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BPS States of Exceptional Non-Critical Strings

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We study the BPS states of non-critical strings which arise for zero size instantons of exceptional groups. This is accomplished by using F-theory and M-theory duals and by employing mirror symmetry to compute the degeneracy of membranes wrapped around 2-cycles of the Calabi–Yau threefold. We find evidence for a number of novel physical phenomena, including having infinitely many light states with the first lightest state including a nearly massless gravitino.

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1. Introduction

One of the most important aspects in the recent developments in string dualities has been the fact that string compactifications which classically look singular have often a reinterpretation in terms of light solitonic states. These light solitonic states can in general have p -spatial coordinates whose ‘tension’ T_p goes to zero. In general if there are many such objects the most relevant light states correspond to the one for which the relevant mass parameter $T_p^{\frac{1}{p+1}}$ is the smallest [1][2]. Typically in string theory when there are multitude of light states all $T_p \propto \epsilon$ as $\epsilon \rightarrow 0$, and thus the lightest states correspond to the solitons with the smallest value of p . For example for type IIA and type IIB strings the relevant light states typically correspond to D-branes wrapped around vanishing cycles whose volume is proportional to ϵ . Given the fact that the type IIA (type IIB) has even (odd) D-branes [3] we see that the most relevant lightest states are either massless particles ($p = 0$) or tensionless strings ($p = 1$).

The case of massless solitonic particles is easier to understand from the viewpoint of quantum field theories. Examples of this class include type IIA compactifications near an ADE singularity of $K3$ where the resulting solitons are responsible for the enhanced ADE gauge symmetry [4] or type IIB near a conifold singularity of a Calabi-Yau threefold which leads to a massless hypermultiplet [5]. Another example of this is when $SO(32)$ instantons shrink to zero size which leads to enhanced gauge symmetry with some matter multiplets [6]. This is not to say that all the cases with massless solitonic particles are easy to understand in terms of local Lagrangians, as for example one can encounter simultaneous massless electric and magnetic particles [7].

The case with $p = 1$ is more unfamiliar, and probably physically more interesting, as it may signal the appearance of new critical quantum field theories based on strings rather than particles, even if one ignores gravitational effects. An example of this includes type IIB near an ADE singularity of $K3$ [1][8]. Given our relative lack of familiarity with quantum field theories based on loops it is natural to try to get a first order understanding of these theories by relating them to the cases where the relevant light degrees of freedom are again particles. An attempt in this direction is to further compactify on a circle. In the resulting theory the most relevant lightest states are now again particles corresponding to the tensionless string wrapped around the circle. In this way for example the physics of type IIB near an ADE singularity times a circle becomes understandable as an ordinary Higgs mechanism in the resulting five dimensional theory.

From this analysis one may get the wrong impression that by a further compactification on a circle we will always obtain a situation with ‘simple physics’ in one lower

dimension. This turns out not to be the case. Roughly speaking what happens sometimes is that in the one lower dimensional theory there are infinitely many massless particles interacting with a tensionless string. At first sight this may appear puzzling as a string wrapped around a circle does not seem to have infinitely many light degrees of freedom in store. Moreover it appears to be more relevant than the unwrapped string. Technically the way this comes about is that due to a quantum effect the wrapped string has a smaller tension than the unwrapped string and thus as the wrapped string becomes tensionless, the unwrapped string still has a positive tension. However one can pass through this transition point beyond which the wrapped string formally acquires a *negative* tension. As we hit the second, and more interesting, transition point the unwrapped string becomes tensionless. Also at this point infinitely many of the BPS states of the wrapped string become massless. Moreover the tensionless string is as relevant as these massless particles. Thus the situation appears roughly as a tensionless string interacting with infinitely many light particles resulting from wrapped states of the same string which due to quantum effects has a negative tension. From the fact that after the first transition and before the second transition the wrapped string acquires a negative tension, one sees that one needs a better “dual picture” and this turns out to be provided by the geometry of special singularities of Calabi–Yau manifolds in the context of M-theory. A study of an example of this situation is one of the main aims of the present paper.

The class of theories we study correspond to $N = 1$ theories in $d = 6$, as is the case for instance for $E_8 \times E_8$ heterotic strings compactified on $K3$. It is natural to wonder what happens when an E_8 instanton shrinks to zero size. In this case one can use M-theory description of $E_8 \times E_8$ heterotic string [9] to gain insight into the nature of the singularity [10]: As an E_8 instanton shrinks it can be locally represented as a 5-brane residing on the 9-brane boundary corresponding to the E_8 whose instanton is shrinking in size. If we deform this situation by moving the 5-brane off the 9-brane one can see there is a tensionless string: Since the membrane of M-theory can end on both the 9-brane [9] and the 5-brane [11][12], in the limit the 5-brane and 9-brane touch we get a tensionless string corresponding to the boundary of the membrane in the common 6-dimensional space-time.

On the other hand heterotic strings on $K3$ are dual to F-theory compactifications on Calabi-Yau threefolds [13][14], and it is possible to isolate what corresponds to an E_8 instanton of zero size [15][14]. We will use this duality to study the spectrum of BPS states for heterotic strings on $K3 \times S^1$ when an E_8 instanton shrinks to zero size and compare it with the predictions of tensionless strings. To do this study we need to count the number of solitonic membranes of M-theory wrapped holomorphically around cycles

of the Calabi-Yau manifold. We use mirror symmetry to accomplish this. We find perfect agreement with predictions based on tensionless strings, subject to some very important subtleties. In particular the Calabi-Yau geometry refines the description of the tensionless string suggesting certain quantum corrections to the classical picture. We will also study the conjectured duals for E_d instantons of zero size [14] and find that they are naturally understandable as the E_8 non-critical string propagating in the presence of Wilson lines breaking E_8 to E_d (up to $U(1)$ factors).

The organization of this paper is as follows: In section 2 we set up the predictions of BPS states based on tensionless strings. In section 3 we set up the question of the BPS spectrum on the type II side (in the context of F-theory and M-theory) in the context of counting curves in del Pezzo surfaces sitting in the Calabi-Yau. In section 4 we compare the predictions and find agreement with expectations based on the tensionless strings. We also discuss some aspects of the BPS spectrum on the type II side which points towards new physics. In an appendix we discuss some technical aspects of the relevant singularity of the Calabi-Yau threefolds.

As we were completing this work an interesting paper appeared [16] which has some overlap with the present work.

2. BPS states from non-critical E_8 string

As mentioned in the last section in the context of M-theory when a 5-brane meets the 9-brane we have a situation dual to a small E_8 instanton of heterotic string. Actually, this is in the phase where the instanton has shrunk to zero size and we have ‘nucleated’ a 5-brane which has departed from the 9-brane world volume of the E_8 . A membrane stretched between the 5-brane and 9-brane lives as a string on the common 6-dimensional space-time. The tension for the string is proportional to the distance between the 5-brane and the 9-brane.

To better understand the properties of this non-critical string let us recall that if we have two parallel 9-branes the resulting string will be the heterotic string, where on each 9-brane lives an E_8 gauge symmetry, each inducing a (say) left-moving E_8 current algebra on the string. Recall also that if we have two parallel 5-branes we get a string in 6-dimension, which couples to an $N = 2$ tensor multiplet. The degrees of freedom on this string is best described by Green-Schwarz strings in 6 dimensions [17][18]. The light cone degrees of freedom are described by 4 left-moving fields and 4 right-moving fields each transforming in the spinor representation of the light cone group $O(4)$.

In the case at hand we have a non-critical string which has half the supersymmetry of the above six dimensional string resulting from stretched membrane between the two 5-branes, as well as having half the E_8 current algebra of the string resulting from a membrane stretched between two 9-branes. In fact we have one left-moving E_8 current algebra at level one and one right-moving spinor of $O(4)$ in the light cone gauge. In addition we have, in the light cone gauge the usual 4 transverse bosonic oscillators.

2.1. Prediction for BPS States

Now we further compactify on a circle and ask what are the BPS states which carry a winding charge of this non-critical string? This is a familiar situation encountered in the study of critical strings [19]. In the case at hand we do not know enough about the properties of the resulting non-critical string to rigorously derive the spectrum of BPS states, but we will follow the same line of argument as in the critical string case and derive what seems to be the reasonable BPS spectrum of states: Let (P_L, P_R) denote the left- and right-moving momenta of the string on the resulting circle. We have

$$(P_L, P_R) = \frac{1}{\sqrt{T}} \left(\frac{n}{2R} - mRT, \frac{n}{2R} + mRT \right)$$

where (n, m) denote the momentum and winding of the string around the circle and T denotes its tension. The overall factor of $\frac{1}{\sqrt{T}}$ in front is put to make (P_L, P_R) dimensionless. Note that

$$\frac{1}{2}(P_R^2 - P_L^2) = n \cdot m .$$

Let (L_0, \bar{L}_0) denote the left- and right-moving Hamiltonians corresponding to oscillating states of the strings. Since we are after BPS states we restrict to ground state oscillator states for the right-movers but arbitrary states on the left-movers. We also need to impose the equality of $L_0 = \bar{L}_0$. Assuming free oscillating states we have

$$L_0 = \frac{1}{2}P_L^2 + L_0^I + N$$

$$\bar{L}_0 = \frac{1}{2}P_R^2 + \bar{N}$$

where L_0^I corresponds to the internal degrees of freedom of the E_8 current algebra and N denotes the contribution of oscillators from 4 transverse bosonic states. Similarly \bar{N} denotes the oscillators contribution from 4 bosonic and 4 fermionic right-moving oscillator states. In principle there could have been a constant addition to L_0 as is for instance for bosonic strings. We will see that in the case at hand to agree with predictions based on

the type II side we do not need any shifts. To have a BPS state we set $\overline{N} = 0$ and we thus see that

$$N + L_0^I = \frac{1}{2}(P_R^2 - P_L^2) = n \cdot m \quad (2.1)$$

The mass of the corresponding BPS state is given by

$$M = |\sqrt{T}P_R| = \left| \frac{n}{2R} + mRT \right| \quad (2.2)$$

With more than one unit of winding, since we do not know enough about the non-critical string, it is difficult to decide whether we have stable new states at higher winding numbers, as would be the case in critical string theories or that the multiply wound state decays to singly wound states (we shall find in section 4, using the type II dual that they indeed do not form such bound states). To avoid this complication we concentrate on the case $m = 1$. Setting $m = 1$ in (2.2) we have

$$M_n = |\sqrt{T}P_R| = \left| \frac{n}{2R} + RT \right| \quad (2.3)$$

We see from (2.1) that if $d(n)$ denotes the degeneracy of BPS states with n -units of momentum around the circle with winding number one we have

$$q^{-\frac{1}{2}} \sum_{n=0}^{\infty} d(n)q^n = \frac{\chi_{E_8}(q)}{\eta(q)^4} = \frac{\theta_{E_8}(q)}{\eta(q)^{12}}$$

where we have used the fact that the internal L_0^I corresponds to a level one E_8 Kac-Moody algebra and can be viewed in the bosonized form as corresponding to 8 bosons compactified on the E_8 root lattice. We thus have 12 oscillators 4 of which are space-time oscillators and 8 are internal and are scalar. Note that we have thus learned that

$$d(0) = 1, d(1) = 252, d(2) = 5130, \dots$$

with masses given by

$$M_0 = RT, M_1 = RT + \frac{1}{R}, M_2 = RT + \frac{2}{R}, \dots$$

What about their space-time quantum numbers? Again because of our ignorance about non-critical strings it is difficult to judge a priori what the right-moving supersymmetric ground state should be. Formally one would think that the right-moving oscillator zero mode consists of a spinor of $O(4)$ in the light cone and so the space-time quantum numbers of the right-moving ground state must be half a hypermultiplet which transforms as $2(0,0) \oplus (1/2,0)$ of $O(4)$. Together with the inversely wound states, this would form a

full hypermultiplet as the ground state on the right-moving side. To get the full quantum number of the above BPS states we have to tensor this with the left-moving quantum numbers. There are two types of quantum numbers, corresponding to space-time as well as E_8 representations. The fact that they form E_8 representations is natural when we recall that locally when an E_8 instanton shrinks we expect at least *locally* to restore the E_8 and so all the states should form E_8 representations. The E_8 content of the BPS states can be easily deduced from the corresponding affine Kac-Moody degeneracies. As for the space-time quantum numbers, we have four transverse bosonic oscillators which transform as $(\frac{1}{2}, \frac{1}{2})$ of $O(4)$. This thus allows us to find the quantum number of all BPS states. For example the state with $n = 0$ is a hypermultiplet singlet of E_8 . The state with $n = 1$ is given by

$$[\mathbf{248}; 4(0,0) \oplus (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})] + [\mathbf{1}; 4(\frac{1}{2}, \frac{1}{2}) \oplus (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)] \quad (2.4)$$

where the first entry denotes the E_8 content of the state. Note that the spin of these states goes all the way up to spin 3/2 (in four dimensional terms). Similar decompositions can be done for all higher values of n as well. Note that as we start decreasing the tension towards zero the once wrapped state with no momentum, which is the lightest state becomes a massless hypermultiplet. The rest of the BPS states, including the one with $n = 1$ are still massive in this limit. So far we do not see any exotic physics and, modulo the question of multiple windings, the tensionless string seems to have produced only a massless hypermultiplet. As we will discuss in the next section there is an important subtlety which is difficult to see in this setup. We will find that the interesting physics is associated not with the point where the wrapped string becomes tensionless with $T = 0$ but actually when T of the wrapped string becomes *negative*. These are more clear from the viewpoints of F-theory and M-theory to which we turn to in the next section.

It is natural to ask what happens when instantons for other gauge groups shrink, for example that of other exceptional groups E_d . In the most standard form of the question this issue does not naturally arise in the six dimensional theories¹, because the only relevant gauge groups are $E_8 \times E_8$ or $SO(32)$. But the issue can naturally arise if we go to 5 dimensions.

A simple way to see this is to suppose we start with the $E_8 \times E_8$ heterotic strings in 10 dimensions and first go from 10 to 9 dimensions on a circle where we turn on a Wilson

¹ Actually in cases where we choose the instantons in a sub-bundle of E_8 the other cases may also occur in some special cases.

line which breaks one of the E_8 's to a smaller group² say $E_d \times U(1)^{8-d}$. The choices for such Wilson lines are parametrized by $8 - d$ real parameters A_i denoting the choice of the Wilson line in each of the $U(1)$'s. Now if we further compactify the heterotic string on $K3$ all the instantons will now reside in E_d . In such a case the question of E_d small instantons and their physical interpretation arises.

We can use the picture of non-critical tensionless E_8 string to gain insight into this situation as well. The idea is to consider the point at which the string is exactly tensionless with $T = 0$. At this point the E_8 gauge symmetry is locally restored as the E_8 instantons have zero size. Now it makes sense to turn on the Wilson lines and break $E_8 \rightarrow E_d \times U(1)^{8-d}$. This situation seems to be indistinguishable from having had started with E_d instantons and making their size shrink. Even though this is not a proof it seems very plausible that the two descriptions are identical. If so then we can learn about how the small E_d instantons behave as far as the spectrum of BPS states are concerned. They are simply the ones for E_8 strings deformed by the Wilson lines on the circle.

Let us denote the relevant Wilson lines as a vectors \mathbf{W}_α , $\alpha = 1 \dots 8 - d$ in the E_8 Cartan Lie algebra. As noted above the BPS states form representations of E_8 . Let Λ denote a weight of a state of one of the BPS states. Suppose it originally had mass M . After turning the Wilson line the mass shifts in the usual way by

$$M \rightarrow M + \sum_{\alpha} \Lambda \cdot \mathbf{W}_{\alpha} \tag{2.5}$$

This now splits the BPS states into states which form representations of $E_d \times U(1)^{8-d}$.

3. F-theory and M-theory Descriptions

We now analyze the same situation from a dual viewpoint using the duality between heterotic string on $K3$ and F-theory on elliptically fibered Calabi-Yau threefolds [13][14]. The transition from a 5-brane approaching a 9-brane to having an instanton of finite size is locally the same transition in the Calabi-Yau language as going from an elliptically fibered Calabi-Yau with base making a blow down from $\mathbf{F}_1 \rightarrow \mathbf{P}^2$ [6][14], where \mathbf{F}_1 is the Hirzebruch surface of degree 1. This turns out to be the same transition which occurs for strong coupling for heterotic strings with instanton numbers (11,13) (which is dual to

² The following discussion is unaltered even if we have other unbroken groups. We use this choice for later comparison with the F-theory conjecture for such cases [14].

F-theory on Calabi–Yau over \mathbf{F}_1) giving us another realization of the transition we are considering. At the transition point an “exceptional divisor” D which is a two sphere with self-intersection -1 shrinks to zero size [13]. The tensionless string in six dimensions is to be identified with the three brane of type IIB wrapped around D [8].

There are two relevant Kähler classes for this transition of the Calabi-Yau: the Kähler class of D which we denote by k_D and the Kähler class of the elliptic curve k_E . Clearly the tension of the string in six dimensions $T \propto k_D$. In the F-theory limit the Kähler class of the elliptic curve is not dynamical and can be thought of as formally being put to zero. More precisely [20] if we compactify further on a circle of radius R down to five dimensions we have an equivalence with M-theory on the same elliptic Calabi-Yau with $k_E \propto 1/R$. In the limit as $R \rightarrow \infty$ we obtain the F-theory compactification in six dimensions.

As it turns out the nature of the above transition is different between 6 and 5 dimensions (suggesting that there is a quantum correction in the 5-dimensional theory) as has been elaborated in [14] which we will now review. In the six dimensional case there is only one class k_D , as $k_E = 0$ and the transition takes place when $k_D = 0$. In the five dimensional theory there are actually two transition points: Let us fix the radius (or equivalently fix k_E) and start decreasing k_D to approach the transition point (see Fig. 1). When $k_D = 0$, D has shrunk to zero size and we have the tensionless string in 6 dimensions. But the interesting transition occurs further down when we take $k_D < 0$. What this means is that we have done a ‘flop’ on D . The actual transition point is when we reach $k_D = -k_E$. At this point an entire 4-cycle which is the E_8 del Pezzo surface (\mathbf{P}^2 blown up at 8 points) which we denote by \mathbf{B}_8 has shrunk to zero size. In terms of the wrapped string this means formally setting the tension to a negative value. We identify

$$k_E = \frac{1}{2R}, \quad k_D = RT \tag{3.1}$$

Note that there really are two physical transitions. At the first transition point where $k_D = 0$ we have a 2-sphere shrunk to zero size. In M-theory the membrane can wrap around this two sphere and give rise to a massless particle. In fact this situation has already been analyzed [15] with the result that at this point one has a single massless hypermultiplet. In fact this is what was expected based on the tensionless string description discussed in the previous section where one obtains a massless hypermultiplet. However as we have already emphasized the interesting transition is associated with a different point at which at least formally this string acquires a *negative tension*. This is also the point where the unwrapped string becomes tensionless. If we analytically continue the formula obtained from the viewpoint of our original string we see that among the BPS states we

considered the string which wraps around the circle *and* carries one unit of momentum about the circle now becomes massless, i.e. one would expect that at this transition point there are 252 massless states. Using the BPS counting of states also for the type II side we will verify below that this is indeed a correct supposition. At this second transition point we can wrap membranes about any 2-cycle on \mathbf{B}_8 and get a massless particle; the 252 states just discussed should be among them. At this point the unwrapped tensionless string is associated with the 5-brane of M-theory wrapped around \mathbf{B}_8 . If we denote the volume of \mathbf{B}_8 by ϵ^2 , then the tension of the resulting string goes as $T \sim \epsilon^2$. As far as dimensional argument the resulting mass scale $T^{1/2} = \epsilon$ is the same as the mass scale for the membranes wrapped around the 2-cycles of \mathbf{B}_8 and so are equally relevant [15]. If we compactify further to 4 dimensions where we get equivalence with type IIA theory on the same Calabi-Yau, we can have three different light states: a 5-brane wrapped around vanishing 4-cycle, a Dirichlet 4-brane wrapped around the vanishing 4-cycle and finally a Dirichlet 2-brane wrapped around any vanishing 2-cycle in \mathbf{B}_8 . It turns out that the simple dimensional analysis now suggests that the relevant light states are the 4-brane wrapped around vanishing 4-cycle which lead to massless particles; thus there may well be a local Lagrangian formulation of this theory in the four dimensional case. This should be very interesting to identify.

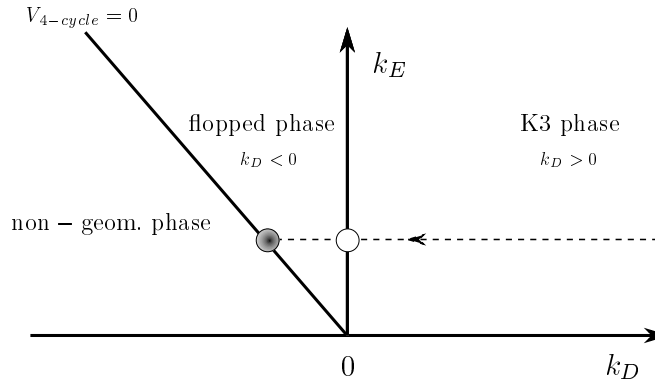


Fig.1: Phases of the Kähler moduli space. In 5d ($k_E \neq 0$) there is a new phase where the string tension of the wrapped string $\sim k_D$ becomes formally negative. Decreasing the tension of the wrapped string following the dashed line it becomes zero at the first transition point (white circle) where one gets only a massless hypermultiplet from winding the string around the circle. Continuing further to negative values one hits the second transition point (black circle) where the unwrapped string becomes tensionless. In addition there are infinitely many massless point like objects arising from wrapped states of the negative tension string interacting with the magnetically charged tensionless string.

Similarly it has been conjectured [14] that the small E_d instantons which one encounters in heterotic string compactifications below 6 dimensions are dual to Calabi-Yau threefolds where a del Pezzo surface of type \mathbf{B}_d (to be reviewed in the next subsection) shrink to zero size. In fact it should be possible to derive this conjecture directly from the case of small E_8 instantons following the physical idea of turning on Wilson lines discussed at the end of the next section and following the parallel geometric description of this operation (as has been done in similar contexts in [21]). At any rate we will find evidence for the above conjecture when we compare the predictions based on the tensionless strings and the geometry of the del Pezzo \mathbf{B}_d .

3.1. Geometry of the del Pezzo Surfaces

In the following we describe some aspects of the geometry involving the vanishing 2- and 4-cycles associated to a del Pezzo Surface sitting in the Calabi-Yau threefold. Although the properties of the tensionless strings associated to them are governed by the local geometry, the choice of an appropriate global embedding will be important in order to be able to extract important physical quantities such as the number of BPS states. In order to do that we have to identify the homology of the vanishing 4-cycle within that of a given Calabi-Yau threefold.

The relevant del Pezzo surfaces³ \mathbf{B}_k can be constructed by blowing up k generic points P_i on \mathbf{P}^2 as i runs from 1 to k where $1 \leq k \leq 8$. The divisor classes of \mathbf{B}_k are thus the class of lines l in \mathbf{P}^2 and the k exceptional divisors D_i lying above the points P_i , whose class we denote by e_i . The number of nontrivial homology elements is $h^{0,0} = h^{2,2} = 1$ and $h^{1,1} = 1 + k$, which gives $k + 3$ as the Euler number. The non zero intersections in the (l, e_1, \dots, e_k) basis of $H^{1,1}(\mathbf{B}_k)$ are $l^2 = 1$ and $e_i^2 = -1$; moreover the anti-canonical class $\bar{\mathcal{K}}_k$ is $\bar{\mathcal{K}}_k = c_1(\mathbf{B}_k) = 3l - \sum_{i=1}^k e_i$.

A curve \mathcal{C} in the homology class $al + \sum_{i=1}^k b_i e_i$ intersects the line l a times and moreover passes b_i times through the points P_i . Its degree is

$$d_{\mathcal{C}} = \bar{\mathcal{K}}_k \cdot \mathcal{C} = 3a - \sum_{i=1}^k b_i \quad (3.2)$$

and its arithmetic genus can be obtained from the Plücker formula [23] as

$$g_{\mathcal{C}} = \frac{(a-1)(a-2)}{2} - \delta = 1 + \frac{1}{2}(\mathcal{C} \cdot \mathcal{C} - \bar{\mathcal{K}}_k \cdot \mathcal{C}), \quad (3.3)$$

³ See [22] Chap. V, Sect. 4 for a thorough exposition of this construction.

where δ is the number of nodes (ordinary double points). At each point P_i one has $b_i(b_i - 1)/2$ nodes and we assume the curves to be otherwise smooth. If there are δ' (κ) additional nodes (cusps) outside the P_i (3.3) is reduced by δ' (κ), hence (3.3) is merely an upper bound on the genus, known as arithmetic genus.

To count the number of irreducible curves of genus 0 note that for such curves $b_i \leq a$ as an irreducible curve of degree a in \mathbf{P}^2 cannot pass more than a times through any given point. To obtain the number of lines one enumerates therefore solutions of (3.2),(3.3) with $d_C = 1$, $g_C = 0$ and $b_i \leq a$. E.g. for \mathbf{B}_8 one finds seven classes $(a; b_1, \dots, b_8) = (0; -1, 0^7)$, $(1; 1^2, 0^6)$, $(2; 1^5, 0^3)$, $(3; 2, 1^6, 0)$, $(4; 2^3, 1^5)$, $(5; 2^6, 1^2)$ and $(6; 3, 2^7)$. Counted with the obvious multiplicities due to the permutation of the points P_i these are $240 = 8 + 28 + 56 + 56 + 56 + 28 + 8$ lines. By the same method the number of lines on the other \mathbf{B}_k $i = 7, \dots, 1$ turns out to be 56, 27, 16, 10, 6, 3, 1. A thorough counting of rational curves on B_6 has been performed in ref. [24] under the requirement that the curves pass through a sufficient number of points to avoid the problem of continuous moduli.

The fact that the lines are in representations of the Weyl groups of E_k for $k \geq 3$ can be recognized by looking at the classes $\mathcal{E} \in H^{1,1}(\mathbf{B}_k)$ fulfilling $\bar{\mathcal{K}}_k \cdot \mathcal{E} = 0$. These can be generated by $\mathcal{E}_1 = l - e_1 - e_2 - e_3$ and $\mathcal{E}_i = e_i - e_{i+1}$, which span the root lattice of E_k . Therefore one has an action of the Weyl group on $H^{1,1}(\mathbf{B}_k)$ and curves \mathcal{C} of given degree d_C must be organized representations of the Weyl group.

The del Pezzo surface \mathbf{B}_9 can be understood as the blow up of \mathbf{P}^2 at the nine intersection points of two cubic curves; this defines an elliptic fibration over \mathbf{P}^1 which can be described generically by the equation

$$y^2 = x^3 + x f_4(z') + g_6(z') \tag{3.4}$$

where z' is the coordinate on the base \mathbf{P}^1 and y, x the coordinates on the fibre. There are 12 singular fibres above the discriminant locus $\Delta = 4f_4^3 + 27g_6^2 = 0$; therefore the Euler characteristic χ is 12. Roughly speaking this is the structure of half a K3; in that case f_4, g_6 are replaced by polynomials of the double degree f_8, g_{12} leading to 24 singular fibres and two copies of E_8 in the intersection matrix.

There are $k + 1$ real parameters associated to the volumes of the independent holomorphic two cycles of \mathbf{B}_k . However not all of them will descend to Kähler moduli of the Calabi–Yau threefold X , the actual number depends on the rank r of the map $H^{1,1}(\mathbf{B}_k) \rightarrow H^{1,1}(X)$. To contract \mathbf{B}_k one has to shrink all the volumes of its holomorphic 2-cycles to zero; in particular this means that one has to restrict a codimension r locus in the Kähler moduli space of X . Moreover, if the class which measures the size of the

elliptic fibre of the del Pezzo is the same as that of the generic fibre of the elliptically fibred three-fold, the contraction necessarily collapses X to its base B . Whereas this is the usual limit one has to take in the F-theory compactification, this situation is clearly inappropriate in the context of M-theory compactifications to 5 dimensions.

In order to obtain contractions at codimension 1 in the moduli space which do not change the dimension of X as discussed before one has to perform flops on 2-cycles in the del Pezzo which support Kähler classes of X . This operation effectively contracts the exceptional divisors; in particular a flop of an exceptional divisor D_i results in a transition from \mathbf{B}_k to \mathbf{B}_{k-1} . On the other hand if the elliptic fibre of \mathbf{B}_k coincides with that of the elliptic fibration X , as in the cases considered below, one has always to flop the base \mathbf{P}^1 because of the variation of the fibre type above the base.

An important aspect of the global embedding of the vanishing cycle in the Calabi–Yau context is the fact that we can measure their volume, and therefore the mass of the associated BPS state. Let $K = \sum J_i t_i$ denote the volume form of the Calabi–Yau threefold X , where J_i are the Kähler classes and t_i special coordinates on the Kähler moduli space and let \hat{C}_i denote the dual homology classes fulfilling $J_i \hat{C}_j = \delta_{ij}$. Classically the area of a cycle in the class $\sum_i c_i \hat{C}_i$ is then given by $\sum_i c_i t_i$. In compactifications to four dimensions instanton corrections may shift the position of the singularity associated to a vanishing cycle away from the locus $\sum_i c_i t_i = 0$; as a consequence multiples $n\hat{C}_0$ of the class \hat{C}_0 of a vanishing cycle will have non-zero volume if this happens. In particular, in order that the volumes of 2-cycles in a vanishing 4-cycles indeed all vanish, it will be important that there is no such shift due to instanton corrections. The absence of a shift in the five-dimensional case can be inferred from the independence of the vector moduli space on the overall volume modulus of the Calabi–Yau manifold, which sits in a hypermultiplet and scales the action of the worldsheet instantons [25]. In the four-dimensional type IIA compactification one has to establish the coincidence of the classical volume with the exact vanishing period to exclude a quantum split of the areas associated to multiples of a given class.

In order to be a valid F-theory compactification we will choose our Calabi–Yau threefold X to be elliptically fibred; to make contact with the heterotic picture we will further require it to be a K3 fibration. As discussed previously these restrictions are not necessary conditions for the existence of the type of tensionless string transition we consider, which requires only the local geometry being that of a vanishing 4-cycle of the appropriate type and neither a global elliptic or K3 fibred structure. Rather these restrictions provide the appropriate global embedding for the local geometry which is convenient for the counting

of the BPS spectrum and the physical interpretation. In particular we take the elliptic fibre of the del Pezzo representing the vanishing 4-cycle to be that of the elliptic fibred Calabi–Yau. This is only possible if the base of the elliptic fibred Calabi–Yau is the Hirzebruch surface \mathbf{F}_1 [14]. The Weierstrass model for the elliptic fibred three-fold with base \mathbf{F}_n is defined by the equation

$$E_8 : y^2 = x^3 + x \sum_{k=-4}^{k=4} z^{4-k} f_{8-nk}(z') + \sum_{k=-6}^{k=6} z^{6-k} f_{12-nk}(z') \quad (3.5)$$

where $n = 1$ for \mathbf{F}_1 , y, x are the coordinates on the elliptic fibre E , z is the coordinate on the fibre F of \mathbf{F}_1 and z' the coordinate on its base B . Note that F is the base of a elliptic fibred K3, fibred itself over the base B . Note that the elliptic fibre is described by the simple elliptic singularity $\mathbf{P}^{1,2,3}[6]$ of E_8 type.

To extend the global description to the del Pezzo surfaces \mathbf{B}_d , $d = 6, 7$, consider the local form of the singularity of the vanishing 4-cycle in C^4 [26]:

$$\begin{aligned} E_8 : y^2 &= x^3 + x^2 f_2 + x f_4 + f_6 \\ E_7 : y^2 &= x^4 + x^3 f_1 + x^2 f_2 + x f_3 + f_4 \\ E_6 : \sum y^k x^l f_{3-k-l} &= 0, \end{aligned} \quad (3.6)$$

where f_n are homogeneous polynomials of degree n in two variables z' and z'' . Note that the elliptic fibre defined by setting z' and z'' constant is no longer of the generic type for $d < 8$ but corresponds to a symmetric torus. In other words, if we insist to keep the elliptic fibre of the del Pezzo to be identical to that of the Calabi–Yau fibration we have to consider elliptic fibred threefolds with the corresponding fibres of E_7 and E_6 type, $\mathbf{P}^{1,1,2}[4]$ and $\mathbf{P}^{1,1,1}[3]$, respectively.

After choosing a section and restricting to a patch, the defining equation of the elliptic fibred threefold with base \mathbf{F}_n replacing (3.5) becomes:

$$\begin{aligned} E_7 : y^2 &= x^4 + x^2 \sum_k f_{4-nk} z^{2-k} w^{2+k} \\ &+ x \sum_k f_{6-nk} z^{3-k} w^{3+k} + \sum_k f_{8-nk} z^{4-k} w^{4+k} \\ E_6 : y^3 + x^3 &= y \sum_{k,l} x^l w^{2-l+k} z^{2-l-k} f_{4-2l-nk} \\ &+ x \sum w^{2+k} z^{2-k} f_{4-nk} + \sum w^{3+k} z^{3-k} f_{6-nk} \end{aligned} \quad (3.7)$$

where f_n are homogeneous polynomials of the base variables z' , z'' . The terms which appear in (3.7) are restricted to respect the C^* symmetries

$$\begin{aligned} E_8 &:(6 - 3n, 4 - 2n, 1, 1, -n, 0), (6, 3, 0, 0, 1, 1) \\ E_7 &:(4 - 2n, 2 - n, 1, 1, -n, 0), (4, 2, 0, 0, 1, 1) \\ E_6 &:(2 - n, 2 - n, 1, 1, -n, 0), (2, 2, 0, 0, 1, 1) \end{aligned}$$

acting on (y, x, z', w', z, w) , where y, x are the coordinates of the elliptic fibre, z', w' those of the base of \mathbf{F}_n and z, w those of the fibre of \mathbf{F}_n . In this way one obtains series of Calabi–Yau threefolds with the corresponding elliptic fibres and base \mathbf{F}_n . Most of them have a simple representation in terms of hypersurfaces in weighted projective spaces

$$\begin{aligned} E_8 &:\mathbf{P}^{1,1,n,4+2n,6+3n}[12 + 6n] \\ E_7 &:\mathbf{P}^{1,1,n,2+n,4+2n}[8 + 4n] \\ E_6 &:\mathbf{P}^{1,1,n,2+n,2+n}[6 + 3n] \end{aligned}$$

respectively; in general the appropriate description is in terms of toric varieties as described in the appendix. For \mathbf{F}_1 one recognizes the singularities (3.6) as the local equations in the neighborhood of $z' = z'' = 0$. The elliptic fibred Calabi–Yau threefolds obtained in this way have hodge numbers $(h^{1,1}, h^{2,1}) = (4, 148)$ and $(5, 101)$, respectively. As described in [14][15] these fibrations based on \mathbf{F}_1 have Higgs branches which describe F-theory on an elliptically fibred threefold with base \mathbf{P}^2 and hodge numbers related to the one with base \mathbf{F}_1 by [14]

$$h^{1,1} = h^{1,1} - k, \quad h^{2,1} = h^{2,1} + c_d - k$$

In the present case we have $k = 1$ and therefore the spectrum on \mathbf{P}^2 is expected to be $(3, 165)$ and $(4, 112)$, respectively; it is straightforward to check that the transition leads to corresponding elliptic fibred threefolds are described by hypersurfaces in $\mathbf{P}^{1,1,1,3,6}$ and $\mathbf{P}^{1,1,1,3,3}$, respectively.

In fact one can show using methods similar to [27] that the three Calabi–Yau manifolds $X_{(3,243)}^{(3)}$, $X_{(4,148)}^{(4)}$ and $X_{(5,101)}^{(5)}$ are connected by extremal transitions where precisely the right number of rational curves are blown down to provide the new hypermultiplets on the side with the larger number of complex structure moduli. From the viewpoint of F-theory, in the six dimensional limit, where the size of the elliptic fiber goes to zero, the $(4, 148)$ model and $(5, 101)$ model can be viewed as a subset of the $(3, 243)$ model where we have tuned the complex structure on the threefold in a particular way. To make a transition we have to give the elliptic fiber a finite size which is only allowed if we further compactify on a circle to 5 dimensions. This is the geometrical realization of turning on the Wilson lines on the circle. A more detailed description of these manifolds for the E_7 and E_6 cases as toric varieties is given in the appendix.

4. Counting holomorphic curves with mirror symmetry

We are interested in BPS states, which arise from wrapping membranes around supersymmetric two cycles. They correspond to the holomorphic curves in the Calabi-Yau manifold X [28]. One can naturally ask whether the BPS states are represented by higher genus curves? This for instance is naturally the case for holomorphic 2-cycles in $K3$ which are represented by higher genus curves [29]. However even in that case it has been shown in [30] that if one allows genus 0 curves with nodal singularities, it will in effect have the full informations about the higher genus cases as well. The results we shall find below by studying the genus 0 curves in the case of del Pezzo also seem to take account of such singular maps automatically and thus in effect contain the information about the higher genus curves as well.

There is also the question of whether the holomorphic maps come in isolated sets or are part of a continuous moduli space. In the latter case mirror map computes the Euler class of an appropriate bundle (of ‘anti-ghost zero modes’) on moduli space of holomorphic cycles. If the bundle coincides with the tangent bundle of moduli space, this would simply give the Euler class of moduli space, which can be interpreted as the ‘net number’ of BPS states (the notion of ‘net number’ can be defined taking into account that some can pair up to become non-BPS representations). We have not proven that when we have a family of holomorphic curves, mirror map computes the relevant topological number for the net number of BPS states, but in some cases that we could check this, it turned out to be so, which leads us to speculate about the general validity of such a statement.

First consider the Calabi-Yau $X^{(3)}$ with hodge numbers ($h^{1,1} = 3, h^{2,1} = 243$), which is an elliptic fibration with a single section over the Hirzebruch surface \mathbf{F}_1 . This Calabi-Yau has two geometrical phases, which we have discussed in the previous sections and will also be described explicitly in terms of toric geometry in the appendix .

In the first phase the manifold is a K_3 fibration and the Kähler cone is spanned by three classes: i) C_F the fibre \mathbf{P}^1 of the \mathbf{F}_1 , ii) C_D the exceptional section \mathbf{P}^1 of the \mathbf{F}_1 and iii) C_E , a curve in the elliptic fibre. The Gromov-Witten invariants of rational curves, i.e. the number of holomorphic spheres with degree d_F, d_D, d_E w.r.t. these classes, n_{d_F, d_D, d_E} , can be counted (modulo the subtleties noted above) using mirror symmetry. As discussed before the relevant classes for the counting of states of the tensionless string are C_D and C_E and we will sometimes denote the relevant n_{0, d_D, d_E} by n_{d_D, d_E} . Here we have the following invariants.

	d_E	0	1	2	3	4	5	6
d_D								
0			480	480	480	480	480	480
1		1	252	5130	54760	419895	2587788	13630694
2				-9252	-673760	-20534040	-389320128	-5398936120
3					848628	115243155	6499779552	219488049810
4						-114265008	-23064530112	-1972983690880

Table 1: Invariants of $X^{(3)}$ for rational curves with degree $d_F = 0$.

Note that in terms of the tensionless string description d_D and d_E denote the winding number and the momentum quantum of the nearly tensionless string wrapped around the circle. The first line of this table is special in that $d_D = 0$ corresponds to no winding of tensionless strings. So this part is just the Kaluza-Klein momentum excitations on circle of the corresponding massless states in the 6-dimensional theory. Given the fact that the net number of hypermultiplets minus the vector multiplets (in 4-dimensional terms) is $-\chi/2$ and that for the manifold with $(h^{11}, h^{21}) = (3, 243)$ we have $-\chi = 480$ we get a perfect match with the corresponding computation from the mirror map⁴. This is a case where in fact one can show that the moduli space of holomorphic curves is not isolated and is in fact a copy of the Calabi-Yau itself making us gain faith in the meaning of the numbers computed by the mirror map. Note that (because of the appearance of the Euler number) this first row of the table is not universal and depends on which Calabi-Yau we used to realize this transition. This is *not* the case with the other rows in the table, corresponding to non-vanishing winding states of the tensionless strings which turn out to be universal. In fact we have checked that in the class of elliptic fibred threefolds whose elliptic fibre is at the same time the fibre of the del Pezzo (as is necessary for the above interpretation of the BPS states), independently of how the corresponding transition is embedded in the Calabi-Yau, these are unaffected⁵.

Now we come to the more interesting numbers in the above table. First of all note that for the singly wound state $d_D = 1$ with no momentum $d_E = 0$ we have one BPS state but for all $d_D > 1$ with $d_E = 0$ there are none. This implies that the multiply wound tensionless

⁴ Note that the factor of 2 is there to make a full hypermultiplet on the supersymmetry side (which for $d_D \neq 0$ is effected by the negative reflection of d_D).

⁵ The universal invariants of the local geometry of the vanishing 4-cycles without the above mentioned restriction are discussed in the appendix.

string with no momentum does not form a bound state. For the winding number one and momentum quantum k we can read off the spectrum of BPS states from the second line of the above table and is precisely given by

$$\begin{aligned}\hat{\Lambda}_{E_8} &= \frac{1}{2} \sum_{\alpha=\text{even}} \frac{\theta_{\alpha}^8(\tau)}{q^{-\frac{1}{2}}\eta(\tau)^{12}} \\ &= 1 + 252q + 5130q^2 + 54760q^3 + 419895q^4 + 2587788q^5 + \dots\end{aligned}\tag{4.1}$$

Here θ_{α} are the Jacobi Theta functions and the η is the Dedekind eta-function

$$\begin{aligned}\frac{1}{2} \sum_{\alpha=\text{even}} \theta_{\alpha}^8(\tau) &= 1 + 240q + 2160q^2 + 6270q^3 + 17520q^4 + 30240q^5 + \dots \\ q^{\frac{1}{2}}\eta(\tau)^{-12} &= 1 + 12q + 90q^2 + 520q^3 + 2535q^4 + 10908q^5 + \dots\end{aligned}$$

We have checked the agreement of these functions and the coefficients of the instanton expansions up to 12th order in q . This result is in perfect accord with expectations based on tensionless strings discussed in section 2.

Note that we actually have more information here for multiple winding states. Consider for example the third row in the above table which corresponds to the BPS states with double winding numbers of the tensionless string. First of all, the fact that the numbers are negative implies that we are dealing with moduli space of holomorphic maps rather than isolated numbers. We should thus interpret these number, as in the first row, as the net number of BPS states. The net number of double wound states exhibit the following quadratic relation in terms of the single wound states

$$8 n_{2,k} = -(k-1) \sum_{i=0}^k n_{1,k-i} n_{1,i} - \delta_{(k \bmod 2),0} n_{1,\frac{k}{2}}.$$

The interpretation of this sum rule is a very interesting one for which we do not know the answer (though we present some speculations in the next section). For example the first non-vanishing number -9252 in the third row can be viewed as

$$8(9252) = (252 \cdot 252) + 2(1 \cdot 5130) + 252$$

We also expect, though have not checked, that the higher winding number BPS states can also be factorized in terms of other BPS degeneracies with total winding number adding up to the winding number of BPS state in question.

4.1. New Physics at the Second Transition

As discussed in section 3 the interesting transition, corresponds to the second point where the class $k_D + k_E = 0$. In particular at this point the 252 states corresponding to $d_D = d_E = 1$ become massless. As discussed in section 2 the quantum number of these states include fields with spin up to spin $3/2$. We are thus seeing a massless gravitino at this point, therefore suggesting enhanced local supersymmetry at this transition point! This is novel in that this is happening at a *finite* distance in moduli space (see also [31]). This is quite remarkable. One may wonder whether we can prove that the corresponding BPS state is stable. Before we approach the first transition point any BPS state with a fixed (d_D, d_E) has the same mass as the sum of masses of BPS states whose (d_D^i, d_E^i) add up to it. So in principle there is a channel were they could have decayed. We do not believe this is the case and believe that they form bound states at threshold; for example the fact that the mirror map does not predict any BPS state for $d_D > 1, d_E = 0$ already suggests that mirror map, which counts only primitive instantons (since we have subtracted multi-cover contributions) only counts states which are stable. At any rate we can actually make this rigorous at least for the 252 states in the second transition. In principle the 252 states can decay to the combination of $d_D = 1, d_E = 0$ states and $d_D = 0, d_E = 1$ states. However after the first transition point where the sign of k_D flips the situation changes. In particular the BPS state corresponding to $d_D = 1, d_E = 0$ at this second transition has negative *mass* but positive $(mass)^2$ and so it is an ordinary massive BPS state [15]. Thus energetics forbid the decay of the 252 states. We are thus rigorously predicting the existence of a massless stable particle with spin $3/2$ at the second transition point. This is however not the end of the story. In fact all the states with $(d_D, d_E) = (n, n)$ are massless at the second transition point. Even though now we cannot prove they are stable (as they can in principle decay to n copies of $(d_D, d_E) = (1, 1)$ states) based on what we said above we believe they represent BPS states which are bound states at threshold. Thus the massless 252 states are just the tip of the iceberg and we seem to see an infinite tower of massless states, with net BPS degeneracies $252, -9252, 848628, \dots$, as had been anticipated in [14][15]. As mentioned before we also have a tensionless magnetic string at this point interacting with all these massless states. This suggests that the totality of BPS state $252, -9252, \dots$ form some kind of “representation” for this new string. The analogy we have in mind is that if we have a gauge particle for some group G it can interact with massless particles which form non-trivial representations of G . Here we believe we have a non-critical string version of this situation at hand where a non-critical tensionless E_8 string interacts with the correspondingly infinite tower of BPS states which can themselves be

viewed as negative tension E_8 strings wrapped around the circle n times with momentum n around the circle. This also suggests that a quantum correction is responsible for shifting the tension of the wrapped string from that of the unwrapped one. Clearly there is a lot of new physics hidden here remaining to be explored. We cannot hesitate to speculate about the implications of a better understanding of this second transition point for the question of supersymmetry breaking, given the fact that we seem to have found as a ‘minor’ part of the infinitely many light BPS states, a nearly massless spin 3/2 state!

4.2. Other E_d

Next we consider the elliptically fibred Calabi–Yau $X^{(4)}$ with base \mathbf{F}_1 and generic fibre of the type $P^{1,1,2}[4]$ with hodge numbers $(4, 148)$. The four Kähler classes are supported by the following classes of curves: i) C_F , the fibre \mathbf{P}^1 of \mathbf{F}_1 , ii) C_D , the exceptional section of \mathbf{F}_1 , iii) C_E , a curve in the elliptic fibre and iv) a new class which will be denoted by C_W , which is introduced by the second section of the fibration. In fact this manifold is connected to the Calabi–Yau threefold $X^{(3)}$ by a transition which contracts the \mathbf{P}^1 of the new Kähler class, $t_W = 0$. From $n_{0,0,0,1} = 96$, $n_{0,0,0,k} = 0$ for $k > 0$ we see that there are 96 2-cycles contracted by the transition, thus explaining the change of the number of complex structure deformations $243 - 148 + 1 = 96$. Moreover the Gromov–Witten invariants sum up as $n_{i,j,k}^{(3)} = \sum_l n_{i,j,k,l}^{(4)}$ for $t_W = 0$.

The rational curves $n_{0,1,k}^{(3)}$ generating the partition function (4.1) split according to the $U(1)$ charges in the decomposition $E_8 \subset U(1) \times E_7$. The relevant Gromov–Witten invariants $n_{n_F, n_D, n_E, n_W}^{(4)}$ are shown in table 2.

	d_W	0	1	2	3	4	5	6	7	8	9	Σ
d_E												
0		1										1
1		1	56	138	56	1						252
2				138	1248	2358	1248	138				5130
3					56	2358	13464	23004	13464	2358	56	54760
4						1	1248	23004	103136	165117	103136	419895

Table 2: Invariants of $X^{(4)}$ for rational curves with degree $d_F = 0$, $d_D = 1$.

Note that the splitting of the E_8 states into E_7 states is in perfect accord with the idea explained in section 2 in the context of tensionless strings and turning on Wilson lines of $U(1)$ on the circle. Moreover we should identify the Wilson line of the $U(1)$ with the

Kähler class of W : $k_W = W$. Also we have to identify $k_E + 2k_W = \frac{1}{R}$; this is consistent with the fact that a curve of type this type is also in the elliptic fibre.

The states of the $X^{(3)}$, which are multiply wound around the circle, have a completely analogous group theoretical decomposition into the classes of states of $X^{(4)}$. For example for the doubly wound states $d_D = 2$ one finds

	d_W	2	3	4	5	6	7	Σ
d_E								
2		-272	-2272	-4164	-2272	-272		-9252
3			-2272	-38088	-165600	-261840	-165600	-673760
4				-4164	-165600	-1484256	-4961952	-20534040

Table 3: Invariants of $X^{(4)}$ for rational curves of degree $d_F = 0$, $d_D = 2$.

Interestingly one can identify the following symmetries of the instanton numbers suggesting a kind of T-duality of the tensionless string:

$$\begin{aligned}
t_E \rightarrow t_E + 4t_W, \quad t_W \rightarrow -t_W &: \frac{1}{R} \rightarrow \frac{1}{R}, \quad W \rightarrow -W \\
t_E \rightarrow -t_E, \quad t_W \rightarrow t_W + t_E, \quad t_D \rightarrow t_D + t_E &: \frac{1}{R} \rightarrow \frac{1}{R}, \quad W \rightarrow W + \frac{1}{R}, \\
RT \rightarrow RT + \frac{1}{R} &
\end{aligned} \tag{4.2}$$

Whereas the first symmetry related to a Weyl symmetry of the underlying E_8 is present for all values of n_F , the second, more remarkable one, holds only for $n_F = 0$. It points to a duality symmetry of the enlarged charge lattice $\Gamma^{1,9}$ (note however that we have found no indication of a $R \rightarrow 1/R$ symmetry as is expected from the zero tension limit). We expect the symmetries of the instantons to correspond to monodromies of the periods around singular loci in the moduli space.

The E_6 case is realized in the Calabi–Yau threefold where the generic elliptic fibre over the \mathbf{F}_1 base is of type $P^{1,1,1}[3]$; its Hodge numbers are $(5, 101)$. The five Kähler classes supported by the classes C_F , C_D , C_E as well as two additional classes which we denote by W_1 and W_2 . With respect to these classes the invariant of the E_8 case split according to the decomposition $E_8 \subset E_6 \times U(1)_1 \times U(1)_2$ as shown in table 4.

	d_{W_1}	0	1	2	3	4	5	6	Σ
d_{W_2}									
0		1							1
1		1	27	27	1				56
2				27	84	27			138
3					1	27	27	1	56
4								1	1

Table 4: Invariants of $X^{(5)}$ for rational curves of degree $d_F = 0$, $d_E = 1$, $d_D = 1$.

	d_{W_1}	2	3	4	5	6	7	8	Σ
d_{W_2}									
2		27	84	27					138
3			84	540	540	84			1248
4				27	540	1224	540	27	2358

Table 5: Invariants of $X^{(5)}$ for rational curves of degree $d_F = 0$, $d_E = 2$, $d_D = 1$.

The results are once again in perfect agreement with the splitting of states based on Wilson lines the identification being $k_E + 2k_{W_2} + 3k_{W_1} = \frac{1}{R}$, $k_{W_1} = W_1$, $k_{W_2} = W_2$ and $k_D = RT$. Remarkably there are similar symmetries of the instanton numbers as in the E_7 case, moreover there is again an analogous decomposition of the multiple wound states $d_D > 1$ of $X^{(4)}$ into those of $X^{(5)}$.

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Appendix A. Toric description of the threefolds $X^{(i)}$

In this appendix we will give the data which specify the Calabi-Yau threefolds as hypersurfaces in toric varieties. The toric varieties can be described by pairs of reflexive polyhedra (Δ, Δ^*) in a four-dimensional lattice. The canonical hypersurfaces (X, X^*) in the projective toric varieties $(\mathbf{P}_\Delta, \mathbf{P}_{\Delta^*})$ give rise to mirror pairs of Calabi-Yau threefolds [32]. The exact worldsheet instanton corrections on X can be expressed in terms of the periods of the mirror manifold X^* . Explicit formulas for the periods at points of large radii are given in terms the topological data and the Mori generators of X in⁶ [33]. The

⁶ We will adapt to the notation of [33]

later data are most easily calculable from the dual polyhedron Δ^* , which we give in the following.

For the $X^{(3)}$ case the the dual polyhedron is the convex hull of the following points

$$\begin{aligned}\nu_0^* &= [0, 0, 0, 0], & \nu_1^* &= [1, 0, 0, 0], & \nu_2^* &= [-1, -1, -6, -9], \\ \nu_3^* &= [0, 1, 0, 0], & \nu_4^* &= [0, -1, -4, -6], & \nu_5^* &= [0, 0, 1, 0], \\ \nu_6^* &= [0, 0, 0, 1], & \nu_7^* &= [0, 0, -2, -3].\end{aligned}\tag{A.1}$$

The hypersurface X in the \mathbf{P}_Δ can be represented in the Batyrev-Cox variables [34] x_0, \dots, x_7 as vanishing of the polynomial

$$P = x_0[x_5^3 + x_6^2 + x_7^6(x_4^{12}(x_1^{18} + x_2^{18}) + x_3^{12}(x_1^6 + x_2^6))].\tag{A.2}$$

The model exhibits two geometrical phases, which correspond to two regular triangulations of Δ^* involving all points, and one non-geometrical phase. The first geometrical phase admits a K_3 -fibration as well as an elliptic fibration and the generators of the Mori cone are:

$$\begin{aligned}l^{(F)} &= [0; 0, 0, 1, 1, 0, 0, -2] \\ l^{(D)} &= [0, 1, 1, 0, -1, 0, 0, -1] \\ l^{(E)} &= [-6; 0, 0, 0, 0, 2, 3, 1],\end{aligned}$$

where the indices refer to the classes of the curves, which bound dual vector in the Kähler cone. The classical triple intersections and the integrals involving the second Chern class are

$$\begin{aligned}\mathcal{R} &= 8J_E^3 + 3J_E^2J_F + J_EJ_F^2 + 2J_E^2J_D + J_DJ_EJ_F \\ \int_M c_2J_E &= 92, & \int_M c_2J_F &= 36, & \int_M c_2J_D &= 24\end{aligned}$$

The differential equations for the periods in the complex structure variables of X^* are generalized hypergeometric systems⁷. In the case at hand we have the differential operators ($\theta := z \frac{d}{dz}$):

$$\begin{aligned}\mathcal{L}_1 &= \theta_E(\theta_E - 2\theta_F - \theta_D) - 12(6\theta_E - 5)(6\theta_E - 1)z_E \\ \mathcal{L}_2 &= \theta_F(\theta_F - \theta_E) - (2\theta_F + \theta_D - \theta_E - 2)(2\theta_F + \theta_D - \theta_E - 1)z_F \\ \mathcal{L}_3 &= \theta_D^2 - (\theta_D - 1 - \theta_F)(2\theta_F + \theta_D - \theta_E - 1)z_D\end{aligned}$$

⁷ Properties of these systems were studied intensively by Gel'fand-Kapranov-Zelevinskii [35]

From the periods and the discriminant of X^*

$$\begin{aligned}\Delta_1 &= (1 - z_E)^3 (1 + z_E - z_E z_D) + z_E^2 z_F (8(1 - z_E)^2 - 16z_E^2 z_F + 36z_E z_D \\ &\quad - 36z_E^2 z_D + 27z_E^2 z_D) \\ \Delta_2 &= (1 - 4z_F)^2 - z_D + 36z_F z_D - 27z_F z_D^2.\end{aligned}$$

one can calculate F_{top}^g which enjoys an expansion in terms of the Gromov-Witten invariants for elliptic and higher genus curves respectively [36]. The behavior of F_{top}^1 near the discriminants is $z_E^{-\frac{26}{3}}$, z_F^{-4} , z_D^{-3} , $\Delta_1^{-\frac{1}{6}}$ and $\Delta_2^{-\frac{1}{6}}$. For the elliptic curves one obtains

$d_F = 0$	d_E	0	1	2	3	4	5	6
	d_D							
	0		4					
	1		-2	-510	-11780	-142330	-1212930	-8207984
	2			762	205320	11361870	31746948	5863932540
	3				-246788	-76854240	-6912918432	-32516238180
	4					76413073	278663327760	348600115600

Table 6: Gromov-Witten invariants for the genus one curves.

The second phase is connected to the first phase by a flop of the class D , which is the \mathbf{P}^1 in the del Pezzo, whose complexified size parameter in the K_3 phase was the modulus of the dilaton of the heterotic theory, i.e. $\tilde{l}^{(1)} = l^{(E)} + l^{(D)}$, $\tilde{l}^{(2)} = l^{(F)} + l^{(D)}$, $\tilde{l}^{(3)} = -l^{(D)}$

The topological data in this phase are given by

$$\begin{aligned}\mathcal{R} &= 8\tilde{J}_1^3 + 3\tilde{J}_2\tilde{J}_1^2 + \tilde{J}_2^2\tilde{J}_1 + 9\tilde{J}_1^2\tilde{J}_3 + 3\tilde{J}_2\tilde{J}_3\tilde{J}_1 + \tilde{J}_2^2\tilde{J}_3 + 9\tilde{J}_3^2\tilde{J}_1 + 3\tilde{J}_2\tilde{J}_3^2 + 9\tilde{J}_3^3 \\ \int_M c_2\tilde{J}_1 &= 92, \quad \int_M c_2\tilde{J}_2 = 36, \quad \int_M c_2\tilde{J}_3 = 102\end{aligned}$$

The transition shrinking the 4-cycle corresponds to the limit $\tilde{z}_1 = 0$, $\tilde{z}_3 = \infty$ where $\tilde{z}_1\tilde{z}_3$ held fixed. This can be seen from the relation of the $\tilde{l}^{(i)}$

$$\hat{l}^{(1)} = \tilde{l}^{(1)} + \tilde{l}^{(3)} = (-6; 0, 0, 1, 0, 2, 3, 0), \quad \hat{l}^{(2)} = \tilde{l}^{(2)} = (0; 1, 1, 1, 0, 0, 0, -3)$$

to the Mori generators $\hat{l}^{(i)}$ of $X^{1,1,1,6,9}$ [18] model, whose polyhedron Δ^* is given by the convex hull of the points in (A.1), with ν_4^* omitted. In the analytic continuation to the new variables

$$\hat{z}_1 = \frac{1}{\tilde{z}_1}, \quad \hat{z}_2 = \tilde{z}_2, \quad \hat{z}_3 = \tilde{z}_1\tilde{z}_3$$

five of the eight periods remain unchanged. Furthermore t_1 and its dual are analytically continued into solutions $\omega^{1/6,0,0} \propto \hat{z}_1^{\frac{1}{6}} + \dots$, $\omega^{5/6,0,0} \propto \hat{z}_1^{\frac{5}{6}} + \dots$, which vanish at $\hat{z}_i = 0$; this proves that the volumes of the 2-cycle and 4-cycle vanish also after including the quantum corrections in the four-dimensional theory obtained by a type IIA compactification.

The structure for the threefolds $X^{(4)}$ and $X^{(5)}$ is similar and we omit the details. The polyhedra Δ^* for $X^{(4)}$ and $X^{(5)}$ can be obtained by adding the point $\nu_8^* = (0, 0, 1, 1)$ and $\nu_9^* = (0, 0, 1, 2)$ to the points in (A.1) respectively. For the $X^{(4)}$ we choose a phase in which the Mori generators $l^{(D)}$, $l^{(E)}$, $l^{(F)}$ and $l^{(W)}$ are related to the Mori generators of $X^{(3)}$ by $l^{(E)} = l'^{(E)} + 3l'^{(W)}$, $l^{(D)} = l'^{(D)}$ and $l^{(F)} = l'^{(F)}$. Likewise the $X^{(5)}$ model exhibits a phase in which the transition to $X^{(4)}$ is apparent as the Mori generators $\hat{l}^{(D)}$, $\hat{l}^{(E)}$, $\hat{l}^{(F)}$, $\hat{l}^{(W_1)}$ and $\hat{l}^{(W_2)}$ are related to the one of the $X^{(4)}$ model by $l'^{(E)} = \hat{l}^{(E)} + \hat{l}^{(W_1)}$, $l'^{(W)} = 2\hat{l}^{(W_2)} + \hat{l}^{(W_1)}$, $l'^{(D)} = \hat{l}^{(D)}$ and $l'^{(F)} = \hat{l}^{(F)}$.

Appendix B. Universal structure of the 4-cycle singularity

In the following we describe the structure of the differential equations and their solutions which govern the Gromov–Witten invariants of the local vanishing 4-cycle \mathbf{B}_k independently of the global embedding in the Calabi–Yau. Specifically we study the dependence of the mirror maps on the Kähler modulus of the collapsing 4-cycle. The restricted one modulus system extracts the universal piece of the invariants associated to the vanishing 2-cycle inside the 4-cycle.

The universal behavior of the singularity is completely governed by a Mori vector l which we can associate to the local form (3.6). For E_k , $k = 8, 7, 6, 5$ one has

$$\begin{aligned}
 E_8 : l &= (-6 | -1, 3, 2, 1, 1, 0, \dots), \\
 E_7 : l &= (-4 | -1, 2, 1, 1, 1, 0, \dots), \\
 E_6 : l &= (-3 | -1, 1, 1, 1, 1, 0, \dots), \\
 E_5 : l &= (-2, -2 | -1, 1, 1, 1, 1, 1, 0, \dots)
 \end{aligned}
 \tag{B.1}$$

This can be understood as follows: the rational curve \hat{C} dual to l is contained in the divisor $D_1 : \{x_1 = 0\}$. The polynomial restricted to D_1 has a C^* symmetry given by the remaining entries of l which implies a form consistent with the local description of the singularity (3.6).

Let t denote the relevant Kähler modulus and $z = z(t)$ the mirror map. From the vectors l we obtain a differential operators \mathcal{L} which govern the mirror map z in the limit

where all other volumes become large compared to it. The differential operator \mathcal{L} turns out to be closely related to the differential operator of the elliptic fibre \mathcal{L}_{ell} :

$$\mathcal{L} = \mathcal{L}_{ell} \theta$$

where $\theta = z \frac{d}{dz}$ and \mathcal{L}_{ell} are the differential operators of the elliptic curves $P^{1,2,3}[6]$, $P^{1,1,2}[4]$, $P^{1,1,1}[3]$, $P^{1,1,1,1}[2, 2]$, (see e.g [37]):

$$E_8 : \mathcal{L}_{ell} = \theta^2 - 12z(6\theta + 5)(6\theta + 1)$$

$$E_7 : \mathcal{L}_{ell} = \theta^2 - 4z(4\theta + 3)(4\theta + 1)$$

$$E_6 : \mathcal{L}_{ell} = \theta^2 - 3z(3\theta + 2)(3\theta + 1)$$

$$E_5 : \mathcal{L}_{ell} = \theta^2 - 4z(2\theta + 1)^2$$

Given a solution ω of \mathcal{L} we have also a solution $\omega_{ell} = \theta \omega$ of \mathcal{L}_{ell} . On the other hand the fundamental period of \mathcal{L} [33]:

$$w_0(z, \rho) = \sum_n c(n + \rho) z^{n+\rho}$$

is given by the constant solution, whereas the mirror map z is given by the single logarithmic solutions. The expression for the instanton corrected Yukawa couplings can be expressed in terms of w_0 as

$$K(t) = \partial_t \partial_t \left(\frac{1}{2w_0} K^0 \partial_\rho \partial_\rho w_0 |_{\rho=0} \right) (t)$$

and subtracting multiple covers we obtain the following Gromov–Witten invariants:

	d	1	2	3	4	5	6
k							
8		252	-9252	848628	-114265008	18958064400	-3589587111852
7		56	-272	3240	-58432	1303840	-33255216
6		27	-54	243	-1728	15255	-153576
5		16	-20	48	-192	960	-5436
0		3	-6	27	-192	1695	-17064

Table 6: Invariants of the vanishing 4-cycles for $k = 0, 5, 6, 7, 8$.

As suggested previously, these would correspond to the net degeneracies of massless BPS states at the point where a 4-cycle of type E_d shrinks to zero size. Amusingly the E_6 series turns out to be precisely 9 times the series one obtains for \mathbf{P}^2 , shown in the last row.

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