Aspects of Space-Time Dualities

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ABSTRACT

Duality groups of Abelian gauge theories on four manifolds and their reduction to two dimensions are considered. The duality groups include elements that relate different space-times in addition to relating different gauge-coupling matrices. We interpret (some of) such dualities as the geometrical symmetries of compactified theories in higher dimensions. In particular, we consider compactifications of a (self-dual) 2-form in $6 - D$, and compactifications of a self-dual 4-form in $10 - D$. Relations with a self-dual superstring in $6 - D$ and with the type IIB superstring are discussed.

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1   Introduction

Electric-magnetic dualities in gauge theories and string duality symmetries have been studied extensively recently (for a review, see for instance [1, 2, 3, 4] and references therein). Sometimes, by using string dualities, electric-magnetic duality can be related to a geometrical symmetry of the internal space in some string compactification.

In this work, we consider duality symmetries in $4 - D$, Abelian gauge theories which involve also the (Euclidean, compact) space-time M^4. Such dualities – rather mysterious from the $4 - D$ point of view – are better understood as geometrical symmetries of theories in higher dimensions, compactified to M^4 on some internal space.

Explicitly, we find dualities which relate a pair (M^4, τ) to a different pair $(\tilde{M}^4, \tilde{\tau})$, where τ is the (complex) coupling constant matrix of a $U(1)^r$ gauge theory on M^4, and $\tilde{\tau}$ is the dual coupling constant matrix of a $U(1)^{\tilde{r}}$ gauge theory on $\tilde{M}^4$. In general, M^4 and $\tilde{M}^4$ not only have a different geometry, but also a different topology; in this case, $\tilde{r} \neq r$.[3]

Some of the dualities considered here can be understood as the consequence of string dualities, in the limit where gravity is decoupled.

Let us start with a simple example. String-string triality relates the heterotic string compactified on $T^4 \times T^2$ to type IIA and type IIB strings compactified on $K^3 \times T^2$. The duality group of the heterotic string includes, in particular, the $SL(2, \mathbf{Z})_S \times SL(2, \mathbf{Z})_U \times SL(2, \mathbf{Z})_T$ acting on the dilaton-axion field S, the complex structure of the two-torus U and its complex Kähler structure T, respectively, as well as the $\mathbf{Z}_2$ factorized duality (mirror symmetry) interchanging $U \leftrightarrow T$ [3]. String-string duality involves also the $\mathbf{Z}_2$ interchanging $U \leftrightarrow S$ or $T \leftrightarrow S$. The low-energy effective field theory, in the infinite Planck-mass limit, is an $N = 4$ supersymmetric Yang-Mills (YM) theory on Minkowski space. At a generic point in moduli space, it is an Abelian gauge theory, including a $U(1)^4$ gauge group originating from the internal torus T^2. One expects the dualities described above to be manifest in the $N = 4$ YM theory. Indeed, the $SL(2, \mathbf{Z})^3$ and $\mathbf{Z}_2$'s, corresponding to $U \leftrightarrow T$, $U \leftrightarrow S$ and $T \leftrightarrow S$, are part of the $Sp(8, \mathbf{Z})$ duality transformations acting on the 4×4 gauge coupling matrix τ of the $U(1)^4$ gauge theory.

Our second example, described in section 2, involves also space-time. We argue that an $SU(2)$, $N = 4$ YM theory, broken to $U(1)$ at large scalar VEVs, is invariant under an $O(2, 2, \mathbf{Z})$ duality group acting on the complex gauge coupling S and the complex structure U of two space directions. This duality group includes the well known $SL(2, \mathbf{Z})_S$ S-duality group, the geometrical $SL(2, \mathbf{Z})_U$ symmetries and, in addition, a duality which interchanges the gauge coupling with the complex structure of space: $S \leftrightarrow U$. In section 2 we also explain the origin of this duality from the manifest geometrical symmetry of a $6 - D$ self-dual superstring, compactified to $4 - D$ on T^2.

In section 3, we consider a compactified theory of (self-dual) 2-form in $6 - D$ and show that its geometrical symmetries lead to the dualities of section 2, as well as several other dualities appearing in string theory.

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[3]In this case, the duality is expected to be only a symmetry of the classical part of the theory.

In section 4, we discuss more dualities in Abelian gauge theories which relate different pairs of (M^4, τ). Moreover, we show how such dualities are a consequence of geometrical symmetries in a theory of self-dual 4-form in $10 - D$, compactified on $M^4 \times T^2 \times \tilde{M}^4$.

2 $S \leftrightarrow U$ Duality in $N = 4$ Yang-Mills Theory and the Self-Dual Superstring in $6 - D$

Our second example involves a compact space-time. Consider an $SU(2)$, $N = 4$ YM theory on $M^4 = S_\beta^1 \times S_R^1 \times T_U^2$. Here S_β^1 is a compact time at radius β ($=$ the inverse temperature), S_R^1 is a compact space coordinate on a circle with radius R, and the other two space coordinates are compactified on a two-torus T_U^2 with complex structure $U = U_1 + iU_2$. We consider the limit in which $\beta, R \to \infty$ such that their ratio is finite, say $\beta/R = 1$. At large vacuum expectation values of the scalar fields $\langle \Phi \rangle \to \infty$, a simple generalization of the computation in [5] gives the partition function

$$Z(S, U) = c \sum_n \exp\{-\pi n^t M(U) \otimes M(S) n\}. \tag{2.1}$$

Here c is an S–independent and U–independent factor, n is a 4-vector of integers, $S = S_1 + iS_2 = \theta/2\pi + i4\pi/g^2$ is the complex gauge coupling and

$$M(S) = \frac{1}{S_2} \begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix}. \tag{2.2}$$

The 2×2 matrix $M(U)$ is given by (2.2) with S replaced by U.

The partition function in (2.1) is invariant under the $O(2, 2, \mathbf{Z})$ duality group acting on U, S [3]. This duality group is the product of $SL(2, \mathbf{Z})_S$, $SL(2, \mathbf{Z})_U$ and the $\mathbf{Z}_2$ interchanging them. The $SL(2, \mathbf{Z})_S$ is the well-known S-duality group acting on S. The $SL(2, \mathbf{Z})_U$ is the geometrical symmetry of the torus T_U^2, acting on its complex structure U. The rather surprising duality transformation is the one interchanging the gauge coupling with the structure of space $S \leftrightarrow U$! It is a mysterious symmetry from the $4 - D$ gauge theory point of view.

We can understand the origin of this duality from the manifest geometrical symmetry of a self-dual superstring in $6 - D$ [6, 7] compactified on $S^1 \times S^1 \times T^2 \times T^2$. This string theory is non-gravitational, defined in rigid $6 - D$ space-time, and is supposed to give a theory with chiral $N = 2$ supersymmetry in $6 - D$. The string is coupled to a self-dual 2-form B (namely, B with a self-dual field strength H ($H = dB = *H$)). It was argued that compactifying this theory to $4 - D$ on $M^4 \times T^2$ gives rise to an $N = 4$ supersymmetric gauge theory [6, 7]; the B field gives rise to a $U(1)$ gauge field in four dimensions. The winding numbers of the string around the two independent cycles of the torus correspond to electric and magnetic charges. The S-duality symmetry of the $4 - D$, $N = 4$ theory is a simple consequence of the geometrical symmetry of the torus. In the decompactification limit, all the charged states become infinitely massive, and one is left with a free Abelian theory in four dimensions.

Consider first the $6-D$ self-dual superstring compactified to $4-D$ on T_S^2, and with $M^4 = S_\beta^1 \times S_R^1 \times T_U^2$. Recall that the subscripts S and U denote the complex structures of T_S^2 and T_U^2, respectively. At the decompactification limit, the partition function of the theory is $Z(S, U)$.

The $S \leftrightarrow U$ duality is manifest if instead we consider the compactification of the $6 - D$ theory to $4 - D$ on T_U^2, and with $\tilde{M}^4 = S_\beta^1 \times S_R^1 \times T_S^2$. Again, at the decompactification limit, the theory is a free $D = 4$, $N = 4$ supersymmetric $U(1)$ gauge theory, but this time with a gauge coupling U, and a complex structure S in space-time. Therefore, the partition function of the theory is $Z(U, S)$.

Since both theories are the same $6 - D$ self-dual theory, we must find

$$Z(S, U) = Z(U, S). \tag{2.3}$$

This duality indeed obtains in the partition function eq. (2.1), but the argument given before has a subtlety: no manifestly Lorentz-invariant action for a self-dual two-form exists. Thus, while the $S \leftrightarrow U$ duality of the equations of motion of an Abelian $4 - D$ gauge theory can be deduced along the previous lines, the duality of the partition function itself must be proven in another way. This problem is studied in the next section.

3   Compactifications of 2-Form Theories in $6-D$ and Duality

Let us consider in more detail the result of the previous section. We want to compactify a $6-D$ two-form on the manifold $T_T^2 \times T_U^2 \times T_S^2$, and impose an appropriate condition of self-duality. In the previous section the modular parameter T was $T = i\beta/R$.

The six-dimensional action is [8]

$$S_{6-D} = \frac{1}{2\pi i} \int dB \wedge H - \frac{1}{4\pi} \int *H \wedge H. \tag{3.1}$$

We denote by a_S and b_S the cycles of the torus T_S^2, obeying

$$\int_{T^2} a_S \wedge *a_S = \frac{|S|^2}{S_2}, \quad \int_{T^2} a_S \wedge *b_S = \frac{S_1}{S_2}, \quad \int_{T^2} b_S \wedge *b_S = \frac{1}{S_2}, \quad S = S_1 + iS_2. \tag{3.2}$$

The self-duality condition is imposed as in [8, 9] by choosing a global vector field V in T_S^2 such that $i_V b_S = 0$, and setting half of the components of the field-strength H equal to dB:

$$i_V(H - dB) = 0. \tag{3.3}$$

The resulting $4 - D$ action is an Abelian gauge theory with coupling S [8]. Compactifying its action on T_U^2 gives rise to a $2 - D$ sigma model. Obviously, we could have reduced the $6 - D$ theory directly to $2 - D$ on $T_S^2 \times T_U^2$. Indeed, the reduction was performed in [8] in the case of a generic four-manifold M^4 with equal number of self-dual and anti-self-dual two-forms ($b^+ = b^-$).

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Following [8], we denote by α_i, β^i, $i = 1, .., b^+$ a basis of $H_2(M^4, \mathbf{Z})$ such that $\int_{M^4} \alpha_i \wedge \beta^j = \delta_i^j$, and we define

$$\int_{M^4} \beta^i \wedge *\beta^j = G^{ij}, \quad \int_{M^4} \alpha_i \wedge *\beta^j = B_{il}G^{lj}, \quad \int_{M^4} \alpha_i \wedge *\alpha_j = G_{ij} - B_{il}G^{lm}B_{mj}. \tag{3.4}$$

The self-duality condition is always given as in eq. (3.3), with the global vector V obeying $i_V\beta^i = 0$ for all i. The $2-D$ sigma model obtained by reducing eq. (3.1) on such manifold is [8]

$$S_{2-D} = \frac{1}{2\pi} \int dX^i \wedge (G_{ij} * dX^j - iB_{ij}dX^j). \tag{3.5}$$

This is the action of a sigma model on the toroidal background with metric G_{ij} and antisymmetric tensor B_{ij}.

In our case $M^4 = T_S^2 \times T_U^2$, and a basis of $H_2(M^4, \mathbf{Z})$ is

$$\alpha = a_S \wedge a_U, \quad a_S \wedge b_U, \quad a_S \wedge b_S,$$
$$\beta = b_U \wedge b_S, \quad b_S \wedge a_U, \quad a_U \wedge b_U. \tag{3.6}$$

The vector V is the same as before: it is oriented along a cycle of T_S^2 and obeys $i_V b_S = 0$. The metric G_{ij} and antisymmetric tensor B_{ij} describe the space $T^2 \times S^1$, where T^2 is a torus with complex structure U and complex Kähler modulus S, while the radius of S^1 is $\sqrt{V_U/V_S}$, where V_U (V_S) is the volume of T_U^2 (T_S^2):

$$G_{ij} = \begin{pmatrix} S_2|U|^2/U_2 & S_2U_1/U_2 & 0 \\ S_2U_1/U_2 & S_2/U_2 & 0 \\ 0 & 0 & V_U/V_S \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & S_1 & 0 \\ -S_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.7}$$

Notice that $\sqrt{V_S/V_U}$ is the coupling constant of the sigma model on S^1, thus, in the limit $V_S/V_U \to \infty$, S^1 decouples and one is left with a $2-D$ sigma model on T^2. The S^1 sigma model possesses a T-duality which inverts the radius. Thus, S^1 decouples also in the limit $V_S/V_U \to 0$. On T^2, the interchange $S \leftrightarrow U$ is simply the mirror symmetry of the torus [4]. Thus, upon further compactification to $0 - D$ on T_T^2, mirror symmetry implies that the partition function obey $Z(S, U, T) = Z(U, S, T)$, as announced in eq. (2.3) (for $T = i$, and after a Poisson resummation, the partition function can be brought into the form (2.1)). The symmetry $U \leftrightarrow T$, leading to a triality, is manifest in the partition function, since our compactification on $T_S^2 \times T_U^2 \times T_T^2$ is manifestly symmetric in the interchange $T_U^2 \leftrightarrow T_T^2$. This triality of the (classical part) of the 1-loop partition function $Z(S, U, T)$ of a $2-D$ sigma model on T^2 was observed in [10]. In string theory, this triality is rather mysterious because T is the complex structure of the world-sheet torus while U and S are the complex structure and Kähler structure of the target-space torus T^2. However, for the 2-form theory on $T_S^2 \times T_U^2 \times T_T^2$ this triality is the geometrical symmetry permuting the three two-tori.

Two generalizations of our results are immediate.

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[4]The $S \leftrightarrow U$ duality holds for any V_U, V_S; but for triality, considered below, one should take $V_S \to \infty$ or 0.

First of all, the symmetry $S \leftrightarrow U$ holds for a compactification $T_S^2 \times T_U^2 \times \Sigma$, with Σ any Riemann surface.

Secondly, the previous result can be further generalized to a generic compactification $T_S^2 \times \Sigma \times \tilde{\Sigma}$, with Σ and $\tilde{\Sigma}$ any two Riemann surfaces, with the same choice for V as in the previous examples. The compactification is manifestly invariant under the interchange $\Sigma \leftrightarrow \tilde{\Sigma}$. By performing the compactification in two stages, first from $6 - D$ to $2 - D$ on $T_S^2 \times \Sigma$, and then form $2 - D$ to $0 - D$ on $\tilde{\Sigma}$, the previous symmetry gives rise to a symmetry under the interchange of the world-sheet with the target space of a toroidal sigma model. In detail, let us denote by A_I and B^I, $I = 1, .., g$, the 1-cycles of the Riemann surface Σ, of genus g. The complex structure of the surface is determined by the data G, B:

$$\int_\Sigma B^I \wedge *B^J = \mathrm{G}^{IJ}, \quad \int_\Sigma A_I \wedge *B^J = \mathrm{B}_{IL}\mathrm{G}^{LJ}, \quad \int_\Sigma A_I \wedge *A_J = \mathrm{G}_{IJ} + \mathrm{B}_{IL}\mathrm{G}^{LM}\mathrm{B}_{MJ}, \quad \mathrm{B}_{IJ} = \mathrm{B}_{JI}. \tag{3.8}$$

We define the 2-cycles of $T_S^2 \times \Sigma$ as

$$\alpha = a_S \wedge A_I, \quad a_S \wedge B^I, \quad a_S \wedge b_S,$$
$$\beta = B^I \wedge b_S, \quad b_S \wedge A_I, \quad \omega_\Sigma. \tag{3.9}$$

Here ω_Σ is the generator of $H_2(\Sigma, \mathbf{Z})$. Following the same steps as before, we find a $2 - D$ sigma model, propagating on the world-sheet $\tilde{\Sigma}$, with constant background metric and antisymmetric tensor describing a torus $T^{2g} \times S^1$:

$$G_{ij} = S_2 \begin{pmatrix} \mathrm{G} + \mathrm{B}\mathrm{G}^{-1}\mathrm{B} & \mathrm{B}\mathrm{G}^{-1} & 0 \\ \mathrm{G}^{-1}\mathrm{B} & \mathrm{G}^{-1} & 0 \\ 0 & 0 & S_2^{-1}V_\Sigma/V_S \end{pmatrix}, \quad B_{ij} = S_1 \begin{pmatrix} 0 & \mathrm{I} & 0 \\ -\mathrm{I} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

where I is the $g \times g$ identity matrix and V_Σ is the volume of Σ. In the two limits $V_S/V_\Sigma \to \infty$, $V_S/V_\Sigma \to 0$, the sigma model on S^1 decouples as remarked after equation (3.7). Inserting the background (3.10) in the action (3.5) one finds that the "leftover," i.e. a sigma model on T^{2g} propagating on $\tilde{\Sigma}$, is the same as a $2\tilde{g}$-dimensional toroidal sigma model propagating on the world-sheet Σ, with background given by eq. (3.10) with G, B replaced everywhere by the data of $\tilde{\Sigma}$: $\tilde{\mathrm{G}}$ and $\tilde{\mathrm{B}}$. Notice that, when the genus of the two Riemann surfaces is different, $\tilde{g} \neq g$, and the two sigma models dual to each other contain a different number of fields.

Other dualities of the classical partition function, interchanging the world-sheet with the target space, were considered in ref. [11]. One of them can be interpreted geometrically by slightly modifying the construction presented above. We still compactify a $6 - D$ two-form, with action given in eq. (3.1), on $T_S^2 \times \Sigma \times \tilde{\Sigma}$, with two changes. First of all we choose the following two-cycle basis for $T_S^2 \times \Sigma$:

$$\alpha = a_S \wedge A_I, \quad b_S \wedge A_I, \quad a_S \wedge b_S,$$
$$\beta = B^I \wedge b_S, \quad a_S \wedge B^I, \quad \omega_\Sigma. \tag{3.11}$$

Secondly, instead of eq. (3.3), we impose

$$\int_\Sigma B^I \wedge (H - dB) = 0, \quad I = 1, ..., g. \tag{3.12}$$

The configurations that minimize the classical action eq. (3.1) are harmonic (they obey $dH = 0$, $d * H = 0$), therefore, they can be expanded in the basis α, β:

$$H = \alpha_i \wedge \Pi^i + \beta^i \wedge \Pi_i^D, \quad B = \alpha_i \wedge X^i + \beta^i \wedge X_i^D. \tag{3.13}$$

Eq. (3.12) implies $\Pi^i = dX^i$. Standard manipulations [8] lead to the sigma-model action eq. (3.5), describing the target space $T^{2g} \times S^1$, with metric and antisymmetric tensor given by

$$G_{ij} = \begin{pmatrix} M(S) \otimes \mathrm{G} & 0 \\ 0 & V_\Sigma/V_S \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} \varepsilon \otimes \mathrm{B} & 0 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.14}$$

where $\mathrm{G}_{IJ}, \mathrm{B}_{IJ}$ are defined in (3.8) and $M(S)$ is defined in (2.2). When $V_\Sigma/V_S \to \infty, 0$, the sigma model on S^1 decouples and one is left as before with a $2g$-dimensional sigma model on the toroidal background T^{2g}. Upon compactification to $0 - D$ on $\tilde{\Sigma}$, the classical equations of motion of H and the constraint (3.12) imply [5]

$$\int_{\tilde{\Sigma}} \tilde{B}^I \wedge (H - dB) = 0. \tag{3.15}$$

The "classical partition function," $Z(\mathrm{G}, \mathrm{B}, \tilde{\mathrm{G}}, \tilde{\mathrm{B}}, S)$, is the sum over all solutions of the classical equations of motion of $\exp(-S_{2-D}|_{class})$, where $S_{2-D}|_{class}$ is the two-dimensional action evaluated at the classical solution. *On the classical solutions* (but not in general!) our dimensional reduction is symmetric in the interchange of of the target-space data G, B with the world-sheet data $\tilde{\mathrm{G}}, \tilde{\mathrm{B}}$ (cfr. eqs. (3.12,3.15)); this can be made explicit by performing a Poisson resummation in the partition function. This symmetry is one of the target space for world-sheet dualities observed (at $S = i$) in ref. [11].

4 $(M^4, \tau) \leftrightarrow (\tilde{M}^4, \tilde{\tau})$ Duality and Compactifications of a Self-Dual 4-Form Theory in $10 - D$

In ref. [8], it was shown that by compactifying a $10 - D$ self-dual four-form, B, on $K^3 \times T_S^2$, one obtains a $4 - D$ Abelian gauge theory with group $U(1)^{b_2}$ (where b_2 is the second Betti number of K^3). This result can be generalized to a compactification $M^4 \times T_S^2$, where M^4 is any smooth manifold with $b_1 = 0$. Let us denote by γ_I, $I = 1, .., b_2$, a basis for $H_2(M^4, \mathbf{Z})$, and define

$$G_{IJ} = \int_{M^4} \gamma_I \wedge *\gamma_J, \quad Q_{IJ} = \int_{M^4} \gamma_I \wedge \gamma_J. \tag{4.1}$$

When $b_1(M^4) = 0$, the dimensional reduction of a five-form field strength H reads

$$H = \alpha_I \wedge F^I + \beta^I \wedge F_I^D, \quad B = \alpha_I \wedge A^I + \beta^I \wedge A_I^D, \quad \alpha_I = \gamma_I \wedge a_S, \quad \beta^I = \gamma_I \wedge b_S. \tag{4.2}$$

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[5]We implicitly assumed throughout the paper that B, upon compactification on M^6, is periodic up to integral elements of $H_3(M^6, \mathbf{Z})$, i.e. $[dB] \in H_3(M^6, \mathbf{Z})$.

Choosing as usual a global vector field $V \in T_S^2$, such that $i_V \beta^I = 0$, one finds a $4-D$ action for $b_2(M^4)$ Abelian gauge fields [8]

$$S_{4-D} = \frac{1}{g^2} \int F^I \wedge G_{IJ} * F^J - i\frac{\theta}{8\pi^2} \int F^I \wedge Q_{IJ} F^J. \tag{4.3}$$

Recall that $S = \theta/2\pi + 4\pi i/g^2$. Upon further compactification on $M^4 \times T_S^2 \times \tilde{M}^4$, one finds a manifest symmetry under the interchange of G, Q with $\tilde{G}, \tilde{Q}$. In other words, one finds a duality between the geometrical data of the manifold $\tilde{M}^4$ and the coupling-constant matrix $\tau_{IJ} = \theta Q_{IJ}/2\pi + 4\pi i G_{IJ}/g^2$.

This symmetry, obvious in our construction, would look rather puzzling from the four-dimensional point of view. Explicitly, in four dimensions, this duality relates a theory on a space-time manifold M^4 and with a coupling-constant matrix τ to a theory on a *different* four-manifold and with a different gauge-coupling matrix:

$$\{M^4(G,Q), \tau(\tilde{G}, \tilde{Q})\} \leftrightarrow \{M^4(\tilde{G}, \tilde{Q}), \tau(G, Q)\}. \tag{4.4}$$

The construction explained here sometimes works for more than the partition function of an Abelian gauge theory.

In particular, when both M^4 and $\tilde{M}^4$ are topologically K^3 surfaces, our construction can be embedded in an $N = 4$ compactification of the Type IIB string, since this string contains among its massless fields a self-dual $10-D$ four-form. Our construction, in other words, computes the large-volume limit $V_{M^4} \to \infty$ of the partition function of a type IIB string on $K^3 \times T_S^2 \times \tilde{K}^3$. For small values of the type IIB string coupling constant [6] the gravitational sector of the theory decouples, and the partition function becomes that of a rigid $N = 4$ supersymmetric Yang-Mills theory compactified on $\tilde{K}^3$. This partition function has been studied (for an $SU(2)$ gauge group) in ref. [12]. It would be interesting to see in detail how the space-time $\leftrightarrow$ gauge-coupling duality works in this case.

Finally, we should mention more duality symmetries of the classical partition sum of free $U(1)^r$ gauge theories on four-manifolds M^4. In [8, 13], it was shown that such partition functions are formally equal to those of a $2-D$ toroidal model with a genus-r world-sheet, and with left-handed and right-handed $2-D$ momenta+windings in a self-dual Lorentzian lattice Γ^{b_+, b_-}, with signature (b_+, b_-), where b_+ (b_-) is the number of (anti-)self dual harmonic two-forms in M^4. The data of M^4 are encoded in Γ^{b_+, b_-}, while the gauge-couplings data are encoded in the period matrix τ of the world-sheet (τ can be extended to a general positive-definite symmetric complex matrix). Such partition functions have many duality symmetries which mix the space-time data with the world-sheet data, similar to the target-space $\leftrightarrow$ world-sheet dualities considered in [11]. As discussed in this work, some of these dualities can be interpreted as the geometrical symmetries of compactified theories in higher dimensions. More dualities could have their geometrical origin in some simple generalizations of the compactifications considered here. One such instance is the compactification of a $10-D$ self-dual 4-form theory on $M^4 \times \Sigma \times \tilde{M}^4$, with Σ any Riemann surface (this has S-duality only if $\Sigma = T^2$).

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[6]Equivalently, because of Type IIB strong-weak coupling duality, for large values of the coupling constant.

To recover all the symmetries of $4 - D$ gauge theories, and their possible geometrical origin from higher dimensions, is an interesting problem which may shed more light on the nonperturbative dynamics of supersymmetric gauge theories and strings. In this paper, we considered aspects of this problem in some simple, yet probably significant cases.

Acknowledgements

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