))\right), \tag{2.4}$$

and coincides with the Kähler potential associated with the target–space metric for the complex fields z^A.

As mentioned above, the sections are defined projectively, i.e., modulo multiplication by an arbitrary holomorphic function,

$$X^I(z) \longrightarrow e^{f(z)}\,X^I(z)\,. \tag{2.5}$$

[4] We use the standard notation where $F_{IJ\ldots}$ denote multiple derivatives of $F(X)$ with respect to X.

These projective transformations induce corresponding Kähler transformations on the Kähler potential,

$$K(z,\bar{z}) \longrightarrow K(z,\bar{z}) - f(z) - \bar{f}(\bar{z})\,. \tag{2.6}$$

On the original quantities X^I the transformation (2.5) induces a phase transformation. This $U(1)$ transformation acts on all quantities that carry nonzero chiral weight. Obviously, the consistency of the above formulation depends on the presence of the aforementioned local $U(1)$.

Eventually one has to fix the parametrization of the holomorphic sections (i.e., impose a gauge), after which the freedom to perform the transformations (2.5) disappears. A convenient way to do this, is by choosing so-called *special* coordinates corresponding to $X^0(z) = 1$ and $X^A(z) = z^A$. In that case we have $|X^0|^2 = m^2_{\rm Planck} \exp\left[K(z,\bar{z})\right]$. In the context of a specific holomorphic parametrization, certain transformations of the holomorphic sections will be accompanied by corresponding projective transformations in order to ensure that one remains within the chosen gauge.

Without the dependence on W^2 the $U(1)$ gauge field can be integrated out straigthforwardly, which leaves the local invariance intact. With the interactions to W^2 the integration over auxiliary fields is more subtle and can be done iteratively order–by–order in the inverse Planck mass. To preserve supersymmetry, elimination of the auxiliary fields should be postponed until the end. For future reference we give the value of the auxiliary field T^{ij}_{ab},

$$T^{ij}_{ab} = -\frac{2i\varepsilon^{ij}}{m^2_{\rm Planck}} \left[X^I \bar{\mathcal{N}}_{IJ} F^{-J}_{ab} - F_I F^{-I}_{ab}\right] + \cdots\,. \tag{2.7}$$

Here $\mathcal{N}$ is the matrix that appears in the kinetic terms for the gauge fields, which satisfies $\mathcal{N}_{IJ}X^J = F_I$. Note that the first term is of order $1/m_{\rm Planck}$ by virtue of (2.3); the ellipses denote fermionic terms and terms that are suppressed by additional negative powers of $m_{\rm Planck}$; the latter arise as a result of the nonminimal supergravity interactions. Because T is a superconformal background field, the expression (2.7) should be insensitive to symplectic transformations of the vector supermultiplets, which we will discuss in due course. This is indeed the case, because both (X^I, F_J) and $(F^{-I}_{ab}, \bar{\mathcal{N}}_{JK}F^{-K}_{ab})$ transform as symplectic vectors. While F^{-I}_{ab} corresponds to the (generalized) electric and magnetic induction fields, $\bar{\mathcal{N}}_{IJ}F^{-J}_{ab}$ describes the (generalized) electric displacement and magnetic fields.

Under supersymmetry the gravitino fields do not transform directly into the vector fields, but into the auxiliary field T. Therefore, the field-dependent linear combination of the fields strengths given in (2.7) defines the graviphoton field strength.

Let us now return to the case where interactions with $[W^2]^g$ are taken into account. For

$g > 1$ the presence of the vector multiplets is crucial in order to compensate for the lack of conformal invariance of $[W^2]^g$. The Lagrangian based on the chiral superspace integral of just W^2 is superconformally invariant; its full nonlinear component form can be found in [26]. The real part leads to the supersymmetric extension of the square of the Weyl tensor and should be regarded as the action of conformal supergravity. Its imaginary part is a total divergence whose space–time integral leads to a topological quantity, the Hirzebruch signature. The action contains the standard gauge–invariant kinetic terms for the $SU(2) \times U(1)$ gauge fields and a kinetic term for the tensor field $T^{ij}_{\mu\nu}$. There is only one (scalar) field that appears in this action as an auxiliary field without derivatives.

With the interactions to the vector multiplets we introduce further modifications. The square of the Weyl tensor now acquires modifications by vector multiplet scalars and the tensor field of the form $F^{(g)}(X)\,(T^{ij}_{cd}\varepsilon_{ij})^{2(g-1)}\,R^2$, but there will also be modifications of the kinetic terms for the vector fields proportional to $(T^{ij}_{cd}\varepsilon_{ij})^{2g}\,F^{(g)}_{IJ}(X)\,F^{-I}_{ab}F^{-J}_{ab}$.

When substituting (2.3) into the various coefficient functions, the $F^{(g)}$ are replaced by sections according to

$$\mathcal{F}^{(g)}(z) = i\,[m^2_{\rm Planck}]^{g-1}\,e^{-(1-g)K(z,\bar{z})}\,F^{(g)}(X)\,. \tag{2.8}$$

With these definitions the Einstein–Hilbert action and the nonlinear sigma model action of the Kähler manifold acquire an explicit factor $m^2_{\rm Planck}$. The nonminimal interactions that involve the square of the Riemann tensor are multiplied by the holomorphic sections $\mathcal{F}^{(g)}$ times an explicit factor $(m^2_{\rm Planck})^{1-g}$.

Finally we note that under projective transformations these sections transform as

$$\mathcal{F}^{(g)}(z) \longrightarrow e^{2(1-g)f(z)}\,\mathcal{F}^{(g)}(z)\,. \tag{2.9}$$

3 Symplectic transformations of the $\mathcal{F}^{(g)}(z)$ and holomorphic anomalies

One of the aims of this paper is to investigate whether we can constrain the quantities $\mathcal{F}^{(g)}$ for certain string compactifications by imposing the requirement of target–space duality. This approach turned out to be successful for the minimal supergravity interactions encoded in the W^2-independent part of $F(X, W^2)$, as was described in [17, 18]. Here it is important to appreciate that the target–space duality group is part of the $Sp(2n+2;\mathbf{Z})$ group of symplectic reparametrizations of the vector supermultiplets.[5] Under these reparametrizations $F(X, W^2)$ does not transform as a function; the new function assigned

[5] As usual the classical action allows $Sp(2n+2;\mathbf{R})$ transformations, but nonperturbatively this group is restricted to an integer-valued subgroup.

to it takes a more complicated form, as we will show below. This implies that invariance requirements cannot be directly imposed on the sections $\mathcal{F}^{(g)}$. As it turns out this feature is generically intertwined with the presence of non-holomorhic additions to the $\mathcal{F}^{(g)}$ [24]. These non-holomorphic terms lead to a so-called holomorphic anomaly [28, 21, 22], which can be understood from the contributions due to massless fields. The purpose of the text below is to elucidate this.

On the scalar fields the symplectic transformations act according to

$$\begin{pmatrix} X^I \\ F_I(X,W^2) \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I(\tilde{X},W^2) \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I(X,W^2) \end{pmatrix} . \tag{3.1}$$

There are similar transformations on the (abelian) field strengths $F^{\pm I}_{\mu\nu}$ and $G^{\pm}_{\mu\nu\, I} = \mathcal{N}_{IJ}F^{\pm J}_{\mu\nu} + \mathcal{O}^{\pm}_{\mu\nu\, I}$ where $\mathcal{O}^{\pm}_{\mu\nu\, I}$ represents moment couplings to the vector fields, such that the Bianchi identities and the field equations read $\partial^\nu(F^+ - F^-)^I_{\mu\nu} = \partial^\nu(G^+ - G^-)_{\mu\nu\, I} = 0$. Note that it is crucial here to include the full dependence on the Weyl multiplet, also in the tensors $G^{\pm}_{\mu\nu\, I}$ and $\mathcal{O}^{\pm}_{\mu\nu\, I}$. The reason is that the symplectic transformations are linked to invariances of the full equations of motion for the vector fields (which involve the Weyl multiplet) and not of (parts of) the Lagrangian. Note that the transformations are W–dependent and holomorphic (both in X and W), but that W itself does not transform under the symplectic transformations.

The matrix in (3.1) constitutes an element of $Sp(2n+2;\mathbf{Z})$. The transformation rule (3.1) specifies the reparametrization of the fields $X^I \to \tilde{X}^I$ and express the change of the first derivatives of the function $F(X,W^2)$. Owing to the symplectic nature of the transformation, the change in the latter is such that the new quantities $\tilde{F}_I$ are again the derivative of some new function, which we denote by $\tilde{F}(\tilde{X},W^2)$. It is possible to integrate the expression for the $\tilde{F}_I$ and determine $\tilde{F}(\tilde{X},W^2)$, up to certain integration constants. The result reads

$$\begin{aligned} \tilde{F}(\tilde{X},W^2) \;&=\; F(X,W^2) - \tfrac{1}{2}X^I F_I \\ &\quad + \tfrac{1}{2}\Big[(U^{\mathrm{T}}W)_{IJ}X^I X^J + (U^{\mathrm{T}}V + W^{\mathrm{T}}Z)_I{}^J X^I F_J + (Z^{\mathrm{T}}V)^{IJ}F_I F_J\Big] , \end{aligned} \tag{3.2}$$

where F_I denotes the derivative of $F(X,W^2)$ with respect to X^I. Clearly $F(X,W^2)$ itself does not transform as a function, although the combination $F(X,W^2) - \frac{1}{2}X^I F_I(X,W^2)$ does. The symplectic reparametrization constitutes an invariance of the equations of motion, if the new function is identical to the old on, i.e., iff

$$\tilde{F}(\tilde{X},W^2) = F(\tilde{X},W^2) . \tag{3.3}$$

Again, the above equation is *not* equivalent to the requirement that $F(X,W^2)$ is an invariant *function*.

By differentiating (3.2) with respect to W^2 and putting $W^2 = 0$ at the end, one can derive the transformation rules for the coefficient functions $F^{(g)}(X)$. In this way one establishes that, with the exception of $g = 1$, none of the coefficient functions transforms as a function. More explicitly, $F^{(1)}$ changes under symplectic reparametrizations according to

$$\tilde{F}^{(1)}(\tilde{X}) = F^{(1)}(X)\,, \tag{3.4}$$

while the $F^{(g>1)}(X)$ transform in a rather complicated way that involves all lower-g functions as well [24]. Observe that we are still formulating the transformation rules for expressions depending on the X^I rather than on the sections $X^I(z)$, but we shall turn to this aspect shortly.

It is possible to introduce modifications to the $F^{(g)}$, such that they become symplectic functions. As it turns out, these modifications are necessarily non-holomorphic, although their precise form cannot be determined solely by the requirement of symplectic covariance, because one can always consider the addition of other symplectic functions. Hence, there is no contradiction between the above result, which identifies the Wilsonian coefficient function $F^{(1)}$ as both symplectic and holomorphic, while we know, for instance from string theory, that it acquires an antiholomorphic contribution. Presumably this simply implies that an independent symplectic but nonholomorphic function must be added. Obviously, for the generic $F^{(g)}$ the non-holomorphic corrections will always transform into holomorphic terms, so as to compensate for the previous lack of symplectic covariance.

The minimal nonholomorphic modifications that are required to turn the Wilsonian coefficient functions into symplectic functions, can be written down systematically by making use of the following derivative $\mathcal{D}$ [24],

$$\mathcal{D} \equiv \frac{\partial}{\partial W^2} + i\frac{\partial^2 F(X,W^2)}{\partial W^2\,\partial X^I}\,N^{IJ}(X,\bar{X},W^2,\bar{W}^2)\,\frac{\partial}{\partial X^J}\,, \tag{3.5}$$

where

$$N_{IJ}(X,\bar{X},W^2,\bar{W}^2) \equiv 2\,\mathrm{Im}\,F_{IJ}(X,W^2)\,, \qquad N^{IJ} \equiv [N^{-1}]^{IJ}\,, \tag{3.6}$$

so that $\mathcal{D}$ is non–holomorphic in both X and W. The derivative (3.5) is constructed such that when acting on a quantity $G(X,\bar{X},W^2,\bar{W}^2)$ that transforms as a function under symplectic transformations, i.e., as

$$\tilde{G}(\tilde{X},\bar{\tilde{X}},W^2,\bar{W}^2) = G(X,\bar{X},W^2,\bar{W}^2)\,, \tag{3.7}$$

then also $\mathcal{D}G(X,\bar{X},W^2,\bar{W}^2)$ will transform as a symplectic function. Using (3.5) one can thus write down a hierarchy of functions which are modifications of the Wilsonian coefficient functions $F^{(g)}$,

$$F^{(g)\,\mathrm{cov}}(X,\bar{X}) \equiv \frac{1}{g!}\left[\mathcal{D}^{g-1}\,\frac{\partial F(X,W^2)}{\partial W^2}\right]_{W^2=0}\,, \tag{3.8}$$

where we included an obvious normalization factor. All the $F^{(g)\,\mathrm{cov}}$ transform as functions under symplectic reparametrizations. Except for $g = 1$, they are not holomorphic. The lack of holomorphy is governed by the following equation ($g > 1$),

$$\frac{\partial F^{(g)\,\mathrm{cov}}}{\partial \bar{X}^I} = \tfrac{1}{2} \bar{F}_I{}^{JK} \sum_{r=1}^{g-1} \frac{\partial F^{(r)\,\mathrm{cov}}}{\partial X^J} \frac{\partial F^{(g-r)\,\mathrm{cov}}}{\partial X^K}\,, \tag{3.9}$$

where $\bar{F}_I{}^{JK} = \bar{F}_{ILM}\, N^{LJ} N^{MK}$. Observe that this equation remains the same under $F^{(g)\,\mathrm{cov}} \to \rho\, \mu^{2g}\, F^{(g)\,\mathrm{cov}}$, corresponding to a rescaling

$$F(X, W^2) \to \rho\, F(X, \mu^2\, W^2)\,, \tag{3.10}$$

for arbitrary ρ and μ.

The integrability of (3.9) imposes a condition on $F^{(1)}$,

$$\Big(\frac{\partial^2 F^{(1)\,\mathrm{cov}}}{\partial \bar{X}^I \partial X^K}\, \bar{F}_J{}^{KL} - (I \leftrightarrow J)\Big) \frac{\partial F^{(g-1)\,\mathrm{cov}}}{\partial X^L} = 0\,. \tag{3.11}$$

This condition is trivially satisfied by (3.8) as $\mathcal{F}^{(1)\,\mathrm{cov}}$ defined by (3.8) is holomorphic. An alternative solution of this equation is $F^{(1)\,\mathrm{cov}} \propto i[\bar{X}^I\, F_I(X) - X^I\, \bar{F}_I(\bar{X})]$. Note that the proportionality constant in this solution can be adjusted by a rescaling of the $F^{(g)\,\mathrm{cov}}$ of the type indicated above.

The above equations are applicable to both rigid and local supersymmetry. In the general case the coefficient functions are just the derivatives of the full function F with respect to a chiral background field and the above results remain true even without setting this background equal to zero after applying the differentiations. The latter is not so when we express the above results in terms of the sections $X^I(z)$ defined in (2.3), where homogeneity of the coefficient functions is important, so that we are forced to set W^2 to zero. The homogeneity of the $F^{(g)}$ in X^I then ensures that the $F^{(g)\,\mathrm{cov}}(X, \bar{X})$ are homogeneous of degree $2(1-g)$ in X and of degree 0 in $\bar{X}$. It is thus straightforward to construct sections $\mathcal{F}^{(g)\,\mathrm{cov}}(z, \bar{z})$ according to (2.8), which transform in the same way as the $\mathcal{F}^{(g)}$ under projective transformations (cf. 2.9). The lack of holomorphy is encoded in an equation which is rather similar to (3.9). It can be obtained by multiplying (3.9) by $\partial \bar{X}^I(\bar{z})/\partial \bar{z}^A$ and using the identity

$$N^{IJ} = e^K \Big[g^{A\bar{B}}\, (\partial_A + \partial_A K) X^I(z)\, (\partial_{\bar{B}} + \partial_{\bar{B}} K) \bar{X}^J(\bar{z}) - X^I(z)\, \bar{X}^J(\bar{z}) \Big]\,, \tag{3.12}$$

where $K(z, \bar{z})$ and $g_{A\bar{B}}(z, \bar{z})$ are the Kähler potential and metric, respectively. The resulting equation, which we will refer to as the 'symplectic' anomaly equation to distinguish it from the anomaly equation discussed below, is covariant with respect to projective

transformations and holomorphic diffeomorphisms and reads ($g > 1$)

$$\partial_{\bar{A}} \mathcal{F}^{(g)\,\mathrm{cov}}(z,\bar{z}) = \tfrac{1}{2} e^{2K} \bar{\mathcal{W}}_{\bar{A}}{}^{BC} \sum_{r=1}^{g-1} D_B \mathcal{F}^{(r)\,\mathrm{cov}}(z,\bar{z})\, D_C \mathcal{F}^{(g-r)\,\mathrm{cov}}(z,\bar{z})\,. \tag{3.13}$$

Here indices are raised or lowered by means of the Kähler metric corresponding to (2.4). Covariant derivatives are projectively covariant and defined by $D_A \mathcal{F}^{(g)} = (\partial_A + 2(1-g)\partial_A K)\mathcal{F}^{(g)}$; when acting on tensors they include the Levi-Civita connection. Furthermore we used the definition

$$\mathcal{W}_{ABC}(z) = i F_{IJK}(X(z)) \frac{\partial X^I(z)}{\partial z^A} \frac{\partial X^J(z)}{\partial z^B} \frac{\partial X^K(z)}{\partial z^C}\,. \tag{3.14}$$

Although (3.13) applies only to $g > 1$, its integrability implies a condition for $\mathcal{F}^{(1)\,\mathrm{cov}}(z,\bar{z})$ similar to (3.11), with the X^I replaced by z^A and $\bar{F}_J^{KL}$ by $\bar{\mathcal{W}}_{\bar{A}}{}^{BC}$.

The above anomaly equations (3.13) may be compared to the anomaly equations derived some time ago in [22] for the topological partition functions of twisted Calabi-Yau nonlinear sigma models. They read,

$$\partial_{\bar{A}} \mathcal{F}^{(g)\,\mathrm{cov}} = \tfrac{1}{2} e^{2K} \bar{\mathcal{W}}_{\bar{A}}{}^{BC} \Big[\lambda^{-2}\, D_B D_C \mathcal{F}^{(g-1)\,\mathrm{cov}} + \sum_{r=1}^{g-1} D_B \mathcal{F}^{(r)\,\mathrm{cov}}\, D_C \mathcal{F}^{(g-r)\,\mathrm{cov}} \Big]\,, \tag{3.15}$$

for $g > 1$, whereas for $g = 1$ we have

$$\begin{aligned} \partial_A \partial_{\bar{B}} \mathcal{F}^{(1)\,\mathrm{cov}} &= \lambda^{-2} \Big[\tfrac{1}{2} e^{2K}\, \mathcal{W}_{ACD} \bar{\mathcal{W}}_{\bar{B}}{}^{CD} + (1 - \tfrac{1}{24}\chi)\, g_{A\bar{B}} \Big] \\ &= \lambda^{-2} \Big[-\tfrac{1}{2} R_{A\bar{B}} - \tfrac{1}{24}[\chi - 12(n+3)] g_{A\bar{B}} \Big]\,, \end{aligned} \tag{3.16}$$

where $R_{A\bar{B}}$ denotes the Ricci tensor of the Calabi–Yau moduli space and χ the Euler number of the Calabi-Yau.[6] The value of the coefficient λ^2 depends on the normalization adopted for the $\mathcal{F}^{(g)\,\mathrm{cov}}$, as follows from performing the rescaling (3.10). The constants λ^{2g} measure the strength of the genus-g partition function so that, in the context of type-II string theory, λ^2 can be identified with the type-II string–coupling constant.

Clearly (3.9) may be regarded as a truncation (for instance, arising from the singular limit $\lambda \to \infty$) of (3.15). The sections constructed from (3.8) are a solution of the truncated anomaly equation (3.13) and are unique provided that $\mathcal{F}^{(1)\,\mathrm{cov}}$ is taken to be holomorphic. Non-holomorphic corrections to $\mathcal{F}^{(1)\,\mathrm{cov}}$ can be included separately and they will propagate into the higher-g coefficient functions upon solving the appropriate anomaly equation (i.e., (3.13) or (3.15)). This lack of uniqueness does not represent a problem of principle. The requirement of constituting a function under symplectic reparametrizations cannot uniquely determine the non-holomorphic terms and the construction based

[6] A particular solution of (3.16) is $\mathcal{F}^{(1)\,\mathrm{cov}} = \lambda^{-2} \big[-\tfrac{1}{2} \ln g - \tfrac{1}{24} \big[\chi - 12(n+3)\big] \big] K$, where g is the determinant of the Kähler metric.

on (3.8) generates the non-holomorphic modifications that are minimally required in order to extend the Wilsonian coefficient functions into symplectic functions.

In a given holomorphic parametrization of the sections $X^I(z)$ the symplectic transformations induce a corresponding transformation on the coordinates z. In order to remain within a given gauge, this transformation is accompanied by a projective transformation,

$$X^I(z) \longrightarrow \tilde{X}^I(\tilde{z}) = e^{f(z)} \left[U^I_J \, X^J(z) + Z^{IJ} F_J(X(z)) \right] . \tag{3.17}$$

With these definitions we obtain the following transformation rule for the $\mathcal{F}^{(g)\,\mathrm{cov}}$,

$$\tilde{\mathcal{F}}^{(g)\,\mathrm{cov}}(\tilde{z},\bar{\tilde{z}}) = e^{2(1-g)f(z)} \, \mathcal{F}^{(g)\,\mathrm{cov}}(z,\bar{z}) . \tag{3.18}$$

As mentioned before, the anomaly equation (3.15) was derived for the genus-g partition functions of twisted Calabi–Yau sigma models. They were shown to correspond to certain type-II g-loop string amplitudes in [21]. It is worthwhile to indicate the origin of the various terms in the anomaly equation. Generically the defining expressions for the $\mathcal{F}^{(g)\,\mathrm{cov}}$ are holomorphic. However, when integrating over the moduli space of genus-g Riemann surfaces one encounters boundary terms associated with various pinchings of the Riemann surface. The first term in (3.15) is due to the pinching of one of the homology cycles, which explains why the genus is lowered by one unit; the second term correspond to a pinching that leads to two disconnected surfaces, so that the sum of their genera equals the original genus g. In terms of the effective field theory, these pinchings are identified as the effects of the propagation of massless modes [28, 21]. As is well known, these effects form an obstacle in obtaining a local effective action. The Wilsonian action, on the other hand, is a local effective action, in which the cumbersome effects of the massless modes are avoided by the presence of an infrared cut–off. However, because of this cut–off the Wilsonian action does not fully capture the physics, and certain features (like the presence of certain symmetries) of the underlying model are not always manifest.

The approach based on (3.8) encapsulates some of these features. The Wilsonian coefficient functions are not covariant with respect to the symplectic reparametrizations and therefore will not be invariant under certain subgroups (such as target-space duality), as one would expect from an *ab initio* calculation based on an underlying physical theory that has this invariance. Certain nonholomorphic corrections readjust this situation, but they themselves have no role to play in the Wilsonian set–up. Identifying these nonholomorphic corrections with the contributions from propagating massless modes provides an explanation for this phenomenon. Comparison with the anomaly equation (3.15) of [22] indicates that the approach based on (3.8) correctly takes into acount the massless–tree contributions. The massless loops, while not excluded in this approach, will appear as

separate contributions. As we will discuss in the next section, in the semi-classical limit of the heterotic string only the second term in the anomaly equation (3.15) survives; hence for the perturbative heterotic string the lack of holomorphy is fully governed by (3.13). Interestingly enough, as alluded to earlier, the same effect takes place on the type-II side in the strong-coupling limit. The expressions (3.8) represent an explicit solution to the latter anomaly equation.

To elucidate the construction based on (3.8), we derive the first few covariant functions $\mathcal{F}^{(g)\,\mathrm{cov}}$ for a specific example, where the W^2-independent part of the holomorphic function equals

$$F(X,W^2)|_{W^2=0} = \frac{d_{ABC}\,X^A X^B X^C}{X^0}\,. \tag{3.19}$$

Its corresponding Kähler potential takes the form

$$K(z,\bar z) = -\log\Big(-id_{ABC}\,(z-\bar z)^A(z-\bar z)^B(z-\bar z)^C\Big)\,. \tag{3.20}$$

Here we employ so-called special coordinates z^A by choosing the holomorphic sections $(X^0(z),X^A(z)) = (1,z^A)$. The matrix $N_{IJ} \equiv -i(F_{IJ}-\bar F_{IJ})$ is then equal to $(I=0,A)$

$$N_{IJ} = \begin{pmatrix} 2n_{CD}(z^C z^D + z^C\bar z^D + \bar z^C\bar z^D) & -3n_{BC}(z+\bar z)^C \\ -3n_{AD}(z+\bar z)^D & 6n_{AB} \end{pmatrix}\,, \tag{3.21}$$

where $n_{AB} = -id_{ABC}\,(z-\bar z)^C$, and its inverse equals

$$N^{IJ} = \begin{pmatrix} 2\,e^K & e^K\,(z+\bar z)^B \\ e^K\,(z+\bar z)^A & \frac{1}{6}n^{AB} + \frac{1}{2}e^K\,(z+\bar z)^A(z+\bar z)^B \end{pmatrix}\,. \tag{3.22}$$

So far we put $W^2=0$. We now consider the W^2-dependent terms and construct the covariant coefficient functions by using (3.8). With the exception of the first one,

$$\mathcal{F}^{(1)\,\mathrm{cov}}(z,\bar z) = \mathcal{F}^{(1)}(z)\,, \tag{3.23}$$

all other functions are nonholomorphic. We exhibit the explicit expressions for $\mathcal{F}^{(2)\,\mathrm{cov}}(z,\bar z)$ and $\mathcal{F}^{(3)\,\mathrm{cov}}(z,\bar z)$,

$$\begin{aligned}
\mathcal{F}^{(2)\,\mathrm{cov}}(z,\bar z) &= \mathcal{F}^{(2)}(z) + \tfrac{1}{12}\widehat{n}^{AB}\partial_A\mathcal{F}^{(1)}(z)\,\partial_B\mathcal{F}^{(1)}(z)\,, \\
\mathcal{F}^{(3)\,\mathrm{cov}}(z,\bar z) &= \mathcal{F}^{(3)}(z) + \tfrac{1}{6}\widehat{n}^{AB}\,\partial_A\mathcal{F}^{(2)}(z)\,\partial_B\mathcal{F}^{(1)}(z) \\
&\quad +2e^K\,(z-\bar z)^A\partial_A\mathcal{F}^{(1)}(z)\,\mathcal{F}^{(2)}(z) \\
&\quad +\tfrac{1}{72}\widehat{n}^{AC}\,\widehat{n}^{BD}\,\partial_A\partial_B\mathcal{F}^{(1)}(z)\,\partial_C\mathcal{F}^{(1)}(z)\,\partial_D\mathcal{F}^{(1)}(z) \\
&\quad +\tfrac{1}{6}e^K\,\widehat{n}^{AB}\,(z-\bar z)^C\partial_A\mathcal{F}^{(1)}(z)\,\partial_B\mathcal{F}^{(1)}(z)\,\partial_C\mathcal{F}^{(1)}(z) \\
&\quad +\tfrac{i}{216}d_{ABC}\,\widehat{n}^{AD}\,\widehat{n}^{BE}\,\widehat{n}^{CF}\,\partial_D\mathcal{F}^{(1)}(z)\,\partial_E\mathcal{F}^{(1)}(z)\,\partial_F\mathcal{F}^{(1)}(z)\,,
\end{aligned} \tag{3.24}$$

where $\widehat{n}^{AB} = n^{AB} + 3\,e^K\,(z-\overline{z})^A\,(z-\overline{z})^B$. Not surprisingly, the resulting expressions are rather similar to the ones in the orbifold example in section 7.1 of [22], which are, however, based on a non-holomorphic $\mathcal{F}^{(1)\,\mathrm{cov}}$ and define the solutions of the anomaly equation (3.15) subject to modular invariance.

Unlike the holomorphic quantities $\mathcal{F}^{(g)}$, these non-holomorphic quantities $\mathcal{F}^{(g)\,\mathrm{cov}}$ transform as functions under symplectic reparametrizations. They satisfy the holomorphic anomaly equation (3.13). In appendix B we present the above results for the case of the S-T-U model.

4 The holomorphic anomaly equations and their solutions for the heterotic string

In this section we focus on $N = 2$ supersymmetric models in four space–time dimensions that have both a type-II and a heterotic description [6]. In the type-II description, such models are obtained by compactifications of the type-IIA string on certain Calabi–Yau manifolds. In the heterotic description, they follow from compactifications of the heterotic $E_8 \times E_8$ string on $K_3 \times T_2$. String–string duality then implies that the non-holomorphic couplings $\mathcal{F}^{(g)\,\mathrm{cov}}(z,\bar{z})$ in the type-II and in the heterotic description are related. On the type-IIA side, the z^A denote the Kähler-class moduli, whereas on the heterotic side the z^A correspond to the heterotic dilaton S and to the heterotic moduli T^a (consisting of the toroidal and Wilson-line moduli). Partial evidence for such a string–string duality has been given for the case of 2 moduli in [12, 23, 14], where on the heterotic side we have the complex dilaton field S and a modulus T, and for the case of 3 moduli in [12, 14, 16], with the dilaton S and the two T_2 moduli T and U on the heterotic side. This model, which we will refer to as the S-T-U model, will be discussed in more detail later in this section. More recently, evidence for string–string duality was also obtained for the case of 4 moduli, which on the heterotic side incorporates the two toroidal moduli T and U and a Wilson-line modulus [31].

In the context of type-II compactifications on Calabi–Yau manifolds, it was shown in [21] that the non-holomorphic sections $\mathcal{F}^{(g)\mathrm{cov}}(z,\bar{z})$ obtained from direct string calculations are equal to the topological partition functions, and thus they satisfy the holomorphic anomaly equations (3.15) and (3.16). In type-II string compactifications the contributions to the $\mathcal{F}^{(g)\,\mathrm{cov}}$ originate from g loops in string perturbation theory [21]. This can be seen as follows. According to (2.8), the $\mathcal{F}^{(g)\,\mathrm{cov}}$ are multiplied by a factor $(m^2_{\mathrm{Planck}})^{1-g}$; keeping the string scale rather than the Planck scale fixed yields a factor $[g_{\mathrm{s}}^{-2}]^{g-1}$, where g_{s}^{-2} is proportional to the dilaton and acts as a loop-counting parameter. There can

be no further dependence on the string coupling, as the type-II dilaton resides in a hypermultiplet and neutral hypermultiplets do not affect the vector-multiplet couplings. Thus we are dealing with contributions at precisely g loops. Note that the relevant anomaly equation (3.15) indeed comprises terms of the same loop order. The first term describes the $(g-1)$-loop contribution with a massless loop appended to it, while the second term describes the product of an r–oop and a $(g-r)$-loop contribution.

For type-II models possessing a dual heterotic description, one thus expects that the heterotic couplings satisfy similar holomorphic anomaly equations. As the arguments in the previous section have shown, the existence of such anomaly equations can, at least partially, be understood from arguments based on symplectic reparametrizations, from which one may conclude that certain features concerning the non–holomorphic terms should be generic and independent of the precise model one is considering. This observation will help us to fix the relative normalization between the sections obtained on the heterotic and on the type-II side. In fact, as we will demonstrate shortly, in the relevant limit of a large Kähler-class modulus the type-II and the symplectic anomaly equations become identical.

Later in this section we will turn to the heterotic weak-coupling limit of these holomorphic anomaly equations and we will solve them for $\mathcal{F}^{(2)\,\mathrm{cov}}$ and $\mathcal{F}^{(3)\,\mathrm{cov}}$ in the context of the S-T-U model, for concreteness. The $\mathcal{F}^{(g)\,\mathrm{cov}}$ exhibit singularities at lines/points in the perturbative heterotic moduli space at which one has perturbative gauge–symmetry enhancement. We will show that, in the vicinity of these lines/points of semi-classical gauge symmetry enhancement, the structure of the $\mathcal{F}^{(2)\,\mathrm{cov}}$ and $\mathcal{F}^{(3)\,\mathrm{cov}}$ we obtain precisely agrees with expressions (3.24) (with certain one-loop corrections included) found in the previous section on the grounds of symplectic covariance. Subsequently we analyze the target–space duality properties and exploit the covariance to further restrict the couplings in terms of modular forms.

4.1 The weak-coupling limit in the heterotic string and target–space duality

As discussed above, in type-II string compactifications the contributions to the $\mathcal{F}^{(g)\,\mathrm{cov}}$ originate from g loops in string perturbation theory. For heterotic vacua the counting is different, because the dilaton resides in the vector multiplet sector. First of all, the Kähler potential contains a characteristic term $\ln g_{\mathrm{s}}^2$, so that the factor $(m^2_{\mathrm{Planck}})^{g-1}\exp[-(1-g)K]$ in the definition (2.8) of the holomorphic sections $\mathcal{F}^{(g)}$ depends only on the string scale and no longer on g_{s}. Therefore the dependence of the corresponding couplings on g_{s} is directly related to the explicit dependence of the $\mathcal{F}^{(g)}$ on the dilaton. The presence of

the dilaton is, however, restricted by non-renormalization arguments. As a result, only the first two terms in the expansion (2.2) depend explicitly and linearly on the dilaton field (in perturbation theory), and thus represent tree-level contributions. All other terms contribute at precisely one loop in string perturbation theory [21].

Let us now assume that the heterotic sections $\mathcal{F}^{(g)\,\mathrm{cov}}$ satisfy a holomorphic anomaly equation similar to (3.15) and deduce, on the basis of the counting arguments given above, what the relevant terms will be in the weak-coupling limit $S + \bar{S} \to \infty$. Since generically all the $\mathcal{F}^{(g)}$ are independent of S and thus correspond to one-loop contributions, it follows that the right-hand side of the anomaly equation is generically of two-loop order and will therefore be suppressed by a factor g_{s}^2: the first term in the anomaly equation appends a massless loop to a one–loop term, while the second term consists of products $\mathcal{F}^{(g-r)}\,\mathcal{F}^{(r)}$ of one-loop terms. Nevertheless, one-loop contributions can still emerge from $\mathcal{F}^{(1)\,\mathrm{cov}}$, which is not exclusively the result of a one-loop correction but contains also a term arising from the tree approximation. However, this term is linear in the dilaton, so that the second derivative term in (3.15) cancels and we are left with the truncated equation (3.13). Actually, also this term simplifies, as it generically contributes at the two–loop level, with the exception of the terms proportional to $\partial_S \mathcal{F}^{(1)\,\mathrm{cov}}\,\partial_a \mathcal{F}^{(g-1)\,\mathrm{cov}}$ and, for $g = 2$, $\partial_S \mathcal{F}^{(1)\,\mathrm{cov}}\,\partial_S \mathcal{F}^{(1)\,\mathrm{cov}}$, which can still give rise to one-loop contributions.

To confirm the above argument let us explicitly consider the limit of large $S + \bar{S}$ in the anomaly equation. We consider the dilaton $S = 4\pi/g^2 - i\theta/2\pi$ and an arbitrary number of moduli T^a, which are related to the special coordinates z^A by $iS = z^1$, $iT^a = z^a$. The class of compactifications is defined by the requirement that, up to nonperturbative contributions which take the form of positive powers of $\exp(-2\pi S)$, the associated holomorphic prepotentials are given by

$$\mathcal{F}^{(0)}(S, T^a) = -S\, T^a \eta_{ab} T^b + h(T^a)\,, \tag{4.1}$$

and the Wilsonian couplings $\mathcal{F}^{(g)}$ are given by

$$\begin{aligned} \mathcal{F}^{(1)}(S, T^a)) &= a\, S + h^{(1)}(T^a)\,, \\ \mathcal{F}^{(g>1)}(S, T^a) &= \mathcal{F}^{(g)}(T^a)\,, \end{aligned} \tag{4.2}$$

where a denotes an integer.[7] The Kähler potential for the above models, which is computed from (4.1), is given by

$$K(S, T) = -\log(S + \bar{S} + V(T, \bar{T})) + \hat{K}(T, \bar{T})\,, \tag{4.3}$$

[7]We will, throughout the paper, use the normalization convention $a = 24$ [38], thereby fixing the normalization of W^2.

where the Kähler potential $\hat{K}$ and the corresponding Kähler metric are given by

$$\begin{aligned}
\hat{K}(T,\bar{T}) &= -\log[(T+\bar{T})^a \eta_{ab} (T+\bar{T})^b]\,, \\
\hat{g}_{a\bar{b}} &= -2\eta_{ab}\, e^{\hat{K}} + 4\eta_{ac}\eta_{bd}(T+\bar{T})^c (T+\bar{T})^d\, e^{2\hat{K}}\,, \\
\hat{g}^{a\bar{b}} &= -\tfrac{1}{2}\eta^{ab}\, e^{-\hat{K}} + (T+\bar{T})^a (T+\bar{T})^b\,,
\end{aligned} \tag{4.4}$$

where η^{ab} is the inverse of η_{ab}. The quantity V is the Green-Schwarz term, defined by

$$V(T,\bar{T}) = \frac{2(h+\bar{h}) - (T+\bar{T})^a (h_a + \bar{h}_{\bar{a}})}{(T+\bar{T})^a \eta_{ab} (T+\bar{T})^b}\,. \tag{4.5}$$

This term satisfies the following equation, which will be useful later on,

$$\begin{aligned}
&\Big[\partial_a \partial_{\bar{b}} + \partial_b \partial_{\bar{a}} + 4\eta_{ab}\, e^{\hat{K}} - 16 [\eta(T+\bar{T})]_a [\eta(T+\bar{T})]_b\, e^{2\hat{K}}\Big] V = \\
&-2(h_{ab} + \bar{h}_{\bar{a}\bar{b}}) e^{\hat{K}} + 2\Big[[\eta(T+\bar{T})]_a (h_{bc} + \bar{h}_{\bar{b}\bar{c}}) + (a \leftrightarrow b)\Big]\, (T^c + \bar{T}^c)\, e^{2\hat{K}}\,,
\end{aligned} \tag{4.6}$$

where $[\eta(T+\bar{T})]_a = \eta_{ab}\,(T+\bar{T})^b$.

The behaviour of the Wilsonian couplings $\mathcal{F}^{(g)}$ in the limit $S+\bar{S} \to \infty$ can easily be determined from (4.2). However, the functions entering in the holomorphic anomaly equation (3.15) are not the Wilsonian coupling functions, but rather the full non-holomorphic functions $\mathcal{F}^{(g)\mathrm{cov}}(z,\bar{z})$. Nevertheless, let us momentarily assume that they satisfy the following conditions in the limit $S+\bar{S} \to \infty$, which are somewhat weaker but consistent with what one would derive for the Wilsonian couplings on the basis of (4.2),

$$\begin{aligned}
D_S \mathcal{F}^{(g>1)\,\mathrm{cov}} &\to 0\,, \\
D_S \mathcal{F}^{(1)\,\mathrm{cov}} &\to a\,, \\
D_{\bar{S}} \mathcal{F}^{(g\geq 1)\,\mathrm{cov}} &\to 0\,, \\
D_T \mathcal{F}^{(g\geq 1)\,\mathrm{cov}} &\to f^{(g)}(T,\bar{T})\,,
\end{aligned} \tag{4.7}$$

where $f^{(g)}$ is some arbitrary function. Let us now consider the anomaly equation (3.15) in the limit $S+\bar{S} \to \infty$. Because the only non-zero components of the tensor $\mathcal{W}_{ABC}$ are $\mathcal{W}_{abS}$ and $\mathcal{W}_{abc}$, the anomaly equation takes the form

$$\begin{aligned}
\partial_{\bar{a}} \mathcal{F}^{(g)\,\mathrm{cov}} &= e^{2K} \bar{\mathcal{W}}_{\bar{a}\bar{b}\bar{s}} \Big[D^{\bar{b}} D^{\bar{s}} \mathcal{F}^{(g-1)\,\mathrm{cov}} + \sum_{r=1}^{g-1} D^{\bar{b}} \mathcal{F}^{(r)\,\mathrm{cov}}\, D^{\bar{s}} \mathcal{F}^{(g-r)\,\mathrm{cov}} \Big] \\
&\quad + \tfrac{1}{2} e^{2K} \bar{\mathcal{W}}_{\bar{a}\bar{b}\bar{c}} \Big[D^{\bar{b}} D^{\bar{c}} \mathcal{F}^{(g-1)\,\mathrm{cov}} + \sum_{r=1}^{g-1} D^{\bar{b}} \mathcal{F}^{(r)\,\mathrm{cov}}\, D^{\bar{c}} \mathcal{F}^{(g-r)\,\mathrm{cov}} \Big]\,.
\end{aligned} \tag{4.8}$$

Using the results (A.3) and (A.4) of appendix A and the asymptotic conditions (4.7), it is then straightforward to show that, in the limit $S+\bar{S} \to \infty$, the $D_B D_C \mathcal{F}^{(g-1)\,\mathrm{cov}}$ term

does not contribute anything in the anomaly equation (3.15), which is therefore reduced to the equation (3.13). Actually, this equation reduces to an even simpler form ($g > 1$),

$$\partial_{\bar{a}}\mathcal{F}^{(g)\,\mathrm{cov}} = e^{2\hat{K}}\bar{\mathcal{W}}_{\bar{a}\bar{b}\bar{S}}\,\hat{g}^{\bar{b}c}\Big[a\,(\partial_c + 2(1-(g-1))\partial_c\hat{K})\mathcal{F}^{(g-1)\,\mathrm{cov}} - a^2\,V_c\,\delta_{g,2}\Big]\,, \tag{4.9}$$

where $\partial_a = \partial/\partial T^a$. As exhibited in (A.2), the quantity $-\hat{g}^{\bar{a}b}\,V_b$, which appears only at genus-2, is equal to the inverse metric component $g^{\bar{a}S}$ in the limit of large $S+\bar{S}$. The consistency of the assumption made above that the $\mathcal{F}^{(g)\,\mathrm{cov}}$ exhibit a similar behaviour at large $S+\bar{S}$ as the holomorphic sections $\mathcal{F}^{(g)}$, is confirmed by considering the holomorphic anomaly equation for $\partial_{\bar{S}}\mathcal{F}^{(g)\,\mathrm{cov}}$, whose right-hand side behaves as $(S+\bar{S})^{-2}$ in the large $S+\bar{S}$ limit, again as a result of (A.3), (A.4) and (4.7). For the case of two moduli (i.e. only S and T) the above result (4.9) was already derived in [23].

Of course, when the models that we are considering have both a heterotic and a type-II description, the covariant sections $\mathcal{F}^{(g)\mathrm{cov}}$ are identical to each other, upon a suitable identification of the type-II and the heterotic fields. A priori it may not be clear that the normalization of the covariant sections is the same. However, in the semi-classical limit we have established that they are subject to the same anomaly equation (3.13) which, according to (3.10), allows for different normalizations corresponding to only an overall rescaling of the full Wilsonian function $F(X,W^2)$, and of the field W. The first one is fixed once the W^2-independent part of F is fixed. The second one simply depends on the normalization adopted for W and is thus related to the relative normalization of the topological partition functions (i.e., the parameter λ in (3.15)) vis à vis the normalization of the Weyl multiplet in the Lagrangian for the effective field theory on the heterotic side. On the type-II side this connection is provided by the work of [21], which showed that certain type-II string amplitudes precisely reproduce the topological partition functions. Our analysis is entirely on the heterotic side and we will simply adopt the standard normalization convention for the Weyl multiplet leading to $a = 24$ (cf. footnote 7).

Let us consider the anomaly equation for $\mathcal{F}^{(2)\,\mathrm{cov}}(T,\bar{T})$ in more detail. At one-loop, $\mathcal{F}^{(1)\,\mathrm{cov}}$ will be given by

$$\mathcal{F}^{(1)\,\mathrm{cov}} = aS_{\mathrm{inv}} + h^{(1)\,\mathrm{cov}}_{\mathrm{inv}}(T,\bar{T})\,, \tag{4.10}$$

where S_{inv} denotes the so-called invariant dilaton [17]. It differs from S by a holomorphic function $\sigma(T)$ of the T^a,

$$S_{\mathrm{inv}} - S \equiv \sigma(T) = \frac{1}{2(n+1)}\Big(-\eta^{ab}h_{ab}(T) + L(T)\Big)\,. \tag{4.11}$$

Here L is a holomorphic function of the moduli fields T^a which transforms into imaginary constant shifts under target–space duality transformations. These shifts are associated

with semiclassical monodromies. The new dilaton field $S_{\rm inv}$ is invariant under target–space duality transformations, but it is no longer a *special* coordinate in the context of $N = 2$ supersymmetry. The true loop-counting parameter of the heterotic string, on the other hand, is given by [17]

$$S + \bar{S} + V(T, \bar{T}) = S_{\rm inv} + \bar{S}_{\rm inv} + V_{\rm inv}(T, \bar{T}) \tag{4.12}$$

where $V_{\rm inv}(T, \bar{T})$ denotes the so-called invariant Green–Schwarz term, defined by this equation. $V_{\rm inv}(T, \bar{T})$ is invariant under one–loop target–space duality transformations. It follows from (4.12) that $V(T, \bar{T}) = V_{\rm inv}(T, \bar{T}) + \sigma(T) + \bar{\sigma}(\bar{T})$. Hence,

$$\partial_b \Big[\mathcal{F}^{(1)\,\rm cov} - aV(T, \bar{T}) \Big] = \partial_b \Big[h_{\rm inv}^{(1)\,\rm cov}(T, \bar{T}) - aV_{\rm inv}(T, \bar{T}) \Big] \ . \tag{4.13}$$

The holomorphic anomaly equation for $\mathcal{F}^{(2)\,\rm cov}(T, \bar{T})$ can then be rewritten into

$$\begin{aligned} \partial_{\bar{a}} \mathcal{F}^{(2)\,\rm cov} &= a\, e^{2\hat{K}} \bar{\mathcal{W}}_{\bar{a}\bar{b}\bar{S}}\, \hat{g}^{\bar{b}c}\, \partial_c \Big[\mathcal{F}^{(1)\,\rm cov} - a\, V \Big] \\ &= a\, e^{2\hat{K}} \bar{\mathcal{W}}_{\bar{a}\bar{b}\bar{S}}\, \hat{g}^{\bar{b}c}\, \partial_c \Big[h_{\rm inv}^{(1)\,\rm cov} - a\, V_{\rm inv} \Big] . \end{aligned} \tag{4.14}$$

As the metric and the tensors $\mathcal{W}$ must be target-space duality covariant, this exhibits the manifest covariance of the anomaly equation in the perturbative limit. The correctness of this result can be inferred from the covariance of the original anomaly equation in the context of the $(S + \bar{S}) \to \infty$ limit.

4.2 The anomaly equation for the heterotic S-T-U model

The S-T-U model can be constructed by compactifying the ten-dimensional heterotic string on $T_2 \times K_3$. A compactification of the $E_8 \times E_8$ heterotic string on K_3, with equal $SU(2)$ instanton number in both E_8 factors, gives rise to a model in six space–time dimensions with gauge group $E_7 \times E_7$. For general vacuum expectation values of the massless hypermultiplets this gauge group is completely broken, and one is left with 244 massless hypermultiplets and no massless vector multiplets. Upon a T_2 compactification down to four dimensions, one obtains a model with 244 massless hypermultiplets and with three massless vector multiplets S, T and U, where S denotes the heterotic dilaton and T, U denote the T_2 moduli. This model is the heterotic dual of the type IIA–model [6] based on the Calabi-Yau space $WP_{1,1,2,8,12}(24)$ with $h_{11} = 3$, $h_{21} = 243$ and, hence, with $\chi = -480$.

In the heterotic description this model possesses a target–space duality symmetry $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U \times \mathbf{Z}_2^{T \leftrightarrow U}$ at the perturbative level $(S + \bar{S} \to \infty)$. All elements

of this discrete symmetry group act as symplectic transformations of the form (3.1) subject to condition (3.3). Hence, according to (3.18) the non-holomorphic functions $\mathcal{F}^{(g)\,\mathrm{cov}}$ transform as modular functions of a specific modular weight. As it is well known, the duality transformations $T \to \tilde{T} = \frac{aT-ib}{icT+d}$ induce a particular Kähler transformation of the Kähler potential, $K \to K - f - \bar{f}$, with $f = -\log(icT+d)$. Then, comparison with (3.18) shows that the non-holomorphic $\mathcal{F}^{(g)\,\mathrm{cov}}$ possess the following duality transformation behaviour,

$$\mathcal{F}^{(g)\,\mathrm{cov}}(\tilde{T},\bar{\tilde{T}},U,\bar{U}) \to (icT+d)^{2(g-1)}\mathcal{F}^{(g)\,\mathrm{cov}}(T,\bar{T},U,\bar{U})\,, \tag{4.15}$$

and similarly for $U \to \tilde{U} = \frac{aU-ib}{icU+d}$. Note that, according to (4.10), $h^{(1)\mathrm{cov}}_{\mathrm{inv}}(T,\bar{T},U,\bar{U})$ and $\mathcal{F}^{(1)\,\mathrm{cov}}(T,\bar{T},U,\bar{U})$ are both invariant under target–space duality transformations (4.15).

The perturbative prepotential for the S-T-U model reads

$$\mathcal{F}^{(0)}(S,T,U) = -STU + h(T,U)\ \ , \tag{4.16}$$

where $h(T,U)$ denotes the one-loop contribution to the prepotential; it would transform as a modular form of weight -2, were it not for the presence of an inhomogeneous term in its transformation rule proportional to a polynomial of T and U containing no powers higher than T^2 or U^2. This polynomial is directly related to the semi-classical monodromies. It then follows [17] that $\partial_T^3 h$ and $\partial_U^3 h$ are single-valued modular functions of weight $+4$ and -2, respectively, under $SL(2,\mathbf{Z})_T$, and of weight -2 and $+4$, respectively, under $SL(2,\mathbf{Z})_U$.

Using the results (A.2) given in appendix A and the explicit form for the matrix η_{ab}, one finds that, in the limit $S+\bar{S} \to \infty$,

$$\begin{aligned} g^{S\bar{S}} = (S+\bar{S})^2\ \ ,\ \ g^{T\bar{T}} = (T+\bar{T})^2\ \ ,\ \ g^{U\bar{U}} = (U+\bar{U})^2\ \ , \\ g^{T\bar{S}} = -(T+\bar{T})^2\,\partial_T V(T,\bar{T},U,\bar{U})\ \ ,\ \ g^{U\bar{S}} = -(U+\bar{U})^2\,\partial_U V(T,\bar{T},U,\bar{U})\,, \end{aligned} \tag{4.17}$$

with the Green–Schwarz term $V(T,\bar{T},U,\bar{U})$ given as

$$V(T,\bar{T},U,\bar{U}) \;=\; \frac{2(h+\bar{h}) \;-\; (T+\bar{T})(\partial_T h + \partial_{\bar{T}}\bar{h}) \;-\; (U+\bar{U})(\partial_U h + \partial_{\bar{U}}\bar{h})}{(T+\bar{T})\,(U+\bar{U})}\,, \tag{4.18}$$

as well as $\mathcal{W}_{STU} = -1$. Then, it follows from (4.9) and from (4.14) that

$$\partial_{\bar{T}}\mathcal{F}^{(g)\,\mathrm{cov}} \;=\; -\frac{a}{(T+\bar{T})^2} D_U \mathcal{F}^{(g-1)\,\mathrm{cov}}\,, \quad g>2\,, \tag{4.19}$$

$$\partial_{\bar{T}}\mathcal{F}^{(2)\,\mathrm{cov}} \;=\; -\frac{a}{(T+\bar{T})^2}\partial_U\left(h^{(1)\mathrm{cov}}_{\mathrm{inv}} - a\,V_{\mathrm{inv}}\right)\,, \tag{4.20}$$

and likewise with T and U interchanged. Note that both $h_{\rm inv}^{(1)\rm cov}$ and $V_{\rm inv}$ are target–space duality invariant, so that both (4.19) and (4.20) are consistent with the modular transformation behaviour given in (4.15).

At $T = U$ one has a perturbative gauge–symmetry enhancement, so one expects that the $\mathcal{F}^{(g)\,\rm cov}$ are singular as $T \to U$. The leading singularities will be holomorphic, whereas the non-leading singularities will be both of the holomorphic and of the non-holomorphic type. In the following subsection we turn to the discussion of these leading holomorphic singularities.

4.3 Leading holomorphic singularities in heterotic models

The holomorphic Wilsonian couplings $\mathcal{F}^{(g)}(z)$ are singular at precisely those points in the moduli space where certain string states become massless. In the type-II context, for example, precisely one hypermultiplet becomes massless at the conifold points of the Calabi-Yau moduli space [32], and the leading holomorphic singularity follows from a $c = 1$ matrix model and is given by [33]

$$\mathcal{F}^{(g>1)}(z) = \frac{B_{2g}}{2g(2g-2)} \frac{1}{[\mu(z)]^{2(g-1)}} \quad , \tag{4.21}$$

where $\mu(z)$ is the mass of the hypermultiplet. The B_{2g} are the Bernoulli numbers.

Let us now discuss the leading singularity structure of the perturbative Wilsonian couplings $\mathcal{F}^{(g)}(z)$ in the heterotic context by taking into account the moduli-dependent, elementary string spectrum. We base ourselves on the perturbative prepotential (4.1). The mass formula for a BPS state equals $m_{BPS}^2 \propto m_{\rm Planck}^2\, e^{K(z,\bar{z})} |\mathcal{M}(z)|^2$, where $\mathcal{M}(z)$ is a holomorphic section defined by

$$\mathcal{M}(z) \equiv M_I X^I(z) - N^I F_I(z)\,, \tag{4.22}$$

and M_I and N^I denote the electric and the magnetic charges of a given BPS state. Obviously, $\mathcal{M}(z)$ transforms under projective transformations as $\mathcal{M}(z) \to e^{f(z)}\,\mathcal{M}(z)$. The separation into 'electric' and 'magnetic' charges refers to a specific symplectic basis. In the basis that is relevant for classical string theory [3, 34, 17, 18], the symplectic holomorphic sections read $(\hat{X}^I(z), \hat{F}_I(z)) = (1, \eta_{bc}T^bT^c, iT^a, -iS\,\eta_{bc}T^bT^c - 2ih + iT^b\,\partial_b h, -iS, 2S\,\eta_{ab}T^b - \partial_a h)$. With this result the perturbative holomorphic mass $\mathcal{M}(S, T^a)$ equals

$$\begin{aligned} \mathcal{M}(S,T) \;&=\; M_0 + M_1\,\eta_{ab}T^aT^b + iM_aT^a \\ &\quad + iS(N^0\,\eta_{ab}T^aT^b + N^1 + 2i\eta_{ab}N^a\,T^b) + 2iN^0\,h - (iN^0T^a - N^a)\,\partial_a h\,. \end{aligned} \tag{4.23}$$

For elementary string states (i.e., states whose mass remains finite in the classical limit), $N^I = 0$. Hence, the holomorphic mass $\mathcal{M}(T)$ for elementary string states is given by

$$\mathcal{M}(T) = M_0 + M_1\,\eta_{ab}T^aT^b + iM_a\,T^a\,, \tag{4.24}$$

which is not affected by perturbative corrections.

Now, let us briefly consider a simple heterotic model with just one modulus field T, the so-called S-T model. In this model, the holomorphic mass for elementary string states can vanish at $T = 1$. At this point in the perturbative moduli space, the $U(1)$ associated with the T modulus becomes enhanced to an $SU(2)$, and thus two additional vector multiplets become massless. In the vicinity of $T = 1$, the $\mathcal{F}^{(g)}$ should have a leading holomorphic singularity given by

$$\mathcal{F}^{(g>1)}(T) \propto \frac{1}{\mathcal{M}^{2(g-1)}(T)}\ , \tag{4.25}$$

where $\mathcal{M} \propto T - 1$. We note that (4.25) is consistent with the transformation behaviour of both $\mathcal{F}^{(g)}(z)$ and $\mathcal{M}(z)$ as holomorphic sections. Indeed, it was shown in [23] that the leading holomorphic singularity of the $\mathcal{F}^{(g)}$ on the heterotic side is given by

$$\mathcal{F}^{(g>1)}(T) = -\frac{2}{\pi}\left(\frac{a}{2}\right)^g \frac{B_{2g}}{2g(2g-2)}\frac{1}{[\mu(T)]^{2(g-1)}}\ , \tag{4.26}$$

where $\mu(T) = \frac{1}{\pi}(T-1)$. The identification of μ follows by comparing [23] the singularity of the holomorphic prepotential $\mathcal{F}^{(0)} = -\frac{1}{2}ST^2 + \frac{1}{\pi}(T-1)^2\log(T-1)$ with $2Z_{c=1} = \mu^2\log\mu$. The relative factor of $\frac{1}{\pi}\left(\frac{a}{2}\right)^g$ between (4.26) and (4.21) reflects a different normalization convention adopted in this paper from the one used in [23], whereas the relative factor of 2 reflects the fact that there are two additional vector multiplets becoming massless at $T = 1$. Noting that $\frac{2}{T-1} \approx \partial_T \log(j(T) - j(1))$, where j denotes the modular invariant function $j = E_4^3/\eta^{24}$, it follows that the leading singularity of $\mathcal{F}^{(g)}$ can be rewritten into the following holomorphic manifestly modular covariant form with the correct modular weight

$$\mathcal{F}^{(g>1)}(T) = \frac{1}{\pi}\left(\frac{a}{2}\right)^g \frac{B_{2g}}{2g(2g-2)(2g-2)!}\hat{D}_T^{2(g-1)}\log(j(T) - j(1))\ , \tag{4.27}$$

where the holomorphic modular covariant derivative $\hat{D}_T$ of a weight-ω modular form $P(T)$ is given by [35]

$$\hat{D}_T P(T) = \Big(\partial_T - \omega\,G_2(T)\Big)P(T)\ , \tag{4.28}$$

and where $G_2(T)$ is related to the Dedekind function $\eta(T)$ by $G_2(T) = \partial_T \log\eta^2(T)$. For later use we define the non-holomorphic modular function $\hat{G}_2(T)$ of weight 2,

$$\hat{G}_2(T,\bar{T}) = \frac{1}{T+\bar{T}} + G_2(T)\,. \tag{4.29}$$

For comparison, we also give the Kähler covariant derivative for a section $P(T)$ that transforms under projective transformations with weight w. It reads

$$D_T P(T) \equiv \big(\partial_T + w\,\partial_T \hat{K}\big) P(T) = \big(\partial_T - \frac{w}{T+\bar{T}}\big) P(T)\,. \tag{4.30}$$

As the projective and the modular weights are opposite, $w = -\omega$ so that the Kähler covariant derivative and the holomorphic covariant derivatives differ by a covariant term equal to $\omega\,\hat{G}_2(T,\bar{T})\,P(T)$.

Next, let us investigate the leading holomorphic singularities for the heterotic model with three vector fields S, T and U. In the same way as above, we find the perturbative expression for the holomorphic mass for elementary string states $\mathcal{M}$ for this S-T-U model,

$$\mathcal{M}(T,U) = M_0 + M_1\,TU + iM_2\,T + iM_3\,U \quad . \tag{4.31}$$

The holomorphic mass (4.31) can vanish at certain lines/points in the perturbative moduli space, namely at $T = U \neq 1, e^{\frac{i\pi}{6}}$, at $T = U = 1$ and at $T = U = e^{\frac{i\pi}{6}}$. At these lines/points, the $U(1)^2$ associated with the T and the U moduli get enhanced to $U(1)\times SU(2)$, $SU(2)^2$ and $SU(3)$, respectively. Hence, the number of additional vector multiplets at these lines/points is two, four and six, respectively. As before, one expects the leading holomorphic singularities to be given by (4.25). More precisely, in the chamber $T > U$, one expects that, as $T \to U$,

$$\mathcal{F}^{(g)}(T,U) \propto -2\left(\frac{1}{T-U}\right)^{2(g-1)} \quad , \tag{4.32}$$

whereas as $T \to U = 1$

$$\mathcal{F}^{(g)}(T,U) \propto -4\left(\frac{1}{T-1}\right)^{2(g-1)} \quad , \tag{4.33}$$

and finally, as $T \to U = \rho = e^{\frac{i\pi}{6}}$,

$$\mathcal{F}^{(g)}(T,U) \propto -6\left(\frac{1}{T-\rho}\right)^{2(g-1)} \quad . \tag{4.34}$$

The unique holomorphic modular covariant generalisation of (4.32) transforming as in (4.15) under modular transformations, is as follows (in the chamber $T > U$)

$$\mathcal{F}^{(g>1)}(T,U) = \beta_g\,\hat{D}_T^{g-1}\hat{D}_U^{g-1}\log(j(T)-j(U)) \quad , \tag{4.35}$$

where

$$\beta_g = -\frac{1}{\pi}\left(\frac{a}{2}\right)^g \frac{B_{2g}}{g(2g-2)(2g-2)!}(-)^g \,. \quad (g>1) \tag{4.36}$$

The relative factor of 2 between (4.35) and (4.27) is a reflection of the fact that in the S-T-U model twice as many states as in the S-T model become massless at $T = 1$.

Finally, it should be pointed out that there can also be subleading holomorphic singularities in the Wilsonian couplings $\mathcal{F}^{(g)}$ as well as subleading non-holomorphic singularities in the full non-holomorphic couplings $\mathcal{F}^{(g)\,\mathrm{cov}}$. Clearly, the latter ones are uniquely determined by the anomaly equations, as we will show in the remaining subsections for the case of $\mathcal{F}^{(2)\,\mathrm{cov}}$ and $\mathcal{F}^{(3)\,\mathrm{cov}}$ in the S-T-U model.

4.4 The modular covariant section $\mathcal{F}^{(2)\,\mathrm{cov}}$ in the S-T-U model

In this section, we will solve the holomorphic anomaly equations (4.20) for $\mathcal{F}^{(2)\,\mathrm{cov}}$. First consider the non–holomorphic $\mathcal{F}^{(1)\,\mathrm{cov}}$ in the weak-coupling limit $S+\bar{S}\to\infty$. In the chamber $T>U$ [19, 14, 15],

$$\mathcal{F}^{(1)\,\mathrm{cov}} = aS_{\mathrm{inv}} + h^{(1)\mathrm{cov}}_{\mathrm{inv}}\,, \tag{4.37}$$

where $h^{(1)\mathrm{cov}}_{\mathrm{inv}}$ decomposes into two terms,

$$h^{(1)\mathrm{cov}}_{\mathrm{inv}} = \frac{b_{\mathrm{grav}}}{2\pi}\left(\hat{K}(T,\bar{T},U,\bar{U}) + \log\eta^{-2}(T)\,\eta^{-2}(U)\right) + \beta_1\,\log(j(T)-j(U))\,. \tag{4.38}$$

Note that $\mathcal{F}^{(1)\,\mathrm{cov}}$ solves the heterotic version of the anomaly equation (3.16). The second term is modular invariant, as is the real part of the first term. However, holomorphic derivatives of $h^{(1)\,\mathrm{cov}}_{\mathrm{inv}}$ constitute modular forms, and they are the quantities that will play a role below. Furthermore, b_{grav} denotes the gravitational beta function. In the standard normalization [38], $a=24$ and $b_{\mathrm{grav}}=528$. The last term proportional to β_1 represents the holomorphic singularity of $\mathcal{F}^{(1)\,\mathrm{cov}}$ [19, 14], which is not covered by the arguments of the previous section. The coefficient β_1 equals

$$\beta_1 = \frac{2}{\pi}\,. \tag{4.39}$$

Since the anomaly equation (4.20) and the one that follows from it by interchanging T and U are linear, we can solve them term by term. We distinguish three different terms. The first one requires to solve the equation

$$[\partial_{\bar{T}}\mathcal{F}^{(2)\,\mathrm{cov}}]_1 = \frac{a^2}{(T+\bar{T})^2}\,\partial_U V_{\mathrm{inv}}\,, \qquad [\partial_{\bar{U}}\mathcal{F}^{(2)\,\mathrm{cov}}]_1 = \frac{a^2}{(U+\bar{U})^2}\,\partial_T V_{\mathrm{inv}}\,, \tag{4.40}$$

where $V_{\mathrm{inv}} = V-\sigma-\bar{\sigma}$, as before, with [17]

$$\sigma(T,U) = -\tfrac{1}{2}\partial_T\partial_U h + \tfrac{1}{8}L(T,U)\,, \quad \text{with} \quad L(T,U) = -\frac{4}{\pi}\log(j(T)-j(U))\,. \tag{4.41}$$

We recall that V_{inv} is not only invariant under target–space duality transformations, but also finite everywhere inside the perturbative moduli space.

The integrability of (4.40) is easily shown by using the following identities, which are special cases of (4.6),

$$\partial_{\bar{T}}\partial_T V = \frac{2V + \partial_T\partial_U h + \partial_{\bar{T}}\partial_{\bar{U}}\bar{h}}{(T+\bar{T})^2}\,, \qquad \partial_{\bar{U}}\partial_U V = \frac{2V + \partial_T\partial_U h + \partial_{\bar{T}}\partial_{\bar{U}}\bar{h}}{(U+\bar{U})^2}\,. \tag{4.42}$$

Here we have used the explicit form of V given in (4.18). Note that, although the numerator is modular invariant, it is singular at the lines/points of semi-classical gauge–symmetry enhancement. The above integrability relation is to be expected, as we know that the full anomaly equations are integrable from the very beginning, as we discussed in section 3.

By making use of (4.42), it follows that the following ansatz for $\mathcal{F}^{(2)\,\mathrm{cov}}$,

$$[\mathcal{F}^{(2)\,\mathrm{cov}}]_1 = \tfrac{1}{2}a^2\,\partial_T\partial_U V_{\mathrm{inv}} + \tfrac{1}{8}a^2\left(\hat{G}_2(T,\bar{T})\,\partial_U L + \hat{G}_2(U,\bar{U})\,\partial_T L\right)\,, \tag{4.43}$$

solves (4.40). The modular form $\hat{G}_2$ was defined in (4.29). Note that $\mathcal{F}^{(2)\,\mathrm{cov}}$, given in (4.43), has the appropriate modular weight and that it also does not contribute to the leading holomorphic singularity given in (4.35). Here, we have made use of the freedom of adding holomorphic terms to $\mathcal{F}^{(2)\,\mathrm{cov}}$ in order to arrive at (4.43).

Next, consider solving the second and the third term of the anomaly equation, which correspond to solving

$$[\partial_{\bar{T}}\mathcal{F}^{(2)\,\mathrm{cov}}]_{2+3} = -\frac{a}{(T+\bar{T})^2}\,\partial_U h_{\mathrm{inv}}^{(1)\mathrm{cov}}\,, \quad [\partial_{\bar{U}}\mathcal{F}^{(2)\,\mathrm{cov}}]_{2+3} = -\frac{a}{(U+\bar{U})^2}\,\partial_T h_{\mathrm{inv}}^{(1)\mathrm{cov}}\,, \tag{4.44}$$

where $h_{\mathrm{inv}}^{(1)\mathrm{cov}}$ is given in (4.38). Consider the first term in $h_{\mathrm{inv}}^{(1)\mathrm{cov}}$,

$$\begin{aligned}
[\partial_{\bar{T}}\mathcal{F}^{(2)\,\mathrm{cov}}]_2 &= -\frac{a\,b_{\mathrm{grav}}}{2\pi}\frac{1}{(T+\bar{T})^2}\partial_U\Big(\hat{K}(T,\bar{T},U,\bar{U}) + \log\eta^{-2}(T)\,\eta^{-2}(U)\Big)\,,\\
[\partial_{\bar{U}}\mathcal{F}^{(2)\,\mathrm{cov}}]_2 &= -\frac{a\,b_{\mathrm{grav}}}{2\pi}\frac{1}{(U+\bar{U})^2}\partial_T\Big(\hat{K}(T,\bar{T},U,\bar{U}) + \log\eta^{-2}(T)\,\eta^{-2}(U)\Big)\,,
\end{aligned} \tag{4.45}$$

which is solved by the modular covariant expression

$$[\mathcal{F}^{(2)\,\mathrm{cov}}]_2 = -\frac{a\,b_{\mathrm{grav}}}{2\pi}\hat{G}_2(T,\bar{T})\,\hat{G}_2(U,\bar{U})\,. \tag{4.46}$$

The above expression (4.46) was also recently derived in [36] in the context of heterotic $N=1$ string vacua.

Then consider the second term of $h_{\mathrm{inv}}^{(1)\mathrm{cov}}$,

$$[\partial_{\bar{T}}\mathcal{F}^{(2)\,\mathrm{cov}}]_3 = -\frac{a}{(T+\bar{T})^2}\partial_U\mathcal{F}^{(1)}\,, \qquad [\partial_{\bar{U}}\mathcal{F}^{(2)\,\mathrm{cov}}]_3 = -\frac{a}{(U+\bar{U})^2}\partial_T\mathcal{F}^{(1)}, \tag{4.47}$$

where

$$\mathcal{F}^{(1)} = \beta_1\,\log(j(T) - j(U)). \tag{4.48}$$

One can proceed in two ways. One can either solve (4.47) directly, or one can use the symplectic formalism developed in section 3 in order to construct a modular covariant non-holomorphic solution $\mathcal{F}^{(2)\,\mathrm{cov}}$ to the anomaly equations (4.47). This is so, because the construction of symplectic functions (3.8) is based on the existence of a holomorphic section $\mathcal{F}^{(1)}$, and the $\mathcal{F}^{(1)}$ given in (4.48) is precisely such an object. Thus, we will use the latter strategy in the following.

The explicit expression for $\mathcal{F}^{(2)\,\mathrm{cov}}$, which one obtains from (3.8) for the S-T-U model, is derived in appendix B. In the limit $S+\bar{S}\to\infty$, it is given by

$$\mathcal{F}^{(2)\,\mathrm{cov}}(T,U,\bar{T},\bar{U}) = \mathcal{F}^{(2)}(T,U) + \frac{a}{T+\bar{T}}\partial_U\mathcal{F}^{(1)} + \frac{a}{U+\bar{U}}\partial_T\mathcal{F}^{(1)} \quad , \tag{4.49}$$

where the holomorphic $\mathcal{F}^{(1)}$ is given by (4.48). Note that the non-holomorphic part of (4.49) is not modular covariant. Just as before, in order to obtain a target–space duality covariant expression for $\mathcal{F}^{(2)\,\mathrm{cov}}$, one has to include an appropriate *holomorphic* $\mathcal{F}^{(2)}(T,U)$ in (4.49). One such appropriate $\mathcal{F}^{(2)}(T,U)$ is given by

$$\mathcal{F}^{(2)}(T,U) = a\,G_2(T)\,\partial_U\mathcal{F}^{(1)} + a\,G_2(U)\,\partial_T\mathcal{F}^{(1)} + \beta_2\,\partial_T\partial_U\log(j(T)-j(U)) \quad , \tag{4.50}$$

where G_2 was introduced in (4.28). The last term in (4.50) is the leading holomorphic singularity (4.35). Combining (4.50) with (4.49) yields

$$[\mathcal{F}^{(2)\,\mathrm{cov}}]_3 = a\left(\hat{G}_2(T,\bar{T})\,\partial_U\mathcal{F}^{(1)} + \hat{G}_2(U,\bar{U})\,\partial_T\mathcal{F}^{(1)}\right) + \beta_2\,\partial_T\partial_U\log(j(T)-j(U))\,. \tag{4.51}$$

Expression (4.51) is manifestly modular covariant, and it can be checked in a straightforward way that it solves the anomaly equation (4.47).

Thus, the solution $\mathcal{F}^{(2)\,\mathrm{cov}}$ to the anomaly equations (4.20) is given by the sum of (4.43), (4.46) and (4.51), that is by

$$\begin{aligned}\mathcal{F}^{(2)\,\mathrm{cov}} \;=\;& \tfrac{1}{2}a^2\,\partial_T\partial_U V_{\mathrm{inv}} + \tfrac{1}{8}a^2\left(\hat{G}_2(T,\bar{T})\,\partial_U L + \hat{G}_2(U,\bar{U})\,\partial_T L\right)\\ & -\frac{a\,b_{\mathrm{grav}}}{2\pi}\hat{G}_2(T,\bar{T})\,\hat{G}_2(U,\bar{U})\\ & +a\left(\hat{G}_2(T,\bar{T})\,\partial_U\mathcal{F}^{(1)} + \hat{G}_2(U,\bar{U})\,\partial_T\mathcal{F}^{(1)}\right) + \beta_2\,\partial_T\partial_U\log(j(T)-j(U)).\end{aligned} \tag{4.52}$$

Note that there is no freedom left in adding further holomorphic modular forms to (4.52). The reason is that target-space duality and the knowledge of the leading holomorphic singularities that are associated with known gauge–symmetry enhancement points/lines, fixes the structure of $\mathcal{F}^{(2)\,\mathrm{cov}}$ completely. Thus, the $\mathcal{F}^{(2)\,\mathrm{cov}}$ given in (4.52) is the full solution to (4.20). In appendix C, we will give the power-series expansion of $\mathcal{F}^{(2)\,\mathrm{cov}}(T,\bar{T},U,\bar{U})$ in the limit $\bar{T},\bar{U}\to\infty$.

4.5 The modular covariant section $\mathcal{F}^{(3)\,\mathrm{cov}}$ in the S-T-U model

In this section, we will solve the holomorphic anomaly equations (4.19) for $\mathcal{F}^{(3)\,\mathrm{cov}}$. In order to solve them, we will need to evaluate $D_U\partial_T\partial_U V_{\mathrm{inv}}$. We find that

$$D_U\partial_T\partial_U V_{\mathrm{inv}} = \tfrac{1}{2}D_T^2\partial_U^3 h - \tfrac{1}{8}D_U\partial_T\partial_U L\ . \tag{4.53}$$

We recall that the term $\partial_U^3 h$ appearing on the right hand side of (4.53) transforms covariantly under target–space duality transformations, and that it has modular weights -2 and 4 under $SL(2,\mathbf{Z})_T$ and $SL(2,\mathbf{Z})_U$ transformations, respectively [17]. Thus, both terms appearing on the right hand side of (4.53) have modular weights 2 and 4 under $SL(2,\mathbf{Z})_T$ and $SL(2,\mathbf{Z})_U$ transformations.

We will now solve the anomaly equations (4.19) for $\mathcal{F}^{(3)\,\mathrm{cov}}$ using (4.52) as the input on the right hand side of (4.19). Since these anomaly equations are linear, we will solve them separately for each of the three lines of (4.52). First consider the anomaly equations based on the first line of (4.52). Using (4.53), it follows that these differential equations can be written out as follows

$$\begin{aligned}[\partial_{\bar T}\mathcal{F}^{(3)\,\mathrm{cov}}]_1 \;=\; & -\frac{a^3}{(T+\bar T)^2}\Big[\tfrac{1}{4}D_T^2\partial_U^3 h(T,U) - \tfrac{1}{16}D_U\partial_T\partial_U L(T,U) \\ & +\tfrac{1}{8}D_U\Big(\hat G_2(U,\bar U)\,\partial_T L(T,U) + \hat G_2(T,\bar T)\,\partial_U L(T,U)\Big)\Big]\ ,\end{aligned} \tag{4.54}$$

and likewise with T and U interchanged. Note that (4.54) is modular covariant. It can be solved in a rather straightforward way, and the solution to (4.54) reads

$$\begin{aligned}[\mathcal{F}^{(3)\,\mathrm{cov}}]_1 \;=\; & \tfrac{1}{4}a^3\Big[\hat G_2(T,\bar T)\,D_T^2\partial_U^3 h + \hat G_2^2(T,\bar T)\,D_T\partial_U^3 h + \tfrac{2}{3}\hat G_2^3(T,\bar T)\,\partial_U^3 h \\ & +\hat G_2(U,\bar U)\,D_U^2\partial_T^3 h + \hat G_2^2(U,\bar U)\,D_U\partial_T^3 h + \tfrac{2}{3}\hat G_2^3(U,\bar U)\,\partial_T^3 h\Big] \\ & -\tfrac{1}{16}a^3\Big[\hat G_2(T,\bar T)\,D_U\partial_T\partial_U L + \hat G_2(U,\bar U)\,D_T\partial_T\partial_U L \\ & -2\hat G_2(T,\bar T)\,\hat G_2(U,\bar U)\,\partial_T\partial_U L\Big] \\ & +\tfrac{1}{8}a^3\Big[\hat G_2(T,\bar T)\,D_U\hat G_2(U,\bar U)\,\partial_T L + \hat G_2(U,\bar U)\,D_T\hat G_2(T,\bar T)\,\partial_U L \\ & +\tfrac{1}{2}\hat G_2^2(T,\bar T)\,D_U\partial_U L + \tfrac{1}{2}\hat G_2^2(U,\bar U)\,D_T\partial_T L \\ & -\hat G_2^2(U,\bar U)\,\hat G_2(T,\bar T)\,\partial_T L - \hat G_2^2(T,\bar T)\,\hat G_2(U,\bar U)\,\partial_U L\Big]\ .\end{aligned} \tag{4.55}$$

Next, consider solving (4.19) based on the second line of (4.52). The solution reads

$$\begin{aligned}[\mathcal{F}^{(3)\,\mathrm{cov}}]_2 \;=\; & -\frac{a^2\,b_{\mathrm{grav}}}{4\pi}\Big[\hat G_2^2(T,\bar T)\,D_U\hat G_2(U,\bar U) + \hat G_2^2(U,\bar U)\,D_T\hat G_2(T,\bar T) \\ & -\hat G_2^2(T,\bar T)\,\hat G_2^2(U,\bar U)\Big]\ .\end{aligned} \tag{4.56}$$

Note that one can in principle add further modular covariant holomorphic terms to either (4.55) or (4.56), such as $[G_2^2(T) - \partial_T G_2(T)][G_2^2(U) - \partial_U G_2(U)]$, for example [22].

Finally, consider solving (4.19) based on the third line of (4.52). One can again use the symplectic formalism developed in section 3 in order to construct a modular covariant non-holomorphic solution $\mathcal{F}^{(3)\,\mathrm{cov}}$. The explicit expressions for $\mathcal{F}^{(3)\,\mathrm{cov}}$, which one obtains from (3.8) for the S-T-U model, are presented in appendix B. They are given by

$$\begin{aligned}
\mathcal{F}^{(3)\,\mathrm{cov}}(T,U,\bar{T},\bar{U}) &= \mathcal{F}^{(3)}(T,U) + a\Big[\frac{\partial_T\mathcal{F}^{(2)}}{U+\bar{U}} + \frac{\partial_U\mathcal{F}^{(2)}}{T+\bar{T}} + \frac{2\mathcal{F}^{(2)}}{(T+\bar{T})(U+\bar{U})}\Big] \\
&\quad + \tfrac{1}{2}a^2\Big[\frac{1}{(T+\bar{T})^2}\partial_U^2\mathcal{F}^{(1)} + \frac{1}{(U+\bar{U})^2}\partial_T^2\mathcal{F}^{(1)} \\
&\qquad + \frac{2}{(T+\bar{T})(U+\bar{U})}\partial_T\partial_U\mathcal{F}^{(1)}\Big] \\
&\quad + \frac{a^2}{(T+\bar{T})(U+\bar{U})}\Big[\frac{\partial_U\mathcal{F}^{(1)}}{T+\bar{T}} + \frac{\partial_T\mathcal{F}^{(1)}}{U+\bar{U}}\Big],
\end{aligned} \tag{4.57}$$

where the holomorphic $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are given in equations (4.48) and (4.50), respectively. Again, note that the non-holomorphic part of $\mathcal{F}^{(3)\,\mathrm{cov}}$ is not modular covariant. As before, in order to obtain a target–space duality covariant expression for $\mathcal{F}^{(3)\,\mathrm{cov}}$, one has to include an appropriate holomorphic $\mathcal{F}^{(3)}(T,U)$ in (4.57). One choice is as follows

$$\begin{aligned}
\mathcal{F}^{(3)}(T,U) &= a^2\Big[\tfrac{1}{2}G_2^2(T)\,\partial_U^2\mathcal{F}^{(1)} + \tfrac{1}{2}G_2^2(U)\partial_T^2\mathcal{F}^{(1)} \\
&\qquad -\Big(G_2(U)\,G_2^2(T) - \partial_T G_2(T)\,G_2(U)\Big)\partial_U\mathcal{F}^{(1)} \\
&\qquad -\Big(G_2(T)\,G_2^2(U) - \partial_U G_2(U)\,G_2(T)\Big)\partial_T\mathcal{F}^{(1)} + G_2(T)\,G_2(U)\,\partial_T\partial_U\mathcal{F}^{(1)}\Big] \\
&\quad -a\beta_2\Big[G_2(T)\,G_2(U)\,\partial_T\partial_U\log(j(T)-j(U)) \\
&\qquad -G_2(T)\,\partial_T\partial_U^2\log(j(T)-j(U)) - G_2(U)\,\partial_U\partial_T^2\log(j(T)-j(U))\Big] \\
&\quad +\beta_3\hat{D}_T^2\hat{D}_U^2\log(j(T)-j(U))\,.
\end{aligned} \tag{4.58}$$

The last term in (4.58) is the leading holomorphic singularity (4.35). Inserting (4.58) into (4.57) yields

$$\begin{aligned}
[\mathcal{F}^{(3)\,\mathrm{cov}}]_3 &= a^2\Big[\tfrac{1}{2}\hat{G}_2^2(T)\,D_U^2\mathcal{F}^{(1)} + D_U\left(\hat{G}_2(U)\,\hat{G}_2(T)\,D_T\mathcal{F}^{(1)}\right) \\
&\qquad +\tfrac{1}{2}\hat{G}_2^2(U)\,D_T^2\mathcal{F}^{(1)} + D_T\left(\hat{G}_2(T)\,\hat{G}_2(U)\,D_U\mathcal{F}^{(1)}\right)\Big] \\
&\quad +a\beta_2\Big[\hat{G}_2(T)\,D_T D_U^2\log(j(T)-j(U)) + \hat{G}_2(U)\,D_U D_T^2\log(j(T)-j(U)) \\
&\qquad -2\hat{G}_2(T)\,\hat{G}_2(U)\,D_U D_T\log(j(T)-j(U))\Big]
\end{aligned}$$

$$-a^2\Big[\hat{G}_2(T)\,\hat{G}_2(U)\,D_U D_T\mathcal{F}^{(1)}+\hat{G}_2(T)\,\hat{G}_2^2(U)\,D_T\mathcal{F}^{(1)}$$
$$+\hat{G}_2(U)\,\hat{G}_2^2(T)\,D_U\mathcal{F}^{(1)}\Big]+\beta_3\hat{D}_T^2\hat{D}_U^2\log(j(T)-j(U))\,. \qquad (4.59)$$

The expression (4.59) is modular covariant, and again it can be checked that it solves the holomorphic anomaly equation (4.19) based on (4.51). In principle, one could again add additional holomorphic modular covariant terms of the correct modular weight to (4.59). Such additional terms could, for instance, describe holomorphic subleading singularities. An example of such a singular subleading term is $[G_2^2(T)-\partial_T G_2(T)][G_2^2(U)-\partial_T G_2(U)]\log(j(T)-j(U))$.

Thus, the full solution $\mathcal{F}^{(3)\,\mathrm{cov}}$ (up to possible additional holomorphic modular forms of weight 4) to the anomaly equation (4.19) is given by the sum of (4.55), (4.56) and (4.59).

5 Conclusions

In this paper we have discussed the computation of the moduli-dependent, higher-order gravitational couplings in the perturbative limit of four-dimensional heterotic string compactifications with $N = 2$ space–time supersymmetry. Our method of calculating the couplings $\mathcal{F}^{(g)}$ consisted in solving the anomaly equation (4.9), where we took as an input the known one-loop expressions for the heterotic prepotential $\mathcal{F}^{(0)}$ and the gravitational couplings $\mathcal{F}^{(1)}$, as well as the leading holomorphic singularity structure of the higher $\mathcal{F}^{(g)}$. Subsequently we imposed target–space duality covariance. For the S-T-U model the result can thus be explicitly written in terms of modular forms. The anomaly equation can be derived, either from string–string duality by taking the type-II anomaly equation (3.15) in the limit of a large Kähler-class modulus, so as to make contact with the perturbative heterotic side, or by exploiting arguments based on symplectic covariance. The results of these two approaches coincide and lead to the anomaly equation (4.9).

Let us briefly recall the relevant steps in the derivation of $\mathcal{F}^{(2)\,\mathrm{cov}}$, given in (4.52), and of $\mathcal{F}^{(3)\,\mathrm{cov}}$, given as the sum of eqs. (4.55), (4.56) and (4.59), for the S-T-U model. As already emphasized, the one-loop $\mathcal{F}^{(1)}$ is both holomorphic and duality invariant near the region $T = U$ in the moduli space. We can therefore use the symplectic formalism to compute the corresponding non-holomophic part of $\mathcal{F}^{(2)}$. Adding an appropriate holomorphic function yields a covariant expression for $\mathcal{F}^{(2)}$. Second, $\mathcal{F}^{(1)}$ contains further terms which are non-singular in the limit $T \to U$, namely a one-loop Green–Schwarz piece from the invariant dilaton and a term proportional to b_{grav}, related to the one-loop Kähler

and σ-model anomalies. In order to obtain the covariant $\mathcal{F}^{(2)\,\mathrm{cov}}$ that belongs to these non-singular terms, we explicitly solved the holomorphic anomaly equation. For the higher $\mathcal{F}^{(g)}$, this procedure can be continued. Those terms which are derived from the holomorphic and invariant $\mathcal{F}^{(1)}$ are constructed by using the covariant derivative (3.5). The remaining terms, which are obtained from the Green–Schwarz term and from the term proportional to b_{grav} in $\mathcal{F}^{(1)}$, follow from the holomorphic anomaly equation.

In summary, we have found a very transparent and systematic structure in the process of solving the anomaly equation on the heterotic side and in the form of its solutions. Then, in principle, important information about the higher-genus instanton numbers of the relevant dual Calabi–Yau three-fold can be obtained through the second-quantized mirror map [25].

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A Some asymptotic results for $S + \bar{S} \to \infty$

We wish to consider certain expressions in the limit that $S + \bar{S} \to \infty$, based on the class of functions (4.1). We expand the corresponding Kähler potential (4.3) as

$$K(S, \bar{S}, T, \bar{T}) \approx -\log(S + \bar{S}) + \hat{K}(T, \bar{T}) - \frac{V(T, \bar{T})}{S + \bar{S}} + \cdots , \tag{A.1}$$

where the ellipses denote terms of higher order in $(S + \bar{S})^{-1}$. $\hat{K}$ and the Green-Schwarz term V were already given in (4.4) and (4.5).

First consider the components of the metric in the large-$(S+\bar{S})$ limit,

$$g_{A\bar{B}} = \begin{pmatrix} \dfrac{1}{(S+\bar{S})^2} - \dfrac{2V}{(S+\bar{S})^3} & \dfrac{V_{\bar{b}}}{(S+\bar{S})^2} \\ \dfrac{V_a}{(S+\bar{S})^2} & \hat{g}_{a\bar{b}} - \dfrac{V^{a\bar{b}}}{S+\bar{S}} \end{pmatrix} + \cdots ,$$

$$g^{A\bar{B}} = \begin{pmatrix} (S+\bar{S})^2 + 2(S+\bar{S})V & -V^{\bar{b}} \\ -V^a & \hat{g}^{a\bar{b}} + \dfrac{V^{a\bar{b}}}{S+\bar{S}} \end{pmatrix} + \cdots . \qquad \text{(A.2)}$$

Here $\hat{g}_{a\bar{b}}$ is the metric associated with the Kähler potential $\hat{K}(T,\bar{T})$, which is used to raise and lower indices in the above formula. The explicit expressions are shown in (4.4). Sub- and superscripts $a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$ attached to V denote differentiations of V with respect to $T^a, T^b, \ldots$ and $\bar{T}^a, \bar{T}^b, \ldots$ that are covariant under diffeomorphisms in the Kähler space parametrized by these coordinates. Although these covariantizations do not contribute to the expressions above, they will contribute to some of the formulae below.

Similarly we evaluate the following expressions for large $S+\bar{S}$, without making any assumptions on the large-$(S+\bar{S})$ behaviour of the functions $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}$ on which the various derivatives act,

$$\begin{aligned} e^{2K}\, D^{\bar{S}}\mathcal{F}_1\, D^{\bar{S}}\mathcal{F}_2 &\approx e^{2\hat{K}}(S+\bar{S})^2\Big(\mathcal{D}_S\mathcal{F}_1 - \frac{V^a}{(S+\bar{S})^2}\mathcal{D}_a\mathcal{F}_1\Big)\Big(\mathcal{D}_S\mathcal{F}_2 - \frac{V^a}{(S+\bar{S})^2}\mathcal{D}_a\mathcal{F}_2\Big)\,, \\ e^{2K}\, D^{\bar{a}}\mathcal{F}_1\, D^{\bar{S}}\mathcal{F}_2 &\approx e^{2\hat{K}}\left(V^{\bar{a}}\mathcal{D}_S\mathcal{F}_1 - \mathcal{D}^{\bar{a}}\mathcal{F}_1\right)\Big(-\mathcal{D}_S\mathcal{F}_2 + \frac{V^a}{(S+\bar{S})^2}\mathcal{D}_a\mathcal{F}_2\Big)\,, \\ e^{2K}\, D^{\bar{a}}\mathcal{F}_1\, D^{\bar{b}}\mathcal{F}_2 &\approx \frac{e^{2\hat{K}}}{(S+\bar{S})^2}\left(V^{\bar{a}}\mathcal{D}_S\mathcal{F}_1 - \mathcal{D}^{\bar{a}}\mathcal{F}_1\right)\left(V^{\bar{a}}\mathcal{D}_S\mathcal{F}_2 - \mathcal{D}^{\bar{a}}\mathcal{F}_2\right)\,, \end{aligned} \qquad \text{(A.3)}$$

and

$$\begin{aligned} e^{2K}\, D^{\bar{S}}\, D^{\bar{S}}\mathcal{F} &\approx e^{2\hat{K}}\Big[(S+\bar{S})^2\mathcal{D}_S^2 - 2V^a\,\mathcal{D}_S\mathcal{D}_a \\ &\qquad + \frac{V^aV^b}{(S+\bar{S})^2}\mathcal{D}_a\mathcal{D}_b + 2(S+\bar{S})\mathcal{D}_S + \frac{\mathcal{V}^a}{(S+\bar{S})^2}\mathcal{D}_a\Big]\mathcal{F}\,, \\ e^{2K}\, D^{\bar{a}}D^{\bar{S}}\mathcal{F} &\approx e^{2\hat{K}}\Big[-V^{\bar{a}}\mathcal{D}_S^2 + \mathcal{D}^{\bar{a}}\mathcal{D}_S + \frac{V^{\bar{a}}V^a}{(S+\bar{S})^2}\mathcal{D}_S\mathcal{D}_a \\ &\qquad - \frac{V^a}{(S+\bar{S})^2}\,\mathcal{D}^{\bar{a}}\mathcal{D}_a + \frac{\mathcal{V}^{\bar{a}}}{(S+\bar{S})^2}\mathcal{D}_S - \frac{V^{\bar{a}a}}{(S+\bar{S})^2}\mathcal{D}_a\Big]\mathcal{F}\,, \end{aligned}$$

$$e^{2K}\,D^{\bar a}D^{\bar b}\mathcal{F} \;\approx\; \frac{e^{2\hat K}}{(S+\bar S)^2}\Big[V^{\bar a}V^{\bar b}\mathcal{D}_S^2-(V^{\bar a}\mathcal{D}^{\bar b}+V^{\bar b}\mathcal{D}^{\bar a})\mathcal{D}_S \\ +\mathcal{D}^{\bar a}\mathcal{D}^{\bar b}-\frac{V^{\bar a\bar b}}{(S+\bar S)^2}\mathcal{D}_S\Big]\mathcal{F}\,, \tag{A.4}$$

where $\mathcal{V}^a$ and $\mathcal{V}^{\bar a}$ are more complicated objects quadratic in V and its derivatives. The derivatives $\mathcal{D}$ contain the Levi-Civita connection associated with the Kähler metric $\hat g_{a\bar b}$ and a Kähler connection proportional to $\partial_A K$.

Finally, consider the matrix N_{IJ} and evaluate its leading behaviour for large $S+\bar S$. First we decompose

$$N_{IJ}=(N_0)_{IJ}+(\Delta N)_{IJ}\,, \tag{A.5}$$

where the first term corresponds to the first term in (4.1) and ΔN contains the terms related to the function h. Its nonvanishing matrix elements are

$$\begin{aligned}(\Delta N)_{00} &= -2h+2h_aT^a-h_{ab}T^aT^b+\{\text{h.c.}\}\,,\\ (\Delta N)_{0a}=(\Delta N)_{a0} &= ih_a-ih_{ab}T^b+\{\text{h.c.}\}\,,\\ (\Delta N)_{ab} &= h_{ab}+\{\text{h.c.}\}\,.\end{aligned} \tag{A.6}$$

Ignoring the contributions from the function $h(t)$, which we will deal with later in perturbation theory, we first use evaluate the matrix n^{AB} introduced in section 3. It reads

$$n^{AB}=\begin{pmatrix} 3(S+\bar S)e^{\hat K} & -3(T+\bar T)^b e^{\hat K}\\ -3(T+\bar T)^a e^{\hat K} & \dfrac{-3}{S+\bar S}\Big(\eta^{ab}-(T+\bar T)^a(T+\bar T)^b e^{\hat K}\Big)\end{pmatrix}. \tag{A.7}$$

With the aid of this result and (3.22) we evaluate the inverse of $(N_0)_{IJ}$,

$$(N_0)^{IJ}=\frac{e^{\hat K}}{S+\bar S}\begin{pmatrix} 2 & i(S-\bar S) & i(T-\bar T)^b\\ i(S-\bar S) & 2S\bar S & -S\,T^b-\bar S\,\bar T^b\\ i(T-\bar T)^a & -S\,T^a-\bar S\,\bar T^a & -\frac12\eta^{ab}\,e^{-\hat K}+T^a\bar T^b+\bar T^aT^b\end{pmatrix}. \tag{A.8}$$

Using (A.6) and (A.8) we can easily determine N^{IJ} for large $S+\bar S$.

B A class of non-holomorphic corrections to $\mathcal{F}^{(2)\,\mathrm{cov}}$ and $\mathcal{F}^{(3)\,\mathrm{cov}}$ in the S-T-U model

In (3.24) we gave the expressions for $\mathcal{F}^{(2)\,\mathrm{cov}}$ and $\mathcal{F}^{(3)\,\mathrm{cov}}$ based on a general cubic function $\mathcal{F}^{(0)}$, in terms of the holomorphic Wilson coefficient functions $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ and $\mathcal{F}^{(3)}$. Here we

specialize these expressions to the case of the S-T-U model. These sections are solutions of the truncated anomaly equation (3.13). At the end we exhibit their behaviour for large $S+\bar{S}$. We remind the reader that, according to (3.8), $\mathcal{F}^{(1)\,\mathrm{cov}}$ remains holomorphic.

Let us first give the Kähler potential for this case,

$$K(S,\bar{S},T,\bar{T},U,\bar{U}) = -\log(S+\bar{S}) - \log(T+\bar{T}) - \log(U+\bar{U})\,. \tag{B.1}$$

Furthermore we list the explicit expressions for the inverse of the matrix N_{IJ} defined in (3.6),

$$N^{IJ} = e^K \begin{pmatrix} 2 & i(S-\bar{S}) & i(T-\bar{T}) & i(U-\bar{U}) \\ i(S-\bar{S}) & 2\,S\bar{S} & -ST-\bar{S}\,\bar{T} & -SU-\bar{S}\,\bar{U} \\ i(T-\bar{T}) & -ST-\bar{S}\,\bar{T} & 2\,T\bar{T} & -TU-\bar{T}\,\bar{U} \\ i(U-\bar{U}) & -SU-\bar{S}\,\bar{U} & -TU-\bar{T}\,\bar{U} & 2\,U\bar{U} \end{pmatrix}, \tag{B.2}$$

and the matrix $\widehat{n}^{AB}$ defined in the text below (3.24),

$$\widehat{n}^{AB} = -6\,e^K \begin{pmatrix} 0 & (S+\bar{S})(T+\bar{T}) & (S+\bar{S})(U+\bar{U}) \\ (S+\bar{S})(T+\bar{T}) & 0 & (T+\bar{T})(U+\bar{U}) \\ (S+\bar{S})(U+\bar{U}) & (T+\bar{T})(U+\bar{U}) & 0 \end{pmatrix}. \tag{B.3}$$

With these definitions it is somewhat tedious but straightforward to calculate the expressions for $\mathcal{F}^{(g)\,\mathrm{cov}}$ for this model, by using (3.24). We remind the read that $\mathcal{F}^{(1)\,\mathrm{cov}}$ remains equal to the holomorphic function $\mathcal{F}^{(1)}$. The expression for $\mathcal{F}^{(2)\,\mathrm{cov}}$ takes the form

$$\mathcal{F}^{(2)\,\mathrm{cov}} = \mathcal{F}^{(2)} + \frac{1}{U+\bar{U}}\,\mathcal{F}^{(1)}_S\,\mathcal{F}^{(1)}_T + \frac{1}{T+\bar{T}}\,\mathcal{F}^{(1)}_S\,\mathcal{F}^{(1)}_U + \frac{1}{S+\bar{S}}\,\mathcal{F}^{(1)}_T\,\mathcal{F}^{(1)}_U\,, \tag{B.4}$$

where the subscripts S, T and U denote ordinary partial derivatives with respect to these coordinates. For large $S+\bar{S}$ this reduces to (4.49) provided one makes use of (4.7).

The expression for $\mathcal{F}^{(3)\,\mathrm{cov}}$ is much longer and reads

$$\begin{aligned} \mathcal{F}^{(3)\,\mathrm{cov}} \;=\; & \mathcal{F}^{(3)} + \frac{1}{U+\overline{U}}\left[\mathcal{F}^{(2)}_S\mathcal{F}^{(1)}_T + \mathcal{F}^{(2)}_T\mathcal{F}^{(1)}_S\right] \\ & + \frac{1}{T+\overline{T}}\left[\mathcal{F}^{(2)}_U\mathcal{F}^{(1)}_S + \mathcal{F}^{(2)}_S\mathcal{F}^{(1)}_U\right] + \frac{1}{S+\bar{S}}\left[\mathcal{F}^{(2)}_T\mathcal{F}^{(1)}_U + \mathcal{F}^{(2)}_U\mathcal{F}^{(1)}_T\right] \\ & + \frac{2\mathcal{F}^{(2)}}{(S+\bar{S})(T+\bar{T})(U+\bar{U})}\left[(S+\bar{S})\mathcal{F}^{(1)}_S + (T+\bar{T})\mathcal{F}^{(1)}_T + (U+\bar{U})\mathcal{F}^{(1)}_U\right] \\ & + \frac{\mathcal{F}^{(1)}_{SS}}{2(T+\bar{T})^2(U+\bar{U})^2}\left[(T+\bar{T})\,\mathcal{F}^{(1)}_T + (U+\bar{U})\mathcal{F}^{(1)}_U\right]^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathcal{F}^{(1)}_{TT}}{2(U+\bar{U})^2(S+\bar{S})^2}\Big[(U+\bar{U})\,\mathcal{F}^{(1)}_U+(S+\bar{S})\mathcal{F}^{(1)}_S\Big]^2\\
&+\frac{\mathcal{F}^{(1)}_{UU}}{2(S+\bar{S})^2(T+\bar{T})^2}\Big[(S+\bar{S})\,\mathcal{F}^{(1)}_S+(T+\bar{T})\mathcal{F}^{(1)}_T\Big]^2\\
&+\frac{\mathcal{F}^{(1)}_{ST}}{(S+\bar{S})(T+\bar{T})(U+\bar{U})^2}\Big[(U+\bar{U})^2(\mathcal{F}^{(1)}_U)^2+(U+\bar{U})(T+\bar{T})\,\mathcal{F}^{(1)}_U\mathcal{F}^{(1)}_T\\
&\qquad\qquad+(S+\bar{S})(U+\bar{U})\,\mathcal{F}^{(1)}_S\mathcal{F}^{(1)}_U+(T+\bar{T})(S+\bar{S})\,\mathcal{F}^{(1)}_T\mathcal{F}^{(1)}_S\Big]\\
&+\frac{\mathcal{F}^{(1)}_{TU}}{(T+\bar{T})(U+\bar{U})(S+\bar{S})^2}\Big[(S+\bar{S})^2(\mathcal{F}^{(1)}_S)^2+(U+\bar{U})(T+\bar{T})\,\mathcal{F}^{(1)}_U\mathcal{F}^{(1)}_T\\
&\qquad\qquad+(S+\bar{S})(U+\bar{U})\,\mathcal{F}^{(1)}_S\mathcal{F}^{(1)}_U+(T+\bar{T})(S+\bar{S})\,\mathcal{F}^{(1)}_T\mathcal{F}^{(1)}_S\Big]\\
&+\frac{\mathcal{F}^{(1)}_{SU}}{(U+\bar{U})(S+\bar{S})(T+\bar{T})^2}\Big[(T+\bar{T})^2(\mathcal{F}^{(1)}_T)^2+(U+\bar{U})(T+\bar{T})\,\mathcal{F}^{(1)}_U\mathcal{F}^{(1)}_T\\
&\qquad\qquad+(S+\bar{S})(U+\bar{U})\,\mathcal{F}^{(1)}_S\mathcal{F}^{(1)}_U+(T+\bar{T})(S+\bar{S})\mathcal{F}^{(1)}_T\mathcal{F}^{(1)}_S\Big]\\
&+\Big[\frac{\mathcal{F}^{(1)}_T}{U+\bar{U}}+\frac{\mathcal{F}^{(1)}_U}{T+\bar{T}}\Big]\Big[\frac{\mathcal{F}^{(1)}_U}{S+\bar{S}}+\frac{\mathcal{F}^{(1)}_S}{U+\bar{U}}\Big]\Big[\frac{\mathcal{F}^{(1)}_S}{T+\bar{T}}+\frac{\mathcal{F}^{(1)}_T}{S+\bar{S}}\Big]\\
&+\frac{2\mathcal{F}^{(1)}_S\,\mathcal{F}^{(1)}_T\,\mathcal{F}^{(1)}_U}{(S+\bar{S})(T+\bar{T})(U+\bar{U})}\,.
\end{aligned}
\tag{B.5}
$$

At large $S+\bar{S}$, this expression reduces to the expression (4.57), where again we made use of (4.7).

C Power-series expansion of $\mathcal{F}^{(2)\,\mathrm{cov}}$

In the following, we will consider the power-series expansion of $\mathcal{F}^{(2)\,\mathrm{cov}}$, given in (4.52), in the limit where $\bar{T}\to\infty,\bar{U}\to\infty$. In this limit, $\mathcal{F}^{(2)\,\mathrm{cov}}$ turns into

$$
\begin{aligned}
&\mathcal{F}^{(2,\mathrm{top})}(T,U)= \qquad\qquad (C.1)\\
&\quad\tfrac{1}{4}a^2\partial_T^2\partial_U^2 h+\Big(\frac{a^2}{4\pi}+\beta_2\Big)\,\partial_T\partial_U\log(j(T)-j(U))-\frac{a\,b_{\mathrm{grav}}}{2\pi}G_2(T)G_2(U)\\
&\quad+a\Big(\beta_1-\frac{a}{2\pi}\Big)\Big[G_2(T)\,\partial_U\log(j(T)-j(U))+G_2(U)\,\partial_T\log(j(T)-j(U))\Big]\,.
\end{aligned}
$$

In [17] it was shown that the one-loop correction h to the prepotential satisfies

$$
\partial_T^3 h(T,U)=2\,\frac{E_4(T)\,E_4(U)\,E_6(U)\,\eta^{-24}(U)}{j(T)-j(U)}\,,
\tag{C.2}
$$

as well as a similar equation with T and U interchanged. The exact expression for h was given in [20] in terms of a power-series expansion. Here we draw attention to the fact

that this correction carries a different normalization in [17] and [20] and was denoted by $h^{(1)}$, while in this paper $h^{(1)}$ denotes the one-loop correction to $\mathcal{F}^{(1)}$. Using the result of [20], it follows that the first line of (C.2) has the following power-series expansion

$$\partial_T^2 \partial_U^2 h \;=\; -4\pi \sum_{k,l} k^2\, l^2\, c_1(kl)\, \frac{e^{-2\pi(kT+lU)}}{(1-e^{-2\pi(kT+lU)})^2}\;, \tag{C.3}$$

where the integers k and l can take the following values: either $k=1, l=-1$ or $k>0, l=0$ or $k=0, l>0$ or $k>0, l>0$. The constants $c_1(n)$ are determined by [20]

$$\frac{E_4(T)\, E_6(T)}{\eta^{24}(T)} = \sum_{n=-1}^{\infty} c_1(n)\, q^n\,, \tag{C.4}$$

where q and T are related by $q = e^{-2\pi T}$.

The second line of (C.2), on the other hand, has the following power-series expansion in the chamber $T > U$ [20]

$$\partial_T \partial_U \log(j(T)-j(U)) = -(2\pi)^2 \sum_{k,l} k\, l\, c(kl)\, \frac{e^{-2\pi(kT+lU)}}{(1-e^{-2\pi(kT+lU)})^2}\,, \tag{C.5}$$

where the constants $c(n)$ are determined by

$$j(T) - 744 = \sum_{n=-1}^{\infty} c(n)\, q^n\;, \tag{C.6}$$

and where the integers k and l can take the same values as indicated above.

The third line of (C.2) can be expanded as follows. Using [35]

$$G_2(T) = -\frac{\pi}{6}\left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, q^n\right)\;, \tag{C.7}$$

where $\sigma_1(n)$ denotes the sum of the divisors of n, one finds that

$$G_2(T)\, G_2(U) = \frac{\pi^2}{36}\Big(1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)\, q_1^m (1+q_3^m) + 576 \sum_{n,m=1}^{\infty} \sigma_1(m)\, \sigma_1(n)\, q_1^{n+m}\, q_3^n\Big), \tag{C.8}$$

where $q_i = e^{-2\pi t_i}$, and where $t_1 = U, t_3 = T-U$.

A closer look at the fourth line in (C.2) shows that it is regular as $T \to U$, that is as $q_3 \to 1$. Using that

$$\begin{aligned}
\partial_T \log(j(T)-j(U)) \;&=\; 2\pi\Big(1 + \frac{e^{-2\pi(T-U)}}{1-e^{-2\pi(T-U)}} + \sum_{k,l>0} k\, c(kl)\, \frac{e^{-2\pi(kT+lU)}}{1-e^{-2\pi(kT+lU)}}\Big) \\
\partial_U \log(j(T)-j(U)) \;&=\; 2\pi\Big(-\frac{e^{-2\pi(T-U)}}{1-e^{-2\pi(T-U)}} + \sum_{k,l>0} l\, c(kl)\, \frac{e^{-2\pi(kT+lU)}}{1-e^{-2\pi(kT+lU)}}\Big)\,,
\end{aligned} \tag{C.9}$$

it follows that

$$
\begin{aligned}
&G_2(T)\,\partial_U \log(j(T)-j(U)) + G_2(U)\,\partial_T \log(j(T)-j(U)) = \\
&2\pi\Big(G_2(U) + \frac{q_3}{1-q_3}(G_2(U)-G_2(T)) \\
&\quad + G_2(U)\sum_{k,l>0} k\,c(kl)\frac{q_1^{k+l}q_3^k}{1-q_1^{k+l}q_3^k} + G_2(T)\sum_{k,l>0} l\,c(kl)\frac{q_1^{k+l}q_3^k}{1-q_1^{k+l}q_3^k}\Big)\,.
\end{aligned}
\tag{C.10}
$$

First evaluate

$$
\begin{aligned}
\frac{q_3}{1-q_3}(G_2(U)-G_2(T)) &= 4\pi\frac{q_3}{1-q_3}\sum_{n=1}^{\infty}\sigma_1(n)\,(q_1^n - q_3^n q_1^n) \\
&= 4\pi\sum_{n=1}^{\infty}\sigma_1(n)\,q_1^n\,(q_3+q_3^2+\cdots+q_3^n)\,.
\end{aligned}
\tag{C.11}
$$

Then, it follows that

$$
G_2(U) + \frac{q_3}{1-q_3}(G_2(U)-G_2(T)) = -\frac{\pi}{6}\Big(1 - 24\sum_{n=1}^{\infty}\sum_{s=0}^{n}\sigma_1(n)\,q_1^n\,q_3^s\Big)\,. \tag{C.12}
$$

And finally, using that

$$
\frac{1}{1-q_1^{k+l}q_3^k} = \sum_{m=0}^{\infty} q_1^{m(k+l)}\,q_3^{mk}\,, \tag{C.13}
$$

one finds that

$$
\begin{aligned}
&G_2(U)\sum_{k,l>0} k\,c(kl)\,\frac{q_1^{k+l}\,q_3^k}{1-q_1^{k+l}q_3^k} + G_2(T)\sum_{k,l>0} l\,c(kl)\,\frac{q_1^{k+l}\,q_3^k}{1-q_1^{k+l}q_3^k} = \\
&-\frac{\pi}{6}\sum_{k,l>0}\sum_{m=0}^{\infty}(-)^m\Big[(k+l)\,c(kl)\,q_1^{(m+1)(k+l)}\,q_3^{(m+1)k} \\
&\qquad -24\sum_{n=1}^{\infty} k\,c(kl)\,\sigma_1(n)\,q_1^{(m+1)(k+l)+n}\,q_3^{(m+1)k} \\
&\qquad -24\sum_{n=1}^{\infty} l\,c(kl)\,\sigma_1(n)\,q_1^{(m+1)(k+l)+n}\,q_3^{(m+1)k+n}\Big]\,.
\end{aligned}
\tag{C.14}
$$

Thus, it follows that $\mathcal{F}^{(2,\mathrm{top})}$ has the following power-series expansion

$$
\begin{aligned}
\mathcal{F}^{(2,\mathrm{top})} &= \pi\Big[-a^2\sum_{k,l} k^2\,l^2\,c_1(kl)\,\frac{q_1^{k+l}\,q_3^k}{(1-q_1^{k+l}q_3^k)^2} \\
&\quad -(a^2+4\pi\beta_2)\sum_{k,l} k\,l\,c(kl)\frac{q_1^{k+l}\,q_3^k}{(1-q_1^{k+l}q_3^k)^2} \\
&\quad -\frac{a\,b_{\mathrm{grav}}}{72}\Big(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)\,q_1^m(1+q_3^m) + 576\sum_{n,m=1}^{\infty}\sigma_1(m)\,\sigma_1(n)\,q_1^{n+m}\,q_3^n\Big)\Big]
\end{aligned}
\tag{C.15}
$$

$$-\frac{\pi a(2\pi\beta_1 - a)}{6}\Big[1 - 24\sum_{n=1}^{\infty}\sum_{s=0}^{n}\sigma_1(n)\,q_1^n\,q_3^s + \sum_{k,l>0}\sum_{m=0}^{\infty}(-)^m\Big[(k+l)\,c(kl)\,q_1^{(m+1)(k+l)}\,q_3^{(m+1)k} - 24\sum_{n=1}^{\infty}k\,c(kl)\,\sigma_1(n)\,q_1^{(m+1)(k+l)+n}\,q_3^{(m+1)k} - 24\sum_{n=1}^{\infty}l\,c(kl)\,\sigma_1(n)\,q_1^{(m+1)(k+l)+n}\,q_3^{(m+1)k+n}\Big]\Big]\,.$$

Thus, we see that the instanton expansion of $\mathcal{F}^{(2,\mathrm{top})}$ is determined in terms of known coefficients $c_1(n)$, $c(n)$ and $\sigma_1(n)$.

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