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CONVERGENCE OF TRACKING CODES FOR COUPLED BETATRON MOTION IN THICK MAGNETS

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Tracking of transverse motion of charged particles in storage ring lattices which contain nonlinear elements may diverge if mappings are applied over too large time steps. The convergence limits depend on the strength of the nonlinearities as well as on the initial conditions. For one-dimensional motion, where an exact solution can be found, these limits are given by the poles in the complex time plane. For two-dimensional motion in realistic magnets, for which no analytic solution is known, viable tests to determine this limit do not exist except repeated computer runs. Here we find the limits of convergence of tracking of coupled betatron motion in thick nonlinear lenses by expansion of the unknown solution into a Laurent series. The location of the "moving singularities" — which determine the convergence limits — can then be obtained from the solution of a set of polynomial equations. The method is applied to the case of LHC dipole magnets with quadrupolar and sextupolar field errors. In the appendix, we discuss a special case of transverse motion in nonlinear elements which is bounded in all directions.

KEY WORDS: Coupled betatron motion, magnets

1 INTRODUCTION

We investigate the convergence properties of particle tracking through focusing structures of storage rings by repeated application of Taylor series maps. To our knowledge, the problem has first been raised in Ref. 1. In that paper, the author applied a mapping technique to the case of the mathematical pendulum which is solvable in closed form using Jacobian elliptic functions. He expanded the solution of the one-dimensional problem

$$\ddot{x} + \sin x = 0 \tag{1}$$

into a two dimensional Taylor series in the phase space variables x and $\dot{x}$. From this series he defined the mapping

$$\begin{pmatrix} x_{n+1} \\ \dot{x}_{n+1} \end{pmatrix} = \sum_k \sum_l \vec{f}_{kl}(\Delta t) x_n^k \dot{x}_n^l \tag{2}$$

where $x_n = n\Delta t$, and Δt is the time step over which the map is calculated. In his paper, Talman used a map for a rather large time step — about 0.8 times the small amplitude period of the pendulum. In accelerator physics, this would correspond to a map over almost a full betatron period. Even with rather high-order Taylor maps, the results obtained diverged quite rapidly — depending on the choice of the initial angle — from the exact solution given by

$$x(t) = 2\arcsin[k \cdot sn(K(m) + t|m)]; \quad k^2 = m = \sin^2 \frac{x_o}{2} \tag{3}$$

On the other hand, we found that, by using sufficiently small time steps Δt, one gets excellent agreement of tracking with the exact solution of the pendulum motion over many periods. Talman's conclusion from his observation was that *"map iteration is not promising for long-term predictions — unless the pendulum system is atypical"*. He further stated that it would be desirable *"to show the characteristics of nonlinear systems that allow useful application of map-prediction techniques"*. This was done in a companion paper which appeared in the same issue,[2] where it was shown that there exists a maximum time step for the mapping beyond which the results diverge.

To find the limiting step size, the exact solution of the pendulum equation was expanded into a one-dimensional Taylor series in the time step Δt. For such a series, *the radius of convergence is given by the distance from the expansion point to the closest pole of the exact solution of the problem in the complex plane*. Applying this criterion, due to Cauchy and Weierstrass, to Equation (3), one finds the convergence limit simply from the complex poles of the Jacobian elliptic functions.[3] It is thus possible to give an analytic expression for the radius of convergence as function of the initial conditions, and it was found to be always less than half the small amplitude period of the pendulum. Since the time step used by Talman was larger, this led to his divergent results.

In Ref. 2, the same approach was applied also to the one-dimensional motion of charged particles in thick sextupoles and octupoles. Exact solutions for these can again be expressed in terms of elliptic functions, as shown in Appendix A. For the corresponding Taylor series maps, the convergence depends not only on the size of the time step, but also on the strength of the nonlinearity.

Although these results are very instructive, they give the convergence limits only for those cases for which the exact solution is already known. It would be more useful to extend the method to dynamical systems for which the solution cannot be found analytically, because only for such systems a numerical integration is really necessary. A first step in this direction was taken in Ref. 4. Besides several other interesting considerations, the authors use the *quotient criterion* to obtain the convergence limits of infinite series describing betatron motion in lattices containing nonlinear magnetic elements. For this purpose, they examine the behaviour of the expansion coefficients of the numerical solutions with increasing step size. They actually get quite good agreement with tracking for several one and two-dimensional cases. However, obtaining these coefficients requires repeated computer runs, and the criterion is only rather approximate.

In the present paper we derive a direct estimate for the convergence limit, again using the theory of Cauchy and Weierstrass. We develop a method to determine poles of the solutions of differential equations *which are not known in explicit form*, and we apply the method to several cases of nonlinear focusing magnets used in storage ring lattices.

In Section 2 we define the problem, and we demonstrate that Taylor series expansions are equivalent when expressed *in powers of the time step* Δt or *in the phase coordinates (displacements and their derivatives at an initial position)*. The representation in the time step has the big advantage to remain one-dimensional in all cases, even if the underlying problem has more than one dimension in real space. Thus one can always use the theory of one-dimensional Taylor series, which is much more straightforward than the equivalent theory in a higher number of expansion variables.

In Section 3 we introduce *Laurent series* to expand the solution around unknown poles in the complex time domain. We introduce the concept of **moving poles**, i.e. poles whose positions depend on the initial conditions of the dynamical system. We apply the theory to simple examples of first and second order differential equations.

In Section 4 we apply these results to the case of the superconducting LHC dipole magnets with strong nonlinear multipole components. We study the problem in two real space dimensions (coupled horizontal and vertical motion), for which no analytic solution of the equations of motion is known. We derive expressions for the maximum permissible step size of tracking codes as function of the multipole components and of the betatron amplitudes in both planes. For the estimated random and systematic quadrupolar and sextupolar components of an LHC dipole, the approximate convergence limits — expressed in the initial transverse coordinates — are found to be safely outside the vacuum chamber. Although it has not been shown in detail, these results justify the replacement of a long dipole by one or two thin lens kicks. However, concatenation of several elements could lead to divergence unless proper techniques are used to guarantee both the symplecticity and the convergence of repeated application of the maps.

2 EQUATIONS OF MOTION AND MAPPINGS

Describing dynamical systems like the focusing structure of an accelerator lattice can be done by a system of nonlinear, second order, differential equations. Since such equations are in general not solvable analytically, one has to revert to numerical methods to evaluate the solution as function of the initial conditions. All known numerical integrators work by reducing a differential equation

$$\ddot{x} = P(x) \tag{4}$$

(or the equivalent system in case of higher dimensions) to an algebraic mapping of the form

$$\begin{aligned} x_{n+1} &= f(x_n, \dot{x}_n) \\ \dot{x}_{n+1} &= g(x_n, \dot{x}_n) \end{aligned} \tag{5}$$

where x_n and $\dot{x}_n$ denote value and derivative of the function $x(t)$ at times

$$t_n = n\Delta t \tag{6}$$

when Δt is a finite time step. In applications related to the motion of charged particles in focusing and bending magnets of accelerators, $P(x)$ in Equation (4) can be approximated by a polynomial

$$P(x) = \sum_{n=1}^{N} a_n x^n \tag{7}$$

for which $P(0) = 0$, because the equations of betatron motion inside a focusing magnet just describe small deviations from an ideal orbit normalized to $x = 0$. In this situation there are basically two possibilities to construct mappings of the form of Equations (5), (6). We may either expand the solution $x(t, x_n, \dot{x}_n)$ into a one dimensional Taylor series with respect to the time steps Δt, or into a two dimensional series with respect to the coordinates x_n and $\dot{x}_n$.

For the Taylor series in time steps this is a rather straightforward procedure. We may write

$$\begin{aligned} x_{n+1} &= x_n + \dot{x}_n \Delta t + \sum_{k=2}^{\infty} \frac{1}{k!} \left.\frac{d^k x}{dt^k}\right|_{t=0} \Delta t^k \\ \dot{x}_{n+1} &= \dot{x}_n + \sum_{k=2}^{\infty} \frac{1}{k!} \left.\frac{d^k x}{dt^k}\right|_{t=0} k \Delta t^{k-1} \end{aligned}$$

This mapping, although it contains all derivatives of the solution, is only a function of x_n and $\dot{x}_n$ since all higher derivatives can recursively be expressed by the two lowest ones. This can be demonstrated by successive differentiation of Equation (4):

$$\begin{aligned} f_2 &= \frac{d^2 x}{dt^2} = P(x) \\ f_3 &= \frac{d^3 x}{dt^3} = \dot{x} P'(x) \\ f_4 &= \frac{d^4 x}{dt^4} = \ddot{x} P'(x) + \dot{x}^2 P''(x) = P(x) P'(x) + \dot{x}^2 P''(x) \\ &\vdots \end{aligned} \tag{9}$$

Provided that Δt is smaller than the convergence radius of the series Equation (8), this mapping represents the exact solution to arbitrary precision. The expansion w.r.t. the coordinates can be directly derived from the Taylor series representation (8) by expanding the k-th derivative $d^k x/dt^k = f_k(x_n, \dot{x}_n)$ itself into a two dimensional power series w.r.t. the coordinates. Then the Taylor mapping becomes

$$x_{n+1} = x_n + \dot{x}_n \Delta t + \sum_{k=2}^{\infty} \frac{\Delta t^k}{k!} \sum_{N=1}^{\infty} \sum_{M=0}^{N} \binom{N}{M} \frac{\partial^N f_k}{\partial x_n^{N-M} \partial \dot{x}_n^M} x_n^{N-M} \dot{x}_n^M$$

$$\dot{x}_{n+1} = \dot{x}_n + \sum_{k=2}^{\infty} \frac{1}{k!} k \, \Delta t^{k-1} \sum_{N=1}^{\infty} \sum_{M=0}^{N} \binom{N}{M} \frac{\partial^N f_k}{\partial x_n^{N-M} \partial \dot{x}_n^M} x_n^{N-M} \dot{x}_n^M \tag{10}$$

where the partial derivatives of f_k are evaluated at $x_n = \dot{x}_n = 0$. In the convergent region of the Taylor expansion in Δt (i.e. for sufficiently small Δt) one may change the order of the infinite sums to obtain

$$\begin{aligned} x_{n+1} &= x_n \left[1 + \sum_{k=2}^{\infty} \frac{\Delta t^k}{k!} \frac{\partial f_k}{\partial x_n} \right] + \dot{x}_n \left[\Delta t + \sum_{k=2}^{\infty} \frac{\Delta t^k}{k!} \frac{\partial f_k}{\partial \dot{x}_n} \right] \\ &\quad + \sum_{N=2}^{\infty} \sum_{M=0}^{N} \left[\binom{N}{M} \sum_{k=2}^{\infty} \frac{\Delta t^k}{k!} \frac{\partial^N f_k}{\partial^{N-M} x_n \partial^M \dot{x}_n^M} \right] x_n^{N-M} \dot{x}_n^M \\ \dot{x}_{n+1} &= \frac{d}{d(\Delta t)} x_{n+1} \end{aligned} \tag{11}$$

We conclude that expanding the differential equation (4) either in time steps Δt or in the coordinates x_n and $\dot{x}_n$ leads to equivalent mappings. It is straightforward to prove that this equivalence holds also for the case of a system of coupled equations. In the following chapters we use only expansions in the time step, since in this manner the problem can always be treated as one-dimensional, even if two spatial directions are considered. As an illustrative example we present the simple case of one-dimensional motion in a thick sextupole

$$\ddot{x} = -\alpha x^2 \tag{12}$$

where α is a real constant. We first derive a mapping by expanding in the coordinates x_n and $\dot{x}_n$. This is done by rewriting Equation (12) as a system of integral equations

$$\begin{aligned} x_{n+1} &= x_n + \dot{x}_n \Delta t - \alpha \int_0^{\Delta t} \int_0^{\Delta t} x^2 dt dt' \\ \dot{x}_{n+1} &= \dot{x}_n - \alpha \int_0^{\Delta t} x^2 dt \end{aligned} \tag{13}$$

We can solve these Volterra type integral equations approximately with an iteration procedure on x. This consists in choosing an initial solution $x^{(0)}(t)$, inserting it into the integrands and evaluating the quadratures. Then we repeat the same procedure with the new approximation as often as required. In order to obtain results correct up to second order in x_n and $\dot{x}_n$, we must use a starting function $x^{(0)}(t)$ correct up to first order in these variables. This condition is evidently fulfilled by the free-space solution

$$x^{(0)}(t) = x_n + \dot{x}_n \Delta t \tag{14}$$

Inserting Equation (14) into the integral equation then gives the result valid to second order:

$$\begin{aligned} x_{n+1} &= x_n + \Delta t \dot{x}_n - \frac{\Delta t^2}{2}\alpha x_n^2 - \frac{\alpha \Delta t^3}{3} x_n \dot{x}_n - \frac{\alpha \Delta t^4}{12} \dot{x}_n^2 \\ \dot{x}_{n+1} &= \dot{x}_n - \Delta t \alpha x_n^2 - \alpha \Delta t^2 x_n \dot{x}_n - \frac{\alpha \Delta t^3}{3} \dot{x}_n^2 \end{aligned} \tag{15}$$

For the expansion in time steps as in Equation (8), we need the derivatives of x. Here we proceed up to fourth order in Δt. Hence

$$\begin{aligned} f_2 &= \frac{d^2 x}{dt^2} = -\alpha x \\ f_3 &= \frac{d^3 x}{dt^3} = -2\alpha x \dot{x} \\ f_4 &= \frac{d^4 x}{dt^4} = -2\alpha(\dot{x}^2 + x\ddot{x}) = -2\alpha(\dot{x}^2 - \alpha x^3) \end{aligned} \tag{16}$$

The final mapping then reads

$$\begin{aligned} x_{n+1} &= x_n + \Delta t \dot{x}_n - \frac{\Delta t^2}{2}\alpha x_n^2 - \frac{\alpha \Delta t^3}{3} x_n \dot{x}_n - \frac{\alpha \Delta t^4}{12} (\dot{x}_n^2 - \alpha x_n^3) \\ \dot{x}_{n+1} &= \dot{x}_n - \Delta t \alpha x_n^2 - \alpha \Delta t^2 x_n \dot{x}_n - \frac{\alpha \Delta t^3}{3} (\dot{x}_n^2 - \alpha x_n^3) \end{aligned} \tag{17}$$

and we see that the two mappings agree up to the second power in the coordinates.

3 EVALUATION OF SINGULARITIES

The convergence breakdown of Taylor series mappings of the form

$$\vec{X}_{n+1} = \sum_{k=0}^{\infty} \vec{a}_k(\vec{X}_n)(\Delta t)^k \tag{18}$$

was investigated in Ref. 2. Here

$$\vec{X}_n = \vec{X}(n\Delta t) \tag{19}$$

is the solution vector taken after n time steps. According to the Cauchy theorem of functional theory, this series (18) will diverge if the time step Δt becomes larger than the distance from the origin to the closest singularity in the complex time plane. We demonstrated this behaviour for the special cases of simple 1-D differential equations, like the mathematical pendulum or 1-D motion of a charged particle in a thick sextupole, for which it is possible to express the exact solutions in closed form. Evidently in this case also the exact location of singularities in the complex time plane are known and it is easy to predict convergence breakdown of Taylor series mappings related to such equations. However, in practical cases we are normally concerned with equations of motion for which the analytical solution is not known (otherwise we would not need a numeric integrator). Therefore we search for a method to find the location of complex and real poles **without** knowing the solution of the underlying differential equations.

The basic method to perform this task comes once more from functional theory. We use the fact that one always can express a complex function $f(t)$ close to a pole of finite order N using a **Laurent series** of the form

$$f(t) = \sum_{n=1}^{N} \frac{a_{-n}}{(t - t_0)^n} + \sum_{n=0}^{\infty} a_n (t - t_0)^n \tag{20}$$

where t_0 is a pole of order N. The first sum in (20) is called **main part** of the expansion while the second sum is called **Taylor part**. The coefficient a_{-1} in the main part is called **residuum of $f(t)$ with respect to t_0**. We can in principle make use of this expansion by inserting (20) with an unknown pole t_0 of order N into a given differential equation of first order

$$\dot{x} = F(x, t); \quad x = f(t) \tag{21}$$

expanding F into a new Laurent series and comparing like powers of $(t - t_0)$. However, we must also take into account that for general nonlinear equations the complex poles of x will depend on the initial condition $x(0) = x_0$. Such poles are well known in the literature as **moving poles** because they move around the complex plane as x_0 varies.

If we denote the expansion quantity $t - t_0$ in (20) by Δ, Equation (21) can be written as

$$\dot{x} = F(x, t_0 + \Delta) \tag{22}$$

Next we perform the following steps:

- We choose the order N of the unknown pole t_0 we wish to look for. It turns out that only if we chose the order of the unknown pole correctly we obtain Laurent series in which not all the a_n disappear.

- We write x as a Laurent series

$$x(t) = \sum_{n=1}^{N} \frac{a_{-n}}{(t - t_0)^n} + \sum_{n=0}^{\infty} a_n (t - t_0)^n \tag{23}$$

- We formulate the moving pole condition

$$x(0) = x_0 = \sum_{n=1}^{N} \frac{a_{-n}}{(-t_0)^n} + \sum_{n=0}^{\infty} a_n (-t_0)^n \tag{24}$$

  which constitutes the relation between the initial condition and the poles location as discussed above.

- We insert (23) into (22), and expand the resulting right hand side of Equation (22) into a new Laurent series with respect to Δ.
- We compare equal positive and negative powers in Δ to obtain an infinite algebraic system for the expansion coefficients a_n. It is possible to show that this algebraic system is always recursive linear and thus can be solved exactly.
- We finally use the moving pole condition (24) to extract the desired pole t_0.

We demonstrate this method for the example

$$\dot{x} = -2tx^2 = -2t_0x^2 - 2x^2\Delta \tag{25}$$

We look for the location of first order poles, hence $N = 1$. The moving pole condition can be written as

$$x_0 = -\frac{a_{-1}}{t_0} + \sum_{n=0}^{\infty} a_n (-t_0)^n \tag{26}$$

With

$$x = \frac{a_{-1}}{\Delta} + \sum_{n=0}^{\infty} a_n \Delta^n \tag{27}$$

we find

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\Delta} = -\frac{a_{-1}}{\Delta^2} + \sum_{n=1}^{\infty} n a_n \Delta^{n-1} \tag{28}$$

Inserting these expressions into the differential equation (25) we get

$$\frac{1}{\Delta^2}\left[-a_{-1}+2t_0a_{-1}^2\right]+\frac{2a_{-1}^2}{\Delta}+\sum_{n=0}^{\infty}\left[a_n(n+4t_0a_{-1})\Delta^{n-1}+4a_{-1}a_n\Delta^n\right]$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}2a_na_m\left[t_0\Delta^{n+m}+\Delta^{n+m+1}\right]=0 \qquad (29)$$

Comparing equal powers in Δ leads to the following recursive linear system for the a_n:

$$\begin{aligned}
&\Delta^{-2} :\rightarrow -a_{-1}+2t_0a_{-1}^2=0\\
&\Delta^{-1} :\rightarrow 4t_0a_{-1}a_0+2a_{-1}^2=0\\
&\Delta^{0} :\rightarrow a_1+4a_{-1}a_1t_0+2t_0a_0^2=0\\
&\vdots
\end{aligned} \qquad (30)$$

Truncating the expansion at this level we obtain

$$a_{-1}=\frac{1}{2t_0} \qquad a_0=-\frac{1}{4t_0^2} \qquad a_1=\frac{1}{8t_0^3} \qquad (31)$$

Inserting these results into (26) results in a single algebraic equation for the unknown pole t_0 as

$$x_0=-\frac{7}{8t_0^2} \qquad (32)$$

which gives two purely imaginary poles of first order:

$$t_0=\pm i\sqrt{\frac{7}{8x_0}}\approx\pm\frac{0.9354i}{\sqrt{x_0}} \qquad (33)$$

Let us check this result against the exact solution of (25) which can be found by separation of variables to be

$$x(t)=\frac{x_0}{1+x_0t^2} \qquad (34)$$

Its two first order poles are given by the condition

$$1+x_0t^2=0\rightarrow t_0=\pm\frac{i}{\sqrt{x_0}} \qquad (35)$$

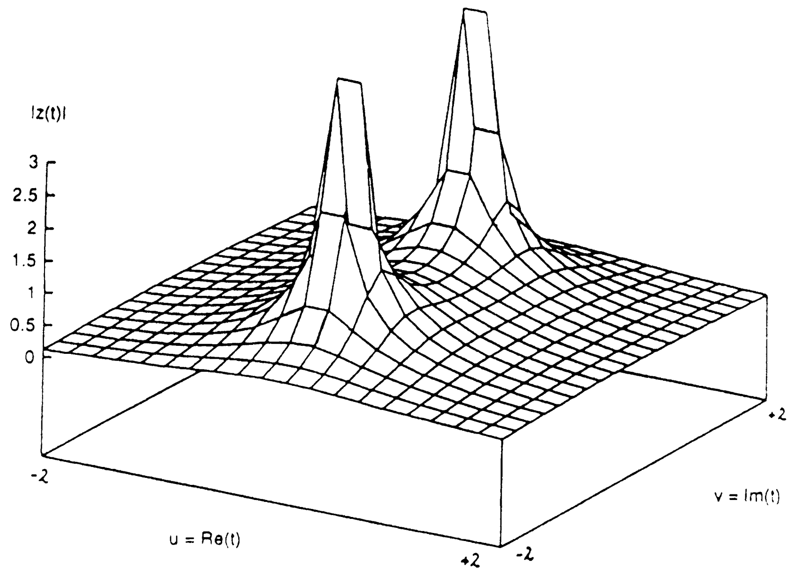

FIGURE 1: Analytic landscape representation of $z(t)=1/(1+t^2)$.

It is straightforward to demonstrate that the infinite system of algebraic equations defined by (29) and (26) gives a sequence of approximations for t_0:

$$t_{0N} = \pm i\sqrt{\frac{2^{N+3}-1}{2^{N+3}x_0}} \tag{36}$$

where N is the index at which the expansion is truncated. For $N \to \infty$, the sequence t_{0N} converges to the correct result given in (35). In Figure 1 we show an analytical landscape representation of the exact solution for $x_0 = 1$ and one can see the two first order poles of $|z(t)|$ at $t = \pm i$ where $|z|$ becomes infinite.

3.1 *Quadrupole with octupole component*

Next we treat the case of a long magnetic quadrupole containing an octupolar component. The equation of motion for a charged particle in the horizontal plane x as function of the longitudinal variable s is given by

$$x'' + K_1 x + \frac{1}{6}K_3 x^3 = 0; \quad x' = \frac{dx}{ds} \tag{37}$$

where K_1 and K_3 are the constant quadrupole and octupole strengths in units of 1/m^2 and 1/m^3 respectively. Although this equation can in principle be solved in closed form in terms of Jacobian elliptic functions (see Appendix A), the expressions are rather complex and a numeric integrator based on a Taylor expansion using moderate truncation order will require less computer time than evaluating the analytic results.

Since Jacobian elliptic functions contain no poles of higher than first order, we may put $N = 1$ and write the Laurent series as in the previous example as

$$x(s) = \frac{a_{-1}}{s - s_0} + \sum_{n=0}^{\infty} a_n (s - s_0)^n \tag{38}$$

Since we are concerned with a second order differential equation (37), the moving singularities will depend on both x_0 and x_0' so that we have to use two conditions:

$$\begin{aligned} x_0 &= -\frac{a_{-1}}{s_0} + \sum_{n=0}^{\infty} a_n (-1)^n s_0^n \\ x_0' &= -\frac{a_{-1}}{s_0^2} + \sum_{n=0}^{\infty} n a_n (-1)^{n-1} s_0^{n-1} \end{aligned} \tag{39}$$

If we wish to truncate the Laurent expansion at a given index $n = M$, we may rewrite Equation (40) as

$$\begin{aligned} x_0 &= -\frac{a_{-1}}{s_0} + \sum_{n=0}^{M-1} a_n (-1)^n s_0^n + a_M (-1)^M s_0^M \\ s_0 x_0' &= -\frac{a_{-1}}{s_0} - \sum_{n=0}^{M-1} n a_n (-1)^n s_0^n - M a_M (-1)^M s_0^M, \end{aligned} \tag{40}$$

where we have taken the M-th terms out of the sums to see that we can eliminate a_M from Equation (41), to get a single equation for the unknown pole:

$$M x_0 + s_0 x_0' = -(M+1)\frac{a_{-1}}{s_0} + \sum_{n=0}^{M} (M - n) a_n (-1)^n s_0^n. \tag{41}$$

After multiplication of this equation with s_0 we find a polynomial algebraic equation of M-th order (note that we redefined the summation index n):

$$M x_0 s_0 + x_0' s_0^2 = -(M+1) a_{-1} - \sum_{n=1}^{M} (M - n + 1) a_{n-1} (-1)^n s_0^n. \tag{42}$$

Hence we need to know the $M + 1$ coefficients $a_{-1}, a_0, a_1, \cdots, a_{M-1}$ in order to determine the solution of (42). In order to obtain the recursive system for determining the a_n we insert

(38) into (37), expand everything w.r.t. $\Delta = (s - s_0)$, and compare equal powers in Δ. The relation for the coefficients a_n then reads:

$$\frac{1}{\Delta^3}\left[2a_{-1} + \frac{K_3}{6}a_{-1}^3\right] + \frac{K_1 a_{-1}}{\Delta}$$
$$+ \sum_{n=0}^{\infty}\left[n(n-1)a_n\Delta^{n-2} + K_1 a_n \Delta^n + \frac{K_3}{2}a_{-1}^2 a_n \Delta^{n-2}\right] \quad (43)$$
$$+ \frac{K_3}{2}a_{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n a_m \Delta^{n+m-1} + \frac{K_3}{6}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} a_n a_m a_l \Delta^{n+m+l} = 0$$

By comparison of equal powers in Δ we find thus

$$\Delta^{-3} : \to 2a_{-1} + \frac{1}{6}K_3 a_{-1}^3 = 0$$
$$\Delta^{-2} : \to \frac{1}{2}K_3 a_{-1}^2 a_0 = 0$$
$$\Delta^{-1} : \to K_1 a_{-1} + \frac{1}{2}K_3 a_{-1}^2 a_1 + \frac{1}{2}K_3 a_{-1} a_0^2 = 0 \quad (44)$$
$$\Delta^{0} : \to 2a_2 + K_1 a_0 + \frac{1}{2}K_3\left(a_{-1}^2 a_2 + 2a_{-1}a_0 a_1 + \frac{1}{3}a_0^3\right) = 0$$

From this system follows for $a_{-1} \neq 0$:

$$a_{-1} = \pm 2i\sqrt{\frac{3}{K_3}}; \quad a_0 = 0; \quad a_1 = \pm i\frac{K_1}{3}\sqrt{\frac{3}{K_3}}; \quad a_2 = 0 \quad (45)$$

Truncating the Laurent expansion (38) at indices increasing from 1 to 3 then results in a sequence of improving approximations for the unknown first order pole s_0 as function of the initial conditions x_0 and x_0' as well as the expansion coefficients. Assuming $x_0' = 0$ we get

$$M = 1 : \quad s_0 x_0 + 2a_{-1} = 0 \to s_0 = -\frac{2a_{-1}}{x_0}$$
$$M = 2 : \quad a_1 s_0^2 + 2s_0 x_0 + 3a_{-1} = 0 \to s_0 = \frac{-4x_0 \pm \sqrt{16x_0^2 - 48a_1 a_{-1}}}{4a_1} \quad (46)$$
$$M = 3 : \quad 2a_1 s_0^2 + 3s_0 x_0 + 4a_{-1} = 0 \to s_0 = \frac{-3x_0 \pm \sqrt{9x_0^2 - 32a_1 a_{-1}}}{4a_1}$$

Substituting for the Laurent coefficients from Equation (45), we obtain

$$M=1: \quad s_0 = -\frac{4i}{x_0}\sqrt{\frac{3}{K_3}} \tag{47}$$

$$M=2: \quad s_0 = \frac{-3i}{4K_1}\sqrt{\frac{K_3}{3}}\left[-4x_0 \pm \sqrt{16x_0^2 + \frac{96K_1}{K_3}}\right]$$

$$M=3: \quad s_0 = \frac{-3i}{4K_1}\sqrt{\frac{K_3}{3}}\left[-3x_0 \pm \sqrt{9x_0^2 + \frac{64K_1}{K_3}}\right]$$

We now use the analytic expression (48) obtained by truncating the Laurent series after the term of order $(s - s_0)^3$ to derive the maximum possible integration step Δs which provides a convergent Taylor series based mapping of the initial value $x_0, x_0' = 0$. The following table shows the maximum $\Delta s = |s_0|$ (which in fact represents the distance of the origin $s = 0$ to the closest pole in the complex plane) as function of x_0 for $K_1 = 1\ \mathrm{m}^{-2}$ and $K_3 = 500\ \mathrm{m}^{-3}$. The numerical results have been obtained by inspecting the absolute values of the Taylor series contributions

$$\beta_N = \left.\frac{d^N x}{ds^n}\right|_{x=x_0, x'=0} \frac{\Delta s^N}{N!} \tag{48}$$

using the approximate quotient criterion

$$\left|\frac{\beta_{14}}{\beta_{12}}\right| = 1 \tag{49}$$

where the contributions $\beta_{12,14}$ have been found as function of x_0 ($x_0' = 0$) by use of symbolic differentiation of the equation of motion.

x_0 [m]	Δs_{an} [m]	Δs_{num} [m]
0.02	2.93	3.48
0.04	2.49	2.74
0.06	2.13	2.26
0.08	1.85	1.92
0.10	1.62	1.66
0.12	1.43	1.45
0.14	1.28	1.29
0.16	1.15	1.16
0.18	1.04	1.05

Let us finally discuss the meaning of the first order pole (48) of the solution of (37). If we consider a focusing quadrupole magnet ($K_1 > 0$) then s_0 will be purely imaginary if also the octupole is focusing ($K_3 > 0$). In this case our theory predicts that inside the magnet the (horizontal) motion will be bounded for all s since no pole exists on the real axes. This fact is well known and can be seen by inspection of the first integral of (37)

$$x'^2 + K_1 x^2 + \frac{K_3}{12} x^3 = C \tag{50}$$

which for $K_{1,3} > 0$ represents a set of closed concentric curves in the phase plane (x, x'). For $K_3 < 0$ we have to distinguish two cases:

1.

$$|x_0| < \frac{8}{3}\sqrt{\frac{K_1}{K_3}} \tag{51}$$

In this case the root in (48) becomes imaginary and s_0 is complex. Also in this case the solution is bounded.

2.

$$|x_0| \geq \frac{8}{3}\sqrt{\frac{K_1}{K_3}} \tag{52}$$

In this case the root is real, and we have a pole on the real axes. This means that the solution becomes infinite at a finite location of s.

4 MODEL FOR THE LHC-DIPOLE MAGNET

Next we consider the example of the LHC-dipole magnet with multipole errors of quadrupole and sextupole components. The equations of motion for the coupled betatron motion in the horizontal and vertical plane x and y neglecting the end fields can be written

$$x'' + \left[\frac{1}{\rho^2} + K_1\right] x + \frac{1}{2} K_2 (x^2 - y^2) = 0$$

$$y'' - K_1 y - K_2 x y = 0 \tag{53}$$

where ρ represents the radius of curvature and K_1 and K_2 are the quadrupolar and sextupolar multipole errors (systematic + random), see Ref. 5 respectively. We split the task into two parts. First we consider the motion in the horizontal plane ($y = 0$) which is one dimensional and is described by

$$x'' + \left[\frac{1}{\rho^2} + K_1\right] x + \frac{1}{2} K_2 x^2 = 0 \tag{54}$$

After that we turn to the fully two dimensional case of the coupled betatron motion. Finally we shall use the measured nonlinear multipole components of the LHC — dipole magnet and compute the maximum length over which a single integration step leads to a meaningful result when using a Taylor series based tracking code.

4.1 The one dimensional case

As can be seen by inspection of Equation (54) the solution $x(s)$ can only have poles of second order. This is due to the fact that the function $1/s^2$ becomes $1/s^4$ when it is squared by the x^2 term in Equation (54) and becomes proportional to $1/s^4$ when being two times differentiated by the x'' term. Thus the comparison of equal powers in $(s - s_0)^{-2}$ will lead to a nonzero result for the Laurent coefficient a_{-2}. Hence we are obliged to use the ansatz:

$$x(s) = \frac{a_{-2}}{(s-s_0)^2} + \frac{a_{-1}}{(s-s_0)} + a_0 + \sum_{n=1}^{\infty} a_n (s-s_0)^n \tag{55}$$

If we truncate the infinite sum at $n = M$, we may write x and x' as

$$x(s) = \frac{a_{-2}}{(s-s_0)^2} + \frac{a_{-1}}{(s-s_0)} + a_0 + \sum_{n=1}^{M-1} a_n (s-s_0)^n + a_M (s-s_0)^M \tag{56}$$

$$x'(s) = -\frac{2a_{-2}}{(s-s_0)^3} - \frac{a_{-1}}{(s-s_0)^2} + \sum_{n=1}^{M-1} n a_n (s-s_0)^{n-1} + M a_M (s-s_0)^{M-1}.$$

Substituting $s = 0$ into these equations results in two moving pole conditions. With $x(0) = x_0$, $x'(0) = x_0'$ we get:

$$M x_0 = \frac{M a_{-2}}{s_0^2} - \frac{M a_{-1}}{s_0} + \sum_{n=0}^{M-1} M a_n (-1)^n s_0^n + M a_M (-1)^M s_0^M$$

$$s_0 x_0' = \frac{2a_{-2}}{s_0^2} - \frac{a_{-1}}{s_0} - \sum_{n=0}^{M-1} n a_n (-1)^n s_0^n - M a_M (-1)^M s_0^M. \tag{57}$$

Adding these equations leads to a single algebraic equation for the unknown pole s_0:

$$M x_0 s_0^2 + s_0^3 x_0' = (M+2) a_{-2} - (M+1) s_0 a_{-1} + \sum_{n=0}^{M-1} a_n (-1)^n (M-n) s_0^{n+2} \tag{58}$$

We just need to know the Laurent coefficients $a_{-2}, a_{-1}, \cdots, a_{M-1}$, insert them into the algebraic equation (29) and solve the equation to obtain s_0. Insertion of the Laurent series into the equation of motion (54) and comparing equal powers in $\Delta = s - s_0$ leads to the recursive system

$$\begin{aligned}
\Delta^{-4}: &\quad a_{-2}\left(\tfrac{1}{2}a_{-2}K_2 + 6\right) = 0 \\
\Delta^{-3}: &\quad a_{-1}\left(\tfrac{1}{2}a_{-2}K_2 + 2\right) = 0 \\
\Delta^{-2}: &\quad a_{-1}^2 K_2 + 2a_{-2}a_0K_2 + 2a_{-2}K_1 = 0 \\
\Delta^{-1}: &\quad a_{-1}a_0K_2 + a_{-1}K_1 + a_{-2}a_1K_2 = 0 \\
\Delta^{0}: &\quad 2a_{-1}a_1K_2 + 2a_{-2}a_2K_2 + a_0^2K_2 + 2a_0K_1 + 4a_2 = 0 \\
\Delta^{1}: &\quad a_{-1}a_2K_2 + a_{-2}a_3K_2 + a_0a_1K_2 + a_1K_1 + 6a_3 = 0 \\
\Delta^{2}: &\quad 2a_{-1}a_3K_2 + 2a_{-2}a_4K_2 + 2a_0a_2K_2 + a_1^2K_2 + 2a_2K_1 + 24a_4 = 0 \\
&\quad \vdots
\end{aligned} \tag{59}$$

For $a_{-2} \neq 0$, the unique solution of this system up to a_3 is

$$a_{-2} = -\frac{12}{K_2} \qquad a_{-1} = 0 \qquad a_0 = -\frac{K_1}{K_2} \qquad a_1 = 0 \qquad a_2 = -\frac{K_1^2}{20K_2} \qquad a_3 = 0 \tag{60}$$

With these results the algebraic equation (58), truncated at $M = 4$, becomes a single biquadratic equation

$$\frac{1}{10}\frac{K_1^2}{K_2}s_0^4 + 4\left(x_0 + \frac{K_1}{K_2}\right)s_0^2 + \frac{72}{K_2} = 0 \tag{61}$$

with the solution

$$s_0 = \pm\frac{2}{K_2}\sqrt{-5x_0K_2 - 5K_1 \pm \sqrt{25x_0^2K_2^2 + 50x_0K_1K_2 - 20K_1^2}} \tag{62}$$

Since we are only interested in the absolute value of s_0, we may omit the $\pm$ sign in front of the first root. In addition we expect the poles which become zero for $x_0 \to \infty$ to be the ones who limit the convergence of a tracking code applied to Equation (54). This is the case if we choose the $+$ sign for the inner root in (62) so that finally

$$s_0 = \frac{2}{K_2}\sqrt{-5x_0K_2 - 5K_1 + \sqrt{25x_0^2K_2^2 + 50x_0K_1K_2 - 20K_1^2}} \tag{63}$$

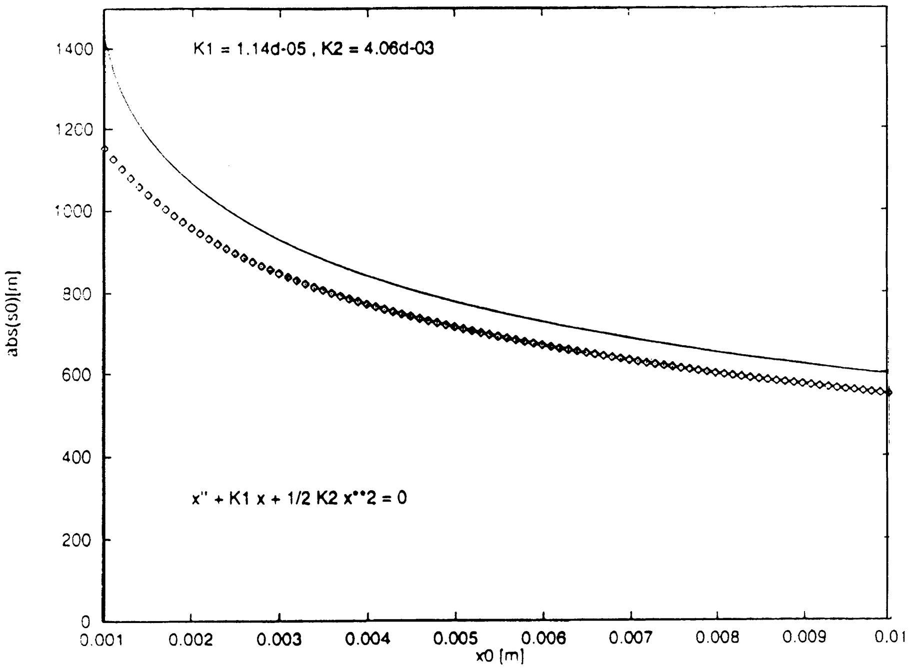

FIGURE 2: Comparison between the analytic (full line) and numeric (diamonds) convergence limit of a tracking code applied to Equation (54) when K_1=1.14·10^{-5} m^{-2} and K_2=4.06·10^{-3} m^{-3}.

For the case that the dimensionless quantity $K_1/(x_0K_2)$ fulfills the condition

$$\left|\frac{K_1}{K_2x_0}\right| \ll 1 \tag{64}$$

we may use a first order Taylor expansion in this quantity, and we obtain the location of the second order pole as

$$|s_0| = \frac{6}{\sqrt{2|x_0K_2|}}\left|2 - \frac{K_1}{2|K_2x_0|}\right| \tag{65}$$

Note that in Ref. 2 we found an exact expression for poles location when $K_1 = 0$

$$|s_0| = \frac{5.87}{\sqrt{2|x_0K_2|}} \tag{66}$$

which is rather close to the present result. In Figure 2 we show a comparison between the analytic result for $|s_0|$ from Equation (63), and the numeric result for the convergence breakdown for which we used the same criterion as in the previous section. For K_1 and K_2

the data for an LHC — dipole magnet[5] have been used. We see a good agreement between theory and experiment for this case. For a betatron amplitude of $x_0 = 1$ cm we find a maximum step size of about 600 m, well beyond the actual magnet length.

4.2 *The two dimensional case*

We now turn to the fully two dimensional case which is described by Equation (53). We have to look for poles in both planes, and denote the unknown poles of second order by s_{0x} and s_{0y} in the horizontal and vertical plane respectively. Each of the two Laurent series will in general contain information about s_{0x} and s_{0y} since Equation (53) are coupled. We therefore have to use the following general ansatz:

$$\begin{aligned} x(s) = {} & \frac{a_{-20}}{(s-s_{0x})^2} + \frac{a_{-21}}{(s-s_{0x})(s-s_{0y})} + \frac{a_{-22}}{(s-s_{0y})^2} \\ & + \frac{a_{-10}}{(s-s_{0x})} + \frac{a_{-11}}{(s-s_{0y})} \\ & + a_0 + a_{10}(s-s_{0x}) + a_{11}(s-s_{0y}) \\ & + a_{20}(s-s_{0x})^2 + a_{21}(s-s_{0x})(s-s_{0y}) + a_{22}(s-s_{0y})^2 \\ y(s) = {} & \frac{b_{-20}}{(s-s_{0x})^2} + \frac{b_{-21}}{(s-s_{0x})(s-s_{0y})} + \frac{b_{-22}}{(s-s_{0y})^2} \\ & + \frac{b_{-10}}{(s-s_{0x})} + \frac{b_{-11}}{(s-s_{0y})} \\ & + b_0 + b_{10}(s-s_{0x}) + b_{11}(s-s_{0y}) \\ & + b_{20}(s-s_{0x})^2 + b_{21}(s-s_{0x})(s-s_{0y}) + b_{22}(s-s_{0y})^2 \end{aligned} \tag{67}$$

Inserting x and y into the equations of motion we obtain, as before, linear recursive systems for the a, b_{mn}. If we use

$$\Delta_x = s - s_{0x} \quad \Delta_y = s - s_{0y} \tag{68}$$

the comparison of equal powers in $\Delta_x^m \Delta_y^n$ gives:

$$\Delta_x^{-4} \rightarrow \begin{cases} \frac{1}{2}a_{-20}^2 K_2 + 6a_{-20} - \frac{1}{2}b_{-20}^2 K_2 = 0 \\ b_{-20}(6 - a_{-20}K_2) = 0 \end{cases} \tag{69}$$

This system for a_{-20} and b_{-20} has two nontrivial solutions:

1.

$$b_{-20} = 0 \quad a_{-20} = -\frac{12}{K_2} \tag{70}$$

Using this solution leads to zero for all b_{mn} and to nonzero values for the a_{mn}. This case corresponds to the one dimensional case treated in the previous section.

2.

$$a_{-20} = \frac{6}{K_2} \qquad b_{-20} = \frac{6\sqrt{3}}{K_2} \tag{71}$$

This case leads to the Laurent coefficients for the fully coupled motion in the x, y plane.

For the remaining coefficients we find the following linear systems:

$$\left.\begin{aligned} -3\sqrt{3}b_{-21} + 4a_{-21} &= 0 \\ -2b_{-21} - 3\sqrt{3}a_{-21} &= 0 \end{aligned}\right\} a_{-21} = b_{-21} = 0 \tag{72}$$

$$\left.\begin{aligned} -3\sqrt{3}b_{-10} + 4a_{-10} &= 0 \\ -2b_{-10} - 3\sqrt{3}a_{-10} &= 0 \end{aligned}\right\} a_{-10} = b_{-10} = 0 \tag{73}$$

$$\left.\begin{aligned} a_{-22} - \sqrt{3}b_{-22} &= 0 \\ \sqrt{3}a_{-22} + b_{-22} &= 0 \end{aligned}\right\} a_{-22} = b_{-22} = 0 \tag{74}$$

$$\left.\begin{aligned} \sqrt{3}b_{-10} + a_{-11} &= 0 \\ b_{-11} + \sqrt{3}a_{-11} &= 0 \end{aligned}\right\} a_{-11} = b_{-11} = 0 \tag{75}$$

$$\left.\begin{aligned} -\sqrt{3}b_0 + a_0 &= -\tfrac{K_1}{K_2} \\ b_0 + \sqrt{3}a_0 &= -\sqrt{3}\tfrac{K_1}{K_2} \end{aligned}\right\} a_0 = -\frac{K_1}{K_2} \qquad b_0 = 0 \tag{76}$$

$$\left.\begin{aligned} -\sqrt{3}b_{10} + a_{10} &= 0 \\ b_{10} + \sqrt{3}a_{10} &= 0 \end{aligned}\right\} a_{10} = b_{10} = 0 \tag{77}$$

$$\left.\begin{aligned} -\sqrt{3}b_{11} + a_{11} &= 0 \\ b_{11} + \sqrt{3}a_{11} &= 0 \end{aligned}\right\} a_{11} = b_{11} = 0 \tag{78}$$

$$\left.\begin{aligned} -6\sqrt{3}b_{20} + 8a_{20} &= \tfrac{1}{2}\tfrac{K_1^2}{K_2} \\ -2b_{20} - 3\sqrt{3}a_{20} &= 0 \end{aligned}\right\} a_{20} = \frac{1}{22}\frac{K_1^2}{K_2} \qquad b_{20} = -\frac{3}{44}\sqrt{3}\frac{K_1^2}{K_2} \tag{79}$$

$$\left.\begin{aligned} -\sqrt{3}b_{21} + a_{21} &= 0 \\ b_{21} + \sqrt{3}a_{21} &= 0 \end{aligned}\right\} a_{21} = b_{21} = 0 \tag{80}$$

$$\left.\begin{aligned} -\sqrt{3}b_{22} + a_{22} &= 0 \\ b_{22} + \sqrt{3}a_{22} &= 0 \end{aligned}\right\} a_{22} = b_{22} = 0 \tag{81}$$

Note that in the case of a vanishing quadrupole component $K_1 = 0$ we find that all coefficients a and b become zero except a_{-20} and b_{-20} which are given in (71). Hence there exists a singular solution in the 2-D case given by

$$x(s) = \frac{6}{K_2(s-s_0)^2}; \quad y(s) = \sqrt{3}x(s) \tag{82}$$

which can easily be verified by inserting these functions into the equations of motion (53). In Appendix B to this paper we demonstrate that every motion starting in the plane $(x\,|\,y = \pm\sqrt{3}x)$ is integrable and can be expressed in terms of known functions.

With these results the two Laurent series for x and y become (truncating at a_{40}, b_{40}):

$$\begin{aligned} x(s) &= \frac{a_{-20}}{\Delta_x^2} + a_0 + a_{20}\Delta_x^2 + a_{40}\Delta_x^4 \\ y(s) &= \frac{b_{-20}}{\Delta_y^2} + b_0 + b_{20}\Delta_y^2 + b_{40}\Delta_y^4 \end{aligned} \tag{83}$$

Using x, $\dot{x}$ and y, $\dot{y}$ together with $s = 0$, we obtain the moving pole conditions (assuming $x_0' = y_0' = 0$)

$$\begin{aligned} 4x_0 s_{0x}^2 &= 6a_{-20} + 4a_0 s_{0x}^2 + 2a_{20}s_{0x}^4 \\ 4y_0 s_{0y}^2 &= 6b_{-20} + 2b_{20}s_{0y}^4 \end{aligned} \tag{84}$$

Using the above values for a_{-20}, a_0, a_{20} as well as b_{-20} and b_{20} results in

$$\begin{aligned} &\frac{1}{11}K_1^2 s_{0x}^4 - 4(x_0K_2 + K_1)s_{0x}^2 + 36 = 0 \\ &-\frac{3}{22}\sqrt{3}K_1^2 s_{0y}^4 - 4y_0K_2 s_{0y}^2 + 36\sqrt{3} = 0 \end{aligned} \tag{85}$$

with the solutions

$$\begin{aligned} s_{0x} &= \frac{1}{K_1}\sqrt{\frac{11}{2}\left[4(x_0K_2 + K_1) - \sqrt{16(x_0K_2 + K_1)^2 - \frac{144}{11}K_1^2}\right]} \\ s_{0y} &= \frac{1}{K_1}\sqrt{\frac{22}{6\sqrt{3}}\left[4y_0K_2 + \sqrt{16y_0^2K_2^2 + \frac{324}{22}K_1^2}\right]} \end{aligned} \tag{86}$$

If the two quantities $K_1/(x_0K_2)$ and $K_1/(y_0K_2)$ are small w.r.t unity we may expand these expressions to obtain in lowest order

$$|s_{0x}| = \frac{3\sqrt{2}}{\sqrt{2|K_2 x_0|}} \left|1 - \frac{K_1}{2x_0 K_2}\right|; \quad |s_{0y}| = \frac{3^{5/4}\sqrt{2}}{\sqrt{2|K_2 y_0|}} \tag{87}$$

For the LHC-dipole magnet these conditions are fulfilled at $x_0 > 0.5$ cm and $y_0 > 0.5$ cm. The convergence limit in Δs for the tracking code is given by the closest pole from the origin, so we must deal with the minimum of s_{0x} and s_{0y}, hence

$$\begin{aligned} \Delta s_{\max} &= \mathrm{Min}(s_{0x}, s_{0y}) \\ &= (3\sqrt{2})\,\mathrm{Min}\left[\frac{1}{\sqrt{2|K_2 x_0|}}\left|1 - \frac{K_1}{2x_0 K_2}\right|, \frac{3^{1/4}}{\sqrt{2|K_2 y_0|}}\right] \end{aligned} \tag{88}$$

In practice it is not possible to use a tracking code with a time step close to the limit of convergence, because the convergence properties then become very bad. This means that we need to go to very high orders which unavoidably leads to long execution times of the codes as well as to serious numerical problems (addition and subtraction of big numbers). Therefore we chose as the practical limit one tenth of the theoretical convergence radius described by Equation (88).

In Figure 3 we show the results for the case of the LHC-dipole magnets with quadrupolar and sextupolar field errors. Each of the rectangles describes the points for which the given magnet length represents one tenth of the convergence radius. The dashed line in this plot represents the actual dimensions of the LHC vacuum chamber. From the figure we realize that according to the theory a mapping for a length of the dipole up to $L = 50$ m would be tolerable, i.e. one may use a tracking code with a time step up to this limit. For the real length of the dipole of 14 m we therefore conclude that it is possible to integrate in one step over the full length with a code using a moderate order of expansion, and that it is therefore also permitted to replace the whole magnet by one (or two) localized kicks in order to save computer time in tracking codes.

5 CONCLUSIONS

In the present paper we develop a method which allows to determine the maximum permitted time step of a numeric integrator, based on a Taylor expansion of the solution of a given equation of motion

$$\dot{\vec{X}} = \vec{F}(\vec{X}, t) \tag{89}$$

The method is based on the theorem of Cauchy and Weierstrass which determines the convergence radius of a Taylor series. For this purpose it is necessary to determine the location of the singularities of the solution in the complex time plane $(\mathrm{Re}(t), \mathrm{Im}(t))$. This task can be achieved by expanding the solution around the unknown poles using a Laurent series. Together with the concept of moving singularities, it is then possible to derive a single polynomial equation for the location of the complex (or real) poles. Comparison with

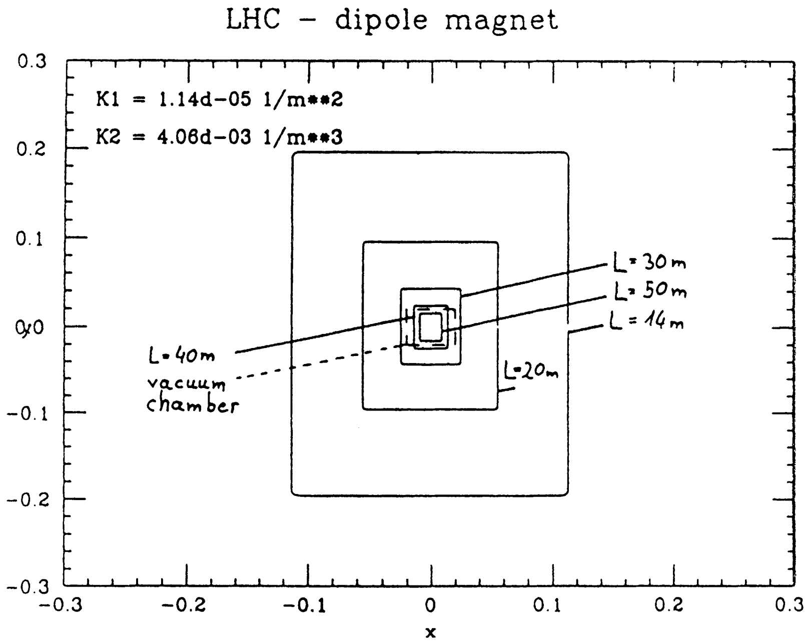

FIGURE 3: Convergence limits for tracking codes applied to LHC-dipole magnets of different lengths.

analytically solvable cases show excellent agreement of the theoretical predictions and the exact location of the poles. The theory was applied to the case of the superconducting LHC dipole magnets with quadrupolar and sextupolar field errors. We found that tracking over the full length of the magnet is possible in one single integration step.

However, more work still remains to be done. First we would like to include higher order multipoles in the theory (multipolar components up to order 22 can be measured, and have been included in tracking codes). In addition skew components should be taken into account.

The main question to be investigated is how many elements of a given nonlinear lattice can be concatenated and described by a single truncated Taylor mapping. In order to answer this question we must extend our method to equations with explicit time (or s) dependence since the magnetic focusing strength varies along the lattice. These investigations are planned for the near future.

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APPENDIX A: SOLUTION OF PARTICLE MOTION IN THICK SEXTUPOLES WITH JACOBIAN ELLIPTIC FUNCTIONS

The equation of 1-D (e.g. horizontal) motion in a thick sextupole is given by the nonlinear, second-order differential equation

$$\frac{d^2x}{dt^2} - K_2x^2 = 0. \tag{90}$$

It can be solved by first rewriting it the differential equation in terms of $v = dx/dt$ and x to yield

$$v\frac{dv}{dx} - K_2x^2 = 0 \tag{91}$$

This can be integrated once to yield

$$\frac{1}{2}v^2 - \frac{K_2}{3}x^3 = \frac{1}{2}C_1 \tag{92}$$

where C_1 is an integration constant to be determined from the initial conditions $x_0 = A, x'(0) = v(0) = B$. We thus find $C_1 = B^2 - 2K_2A^3/3$ and define

$$\alpha^2 = \frac{2}{3}K_2\beta^3 = \frac{3B^2}{2K_2} - A^3 \tag{93}$$

We solve for $v = dx/dt$ and integrate to get

$$\alpha t = \int_x^\beta \frac{dx'}{\sqrt{x^3 + \beta^3}} + C_2 \tag{94}$$

where C_2 is a second integration constant, and the RHS is an *elliptic integral*. The result can be inverted and solved for $x(t)$ in terms of elliptic functions

$$x(t) = \beta \left[1 - \sqrt{3} \frac{1 - \text{cn}(u|m)}{1 + \text{cn}(u|m)} \right] \tag{95}$$

where $\text{cn}(u|m)$ are the elliptic cosines of the argument $u = \gamma s + \delta$ and modulus $m = (2 + \sqrt{3})/4$. The parameters are defined by

$$\gamma^2 = \frac{2\beta}{\sqrt{3}} K_2$$

$$\delta = F(\phi|m)$$

$$\cos\phi = \frac{\beta(\sqrt{3} - 1) + A}{\beta(\sqrt{3} + 1) - A} \tag{96}$$

Although this expression appears rather complicated, it can be readily evaluated by computer.

Similar solutions can also be derived for 1-D particle motion in a thick octupole. There are several different expressions valid for different regions of the initial conditions, all having terms containing elliptic functions, which will not be shown here.

APPENDIX B: BOUNDED TRANSVERSE MOTION OF CHARGED PARTICLES IN THICK SEXTUPOLES

The Hamiltonian for the transverse motion of a charged particle inside a sextupole of strength K_2 can be written

$$H = \frac{x'^2 + y'^2}{2} + \frac{K_2}{6}(x^3 - 3xy^2) \tag{97}$$

from which we obtain the equations of motion

$$x'' + \frac{K_2}{2}(x^2 - y^2) = 0$$

$$y'' - K_2 xy = 0 \tag{98}$$

For motion in a plane we make the ansatz $y = ax$. Substitution into the equations of motion yields 2 equations which are consistent only if $a^2 = 3$ or

$$y = \pm\sqrt{3}x \tag{99}$$

The Hamiltonian then becomes

$$H = 2x'^2 - \frac{4}{3} K_2 x^3 \tag{100}$$

and the equation of motion becomes that of the 1-D motion

$$x'' - K_2 x^2 = 0. \tag{101}$$

For the particular case $H = 0$, the Hamiltonian yields directly

$$\frac{dx}{ds} = 2\alpha_2 K_2 x^3 \tag{102}$$

where

$$\alpha_2^2 = \frac{K_2}{6} \tag{103}$$

The expression for ds can be integrated to yield

$$x(s) = \frac{1}{(\alpha_2 s + C)^2} \tag{104}$$

where

$$C = \frac{1}{\sqrt{x_0}} \tag{105}$$

to fulfill the initial condition $x(0) = x_0$, and hence also $y(0) = \pm\sqrt{3}x_0$. The initial condition for the derivatives then are also determined as $x'(0) = -2\alpha_2 x_0^{3/2}$ and $y'(0) = -\pm 2\sqrt{3}\alpha_2 x_0^{3/2}$. The square roots require that $x > 0$ for $K_2 > 0$ and vice versa.

This solution shows the astonishing property of decreasing monotonically towards zero with increasing s. However, it is only valid for one particular pair of slopes for a pair of initial amplitudes x_0, y_0 located in the two planes $y = \pm\sqrt{3}x$.

The equations can also be solved for more general initial slopes for initial conditions in the same two planes. Dropping the limitation $H = 0$, the equation of motion can be solved in terms of Jacobian elliptic functions. The solution can be expressed in terms of the elliptic cosine function cn as

$$x(t) = \beta\left[1 - \sqrt{3}\frac{1 - \mathrm{cn}(u|m)}{1 + \mathrm{cn}(u|m)}\right] \tag{106}$$

where the parameter β, the argument u and the modulus m were defined in Appendix A. We quote this result only to show the existence of solutions for any initial slope $x'(0)$ at any initial value $x(0)$ — but restricted to the planes $y = \pm\sqrt{3}x$.

The motion of particles with arbitrary initial slopes can also be found directly by numerical integration of Equation (101), and is shown in Figure 4.

As can be seen from the figure, the trajectories near the special solution for $H = 0$ start to converge towards zero, but eventually break away and grow without bounds. If the sextupole is shorter than this turning point, the motion will nevertheless remain bounded in both planes $y = \pm\sqrt{3}x$ for a finite range of initial slopes.

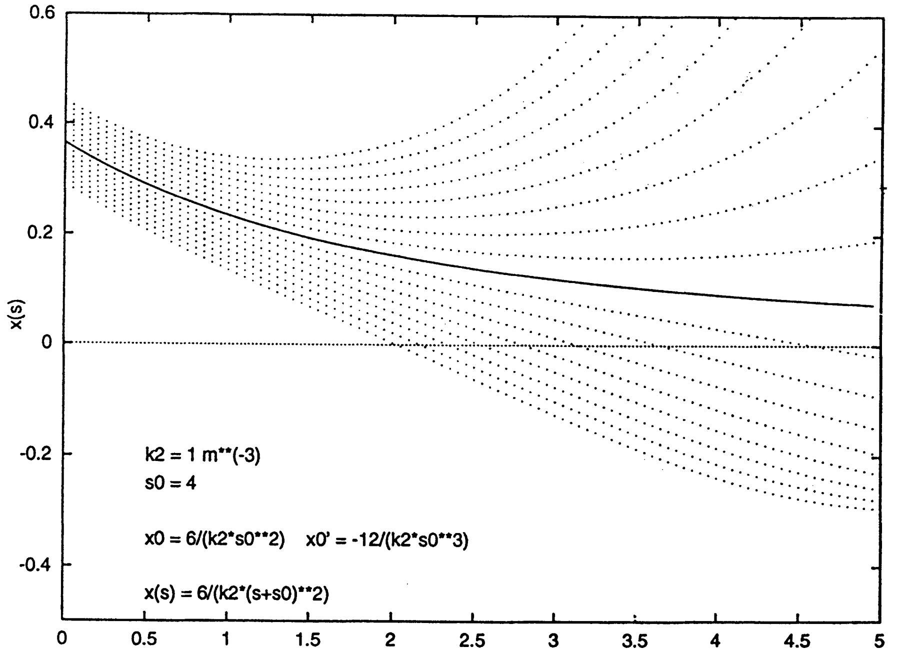

FIGURE 4: Transverse particle motion in a thick sextupole for various initial slopes.

A similar consideration can be made for 2-D motion in octupoles. The Hamiltonian there is

$$H = \frac{x'^2 + y'^2}{2} + \frac{K_3}{24}(x^4 - 6x^2y^2 + y^4) \tag{107}$$

where K_3 is the strength of the octupole. The equations of motion become

$$\begin{aligned} x'' + \frac{K_3}{6}(x^3 - 3xy^2) &= 0 \\ y'' - \frac{K_3}{6}(3x^2y - y^3) &= 0 \end{aligned} \tag{108}$$

The ansatz $y = ax$ leads to consistent results only if $a = \pm 1$. The equation of motion again becomes identical to that for the 1-D motion

$$x'' - \frac{K_3}{3}x^3 = 0 \tag{109}$$

and can be solved in terms of elliptic functions. The particular solution for $H = x'^2 - K_3x^4/6 = 0$ becomes

$$x = \frac{1}{1/x_0 \pm \alpha_3 s} \tag{110}$$

$$\alpha_3^2 = \frac{K_3}{6} \tag{111}$$

One of the solutions diverges for $s = 1/\alpha_3 x_0$ and is not monotonic, but the other one describes bounded motion in octupoles, similar to the one found to exist in sextupoles.