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# A General Algorithm for Calculating Jet Cross Sections in NLO QCD\*

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## Abstract

We present a new general algorithm for calculating arbitrary jet cross sections in arbitrary scattering processes to next-to-leading accuracy in perturbative QCD. The algorithm is based on the subtraction method. The key ingredients are new factorization formulae, called dipole formulae, which implement in a Lorentz covariant way both the usual soft and collinear approximations, smoothly interpolating the two. The corresponding dipole phase space obeys exact factorization, so that the dipole contributions to the cross section can be exactly integrated analytically over the whole of phase space. We obtain explicit analytic results for any jet observable in any scattering or fragmentation process in lepton, lepton-hadron or hadron-hadron collisions. All the analytical formulae necessary to construct a numerical program for next-to-leading order QCD calculations are provided. The algorithm is straightforwardly implementable in general purpose Monte Carlo programs.

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# 1 Introduction

Most of the recent progress in the understanding of strong interaction physics at large momentum transfer has been due to the comparison between precise experimental data and very accurate QCD calculations to higher perturbative orders [1].

The perturbative QCD approach for computing hadronic cross sections is based on the parton model picture. According to this picture, the cross section for any hard-scattering process (i.e. any process involving a large transferred momentum  $Q$ ) can be written as a convolution of structure ( $f_a(x, Q^2)$ ) and fragmentation ( $d_a(x, Q^2)$ ) functions of partons (quarks and gluons) and a hard-cross section factor. The former are non-perturbative quantities but are universal, that is, they are process independent. The latter is instead dominated by momentum regions of the order of  $Q$  and hence, provided that  $Q \gg \Lambda$  (where  $\Lambda$  is the QCD scale), it can be computed in QCD perturbation theory to the lowest order in the ‘small’ (due to asymptotic freedom) running coupling  $\alpha_S(Q) \sim (\beta_0 \ln Q^2/\Lambda^2)^{-1}$ .

This naïve parton model approach corresponds to the so-called leading-order (LO) approximation. It is justified by the high-momentum behaviour of the running coupling  $\alpha_S(Q)$ . However, just because of its perturbative nature, the running of the QCD coupling can be hidden in higher-order corrections by the replacement  $\alpha_S(Q) = \alpha_S^{(0)} [1 + K(Q) \alpha_S(Q) + \dots]$ ,  $\alpha_S^{(0)}$  being the values of  $\alpha_S$  at a fixed (and arbitrary) momentum scale. It follows that a LO calculation predicts only the order of magnitude of a given cross section and the rough features of a certain observable<sup>†</sup>. The accuracy of the perturbative QCD expansion is instead controlled by the size of the higher-order contributions. Any definite perturbative QCD prediction thus requires (at least) a next-to-leading order (NLO) calculation, and NLO definitions of  $\alpha_S$ ,  $f_a(x, Q^2)$  and  $d_a(x, Q^2)$ .

This is the reason why the results of the higher-order QCD calculations which are available at present have been proved to be of vital importance to assess the progress mentioned above.

These higher-order computations have been carried out over a period of about fifteen years, often long after the accuracy of experimental data has made them necessary, because of the difficulties in setting up a general and straightforward calculational procedure. The physical origin of these difficulties is in the necessity of factorizing the long- and short-distance components of the scattering processes and is reflected in the perturbative calculation by the presence of divergences.

In general, when evaluating higher-order QCD cross sections, one has to consider real-emission contributions and virtual-loop corrections and one has to deal with different kind of singularities. The customary *ultraviolet* singularities, present in the virtual contributions, are removed by renormalization. The low-momentum (*soft*) and small-angle (*collinear*) regions instead produce singularities both in the real and in the virtual contributions. In order to handle these divergences, the observable one is interested in has to be properly defined. It has to be a *jet quantity*, that is, a hadronic observable that turns out to be

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<sup>†</sup>Typically, this poor predictivity is quantitatively signalled by a strong dependence on the (unphysical) renormalization and factorization scales.

infrared safe and either collinear safe or collinear factorizable: its actual value has to be independent of the number of soft and collinear particles in the final state (see Sect. 7 for a formal definition). In the case of jet quantities, the coherent sum over different (real and virtual) soft and collinear partonic configurations in the final state leads to the cancellation of soft singularities. The left-over collinear singularities are then factorized into the process-independent structure and fragmentation functions of partons (parton distributions), leading to predictable scaling violations. As a result, jet cross sections are finite (calculable) at the partonic level order by order in perturbation theory. All the dependence on long-distance physics is either included in the parton distributions or in non-perturbative corrections that are suppressed by inverse powers of the (large) transferred momentum  $Q$  that controls the scattering process.

Because of this complicated pattern of singularities, the simplest quantities that can be computed in QCD perturbation theory are fully inclusive. In this case one considers all possible final states and integrates the QCD matrix elements over the whole available final-state phase space. Thus one can add real and virtual contributions before performing the relevant momentum integrations in such a way that only ultraviolet singularities appear at the intermediate steps of the calculation. Owing to this simplification, powerful techniques have been set up [2] to perform analytic calculations up to next-to-next-to-leading order (NNLO), i.e. to relative accuracy  $\mathcal{O}(\alpha_s^3)$  with respect to the lowest-order approximation.

In the case of less inclusive cross sections, QCD calculations beyond leading order (LO) are much more involved. Owing to the complicated phase space for multi-parton configurations, analytic calculations are in practice impossible for all but the simplest quantities. However, the use of numerical methods is far from trivial because real and virtual contributions have a different number of final-state partons and thus have to be integrated *separately* over different phase space regions. Unlike the case of fully inclusive observables, one cannot take advantage of the cancellation of soft and collinear divergences at the integrand level. Soft and collinear singularities, present in the intermediate steps, have to be first regularized, generally by analytic continuation in a number of space-time dimensions  $d = 4 - 2\epsilon$  different from four. Then the real and virtual contributions should be calculated independently, yielding equal-and-opposite poles in  $\epsilon$ . Great progress has been made in recent years in the analytical techniques for calculating the virtual processes [3], but the analytic continuation greatly complicates the Lorentz algebra in the evaluation of the matrix elements and prevents a straightforward implementation of numerical integration techniques. Despite these difficulties, efficient computational techniques have been set up, at least to NLO, during the last few years.

There are, broadly speaking, two types of algorithm used for NLO calculations: one based on the phase-space *slicing* method and the other based on the *subtraction* method<sup>‡</sup>. The main difference between these algorithms and the standard procedures of analytic calculations is that only a minimal part of the full calculation is treated analytically, namely only those contributions giving rise to the singularities. Moreover, for any given process, these contributions are computed in a manner that is independent of the particular jet observable considered. Once every singular term has been isolated and the cancellation/factorization of divergences achieved, one can perform the remaining part of the cal-

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<sup>‡</sup>We refer the reader to the Introduction of Ref. [4] for an elementary description of the basic difference between the two methods.

ulation in four space-time dimensions. Although, when possible, one still has the freedom of completing the calculation analytically, at this point the use of numerical integration techniques (typically, Monte Carlo methods) is certainly more convenient. First of all, the numerical approach allows one to calculate any number and any type of observable simultaneously by simply histogramming the appropriate quantities, rather than having to make a separate analytic calculation for each observable. Furthermore, using the numerical approach, it is easy to implement different experimental conditions, for example, detector acceptances and experimental cuts. In other words, the phase-space slicing and subtraction algorithms provide the basis for setting up a general-purpose Monte Carlo program for carrying out arbitrary NLO QCD calculations in a given process.

Both the slicing [5] and the subtraction [6] methods were first used in the context of NLO calculations of three-jet cross sections in  $e^+e^-$  annihilation. Then they have been applied to other cross sections adapting the method each time to the particular process. Only recently has it become clear that both algorithms are generalizable in a process-independent manner. The key observation is that the singular parts of the QCD matrix elements for real emission can be singled out in a general way by using the factorization properties of soft and collinear radiation [7].

At present, a general version of the slicing algorithm is available for calculating NLO cross sections for production of *any* number of jets both in lepton [8] and hadron [9] collisions. To our knowledge, fragmentation processes have been considered only in the particular cases of prompt-photon production [10] and single- and double-hadron inclusive distributions [11,12]. The complete generalization of this method to include fragmentation functions and heavy flavours is in progress [13].

As for the subtraction algorithm, a general NLO formalism has been set up for computing three-jet observables in  $e^+e^-$  annihilation [6,14] and cross sections up to two final-state jets in hadron collisions<sup>§</sup> [4,15]. Also the treatment of massive partons has been considered in the particular case of heavy-quark correlations in hadron collisions [18].

In this paper, we present a *completely general* version of the subtraction algorithm. This generality is obtained by fully exploiting the factorization properties of soft and collinear emission and, thus, deriving new improved factorization formulae, called *dipole factorization formulae*. They allow us to introduce a set of universal counterterms that can be used for *any* NLO QCD calculation. Therefore, our version of the subtraction method can be compared with those used for three-jet observables in  $e^+e^-$  annihilation [6] and two-jet quantities in hadron collisions [4] (although, in these known cases our treatment turns out to differ in many respects from the previous ones). Moreover, we are able to consider the production of *any* number of jets in lepton and hadron cross sections and to provide a general treatment of fragmentation processes and multi-particle correlations. The inclusion of heavy quarks in the algorithm can also be performed in a completely general and process-independent manner [19]. The extension of our method to polarized scattering is not considered here but it is straightforward.

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<sup>§</sup>After completion of the present work, the method of Ref. [4] has been modified to deal with three-jet cross sections [16]. The formalism presented in Ref. [16] can be extended to  $n$ -jet production both in lepton and hadron collisions. A similar method has been presented in Ref. [17].

Besides discussing in detail our general formalism, we explicitly carry out the  $d$ -dimensional analytical part of the NLO calculation for all the (unpolarized) scattering processes involving massless quarks and gluons. Knowing the relevant QCD matrix elements, the results of our algorithm can be straightforwardly implemented in NLO numerical codes without any additional calculation. Detailed numerical applications to  $e^+e^-$  annihilation [20] and deep inelastic lepton-hadron scattering [21] are presented elsewhere.

We begin in Sect. 2 by giving a brief overview of the general method, describing the subtraction procedure and how our dipole formulae are used to implement it. In Sect. 3 we establish the notation used throughout the paper. In Sect. 4 we review the factorization properties of QCD matrix elements in the soft and collinear limits before presenting, in Sect. 5, our dipole factorization formulae, which smoothly interpolate these two limiting regions. After briefly recalling, in Sect. 6, the precise definitions of QCD cross sections at NLO, we go on to describe in detail our subtraction method for evaluating these cross sections, in Sects. 7–11. In Sect. 12 we summarize and discuss our results. Appendix A gives more details, and some examples, of the necessary colour algebra. In Appendix B we explicitly perform the only difficult integral we encounter. In Appendix C we collect together the main formulae needed to implement our method in specific calculations. Finally in Appendix D we work through a few simple examples of applying our method to specific cross sections.

Since the paper is quite long, readers mainly interested in understanding the general method or in some particular application are advised to first read Sects. 2, 3, 4, 5.1 and 7. Here we discuss in detail our general formalism and its use for processes with no initial-state hadrons like, for instance,  $e^+e^-$  annihilation (in this case, a brief description of our method has already appeared [20]). Sections 8–11 and the other Subsections in Sect. 5 can then be read quite independently from one another. The final formulae that are necessary for the actual numerical implementation of our algorithm in each different scattering process are summarized in a Subsection at the end of each of Sects. 7–11.

## 2 The general method

In this Section we explain the general idea behind our version of the subtraction method by describing the subtraction procedure (Sect. 2.1) and considering mainly the simplified case of jet cross sections in processes with no initial-state hadrons, for instance,  $e^+e^-$  annihilation (Sect. 2.2). A brief description of our method for more complicated scattering processes is sketched in Sect. 2.3.

### 2.1 The subtraction procedure

Suppose we want to compute a jet cross section  $\sigma$  to NLO, namely

$$\sigma = \sigma^{LO} + \sigma^{NLO} . \quad (2.1)$$

Here the LO cross section  $\sigma^{LO}$  is obtained by integrating the fully exclusive cross section  $d\sigma^B$  in the Born approximation over the phase space for the corresponding jet quantity. Suppose also that this LO calculation involves  $m$  partons in the final state. Thus, we write

$$\sigma^{LO} = \int_m d\sigma^B , \quad (2.2)$$

where, in general, all the quantities (QCD matrix elements and phase space) are evaluated in  $d = 4 - 2\epsilon$  space-time dimensions. However, by definition, at this LO the phase space integration in Eq. (2.2) is finite so that the whole calculation can be carried out (analytically or numerically) in four dimensions.

Now we go to NLO. We have to consider the exclusive cross section  $d\sigma^R$  with  $m + 1$  partons in the final-state and the one-loop correction  $d\sigma^V$  to the process with  $m$  partons in the final state:

$$\sigma^{NLO} \equiv \int d\sigma^{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V . \quad (2.3)$$

The two integrals on the right-hand side of Eq. (2.3) are separately divergent if  $d = 4$ , although their sum is finite. Therefore, before any numerical calculation can be attempted, the separate pieces have to be regularized. Using dimensional regularization, the divergences (arising out of the integration) are replaced by double (soft and collinear) poles  $1/\epsilon^2$  and single (soft, collinear or ultraviolet) poles  $1/\epsilon$ . Suppose that one has already carried out the renormalization procedure in  $d\sigma^V$  so that all its ultraviolet poles have been removed.

The general idea of the subtraction method for writing a general-purpose Monte Carlo program is to use the identity

$$d\sigma^{NLO} = [d\sigma^R - d\sigma^A] + d\sigma^A + d\sigma^V , \quad (2.4)$$

where  $d\sigma^A$  is a proper approximation of  $d\sigma^R$  such as to have the same *pointwise* singular behaviour (in  $d$  dimensions) as  $d\sigma^R$  itself. Thus,  $d\sigma^A$  acts as a *local* counterterm for  $d\sigma^R$  and, introducing the phase space integration,

$$\sigma^{NLO} = \int_{m+1} [d\sigma^R - d\sigma^A] + \int_{m+1} d\sigma^A + \int_m d\sigma^V , \quad (2.5)$$

one can safely perform the limit  $\epsilon \rightarrow 0$  under the integral sign in the first term on the right-hand side of Eq. (2.5). Hence, this first term can be integrated numerically in four dimensions.

All the singularities are now associated to the last two terms on the right-hand side of Eq. (2.5). If one is able to carry out analytically the integration of  $d\sigma^A$  over the one-parton subspace leading to the  $\epsilon$  poles, one can combine these poles with those in  $d\sigma^V$ , thus cancelling all the divergences, performing the limit  $\epsilon \rightarrow 0$  and carrying out numerically the remaining integration over the  $m$ -parton phase space. The final structure of the calculation is as follows

$$\sigma^{NLO} = \int_{m+1} \left[ (d\sigma^R)_{\epsilon=0} - (d\sigma^A)_{\epsilon=0} \right] + \int_m \left[ d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0} , \quad (2.6)$$

and can be easily implemented in a ‘partonic Monte Carlo’ program, which generates appropriately weighted partonic events with  $m + 1$  final-state partons and events with  $m$  partons.

Note that the subtracted term  $[d\sigma^R - d\sigma^A]$  in Eq. (2.6) is integrable in four dimensions by definition. The fact that all the divergences cancel in the second term on the right-hand side of Eq. (2.6) is instead not a general feature of all hadronic cross section<sup>¶</sup>. The cancellation of divergences is guaranteed only for the hadronic observables that we are considering in this paper, namely jet observables.

These quantities have to be experimentally (theoretically) defined in such a way that their actual value is independent of the number of soft and collinear hadrons (partons) produced in the final state. In particular, this value has to be the same in a given  $m$ -parton configuration and in all  $m + 1$ -parton configurations that are kinematically degenerate with it (i.e. that are obtained from the  $m$ -parton configuration by adding a soft parton or replacing a parton with a pair of collinear partons carrying the same total momentum). This property can be simply restated in a formal way. If the function  $F_J^{(n)}$  gives the value of a certain jet observable in terms of the momenta of the  $n$  final-state partons, we should have

$$F_J^{(m+1)} \rightarrow F_J^{(m)} , \quad (2.7)$$

in any case where the  $m + 1$ -parton and the  $m$ -parton configurations are kinematically degenerate.

The Born-level cross section  $d\sigma^B$  can be (symbolically) written as a function of the jet-defining function  $F_J^{(m)}$  in the following way

$$d\sigma^B = d\Phi^{(m)} |\mathcal{M}_m|^2 F_J^{(m)} , \quad (2.8)$$

where  $d\Phi^{(m)}$  and  $\mathcal{M}_m$  respectively are the full phase space and the QCD matrix element to produce  $m$  final-state partons. The corresponding expression for the real cross section  $d\sigma^R$  is:

$$d\sigma^R = d\Phi^{(m+1)} |\mathcal{M}_{m+1}|^2 F_J^{(m+1)} . \quad (2.9)$$

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<sup>¶</sup>The presence of singularities in a QCD cross section computed in perturbation theory does not mean that the theory itself is inconsistent. It simply means that one is considering a cross section that cannot be reliably estimated using the perturbative expansion. At any energy scale, it is affected by non-perturbative phenomena that are as big as the perturbative ones.



The structure of Eq. (2.9) and the fundamental property (2.7) are essential for the feasibility of the subtraction procedure described in this Subsection. There are obviously many ways of approximating the matrix element  $\mathcal{M}_{m+1}$  in the neighbourhood of its soft and collinear singularities. Correspondingly, one can approximate  $F_J^{(m+1)}$  and obtain a local counter-term  $d\sigma^A$ . The main point is that, due to the limiting behaviour in Eq. (2.7), one can always find an approximation for  $F_J^{(m+1)}$  such that the one-parton subspace leading to the soft and collinear divergences effectively decouples. Thus, one can perform the integral  $\int_1 d\sigma^A$  and the subtraction formula (2.6) can, *in principle*, always be implemented.

## 2.2 Dipole formulae and universal implementation of the subtraction procedure for jet cross sections

The key for the subtraction procedure to work *in practice* is obviously the actual form of  $d\sigma^A$ . One needs to find an expression for  $d\sigma^A$  that fulfils the following properties: *i*) for any given process,  $d\sigma^A$  has to be obtained in a way that is independent of the particular jet observable considered; *ii*) it has to exactly match the singular behaviour of  $d\sigma^R$  in  $d$  dimensions; *iii*) its form has to be particularly convenient for Monte Carlo integration techniques; *iv*) it has to be exactly integrable analytically in  $d$  dimensions over the single-parton subspaces leading to soft and collinear divergences.

In Ref. [6], a suitable expression for  $d\sigma^A$  for the process  $e^+e^- \rightarrow 3$  jets was obtained by starting from the explicit expression (in  $d$  dimensions) of the corresponding  $d\sigma^R$  and by performing extensive partial fractioning of the  $3 + 1$ -parton matrix elements, so that each divergent piece could be extracted. This is an extremely laborious and ungeneralizable task, in the sense that having done it for  $e^+e^- \rightarrow 3$  jets does not help us to do this for, say,  $e^+e^- \rightarrow 4$  jets or for any other process.

In Ref. [4], the general properties of soft and collinear emission were first used (in the context of the subtraction method) to construct  $d\sigma^A$ , for one- and two-jet production in hadron collisions, in a way that is independent of the detailed form of the corresponding  $d\sigma^R$ .

The central proposal of our version of the subtraction method is that one can give a recipe for constructing  $d\sigma^A$  that is completely *process independent* (and not simply independent of the jet observable). Starting from our physical knowledge of how the  $m + 1$ -parton matrix elements behave in the soft and collinear limits that produce the divergences (see Sect. 4), we derive improved factorization formulae, called dipole formulae (see Sect. 5), that allow us to write:

$$d\sigma^A = \sum_{\text{dipoles}} d\sigma^B \otimes dV_{\text{dipole}} . \quad (2.10)$$

The notation in Eq. (2.10) is symbolic. Here  $d\sigma^B$  denotes an appropriate colour and spin projection of the Born-level exclusive cross section. The symbol  $\otimes$  stands for properly defined phase space convolutions and sums over colour and spin indices. The dipole factors  $dV_{\text{dipole}}$  (which match the singular behaviour of  $d\sigma^R$ ) are instead universal, i.e. completely independent of the details of the process and they can be computed once for all. In particular, the dependence on the jet observable is completely embodied by the factor  $d\sigma^B$

of Eq. (2.10), in the form of Eq. (2.8).

There are several dipole terms on the right-hand side of Eq. (2.10). Each of them corresponds to a different kinematic configuration of  $m + 1$  partons. Each configuration can be thought as obtained by an effective two-step process: an  $m$ -parton configuration is first produced and then one of these partons decays into two partons. It is this two-step pseudo-process that leads to the factorized structure on the right-hand side of Eq. (2.10).

The reason for having several dipoles is that each of them mimics one of the  $m + 1$ -parton configurations in  $d\sigma^R$  that are kinematically degenerate with a given  $m$ -parton state. Any time the  $m + 1$ -parton state in  $d\sigma^R$  approaches a soft and/or collinear region, there is a corresponding dipole factor in  $d\sigma^A$  that approaches the same region with exactly the same probability as in  $d\sigma^R$ . In this manner  $d\sigma^A$  acts as a local counter-term for  $d\sigma^R$ .

Our expression for  $d\sigma^A$  in Eq. (2.10) is completely defined over the full  $m + 1$ -parton phase space (in particular,  $d\sigma^A$  does not depend on any additional phase space cut-off<sup>||</sup>): momentum conservation is exactly implemented in each term on the right-hand side of Eq. (2.10) and there is a one-to-one correspondence between each partonic configuration in  $d\sigma^R$  and (each of the several) in  $d\sigma^A$ . Therefore, in our case,  $[d\sigma^R - d\sigma^A]$  is straightforwardly integrable via Monte Carlo methods: one generates an  $m + 1$ -parton event with weight  $d\sigma^R$  and, correspondingly, one can obtain an  $m + 1$ -parton counter-event with weight  $d\sigma^A$ .

Furthermore, the product structure in Eq. (2.10) allows us a factorizable mapping from the  $m + 1$ -parton phase space to an  $m$ -parton subspace (that identified by the partonic variable in  $d\sigma^B$ ) times a single-parton phase space (that identified by the dipole partonic variables in  $dV_{\text{dipole}}$ ). This mapping makes  $dV_{\text{dipole}}$  fully integrable analytically. We can write (again, symbolically):

$$\int_{m+1} d\sigma^A = \sum_{\text{dipoles}} \int_m d\sigma^B \otimes \int_1 dV_{\text{dipole}} = \int_m [d\sigma^B \otimes \mathbf{I}] \quad , \quad (2.11)$$

where the universal factor  $\mathbf{I}$  is defined by

$$\mathbf{I} = \sum_{\text{dipoles}} \int_1 dV_{\text{dipole}} \quad , \quad (2.12)$$

and contains all the  $\epsilon$  poles that are necessary to cancel the (equal and with opposite sign) poles in  $d\sigma^V$ .

The structure of the final result is given as follows in terms of two contributions  $\sigma^{NLO\{m+1\}}$ ,  $\sigma^{NLO\{m\}}$  (with  $m + 1$ -parton and  $m$ -parton kinematics, respectively) which

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<sup>||</sup>This is quite a non-trivial feature of our approach. The most naïve way of setting up subtraction procedures based on universal properties in the soft and collinear limits would lead to the introduction of energy and angular cut-offs, thus breaking Lorentz covariance at intermediate steps. Alternative and less naïve methods for imposing soft and collinear cut-offs have their own disadvantages, too. For instance, the cut-off can be related to some kinematic invariant of the process (see Ref. [4]), or one can introduce a Lorentz covariant cut-off on parton-parton invariant masses (see Ref. [8]) rather than energies and angles. In the first case one could spoil the universality of the subtraction procedure making it process dependent. In the second case, it is quite difficult to arrange the cut-off in such a way that the subtraction term is exactly integrable analytically to any accuracy in the cut-off itself.

are separately finite (and integrable) in four space-time dimensions:

$$\begin{aligned}\sigma^{NLO} &= \sigma^{NLO\{m+1\}} + \sigma^{NLO\{m\}} \\ &= \int_{m+1} \left[ (d\sigma^R)_{\epsilon=0} - \left( \sum_{\text{dipoles}} d\sigma^B \otimes dV_{\text{dipole}} \right)_{\epsilon=0} \right] + \int_m [d\sigma^V + d\sigma^B \otimes \mathbf{I}]_{\epsilon=0} .\end{aligned}\tag{2.13}$$

Equation (2.13) represents our practical implementation of the general subtraction formula (2.6).

In this paper we provide explicit expressions for both the universal factors  $dV_{\text{dipole}}$  and  $\mathbf{I}$ . Having these factors at our disposal, the only other ingredients necessary for the full NLO calculation, according to Eq. (2.13), are the following (reading Eq. (2.13) from the right to the left):

- a set of independent colour projections\*\* of the matrix element squared at the Born level, summed over parton polarizations, in  $d$  dimensions;
- the one-loop contribution  $d\sigma^V$  in  $d$  dimensions;
- an additional projection of the Born level matrix element over the helicity of each external gluon in four dimensions;
- the real emission contribution  $d\sigma^R$  in four dimensions.

These few ingredients are sufficient for writing, in a straightforward way, a general-purpose NLO Monte Carlo algorithm. Note in particular that there is no need to extract a proper counter-term  $d\sigma^A$  starting from a cumbersome expression for  $d\sigma^R$  in  $d$  dimensions. The NLO matrix element contributing to  $d\sigma^R$  can be evaluated directly in four space-time dimensions thus leading to an extreme simplification of the Lorentz algebra, particularly if one makes use of helicity amplitudes [22] to control the rapid increase in the number of Feynman diagrams as the number of parton grows.

### 2.3 Factorization of collinear singularities and general algorithm for processes with identified hadrons

The discussion in Sects. 2.1 and 2.2 applies to all the processes with no initial-state hadrons (for instance,  $e^+e^-$  annihilation). However, perturbative QCD can be used also for the calculation of jet cross sections in lepton-hadron and hadron-hadron collisions<sup>††</sup>. The main difference is that the presence of initial-state hadrons (partons), carrying a well defined momentum, spoils the cancellation of the collinear singularities arising in the perturbative treatment. The left-over singularities can be factorized and reabsorbed into non-perturbative and universal (process-independent) distribution functions, the parton densities of the incoming hadron. Provided this factorization procedure is consistently carried

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\*\* Actually, if the total number of QCD partons involved in the LO matrix element is less than or equal to three, one simply needs its incoherent sum over the colours (see Appendix A).

<sup>††</sup> Throughout this paper, whenever referring to initial-state hadrons, we implicitly also include hadron-like particles such as photons.

out, one can thus define parton-initiated jet cross sections (see Sect. 6) that are free from singularities and can be computed with a subtraction procedure similar to that described in Sect. 2.1.

Similar features appear when the jet observable depends on the actual value of the momentum of one or more hadrons observed in the final state (the inclusive one-particle distribution in  $e^+e^-$  annihilation is the simplest example). Also in this case there are left-over collinear singularities that can be reabsorbed into non-perturbative and universal distribution functions, the fragmentation functions of the outgoing hadron. As a result, one can again define partonic cross sections that are free from singularities and computable in perturbation theory (see Sect. 6).

Because of these common features, in this paper the processes with initial-state hadrons and those involving fragmentation functions will be referred to as processes with identified hadrons (partons). The hadronic cross section is obtained by convoluting partonic cross sections with non-perturbative distribution functions. As stated above, the NLO partonic cross sections can be evaluated using the subtraction method and, thus, trying to implement the subtraction formula in Eq. (2.6). There are nonetheless some additional complications with respect to the case with no identified particles. These complications regard the construction of the approximated cross section  $d\sigma^A$ .

Just as when there are no identified hadrons, the real cross section  $d\sigma^R$  is singular whenever a pair of the  $m + 1$  final-state partons become collinear. However in addition, it is also singular in the region where one of them becomes collinear to an identified parton. Moreover, the phase space integration has to be performed in the presence of additional kinematic constraints, related to the fact that the momenta of the identified partons have to be kept fixed (or, at most, rescaled by an overall momentum fraction). As for the approximated cross section  $d\sigma^A$ , it follows that, on one side, it should act as a local counter-term also in the new singular regions and, on the other side, its integral  $\int_1 d\sigma^A$  should still be computable analytically even in the presence of the additional phase space constraints.

The dipole formalism presented in this paper provides a simple and general solution to these problems. Indeed, we are able to write the cross section  $d\sigma^A$  in the following form (see Sects. 8–11)

$$d\sigma^A = \sum_{\text{dipoles}} d\sigma^B \otimes (dV_{\text{dipole}} + dV'_{\text{dipole}}) \quad . \quad (2.14)$$

Equation (2.14) is completely analogous to Eq. (2.10). The additional dipole terms  $dV'_{\text{dipole}}$  on the right-hand side match the singularities of  $d\sigma^R$  coming from the region collinear to the momenta of the identified partons. Moreover, these dipole terms are still (i.e. even if the momenta of the identified partons are fixed) fully integrable analytically over the one-parton subspace leading to soft and collinear divergences.

These are peculiar features of the dipole approach. As discussed below Eq. (2.10), each dipole contribution is effectively obtained by first producing an  $m$ -parton configuration and then letting one parton to decay into two partons. The dipole formulae implement this two-step procedure by enforcing exact momentum conservation. Actually there are equivalent ways of doing that, corresponding to different ways of treating the momentum recoil in the  $m$ -parton configuration. This freedom allows us to define alternative versions

of the factorization formulae (Sect. 5) and, hence, different dipole factors (like  $dV_{\text{dipole}}$  and  $dV'_{\text{dipole}}$  in Eq. (2.14)). These differences are then used to match (and overcome) the phase-space constraints that are encountered in the calculation of QCD cross sections with identified particles.

Having introduced the counter-term  $d\sigma^A$  in Eq. (2.14), we can proceed to its integration as in Eq. (2.11). We thus obtain the singular factor  $\mathbf{I}$  in Eq. (2.12) and additional singular terms, which are reabsorbed into the non-perturbative distribution functions.

The final result of our subtraction procedure is given in terms of the NLO partonic cross section  $\sigma^{NLO}(p)$ , where the dependence on the momentum  $p$  symbolically denotes the functional dependence on the momenta of the identified partons. This cross section is obtained by a formula that is similar to Eq. (2.13), namely (see Sects. 8–11)

$$\begin{aligned}
\sigma^{NLO}(p) &= \sigma^{NLO\{m+1\}}(p) + \sigma^{NLO\{m\}}(p) + \int_0^1 dx \hat{\sigma}^{NLO\{m\}}(x; xp) \\
&= \int_{m+1} \left[ \left( d\sigma^R(p) \right)_{\epsilon=0} - \left( \sum_{\text{dipoles}} d\sigma^B(p) \otimes \left( dV_{\text{dipole}} + dV'_{\text{dipole}} \right) \right)_{\epsilon=0} \right] \\
&+ \int_m \left[ d\sigma^V(p) + d\sigma^B(p) \otimes \mathbf{I} \right]_{\epsilon=0} + \int_0^1 dx \int_m \left[ d\sigma^B(xp) \otimes (\mathbf{P} + \mathbf{K} + \mathbf{H})(x) \right]_{\epsilon=0} .
\end{aligned} \tag{2.15}$$

Here, the contributions  $\sigma^{NLO\{m+1\}}(p)$  and  $\sigma^{NLO\{m\}}(p)$  (with  $m+1$ -parton and  $m$ -parton kinematics, respectively) are completely analogous to those in Eq. (2.13).

The last term on the right-hand side of Eq. (2.15) is a finite (in four dimensions) remainder that is left after factorization of initial-state and final-state collinear singularities into the non-perturbative distribution functions (parton densities and fragmentation functions). This term involves a cross section  $\hat{\sigma}^{NLO\{m\}}(x; xp)$  with  $m$ -parton kinematics and an additional one-dimensional integration with respect to the longitudinal momentum fraction  $x$ . This integration arises from the convolution of the Born-type cross section  $d\sigma^B(xp)$  with  $x$ -dependent functions  $\mathbf{P}, \mathbf{K}, \mathbf{H}$  that are similar (but finite for  $\epsilon \rightarrow 0$ ) to the factor  $\mathbf{I}$ . The functions  $\mathbf{P}, \mathbf{K}$  and  $\mathbf{H}$  are universal, that is, they are independent of the detail of the scattering process and of the jet observable: they simply depend on the number of identified partons. Our algorithm provides the explicit expressions for these functions. Therefore, writing a general-purpose NLO Monte Carlo program for processes with identified particles does not require any further conceptual or analytic effort with respect to the case with no identified particles.

### 3 Notation

#### 3.1 Dimensional regularization

In general we use dimensional regularization in  $d = 4 - 2\epsilon$  space-time dimensions and consider  $d-2$  helicity states for gluons and 2 helicity states for massless quarks (i.e. fermions are four-component spinors). This defines the usual dimensional-regularization scheme. Other dimensional-regularization prescriptions can be used. However, since the regularization dependence is unphysical (i.e. it cancels in physical cross sections), within our formalism it is more convenient to parametrize it in terms of simple coefficients that enter in the one-loop contribution (see Sect. 3.3).

The dimensional-regularization scale, which appears in the calculation of the matrix elements, is denoted by  $\mu$ . Physical cross sections do not depend on  $\mu$ , although, when evaluated in fixed-order perturbation theory, they do depend on the renormalization scale  $\mu_R$  and on factorization scales  $\mu_F$ . In other words, the dependence on  $\mu$  cancels after having combined the matrix elements in the calculation of physical cross sections. Therefore, in order to avoid a cumbersome notation, we set  $\mu = \mu_R$ , while  $\mu$  and  $\mu_F$  will differ in general.

The  $d$ -dimensional phase space, which involves the integration over the momenta  $\{p_1, \dots, p_m\}$  of  $m$  final-state partons, will be denoted as follows

$$\left[ \prod_{l=1}^m \frac{d^d p_l}{(2\pi)^{d-1}} \delta_+(p_l^2) \right] (2\pi)^d \delta^{(d)}(p_1 + \dots + p_m - Q) \equiv d\phi_m(p_1, \dots, p_m; Q) . \quad (3.1)$$

When there is no ambiguity on the number of final-state partons, we shall drop the subscript  $m$  in  $d\phi_m$ .

#### 3.2 Matrix elements

Let us first consider processes that involve only final-state QCD partons ( $e^+e^-$ -type processes). Non QCD partons ( $\gamma^*$ ,  $Z^0$ ,  $W^\pm$ ,  $\dots$ ), carrying a total incoming momentum  $Q_\mu$ , are always understood.

The (*tree-level*) matrix element with  $m$  QCD partons in the final state has the following general structure

$$\mathcal{M}_m^{c_1, \dots, c_m; s_1, \dots, s_m}(p_1, \dots, p_m) \quad (3.2)$$

where  $\{c_1, \dots, c_m\}$ ,  $\{s_1, \dots, s_m\}$  and  $\{p_1, \dots, p_m\}$  are respectively colour indices ( $a = 1, \dots, N_c^2 - 1$  different colours for each gluon,  $\alpha = 1, \dots, N_c$  different colours for each quark or antiquark), spin indices ( $\mu = 1, \dots, d$  for gluons,  $s = 1, 2$  for massless fermions) and momenta.

It is useful to introduce a basis  $\{|c_1, \dots, c_m \rangle \otimes |s_1, \dots, s_m \rangle\}$  in colour + helicity space in such a way that

$$\mathcal{M}_m^{c_1, \dots, c_m; s_1, \dots, s_m}(p_1, \dots, p_m) \equiv \left( \langle c_1, \dots, c_m | \otimes \langle s_1, \dots, s_m | \right) |1, \dots, m \rangle_m . \quad (3.3)$$

Thus  $|1, \dots, m \rangle_m$  is a vector in colour + helicity space.

According to this notation, the matrix element squared (summed over final-state colours and spins)  $|\mathcal{M}_m|^2$  can be written as

$$|\mathcal{M}_m|^2 = {}_{m < 1, \dots, m} |1, \dots, m \rangle_m . \quad (3.4)$$

In the following we shall always consider matrix elements squared summed over final-state spins (the generalization to fixed-helicity amplitudes is feasible).

As for the colour structure<sup>‡‡</sup>, it is convenient to associate a colour charge  $\mathbf{T}_i$  with the emission of a gluon from each parton  $i$ . If the emitted gluon has colour index  $c$ , the colour-charge operator is:

$$\mathbf{T}_i \equiv T_i^c |c \rangle \quad (3.5)$$

and its action onto the colour space is defined by

$$\langle c_1, \dots, c_i, \dots, c_m, c | \mathbf{T}_i | b_1, \dots, b_i, \dots, b_m \rangle = \delta_{c_1 b_1} \dots T_{c_i b_i}^c \dots \delta_{c_m b_m} , \quad (3.6)$$

where  $T_{cb}^a \equiv if_{cab}$  (colour-charge matrix in the adjoint representation) if the emitting particle  $i$  is a gluon and  $T_{\alpha\beta}^a \equiv t_{\alpha\beta}^a$  (colour-charge matrix in the fundamental representation) if the emitting particle  $i$  is a quark (in the case of an emitting antiquark  $T_{\alpha\beta}^a \equiv \bar{t}_{\alpha\beta}^a = -t_{\beta\alpha}^a$ ).

The colour-charge algebra is:

$$\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i \quad \text{if } i \neq j; \quad \mathbf{T}_i^2 = C_i, \quad (3.7)$$

where  $C_i$  is the Casimir operator, that is,  $C_i = C_A = N_c$  if  $i$  is a gluon and  $C_i = C_F = (N_c^2 - 1)/2N_c$  if  $i$  is a quark or antiquark.

Note that by definition, each vector  $|1, \dots, m \rangle_m$  is a colour-singlet state. Therefore colour conservation is simply

$$\sum_{i=1}^m \mathbf{T}_i |1, \dots, m \rangle_m = 0 . \quad (3.8)$$

Using this notation, we also define the square of colour-correlated tree-amplitudes,  $|\mathcal{M}_m^{i,k}|^2$ , as follows

$$\begin{aligned} |\mathcal{M}_m^{i,k}|^2 &\equiv {}_{m < 1, \dots, m} | \mathbf{T}_i \cdot \mathbf{T}_k | 1, \dots, m \rangle_m \\ &= \left[ \mathcal{M}_m^{a_1 \dots b_i \dots b_k \dots a_m}(p_1, \dots, p_m) \right]^* T_{b_i a_i}^c T_{b_k a_k}^c \mathcal{M}_m^{a_1 \dots a_i \dots a_k \dots a_m}(p_1, \dots, p_m) . \end{aligned} \quad (3.9)$$

In the case of hard processes with QCD partons in the initial state, in addition to  $m$  final-state partons, the relevant matrix element is:

$$\mathcal{M}_{m,a \dots}^{c_1, \dots, c_m, c_a, \dots; s_1, \dots, s_m, s_a, \dots}(p_1, \dots, p_m; p_a, \dots) , \quad (3.10)$$

---

<sup>‡‡</sup>Within our formalism, there is no need to consider the decomposition of the matrix elements into colour subamplitudes [23], as in Ref. [8].

and the corresponding vector in colour + helicity space will be denoted in the following way

$$|1, \dots, m; a, \dots \rangle_{m,a\dots} \equiv \frac{1}{\sqrt{n_c(a) \dots}} (|c_1, \dots, c_m, c_a, \dots \rangle \otimes |s_1, \dots, s_m, s_a, \dots \rangle) \cdot \mathcal{M}_{m,a\dots}^{c_1, \dots, c_m, c_a, \dots; s_1, \dots, s_m, s_a, \dots}(p_1, \dots, p_m; p_a, \dots) . \quad (3.11)$$

Here the labels  $a, \dots$  refer to the initial-state partons. The normalization of the state vector in Eq. (3.11) is fixed by including a factor of  $1/\sqrt{n_c(a)}$  for each initial-state parton carrying  $n_c(a)$  colours.

Note that the colour-charge operator of an initial-state parton  $a$  is defined by crossing symmetry, that is,  $(\mathbf{T}_a)_{bd}^c = if_{bcd}$  if  $a$  is a gluon and  $(\mathbf{T}_a)_{\alpha\beta}^c = \bar{t}_{\alpha\beta}^c = -t_{\beta\alpha}^c$  if  $a$  is a quark (if  $a$  is an antiquark,  $(\mathbf{T}_a)_{\alpha\beta}^c = t_{\alpha\beta}^c$ ). The analogue of the colour-conservation condition (3.8) is:

$$\left( \sum_{i=1}^m \mathbf{T}_i + \mathbf{T}_a + \dots \right) |1, \dots, m; a, \dots \rangle_{m,a\dots} = 0 . \quad (3.12)$$

Owing to the normalization of the state vector in Eq. (3.11), the square of colour-correlated tree-amplitudes is:

$$\begin{aligned} |\mathcal{M}_{m,a\dots}^{I,J}|^2 &\equiv {}_{m,a\dots} \langle 1, \dots, m; a, \dots | \mathbf{T}_I \cdot \mathbf{T}_J | 1, \dots, m; a, \dots \rangle_{m,a\dots} \\ &= \frac{1}{n_c(a) \dots} \left[ \mathcal{M}_{m,a\dots}^{a_1 \dots b_I \dots b_J \dots}(p_1, \dots, p_m; p_a, \dots) \right]^* T_{b_I a_I}^c T_{b_J a_J}^c \\ &\quad \mathcal{M}_{m,a\dots}^{a_1 \dots a_I \dots a_J \dots}(p_1, \dots, p_m; p_a, \dots) , \end{aligned} \quad (3.13)$$

where the indices  $I, J$  refer either to final-state or initial-state partons.

We refer to Appendix A for more details of the colour algebra.

### 3.3 One-loop matrix elements and scheme (in)dependence

We denote by  $|\mathcal{M}_{m,a\dots}^{(bare)}(p_1, \dots, p_m; p_a, \dots)|_{(1-loop)}^2$  the one-loop correction to the square of the tree-level matrix element in Eq. (3.10). This term enters in the computation of the virtual contribution  $d\sigma^V$  to the NLO QCD cross section. Actually,  $d\sigma^V$  is proportional to the renormalized one-loop correction  $|\mathcal{M}_{m,a\dots}(p_1, \dots, p_m; p_a, \dots)|_{(1-loop)}^2$  and the latter is obtained from the corresponding bare quantity by simply adding an ultraviolet counterterm. More precisely, if the (tree-level) matrix element squared  $|\mathcal{M}_{m,a\dots}(p_1, \dots, p_m; p_a, \dots)|^2$  is proportional to the  $n$ -th power of the QCD coupling  $\alpha_S$ , the renormalized one-loop correction is given by

$$\begin{aligned} |\mathcal{M}_{m,a\dots}(p_1, \dots, p_m; p_a, \dots)|_{(1-loop)}^2 &= |\mathcal{M}_{m,a\dots}^{(bare)}(p_1, \dots, p_m; p_a, \dots)|_{(1-loop)}^2 \\ &\quad - n \frac{\alpha_S}{2\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left( \frac{\beta_0}{\epsilon} + \tilde{\beta}_0 \right) |\mathcal{M}_{m,a\dots}(p_1, \dots, p_m; p_a, \dots)|^2 , \end{aligned} \quad (3.14)$$

where  $\beta_0 = (11C_A - 4T_R N_f)/6$  is the first coefficient of the QCD  $\beta$ -function ( $N_f$  is the number of quark flavours and  $\text{Tr}(t^a t^b) = \delta^{ab} T_R$ , i.e.  $T_R = 1/2$ ) and  $\tilde{\beta}_0$  is an  $\epsilon$ -independent



and process-independent coefficient that defines the renormalization scheme. In Eq. (3.14) and in the rest of the paper,  $\alpha_S$  stands for  $\alpha_S(\mu)$ , the NLO QCD running coupling evaluated at the renormalization scale  $\mu$ . The actual value of the QCD coupling  $\alpha_S(\mu)$  depends on the renormalization scheme. The customary  $\overline{\text{MS}}$  scheme is obtained by setting  $\tilde{\beta}_0 = 0$  in Eq. (3.14).

The detailed expression of  $|\mathcal{M}_{m,a\dots}^{(bare)}(p_1, \dots, p_m; p_a, \dots)|_{(1-loop)}^2$  (and, hence, of  $|\mathcal{M}|_{(1-loop)}^2$ ) depends on the dimensional regularization procedure used for evaluating the loop integral. Since we are using conventional dimensional regularization, we need the result for  $|\mathcal{M}_{m,a\dots}^{(bare)}|_{(1-loop)}^2$  within this regularization scheme. If the one-loop correction is known in a different scheme, we have to introduce a correction factor proportional to the corresponding tree-level amplitude. More precisely, we have to perform the following replacement in Eq. (3.14):

$$|\mathcal{M}_{m,a\dots}(p_1, \dots, p_m; p_a, \dots)|_{(1-loop)}^2 \rightarrow |\mathcal{M}_{m,a\dots}(p_1, \dots, p_m; p_a, \dots)|_{(1-loop)}^2 - \frac{\alpha_S}{2\pi} \left[ \sum_{i=1}^m \tilde{\gamma}_i + \tilde{\gamma}_a + \dots \right] |\mathcal{M}_{m,a\dots}(p_1, \dots, p_m; p_a, \dots)|^2, \quad (3.15)$$

where the universal coefficients  $\tilde{\gamma}_i, \tilde{\gamma}_a, \dots$  depend only on the flavour of the QCD partons. The actual values of these coefficients for several different regularization schemes can be found, for instance, in Ref. [24].

## 4 Factorization in the soft and collinear limits

### 4.1 Soft and collinear singularities in tree-level amplitudes

Let us consider a generic tree-level matrix element  $\mathcal{M}_{m+1,a\dots}$  with  $m + 1$  QCD partons (Fig. 1). The dependence of  $|\mathcal{M}_{m+1,a\dots}|^2$  on the momentum  $p_j$  of a final-state parton  $j$  is singular in two different phase-space regions: a) in the *soft* region, defined by the limit  $p_j = \lambda q$ ,  $\lambda \rightarrow 0$  (where  $q$  is an arbitrary four momentum),  $|\mathcal{M}_{m+1,a\dots}|^2$  behaves as  $1/\lambda^2$ ; in the *collinear* region, defined by the limit  $p_j \rightarrow (1 - z)p_i/z$  (where  $p_i$  is the momentum of another QCD parton in  $\mathcal{M}_{m+1,a\dots}$ ),  $|\mathcal{M}_{m+1,a\dots}|^2$  behaves as  $1/(p_i p_j)$ . This singular behaviour of  $|\mathcal{M}_{m+1,a\dots}|^2$  leads to the soft and collinear divergences of the NLO contribution  $\int_{m+1} d\sigma^R$  in Eq. (2.3) if the phase-space integration over  $p_j$  is performed in four dimensions.

The starting point of our method for constructing the counter-term  $d\sigma^A$  in Eq. (2.5) is the observation that the singular behaviour of  $|\mathcal{M}_{m+1,a\dots}|^2$  is universal, that is, it is not dependent on the very detailed structure of  $\mathcal{M}_{m+1,a\dots}$  itself. The origin of this universality is in the fact that, for its singular terms with respect to the momentum  $p_j$ , the tree amplitude  $\mathcal{M}_{m+1,a\dots}$  can always be considered as being obtained by the insertion of the parton  $j$  over all the possible external legs of a tree-level amplitude  $\mathcal{M}_{m,a\dots}$  with  $m$  QCD partons (Fig. 1). Thus, the singular behaviour of  $\mathcal{M}_{m+1,a\dots}$  is essentially factorizable with respect to  $\mathcal{M}_{m,a\dots}$  and the singular factor only depends on the momenta and quantum numbers of the QCD partons in  $\mathcal{M}_{m,a\dots}$ . Actually, according to the external-leg insertion sketched in Fig. 1, it is evident that the singular factor we are looking for is *quasi-local*, in the sense that it only depends on the momenta and quantum numbers of three partons: the parton  $j$  that is inserted onto  $\mathcal{M}_{m,a\dots}$  and the partons  $i$  and  $k$  in  $\mathcal{M}_{m,a\dots}$ .

This feature of the soft and collinear singularities will be used to obtain factorization formulae with the following symbolic structure (Fig. 2)

$$|\mathcal{M}_{m+1,a\dots}|^2 \rightarrow |\mathcal{M}_{m,a\dots}|^2 \otimes \mathbf{V}_{ij,k} . \quad (4.1)$$

Here  $\mathbf{V}_{ij,k}$  is the singular factor, which depends on the momenta and quantum numbers of the three partons  $i, j, k$ . As explained in detail in Sect. 5, two of these partons (e.g.  $i$  and  $j$ ) will play the role of ‘emitter’ and the third parton (e.g.  $k$ ) that of ‘spectator’.

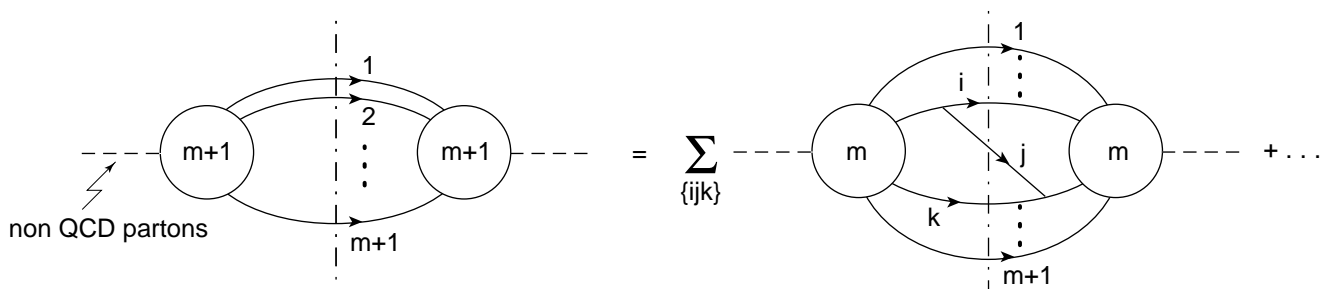


Figure 1: Diagrammatic representation of the external-leg insertion rule. The blobs denote the tree-level matrix elements and their complex conjugate. The dots on the right-hand side stand for non-singular terms both in the soft and collinear limits.

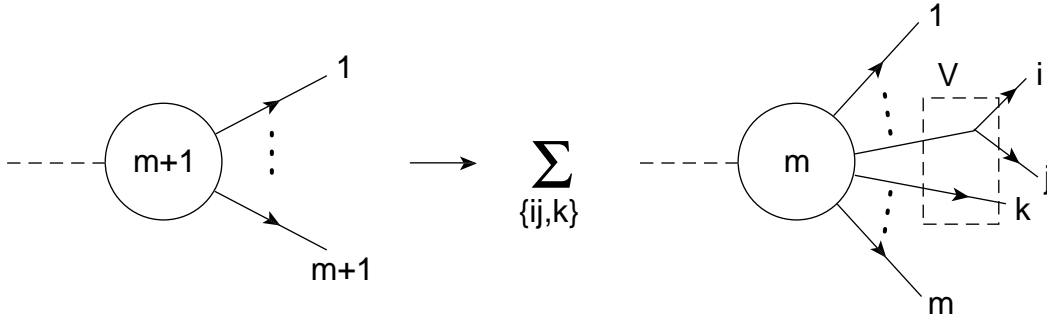


Figure 2: Pictorial representation of the dipole factorization procedure. When the partons  $i$  and  $j$  become soft and/or collinear, the singularities are factorized into the term  $\mathbf{V}_{ij,k}$  (the dashed box on the right-hand side) which embodies correlations with a single additional parton  $k$ .

Because of this structure, the factorization formulae described in Sect. 5 will be called *dipole factorization formulae*. In order to explicitly construct these formulae, we should first recall the known properties [7,25] of the tree-level QCD matrix elements in the soft and collinear limits.

## 4.2 Soft limit

Let us consider the matrix element  $\mathcal{M}_{m+1,a,\dots}$  and the corresponding vector (in colour + helicity space)  $|1, \dots, m+1; a, \dots\rangle_{m+1,a,\dots}$  defined in Eq. (3.11). Let us denote by  $p_j$  the momentum of a final-state gluon in  $\mathcal{M}_{m+1,a,\dots}$ . In the soft limit, which we parametrize in terms of an arbitrary four vector  $q^\mu$  and a scale parameter  $\lambda$ :

$$p_j^\mu = \lambda q^\mu, \quad \lambda \rightarrow 0, \quad (4.2)$$

the matrix element squared behaves as follows [7]

$$\begin{aligned} & {}_{m+1,a,\dots} \langle 1, \dots, m+1; a, \dots | |1, \dots, m+1; a, \dots \rangle_{m+1,a,\dots} \rightarrow \\ & -\frac{1}{\lambda^2} 4\pi\mu^{2\epsilon} \alpha_S {}_{m,a,\dots} \langle 1, \dots, m+1; a, \dots | [\mathbf{J}^\mu(q)]^\dagger \mathbf{J}_\mu(q) |1, \dots, m+1; a, \dots \rangle_{m,a,\dots} \end{aligned} \quad (4.3)$$

Here we have neglected all the contributions less singular than  $1/\lambda^2$ . The  $m$ -parton matrix element on the right-hand side of Eq. (4.3) is obtained from  $\mathcal{M}_{m+1,a,\dots}$  by simply removing the soft gluon  $p_j$ . The term  $\mathbf{J}_\mu(q)$  is the eikonal current for the emission of the soft gluon  $q$ . Its explicit expression is given in terms of the momenta and colour charges of the partons in  $|1, \dots, m+1; a, \dots\rangle_{m,a,\dots}$ :

$$\mathbf{J}^\mu(q) = \sum_i \mathbf{T}_i \frac{p_i^\mu}{p_i \cdot q} + \mathbf{T}_a \frac{p_a^\mu}{p_a \cdot q} + \dots \quad (4.4)$$

The formula in Eq. (4.3) is well known. Here we limit ourselves to recalling a few properties of Eq. (4.3), which are relevant for understanding the structure of the improved factorization formulae that we shall introduce in Sect. 5.

Although first derived in the four-dimensional case, Eq. (4.3) is actually valid in any number  $d = 4 - 2\epsilon$  of space-time dimensions. The only dependence on  $d$  is in the overall scale factor  $\mu^{2\epsilon}$  on the right-hand side.

The  $m$ -parton matrix element is not exactly factorized. Since the eikonal current in Eq. (4.4) depends on the colour charges of the hard partons, it leads to *colour correlations* on the right-hand of Eq. (4.3).

In the actual calculation of cross sections, Eq. (4.3) cannot be used as a true factorization formula not only because of these colour correlations. In fact, the tree-level matrix elements are unambiguously defined only when momentum conservation is fulfilled exactly. Since, in general, the  $m + 1$ - parton phase space does not factorize into an  $m$ -parton times a single-parton phase space, the right-hand side of Eq. (4.3) is unequivocally defined only in the strict soft limit  $\lambda = 0$ . Away from the point  $\lambda = 0$ , care has to be taken in implementing momentum conservation.

The form of the eikonal current in Eq. (4.4) is actually valid both for massless and massive partons. Squaring the current as in Eq. (4.3), and taking the massless limit, one obtains:

$$[\mathbf{J}^\mu(q)]^\dagger \mathbf{J}_\mu(q) = \sum_{k,i} \mathbf{T}_k \cdot \mathbf{T}_i \frac{p_k p_i}{(p_k q)(p_i q)} + \left( 2 \sum_i \mathbf{T}_a \cdot \mathbf{T}_i \frac{p_a p_i}{(p_a q)(p_i q)} + \dots \right) . \quad (4.5)$$

Each term  $p_k p_i / (p_k q)(p_i q)$  on the right-hand side of Eq. (4.5) leads to collinear singularities when the soft momentum  $q$  is parallel either to  $p_i$  or to  $p_k$  or  $p_a$ . These collinear singularities can be disentangled by using the following identity

$$\frac{p_k p_i}{(p_k q)(p_i q)} = \frac{p_k p_i}{p_k \cdot q (p_i + p_k) \cdot q} + \frac{p_k p_i}{p_i \cdot q (p_i + p_k) \cdot q} , \quad (4.6)$$

and likewise for the terms  $p_a p_i / (p_a q)(p_i q)$ ,  $\dots$ . Thus, Eq. (4.3) can be rewritten as follows

$$\begin{aligned} & {}_{m+1,a..} \langle 1, \dots, m+1; a, \dots | | 1, \dots, m+1; a, \dots \rangle_{m+1,a..} \rightarrow -\frac{1}{\lambda^2} 8\pi\mu^{2\epsilon}\alpha_S \\ & \cdot \sum_i \frac{1}{p_i q} \sum_{k \neq i} {}_{m,a..} \langle 1, \dots, m+1; a, \dots | \frac{\mathbf{T}_k \cdot \mathbf{T}_i p_k p_i}{(p_i + p_k)q} | 1, \dots, m+1; a, \dots \rangle_{m,a..} + \dots , \end{aligned} \quad (4.7)$$

where the dots stand for similar contributions that involve the initial-state parton  $a, \dots$ . The dipole structure mentioned at the end of Sect. 4.1 starts to emerge from Eq. (4.7). Each term on the right-hand side of Eq. (4.7) depends on the radiated soft momentum  $q$ , on the ‘emitter’ momentum  $p_i$  (whose direction signals the presence of a collinear singularity) and on the ‘spectator’ parton  $k$  (which accounts for colour correlations).

### 4.3 Collinear limit

Let us consider the momenta  $p_i$  and  $p_j$  of two final-state partons in  $\mathcal{M}_{m+1,a..}$ . The limit where  $p_i$  and  $p_j$  become collinear is precisely defined as follows

$$p_i^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2pn} , \quad p_j^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z} \frac{n^\mu}{2pn} ,$$

$$2p_i p_j = -\frac{k_\perp^2}{z(1-z)} , \quad k_\perp \rightarrow 0 . \quad (4.8)$$

In Eq. (4.8) the light-like ( $p^2 = 0$ ) vector  $p^\mu$  denotes the collinear direction, while  $n^\mu$  is an auxiliary light-like vector which is necessary to specify the transverse component  $k_\perp$  ( $k_\perp^2 < 0$ ) ( $k_\perp p = k_\perp n = 0$ ) or, equivalently, how the collinear direction is approached. In the small- $k_\perp$ -limit (i.e. neglecting terms that are less singular than  $1/k_\perp^2$ ), the  $m+1$ -parton matrix element behaves as follows [25]

$$\begin{aligned} & m_{+1,a..} \langle 1, \dots, m+1; a, \dots | | 1, \dots, m+1; a, \dots \rangle_{m_{+1,a..}} \rightarrow \\ & \frac{1}{p_i p_j} 4\pi \mu^{2\epsilon} \alpha_S \langle m_{a..} \langle 1, \dots, m+1; a, \dots | \hat{P}_{(ij),i}(z, k_\perp; \epsilon) | 1, \dots, m+1; a, \dots \rangle_{m_{a..}} \rangle . \end{aligned} \quad (4.9)$$

The  $m$ -parton matrix element on the right-hand side of Eq. (4.9) is obtained by replacing the partons  $i$  and  $j$  in  $\mathcal{M}_{m_{+1,a..}}$  with a single parton denoted by  $ij$ . This parton carries the quantum numbers of the pair  $i+j$  in the collinear limit. In other words, its momentum is  $p^\mu$  and its other quantum numbers (flavour, colour) are obtained according to the following rule: anything + gluon gives anything and quark + antiquark gives gluon.

The kernel  $\hat{P}_{(ij),i}$  in Eq. (4.9) is the  $d$ -dimensional Altarelli-Parisi splitting function. It depends not only on the momentum fraction  $z$  involved in the collinear splitting  $ij \rightarrow i+j$ , but also on the transverse momentum  $k_\perp$  and on the helicity of the parton  $ij$  in the  $m$ -parton matrix element. More precisely,  $\hat{P}_{(ij),i}$  is a matrix acting on the spin indices of the parton  $ij$  in  $\langle m_{a..} \langle 1, \dots, m+1; a, \dots |$  and  $| 1, \dots, m+1; a, \dots \rangle_{m_{a..}}$ . Because of these *spin correlations*, the square of the  $m$ -parton matrix element cannot be simply factorized on the right-hand side of Eq. (4.9).

The explicit expressions of  $\hat{P}_{ab}(z, k_\perp; \epsilon)$  for the splitting processes

$$a(p) \rightarrow b(zp + k_\perp + \mathcal{O}(k_\perp^2)) + c((1-z)p - k_\perp + \mathcal{O}(k_\perp^2)) \quad (4.10)$$

are as follows

$$\langle s | \hat{P}_{qq}(z, k_\perp; \epsilon) | s' \rangle = \delta_{ss'} C_F \left[ \frac{1+z^2}{1-z} - \epsilon(1-z) \right] , \quad (4.11)$$

$$\langle s | \hat{P}_{qg}(z, k_\perp; \epsilon) | s' \rangle = \delta_{ss'} C_F \left[ \frac{1+(1-z)^2}{z} - \epsilon z \right] , \quad (4.12)$$

$$\langle \mu | \hat{P}_{gq}(z, k_\perp; \epsilon) | \nu \rangle = T_R \left[ -g^{\mu\nu} + 4z(1-z) \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right] , \quad (4.13)$$

$$\langle \mu | \hat{P}_{gg}(z, k_\perp; \epsilon) | \nu \rangle = 2C_A \left[ -g^{\mu\nu} \left( \frac{z}{1-z} + \frac{1-z}{z} \right) - 2(1-\epsilon)z(1-z) \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right] , \quad (4.14)$$

where the spin indices of the parent parton  $a$  have been denoted by  $s, s'$  if  $a$  is a fermion and  $\mu, \nu$  if  $a$  is a gluon.

Equations (4.11)–(4.14) lead to the more familiar form of the  $d$ -dimensional splitting functions only after average over the polarizations of the parton  $a$ . The  $d$ -dimensional average is obtained by means of the factors

$$\frac{1}{2} \delta_{ss'} \quad (4.15)$$

for a fermion, and (the gauge terms are proportional either to  $p^\mu$  or to  $p^\nu$ )

$$\frac{1}{d-2} d_{\mu\nu}(p) = \frac{1}{2(1-\epsilon)} (-g_{\mu\nu} + \text{gauge terms}) , \quad (4.16)$$

with

$$-g^{\mu\nu} d_{\mu\nu}(p) = d-2 , \quad p^\mu d_{\mu\nu}(p) = 0 , \quad (4.17)$$

for a gluon with on-shell momentum  $p$ . Denoting by  $\langle \hat{P}_{ab} \rangle$  the average of  $\hat{P}_{ab}$  over the polarizations of the parton  $a$ , we have:

$$\langle \hat{P}_{qq}(z; \epsilon) \rangle = C_F \left[ \frac{1+z^2}{1-z} - \epsilon(1-z) \right] , \quad (4.18)$$

$$\langle \hat{P}_{qg}(z; \epsilon) \rangle = C_F \left[ \frac{1+(1-z)^2}{z} - \epsilon z \right] , \quad (4.19)$$

$$\langle \hat{P}_{gq}(z; \epsilon) \rangle = T_R \left[ 1 - \frac{2z(1-z)}{1-\epsilon} \right] , \quad (4.20)$$

$$\langle \hat{P}_{gg}(z; \epsilon) \rangle = 2C_A \left[ \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] , \quad (4.21)$$

So far, we have considered the case in which two final-state partons in  $\mathcal{M}_{m+1,a\dots}$  become collinear. In general, one has to deal also with the case in which a final-state parton  $i$  becomes collinear to an initial-state parton  $a$ . Here the collinear limit is defined as follows

$$\begin{aligned} p_i^\mu &= (1-x)p_a^\mu + k_\perp^\mu - \frac{k_\perp^2}{1-x} \frac{n^\mu}{2p_a n} , \\ 2p_i p_a &= -\frac{k_\perp^2}{1-x} , \quad k_\perp \rightarrow 0 , \end{aligned} \quad (4.22)$$

and the corresponding splitting process  $a \rightarrow ai + i$  involves the transition from the initial-state parton  $a$  to the initial-state parton  $ai$  with the associated emission of the final-state parton  $i$ . The quantum numbers of the parton  $ai$  are assigned according to their conservation at the QCD tree-level vertices: if  $a$  and  $i$  are partons of the same species, then  $ai$  is a gluon; if  $a$  is a fermion (gluon) and  $i$  is a gluon (fermion) then  $ai$  is a fermion (antifermion).

The analogue of Eq. (4.9) in the collinear limit (4.22) is the following

$$\begin{aligned} &_{m+1,a\dots} \langle 1, \dots, m+1; a, \dots | | 1, \dots, m+1; ai, \dots \rangle_{m+1,a\dots} \rightarrow \\ & \frac{1}{x} \frac{1}{p_i p_a} 4\pi \mu^{2\epsilon} \alpha_S \, _{m,ai\dots} \langle 1, \dots, m+1; ai, \dots | \hat{P}_{a,(ai)}(x, k_\perp; \epsilon) | 1, \dots, m+1; ai, \dots \rangle_{m,ai\dots} . \end{aligned} \quad (4.23)$$

Now, the  $m$ -parton matrix element on the right-hand side is obtained from  $\mathcal{M}_{m+1,a\dots}$  by removing the final-state parton  $i$  and replacing the initial-state parton  $a$  with the parton  $ai$ . Note two main differences with respect to Eq. (4.9): on the right-hand side of Eq. (4.23) there is an additional factor of  $1/x$  and the initial-state parton  $ai$  carries the momentum  $xp_a^\mu$ .

As in the case of Eq. (4.3), Eqs. (4.9) and (4.23) have to be regarded as limiting formulae rather than factorization formulae. Their implementation in the calculation of QCD cross sections indeed requires a careful treatment of momentum conservation away from the collinear limit.

Note that the splitting functions in Eqs. (4.11–4.14) are divergent for  $z \rightarrow 0, 1$ . These divergences are the soft singularities already discussed in Sect. 4.2. When using Eqs. (4.7) and (4.9) to approximate the singular behaviour of  $\mathcal{M}_{m+1,a\dots}$  care has to be taken in order to avoid double counting the soft and collinear divergences in their overlapping region.

Note also that Eqs. (4.9) and (4.23) do not depend solely on the collinear momenta  $p_i, p_j$  and  $p_i, p_a$ . In fact, the Altarelli-Parisi splitting functions produce spin correlations with respect to the directions of the other momenta in the matrix element. The dipole structure of these limiting formulae is thus hidden in the azimuthal dependence of these correlations.

## 5 Dipole factorization formulae

In this Section we present in detail our basic formalism to construct the local counter-term  $d\sigma^A$  for the NLO cross section in Eq. (2.4). We introduce improved factorization formulae for the QCD matrix elements. These formulae are based on a dipole structure with respect to *colour* and *spin* indices and have the following main features. Our dipole factorization formulae coincide with Eqs. (4.7) and (4.9) (or (4.23)) respectively in the soft and collinear limit. These limits are approached smoothly, thus avoiding double counting of overlapping soft and collinear singularities. This smooth transition is possible because the dipole formulae fulfil exact momentum conservation. Actually, we present several alternative versions of the factorization formulae that differ from one another in the implementation of momentum conservation away from the soft and collinear limits. These differences are then used to match the phase-space constraints that are encountered in the calculation of different kinds of QCD cross sections. In this manner, for any QCD process, we achieve the analytical integrability of the counter-term  $d\sigma^A$  over the single-parton subspace leading to soft and collinear divergences.

We start by considering the case of final-state singularities for matrix elements without (Sect. 5.1) or with (Sect. 5.2) initial-state partons. Sections 5.3 and 5.5 deal with initial-state singularities in the case of one or two initial-state partons, respectively. Cross sections with identified hadrons in the final state require the introduction of fragmentation functions. Factorization formulae suitable for these fragmentation processes are presented in Sects. 5.4 and 5.6.

The main properties of the dipole factorization formulae are considered in detail in Sect. 5.1. In the following Subsections we limit ourselves to writing down the formalism and emphasizing the relevant differences with respect to the case discussed in Sect. 5.1.

### 5.1 Final-state singularities with no initial-state partons

The dipole factorization formula in the limit  $p_i \cdot p_j \rightarrow 0$  for the matrix elements with no partons in the initial state is the following

$${}_{m+1} \langle 1, \dots, m+1 | | 1, \dots, m+1 \rangle_{m+1} = \sum_{k \neq i, j} \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) + \dots \quad (5.1)$$

where  $\dots$  stands for terms that are not singular in the limit  $p_i \cdot p_j \rightarrow 0$  and the dipole contribution  $\mathcal{D}_{ij,k}$  is given by

$$\mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) = -\frac{1}{2p_i \cdot p_j} \cdot {}_m \langle 1, \dots, \tilde{i}j, \dots, \tilde{k}, \dots, m+1 | \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} \mathbf{V}_{ij,k} | 1, \dots, \tilde{i}j, \dots, \tilde{k}, \dots, m+1 \rangle_m . \quad (5.2)$$

The  $m$ -parton matrix element on the right-hand side of Eq. (5.2) is obtained from the original  $m+1$ -parton matrix element by replacing a) the partons  $i$  and  $j$  with a single



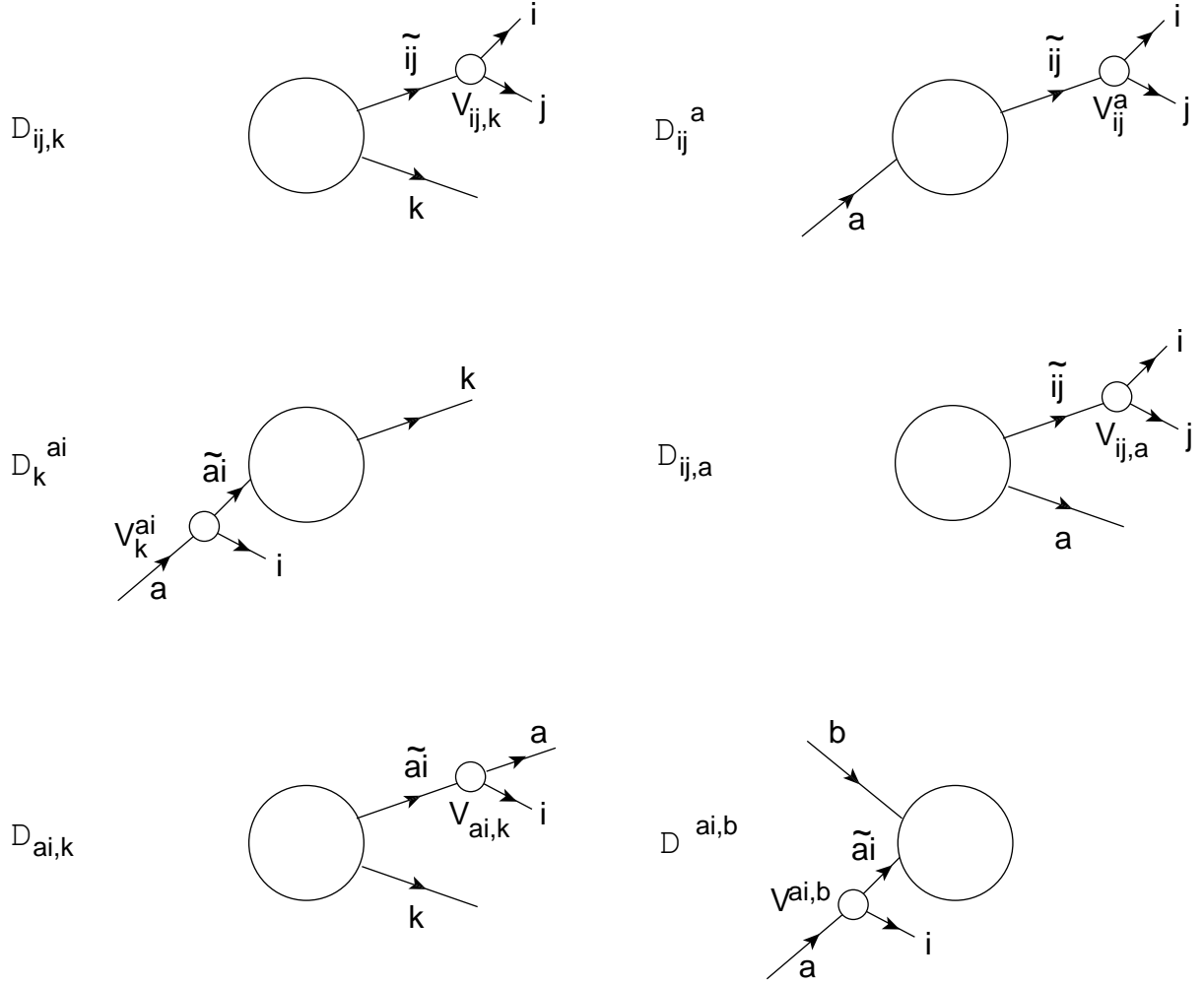


Figure 3: Effective diagrams for the different dipole formulae introduced in Sect. 5. The blobs denote the  $m$ -parton matrix element. Incoming and outgoing lines respectively stand for initial-state and final-state partons.

parton  $\tilde{i}j$  (*the emitter*) and  $b$ ) the parton  $k$  with the parton  $\tilde{k}$  (*the spectator*)\*.

All the quantum numbers except momenta are assigned as follows. The spectator parton  $\tilde{k}$  has the same quantum numbers as  $k$ . The quantum numbers of the emitter parton  $\tilde{i}j$  are obtained according to their conservation in the collinear splitting process  $\tilde{i}j \rightarrow i + j$  (cfr. Sect. 4.3). This rule applies to Eq. (5.2) as well as to all the dipole formulae we shall introduce in the following Subsections.

The momenta of the emitter and the spectator are defined in different ways in different dipole formulae. In Eq. (5.2) we have

$$\tilde{p}_k^\mu = \frac{1}{1 - y_{ij,k}} p_k^\mu, \quad \tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - \frac{y_{ij,k}}{1 - y_{ij,k}} p_k^\mu, \quad (5.3)$$

\*In general, we use the following notation in the dipole formulae (Fig. 3). A pair of indices like  $\tilde{i}j$  denotes the emitter parton and a single index like  $\tilde{k}$  denotes the spectator parton. If these indices appear as subscripts or superscripts, they respectively indicate final-state or initial-state partons.

where the dimensionless variable  $y_{ij,k}$  is given by

$$y_{ij,k} = \frac{p_i p_j}{p_i p_j + p_j p_k + p_k p_i} . \quad (5.4)$$

Note that both the emitter and the spectator are on-shell ( $\tilde{p}_{ij}^2 = \tilde{p}_k^2 = 0$ ) and that, performing the replacement  $\{i, j, k\} \rightarrow \{\tilde{i}, \tilde{j}, \tilde{k}\}$ , momentum conservation is implemented exactly:

$$p_i^\mu + p_j^\mu + p_k^\mu = \tilde{p}_{ij}^\mu + \tilde{p}_k^\mu . \quad (5.5)$$

These are common features of all the dipole formulae in the paper.

In the bra-ket on the right-hand side of Eq. (5.2),  $\mathbf{T}_k$  and  $\mathbf{T}_{ij}$  are the colour charges of the emitter and the spectator and  $\mathbf{V}_{ij,k}$  are matrices in the helicity space of the emitter. These splitting matrices, which depend on  $y_{ij,k}$  and on the kinematic variables  $\tilde{z}_i, \tilde{z}_j$ :

$$\tilde{z}_i = \frac{p_i p_k}{p_j p_k + p_i p_k} = \frac{p_i \tilde{p}_k}{\tilde{p}_{ij} \tilde{p}_k} , \quad \tilde{z}_j = \frac{p_j p_k}{p_j p_k + p_i p_k} = \frac{p_j \tilde{p}_k}{\tilde{p}_{ij} \tilde{p}_k} = 1 - \tilde{z}_i , \quad (5.6)$$

are related to the  $d$ -dimensional Altarelli-Parisi splitting functions in Eqs. (4.11)–(4.14).

For fermion + gluon splitting we have ( $s$  and  $s'$  are the spin indices of the fermion  $\tilde{i}$  in  $\langle \dots, \tilde{i} \dots \rangle$  and  $|\dots, \tilde{i} \dots\rangle$  respectively)

$$\begin{aligned} \langle s | \mathbf{V}_{q_i g_j, k}(\tilde{z}_i; y_{ij,k}) | s' \rangle &= 8\pi\mu^{2\epsilon} \alpha_S C_F \left[ \frac{2}{1 - \tilde{z}_i(1 - y_{ij,k})} - (1 + \tilde{z}_i) - \epsilon(1 - \tilde{z}_i) \right] \delta_{ss'} \\ &\equiv V_{q_i g_j, k} \delta_{ss'} . \end{aligned} \quad (5.7)$$

For quark + antiquark and gluon + gluon splitting we have ( $\mu$  and  $\nu$  are the spin indices of the gluon  $\tilde{i}$  in  $\langle \dots, \tilde{i} \dots \rangle$  and  $|\dots, \tilde{i} \dots\rangle$  respectively)

$$\langle \mu | \mathbf{V}_{q_i \bar{q}_j, k}(\tilde{z}_i) | \nu \rangle = 8\pi\mu^{2\epsilon} \alpha_S T_R \left[ -g^{\mu\nu} - \frac{2}{p_i p_j} (\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu) (\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu) \right] \equiv V_{q_i \bar{q}_j, k}^{\mu\nu} , \quad (5.8)$$

$$\begin{aligned} \langle \mu | \mathbf{V}_{g_i g_j, k}(\tilde{z}_i; y_{ij,k}) | \nu \rangle &= 16\pi\mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( \frac{1}{1 - \tilde{z}_i(1 - y_{ij,k})} \right. \right. \\ &\left. \left. + \frac{1}{1 - \tilde{z}_j(1 - y_{ij,k})} - 2 \right) + (1 - \epsilon) \frac{1}{p_i p_j} (\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu) (\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu) \right] \equiv V_{g_i g_j, k}^{\mu\nu} . \end{aligned} \quad (5.9)$$

### *Soft and collinear limits*

Note that the matrices  $\mathbf{V}_{ij,k}$  do not lead to two-particle singularities in any of the limits  $p_i \cdot p_j, p_i \cdot p_k, p_j \cdot p_k \rightarrow 0$ . This is because the only non-polynomial dependence on  $\tilde{z}_i$  is in the factors

$$\frac{1}{1 - \tilde{z}_i(1 - y_{ij,k})} = \frac{p_i p_j + p_j p_k + p_k p_i}{(p_i + p_k) p_j} . \quad (5.10)$$

Therefore the dipole term on the right-hand side of Eq. (5.1) contains only collinear and soft divergences for  $p_i \cdot p_j \rightarrow 0$ .

In the collinear limit of Eq. (4.8), the dipole variables become:

$$\begin{aligned} y_{ij,k} &\rightarrow -\frac{k_{\perp}^2}{2z(1-z)pp_k}, \quad \tilde{z}_i = 1 - \tilde{z}_j \rightarrow z, \\ \tilde{p}_k^{\mu} &\rightarrow p_k^{\mu}, \quad \tilde{p}_{ij}^{\mu} \rightarrow p^{\mu}. \end{aligned} \quad (5.11)$$

Thus, the  $m$ -parton matrix element in Eq. (5.2) coincides with that in Eq. (4.9). Moreover, the vector  $\tilde{z}_i p_i^{\mu} - \tilde{z}_j p_j^{\mu}$  in Eqs. (5.8,5.9) becomes  $[\tilde{z}_i z - \tilde{z}_j(1-z)]p^{\mu} + k_{\perp}^{\mu}$  and, since its  $p^{\mu}$ -component gives a vanishing contribution to Eq. (5.2) ( $p^{\mu} < \mu || \dots, p, \dots >_m = 0$  because of gauge invariance), the matrices  $\mathbf{V}_{ij,k}$  become proportional to Altarelli-Parisi splitting functions:

$$\mathbf{V}_{ij,k} \rightarrow 8\pi\mu^{2\epsilon}\alpha_S \hat{P}_{(ij),i}(z, k_{\perp}; \epsilon). \quad (5.12)$$

In particular, the only dependence on  $k$  that survives in Eq. (5.2) is that on  $\mathbf{T}_k$ . Therefore, one can perform the sum over the colour charges on the right-hand side of Eq. (5.1) and, using charge conservation ( $\sum_{k \neq i,j} \mathbf{T}_k = -\mathbf{T}_{ij}$ ), one can check that this equation reproduces the collinear behaviour in Eq. (4.9).

In the soft limit of Eq. (4.2), we have

$$\begin{aligned} y_{ij,k} &\rightarrow 0, \quad \tilde{z}_i \rightarrow 1, \quad \tilde{z}_j \rightarrow 0, \\ \tilde{p}_k^{\mu} &\rightarrow p_k^{\mu}, \quad \tilde{p}_{ij}^{\mu} \rightarrow p_i^{\mu}. \end{aligned} \quad (5.13)$$

Thus the  $m$ -parton matrix element in Eq. (5.2) coincides with that in Eq. (4.7). Moreover, the only singular factor in  $\mathbf{V}_{ij,k}$  is due to the term in Eq. (5.10):

$$\frac{1}{1 - \tilde{z}_i(1 - y_{ij,k})} \rightarrow \frac{1}{\lambda} \frac{p_k p_i}{(p_i + p_k)q}, \quad (5.14)$$

which gives

$$\lambda \mathbf{V}_{ij,k} \rightarrow 16\pi\mu^{2\epsilon}\alpha_S \mathbf{T}_{ij}^2 \frac{p_k p_i}{(p_i + p_k)q}. \quad (5.15)$$

Inserting this expression into Eqs. (5.2), we see that the dipole term in Eq. (5.1) reproduces the soft limit in Eq. (4.7).

This discussion proves that the dipole formula (5.1) provides a point-wise approximation of the  $m+1$ -parton matrix element in the singular region  $p_i \cdot p_j \rightarrow 0$ . Note that to achieve this, the helicity dependence of the splitting kernels  $\mathbf{V}_{ij,k}$  is essential. The azimuthal correlations due to this dependence cancel after integration over  $p_i$  (see Eqs. (5.26,5.27)) and hence, provided that the counting of helicity states is consistently performed in  $d$  dimensions, they are not relevant for reproducing the correct poles in  $1/\epsilon$  in the contribution  $\int_{m+1} d\sigma^A$  to the NLO cross section in Eq. (2.5). Nonetheless, these correlations have to be properly taken into account in constructing the *local* counter-term  $d\sigma^A$ , which makes the contribution  $[d\sigma^R - d\sigma^A]$  integrable in four dimensions. Indeed, the parton azimuthal correlations due to this dependence are not only essential in the most general case when the jet cross section explicitly depends on them, but even when it does not<sup>†</sup>.

<sup>†</sup>In this case the evaluation of  $\int_{m+1} d\sigma^R$  in four dimensions usually involves double angular integrals of the type  $\int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi \cos\varphi/(1 - \cos\theta)$ , where  $\varphi$  is the azimuthal angle. These integrals are mathematically ill-defined. If their numerical integration is attempted, one can obtain any answer whatsoever, depending on the detail of the integration procedure. Performing the integral analytically before going to 4 dimensions, one obtains  $\int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi \cos\varphi/(1 - \cos\theta) \sin^{-2\epsilon} \theta \sin^{-2\epsilon} \varphi = 0$ .

### Phase space factorization

The definition (5.3) of the dipole momenta is particularly useful because it allows us to exactly factorize the phase space of the partons  $i, j, k$  into the dipole phase space times a single-parton contribution. Indeed, let us consider the following 3-particle contribution to the final-state phase space:

$$d\phi(p_i, p_j, p_k; Q) = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \frac{d^d p_j}{(2\pi)^{d-1}} \delta_+(p_j^2) \frac{d^d p_k}{(2\pi)^{d-1}} \delta_+(p_k^2) (2\pi)^d \delta^{(d)}(Q - p_i - p_j - p_k). \quad (5.16)$$

In terms of the momenta  $\tilde{p}_{ij}$ ,  $\tilde{p}_k$  and  $p_i$ , this phase-space contribution takes the factorized form:

$$d\phi(p_i, p_j, p_k; Q) = d\phi(\tilde{p}_{ij}, \tilde{p}_k; Q) [dp_i(\tilde{p}_{ij}, \tilde{p}_k)] , \quad (5.17)$$

where

$$[dp_i(\tilde{p}_{ij}, \tilde{p}_k)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \mathcal{J}(p_i; \tilde{p}_{ij}, \tilde{p}_k) , \quad (5.18)$$

and the Jacobian factor is

$$\mathcal{J}(p_i; \tilde{p}_{ij}, \tilde{p}_k) = \Theta(1 - \tilde{z}_i) \Theta(1 - y_{ij,k}) \frac{(1 - y_{ij,k})^{d-3}}{1 - \tilde{z}_i} . \quad (5.19)$$

In terms of the kinematic variables defined earlier, we have

$$\begin{aligned} [dp_i(\tilde{p}_{ij}, \tilde{p}_k)] &= \frac{(2\tilde{p}_{ij}\tilde{p}_k)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} d\tilde{z}_i dy_{ij,k} \Theta(\tilde{z}_i(1 - \tilde{z}_i)) \Theta(y_{ij,k}(1 - y_{ij,k})) \\ &\cdot (\tilde{z}_i(1 - \tilde{z}_i))^{-\epsilon} (1 - y_{ij,k})^{1-2\epsilon} y_{ij,k}^{-\epsilon} , \end{aligned} \quad (5.20)$$

where  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to  $\tilde{p}_{ij}$  and  $\tilde{p}_k$  and thus

$$\int d\Omega^{(d-3)} = \frac{2\pi}{\pi^\epsilon \Gamma(1 - \epsilon)} . \quad (5.21)$$

### Integration of the splitting functions $\mathbf{V}_{ij,k}$

To evaluate the integral  $\int_{m+1} d\sigma^A$  in Eq. (2.5), we can compute the integral of the splitting functions  $\mathbf{V}_{ij,k}$  over  $[dp_i(\tilde{p}_{ij}, \tilde{p}_k)]$  once and for all. The only non-trivial point involved in this integration regards azimuthal correlations.

Note, however, that the spin correlation tensor

$$\frac{1}{p_i p_j} (\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu) (\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu) \quad (5.22)$$

in Eqs. (5.8,5.9) is orthogonal to both  $\tilde{p}_{ij}^\mu$  and  $\tilde{p}_{ij}^\nu$ . Using this property and Lorentz covariance (the integral may depend only on  $\tilde{p}_{ij}$  and  $\tilde{p}_k$ ), it follows that the azimuthal integration of the spin correlation tensor gives a contribution of the type

$$K_{\mu\nu} = A \left[ -g_{\mu\nu} + \frac{\tilde{p}_{ij}^\mu \tilde{p}_k^\nu + \tilde{p}_k^\mu \tilde{p}_{ij}^\nu}{\tilde{p}_{ij} \cdot \tilde{p}_k} \right] + B \tilde{p}_{ij}^\mu \tilde{p}_{ij}^\nu . \quad (5.23)$$

On the other hand, the (on-shell) matrix element  $|1, \dots, \tilde{i}^j, \dots, \tilde{k}, \dots, m+1 \rangle_m$  is conserved (gauge invariance), that is,

$$\tilde{p}_{ij}^\mu \langle \mu | |1, \dots, \tilde{i}^j, \dots, \tilde{k}, \dots, m+1 \rangle_m = 0. \quad (5.24)$$

Thus, only the term  $-A g_{\mu\nu}$  in  $K_{\mu\nu}$  contributes to the dipole formula after integration over  $[dp_i(\tilde{p}_{ij}, \tilde{p}_k)]$ . This term can be singled out by taking the  $d$ -dimensional average over the polarizations of the emitter:

$$\frac{1}{d-2} d^{\mu\nu}(\tilde{p}_{ij}) K_{\mu\nu} = \frac{1}{d-2} d^{\mu\nu}(\tilde{p}_{ij}) [-g_{\mu\nu} A]. \quad (5.25)$$

In conclusion, after integration of the dipole  $\mathcal{D}_{ij,k}(p_1, \dots, p_{m+1})$  over  $[dp_i(\tilde{p}_{ij}, \tilde{p}_k)]$ , only colour correlations survive, in the form:

$$\begin{aligned} & \int [dp_i(\tilde{p}_{ij}, \tilde{p}_k)] \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) \\ &= -\mathcal{V}_{ij,k} \langle m+1, \dots, \tilde{i}^j, \dots, \tilde{k}, \dots, m+1 | \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} | m+1, \dots, \tilde{i}^j, \dots, \tilde{k}, \dots, m+1 \rangle_m, \end{aligned} \quad (5.26)$$

where

$$\mathcal{V}_{ij,k} = \int [dp_i(\tilde{p}_{ij}, \tilde{p}_k)] \frac{1}{2p_i \cdot p_j} \langle \mathbf{V}_{ij,k} \rangle \equiv \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_{ij}\tilde{p}_k} \right)^\epsilon \mathcal{V}_{ij}(\epsilon), \quad (5.27)$$

and  $\langle \mathbf{V}_{ij,k} \rangle$  denotes the average of  $\mathbf{V}_{ij,k}$  over the polarizations of the emitter parton  $\tilde{i}^j$ .

The function  $\mathcal{V}_{ij}(\epsilon)$  depends only on the flavour indices  $i$  and  $j$ . Using Eq. (5.20), from the definition of  $\mathcal{V}_{ij}(\epsilon)$  in Eq. (5.27) we obtain

$$\mathcal{V}_{ij}(\epsilon) = \int_0^1 d\tilde{z}_i (\tilde{z}_i(1-\tilde{z}_i))^{-\epsilon} \int_0^1 \frac{dy}{y} (1-y)^{1-2\epsilon} y^{-\epsilon} \frac{\langle \mathbf{V}_{ij,k}(\tilde{z}_i; y) \rangle}{8\pi\alpha_S\mu^{2\epsilon}}, \quad (5.28)$$

where the spin-averaged splitting functions are:

$$\frac{\langle \mathbf{V}_{qg,k}(\tilde{z}; y) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = C_F \left[ \frac{2}{1-\tilde{z}(1-y)} - (1+\tilde{z}) - \epsilon(1-\tilde{z}) \right], \quad (5.29)$$

$$\frac{\langle \mathbf{V}_{q\bar{q},k}(\tilde{z}; y) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = T_R \left[ 1 - \frac{2\tilde{z}(1-\tilde{z})}{1-\epsilon} \right], \quad (5.30)$$

$$\frac{\langle \mathbf{V}_{gg,k}(\tilde{z}; y) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = 2C_A \left[ \frac{1}{1-\tilde{z}(1-y)} + \frac{1}{1-(1-\tilde{z})(1-y)} - 2 + \tilde{z}(1-\tilde{z}) \right]. \quad (5.31)$$

Performing the integration in Eq. (5.28), we find

$$\mathcal{V}_{qg}(\epsilon) = \frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)} C_F \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \frac{3+\epsilon}{2(1-3\epsilon)} \right] = C_F \left[ \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 5 - \frac{\pi^2}{2} + \mathcal{O}(\epsilon) \right], \quad (5.32)$$

$$\mathcal{V}_{q\bar{q}}(\epsilon) = \frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)} T_R \left[ -\frac{1}{\epsilon} \frac{2(1-\epsilon)}{(1-3\epsilon)(3-2\epsilon)} \right] = T_R \left[ -\frac{2}{3\epsilon} - \frac{16}{9} + \mathcal{O}(\epsilon) \right], \quad (5.33)$$

$$\mathcal{V}_{gg}(\epsilon) = \frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)} 2C_A \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \frac{11-7\epsilon}{2(1-3\epsilon)(3-2\epsilon)} \right] = 2C_A \left[ \frac{1}{\epsilon^2} + \frac{11}{6\epsilon} + \frac{50}{9} - \frac{\pi^2}{2} + \mathcal{O}(\epsilon) \right]. \quad (5.34)$$

Note that in each case, the first result is exact in any number of dimensions,  $d = 4 - 2\epsilon$ , while the latter is valid for  $\epsilon \rightarrow 0$ .

## 5.2 Final-state singularities with initial-state partons

In the presence of initial-state partons  $a, \dots$ , the  $m + 1$ -parton matrix element has both final-state ( $p_i \cdot p_j \rightarrow 0$ ) and initial-state ( $p_a \cdot p_j \rightarrow 0$ ) singularities. Let us first consider the dipole factorization formula for final-state singularities. Neglecting terms that are not singular when  $p_i \cdot p_j \rightarrow 0$ , we have:

$$\begin{aligned}
& {}_{m+1,a..} \langle 1, \dots, m+1; a, \dots | | 1, \dots, m+1; a, \dots \rangle_{m+1,a..} = \sum_{k \neq i, j} \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}; p_a, \dots) \\
& + \left[ \mathcal{D}_{ij}^a(p_1, \dots, p_{m+1}; p_a, \dots) + \dots \right] + \dots \quad .
\end{aligned} \tag{5.35}$$

The first term on the right-hand side is the same contribution as in Eqs. (5.1,5.2), while the terms in the square bracket represent additional dipole contributions in which the emitter is the final-state parton  $\tilde{i}\tilde{j}$  and the spectators are the initial-state partons  $\tilde{a}, \dots$ . These dipole terms are explicitly given by

$$\begin{aligned}
\mathcal{D}_{ij}^a(p_1, \dots, p_{m+1}; p_a, \dots) &= -\frac{1}{2p_i \cdot p_j} \frac{1}{x_{ij,a}} \\
\cdot {}_{m,a..} \langle 1, \dots, \tilde{i}\tilde{j}, \dots, m+1; \tilde{a}, \dots | &\frac{\mathbf{T}_a \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} \mathbf{V}_{ij}^a | 1, \dots, \tilde{i}\tilde{j}, \dots, m+1; \tilde{a}, \dots \rangle_{m,a..} \quad .
\end{aligned} \tag{5.36}$$

In Eq. (5.36) the momenta of the spectator  $\tilde{a}$  and the emitter  $\tilde{i}\tilde{j}$  are defined as follows

$$\tilde{p}_a^\mu = x_{ij,a} p_a^\mu, \quad \tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - (1 - x_{ij,a}) p_a^\mu, \tag{5.37}$$

$$x_{ij,a} = \frac{p_i p_a + p_j p_a - p_i p_j}{(p_i + p_j) p_a}, \tag{5.38}$$

and the corresponding splitting functions are:

$$\begin{aligned}
\langle s | \mathbf{V}_{q_i g_j}^a(\tilde{z}_i; x_{ij,a}) | s' \rangle &= 8\pi \mu^{2\epsilon} \alpha_S C_F \left[ \frac{2}{1 - \tilde{z}_i + (1 - x_{ij,a})} - (1 + \tilde{z}_i) - \epsilon(1 - \tilde{z}_i) \right] \delta_{ss'}, \\
&\equiv V_{q_i g_j}^a \delta_{ss'}, \quad .
\end{aligned} \tag{5.39}$$

$$\begin{aligned}
\langle \mu | \mathbf{V}_{g_i g_j}^a(\tilde{z}_i; x_{ij,a}) | \nu \rangle &= 16\pi \mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( \frac{1}{1 - \tilde{z}_i + (1 - x_{ij,a})} \right. \right. \\
&\left. \left. + \frac{1}{1 - \tilde{z}_j + (1 - x_{ij,a})} - 2 \right) + (1 - \epsilon) \frac{1}{p_i p_j} (\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu) (\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu) \right],
\end{aligned} \tag{5.40}$$

$$\langle \mu | \mathbf{V}_{q_i \bar{q}_j}^a(\tilde{z}_i) | \nu \rangle = 8\pi \mu^{2\epsilon} \alpha_S T_R \left[ -g^{\mu\nu} - \frac{2}{p_i p_j} (\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu) (\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu) \right], \tag{5.41}$$

where

$$\tilde{z}_i = \frac{p_i p_a}{p_i p_a + p_j p_a} = \frac{p_i \tilde{p}_a}{\tilde{p}_{ij} \tilde{p}_a}, \quad \tilde{z}_j = \frac{p_j p_a}{p_j p_a + p_i p_a} = \frac{p_j \tilde{p}_a}{\tilde{p}_{ij} \tilde{p}_a} = 1 - \tilde{z}_i. \tag{5.42}$$

In the definition of the splitting functions  $\mathbf{V}_{ij}^a$ , the kinematic variable  $x_{ij,a}$  plays a role similar to that of  $y_{ij,k}$  in  $\mathbf{V}_{ij,k}$ : it provides a smooth interpolation between the soft and collinear limits. Following the same argument as in Sect. 5.1, it is straightforward to check that Eqs. (5.35,5.36) reproduce the correct soft and collinear behaviour of the  $m+1$ -parton matrix element in the limit  $p_i \cdot p_j \rightarrow 0$ .

Note that, comparing Eqs. (5.39–5.41) with the equivalent expressions when the spectator is a final-state parton, Eqs. (5.7–5.9), they are not symmetric under crossing symmetry,  $p_k \rightarrow -p_a$ . While the non-soft terms do become identical in the collinear limit, the soft terms do not in the soft limit. This is because the crossing-symmetric eikonal term is split between the two dipole terms that, separately, are not crossing symmetric. In fact, after adding Eqs. (5.39–5.41) and the equivalent expressions for initial-state singularities given below in Eqs. (5.65–5.68), the soft terms for  $p_j \rightarrow 0$  do obey crossing:

$$\text{Final-state parton } p_k : \quad \frac{p_k p_i}{p_i p_j (p_i p_j + p_k p_j)} + \frac{p_k p_i}{p_k p_j (p_i p_j + p_k p_j)} = \frac{p_k p_i}{(p_i p_j)(p_k p_j)}, \quad (5.43)$$

$$\text{Initial-state parton } p_a : \quad \frac{p_a p_i}{p_i p_j (p_i p_j + p_a p_j)} + \frac{p_a p_i}{p_a p_j (p_i p_j + p_a p_j)} = \frac{p_a p_i}{(p_i p_j)(p_a p_j)}. \quad (5.44)$$

If, instead, naïve crossing,  $p_k \rightarrow -p_a$ , had been used on the individual terms on the left-hand side of Eqs. (5.43,5.44), spurious singularities would develop at  $p_i p_j = p_a p_j$ . Similar comments concerning crossing apply to the formulae in all later Subsections.

Considering the three-parton phase space

$$d\phi(p_i, p_j; Q + p_a) = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \frac{d^d p_j}{(2\pi)^{d-1}} \delta_+(p_j^2) (2\pi)^d \delta^{(d)}(Q + p_a - p_i - p_j), \quad (5.45)$$

the analogue of the phase-space *factorization* in Eq. (5.17) is the following phase-space *convolution*

$$d\phi(p_i, p_j; Q + p_a) = \int_0^1 dx d\phi(\tilde{p}_{ij}; Q + xp_a) [dp_i(\tilde{p}_{ij}; p_a, x)], \quad (5.46)$$

where

$$[dp_i(\tilde{p}_{ij}; p_a, x)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \Theta(x)\Theta(1-x) \delta(x - x_{ij,a}) \frac{1}{1 - \tilde{z}_i}, \quad (5.47)$$

or, more explicitly, using the kinematic variables in Eqs. (5.38,5.42):

$$\begin{aligned} [dp_i(\tilde{p}_{ij}; p_a, x)] &= \frac{(2\tilde{p}_{ij} p_a)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} d\tilde{z}_i dx_{ij,a} \Theta(\tilde{z}_i(1 - \tilde{z}_i)) \Theta(x(1-x)) \\ &\cdot (\tilde{z}_i(1 - \tilde{z}_i))^{-\epsilon} \delta(x - x_{ij,a}) (1-x)^{-\epsilon}, \end{aligned} \quad (5.48)$$

where  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to  $\tilde{p}_{ij}$  and  $p_a$ . Note that we are interested in using Eqs. (5.46,5.47) for a NLO calculation whose leading-order kinematic is  $Q + p_a = p$  with  $p^2 = 0, p_0 \geq 0$ . Thus the physical constraint  $Q^2 \leq 0$  is always understood in Eqs. (5.46,5.47).

The integration of the splitting function  $\mathbf{V}_{ij}^a$  over the phase space in Eq. (5.47) defines the functions  $\mathcal{V}_{ij}(x; \epsilon)$  (the treatment of the azimuthal correlations is similar to that in

Sect. 5.1):

$$\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_{ij}p_a} \right)^\epsilon \mathcal{V}_{ij}(x; \epsilon) \equiv \int [dp_i(\tilde{p}_{ij}; p_a, x)] \frac{1}{2p_i p_j} \langle \mathbf{V}_{ij}^a(\tilde{z}_i; x_{ij,a}) \rangle, \quad (5.49)$$

that is,

$$\mathcal{V}_{ij}(x; \epsilon) = \Theta(x)\Theta(1-x) \left( \frac{1}{1-x} \right)^{1+\epsilon} \int_0^1 d\tilde{z}_i (\tilde{z}_i(1-\tilde{z}_i))^{-\epsilon} \frac{\langle \mathbf{V}_{ij}^a(\tilde{z}_i; x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}}. \quad (5.50)$$

As usual,  $\langle \mathbf{V}_{ij}^a \rangle$  denotes the average of  $\mathbf{V}_{ij}^a$  over the polarizations of the parton  $\tilde{j}$ .

The  $\epsilon$ -dependence of  $\mathcal{V}_{ij}(x; \epsilon)$  has to be interpreted with care. For  $\epsilon = 0$ ,  $\mathcal{V}_{ij}(x; \epsilon)$  is well-defined everywhere except the point  $x = 1$  where it is singular as  $\frac{\ln(1-x)}{1-x}$ . In order to avoid this singularity, the limit  $\epsilon \rightarrow 0$  has to be taken uniformly in  $x$ . In this manner,  $\mathcal{V}_{ij}(x; \epsilon)$  defines an  $x$ -distribution whose coefficients contain poles in  $1/\epsilon$ . More precisely, we have:

$$\mathcal{V}_{ij}(x; \epsilon) = [\mathcal{V}_{ij}(x; \epsilon)]_+ + \delta(1-x) \int_0^1 dz \mathcal{V}_{ij}(z; \epsilon), \quad (5.51)$$

$$[\mathcal{V}_{ij}(x; \epsilon)]_+ = [\mathcal{V}_{ij}(x; \epsilon = 0)]_+ + \mathcal{O}(\epsilon), \quad (5.52)$$

where the ‘+’-distribution is defined, as usual, by its action on a generic test function  $g(x)$ :

$$\int_0^1 dx g(x) [\mathcal{V}(x)]_+ \equiv \int_0^1 dx [g(x) - g(1)] \mathcal{V}(x). \quad (5.53)$$

Note that in Eq. (5.52), the expansion around  $\epsilon = 0$  is well defined, i.e. the  $\mathcal{O}(\epsilon)$  term on the right-hand side is integrable in  $x$ .

The explicit form of the spin-averaged splitting functions  $\langle \mathbf{V}_{ij}^a \rangle$  is the following:

$$\frac{\langle \mathbf{V}_{qg}^a(\tilde{z}; x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = C_F \left[ \frac{2}{1-\tilde{z}+(1-x)} - (1+\tilde{z}) - \epsilon(1-\tilde{z}) \right], \quad (5.54)$$

$$\frac{\langle \mathbf{V}_{q\bar{q}}^a(\tilde{z}; x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = T_R \left[ 1 - \frac{2\tilde{z}(1-\tilde{z})}{1-\epsilon} \right], \quad (5.55)$$

$$\frac{\langle \mathbf{V}_{gg}^a(\tilde{z}; x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = 2C_A \left[ \frac{1}{1-\tilde{z}+(1-x)} + \frac{1}{\tilde{z}+(1-x)} - 2 + \tilde{z}(1-\tilde{z}) \right]. \quad (5.56)$$

Inserting Eqs. (5.54–5.56) into Eqs. (5.50–5.52) we obtain:

$$\begin{aligned} \mathcal{V}_{qg}(x; \epsilon) &= C_F \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ - \frac{3}{2} \left( \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] \\ &+ \delta(1-x) \left[ \mathcal{V}_{qg}(\epsilon) - \frac{3}{2} C_F \right] + \mathcal{O}(\epsilon), \end{aligned} \quad (5.57)$$

$$\mathcal{V}_{q\bar{q}}(x; \epsilon) = \frac{2}{3} T_R \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \left[ \mathcal{V}_{q\bar{q}}(\epsilon) + \frac{2}{3} T_R \right] + \mathcal{O}(\epsilon), \quad (5.58)$$

$$\begin{aligned} \mathcal{V}_{gg}(x; \epsilon) &= 2C_A \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ - \frac{11}{6} \left( \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] \\ &+ \delta(1-x) \left[ \mathcal{V}_{gg}(\epsilon) - \frac{11}{3} C_A \right] + \mathcal{O}(\epsilon), \end{aligned} \quad (5.59)$$

where  $\mathcal{V}_{ij}(\epsilon)$  are the functions in Eqs. (5.32–5.34). Note that the terms  $\delta(1-x)\mathcal{V}_{ij}(\epsilon)$  account for all the  $\epsilon$ -poles of  $\mathcal{V}_{ij}(x; \epsilon)$ , i.e.  $\mathcal{V}_{ij}(x; \epsilon) - \delta(1-x)\mathcal{V}_{ij}(\epsilon)$  are finite for  $\epsilon \rightarrow 0$ .



### 5.3 Initial-state singularities: one initial-state parton

In the limit  $p_a \cdot p_i \rightarrow 0$ , the dipole factorization formula for the  $m+1$ -parton matrix element with a single initial-state parton  $a$  is the following

$${}_{m+1,a < 1, \dots, m+1; a} | 1, \dots, m+1; a \rangle_{m+1,a} = \sum_{k \neq i} \mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}; p_a) + \dots, \quad (5.60)$$

where the dots stand for non-singular contributions and the first term is the sum of the dipole contributions in which the emitter is the initial-state parton  $\widetilde{ai}$  and the spectator is the final-state parton  $k$ . These dipoles are given by

$$\begin{aligned} \mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}; p_a) &= -\frac{1}{2p_a \cdot p_i} \frac{1}{x_{ik,a}} \\ &\cdot {}_{m,a < 1, \dots, \widetilde{k}, \dots, m+1; \widetilde{ai}} | 1, \dots, \widetilde{k}, \dots, m+1; \widetilde{ai} \rangle_{m,a} \cdot \frac{\mathbf{T}_k \cdot \mathbf{T}_{ai}}{\mathbf{T}_{ai}^2} \mathbf{V}_k^{ai} \end{aligned} \quad (5.61)$$

Note that, unlike the case in which the emitter is a final state  $\widetilde{ij}$ , now the dipole is not symmetric under  $a \leftrightarrow i$ . In fact, the momentum of the emitter parton  $\widetilde{ai}$  is parallel to  $p_a$ :

$$\widetilde{p}_{ai}^\mu = x_{ik,a} p_a^\mu, \quad (5.62)$$

where

$$x_{ik,a} = \frac{p_k p_a + p_i p_a - p_i p_k}{(p_k + p_i) p_a}. \quad (5.63)$$

On the other hand, the momentum of the spectator  $\widetilde{k}$  is not parallel to  $p_k$ :

$$\widetilde{p}_k^\mu = p_k^\mu + p_i^\mu - (1 - x_{ik,a}) p_a^\mu, \quad (5.64)$$

The splitting functions  $\mathbf{V}_k^{ai}$  are matrices in the helicity space of the parton  $\widetilde{ai}$ . Their explicit expression is the following

$$\begin{aligned} \langle s | \mathbf{V}_k^{q_a g_i}(x_{ik,a}; u_i) | s' \rangle &= 8\pi\mu^{2\epsilon} \alpha_S C_F \delta_{ss'} \left[ \frac{2}{1 - x_{ik,a} + u_i} - (1 + x_{ik,a}) - \epsilon(1 - x_{ik,a}) \right], \\ &\equiv V_k^{q_a g_i} \delta_{ss'}, \end{aligned} \quad (5.65)$$

$$\langle s | \mathbf{V}_k^{g_a \bar{q}_i}(x_{ik,a}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S T_R [1 - \epsilon - 2x_{ik,a}(1 - x_{ik,a})] \delta_{ss'} \equiv V_k^{g_a \bar{q}_i} \delta_{ss'}, \quad (5.66)$$

$$\begin{aligned} \langle \mu | \mathbf{V}_k^{q_a q_i}(x_{ik,a}; u_i) | \nu \rangle &= 8\pi\mu^{2\epsilon} \alpha_S C_F \left[ -g^{\mu\nu} x_{ik,a} \right. \\ &\left. + \frac{1 - x_{ik,a}}{x_{ik,a}} \frac{2u_i(1 - u_i)}{p_i p_k} \left( \frac{p_i^\mu}{u_i} - \frac{p_k^\mu}{1 - u_i} \right) \left( \frac{p_i^\nu}{u_i} - \frac{p_k^\nu}{1 - u_i} \right) \right], \end{aligned} \quad (5.67)$$

$$\begin{aligned} \langle \mu | \mathbf{V}_k^{g_a g_i}(x_{ik,a}; u_i) | \nu \rangle &= 16\pi\mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( \frac{1}{1 - x_{ik,a} + u_i} - 1 + x_{ik,a}(1 - x_{ik,a}) \right) \right. \\ &\left. + (1 - \epsilon) \frac{1 - x_{ik,a}}{x_{ik,a}} \frac{u_i(1 - u_i)}{p_i p_k} \left( \frac{p_i^\mu}{u_i} - \frac{p_k^\mu}{1 - u_i} \right) \left( \frac{p_i^\nu}{u_i} - \frac{p_k^\nu}{1 - u_i} \right) \right], \end{aligned} \quad (5.68)$$

where

$$u_i = \frac{p_i p_a}{p_i p_a + p_k p_a} . \quad (5.69)$$

Note that the factor  $1/x_{ik,a}$  on the right-hand side of Eq. (5.61) is relevant for reproducing the collinear limit in Eq. (4.23).

The analogue of the phase-space convolution in Eq. (5.46) is the following identity (similarly to Eqs. (5.46,5.47), we have  $Q^2 \leq 0$  in the physical region of the phase space we are interested in)

$$d\phi(p_i, p_k; Q + p_a) = \int_0^1 dx d\phi(\tilde{p}_k; Q + xp_a) [dp_i(\tilde{p}_k; p_a, x)] , \quad (5.70)$$

where

$$[dp_i(\tilde{p}_k; p_a, x)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \Theta(x)\Theta(1-x) \delta(x - x_{ik,a}) \frac{1}{1-u_i} . \quad (5.71)$$

Using the kinematic variables in Eqs. (5.63,5.69), the phase space in Eq. (5.71) can be written as follows

$$\begin{aligned} [dp_i(\tilde{p}_k; p_a, x)] &= \frac{(2\tilde{p}_k p_a)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} du_i dx_{ik,a} \Theta(u_i(1-u_i)) \Theta(x(1-x)) \\ &\cdot (u_i(1-u_i))^{-\epsilon} \delta(x - x_{ik,a}) (1-x)^{-\epsilon} , \end{aligned} \quad (5.72)$$

where  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to  $\tilde{p}_k$  and  $p_a$ .

It is also useful to introduce the following integral of the splitting function  $\mathbf{V}_k^{ai}$ :

$$\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_k p_a} \right)^\epsilon \mathcal{V}^{a,ai}(x; \epsilon) \equiv \int [dp_i(\tilde{p}_k; p_a, x)] \frac{1}{2p_a p_i} \frac{n_s(\tilde{ai})}{n_s(a)} < \mathbf{V}_k^{ai}(x_{ik,a}; u_i) > , \quad (5.73)$$

where  $n_s(a)$  ( $n_s(\tilde{ai})$ ) is the number of polarizations of the parton  $a$  ( $\tilde{ai}$ ) ( $n_s = 2$  for fermions and  $n_s = d - 2 = 2(1 - \epsilon)$  for gluons) and  $< \mathbf{V}_k^{ai} >$  denotes the average of  $\mathbf{V}_k^{ai}$  over the polarizations of the emitter parton  $\tilde{ai}$ .

From the definition in Eq. (5.73), we have

$$\mathcal{V}^{a,ai}(x; \epsilon) = \Theta(x)\Theta(1-x) \left( \frac{1}{1-x} \right)^\epsilon \int_0^1 \frac{du_i}{u_i} (u_i(1-u_i))^{-\epsilon} \frac{n_s(\tilde{ai})}{n_s(a)} \frac{< \mathbf{V}_k^{ai}(x; u_i) >}{8\pi\alpha_S\mu^{2\epsilon}} . \quad (5.74)$$

The integration over  $u_i$  in Eq. (5.74) leads to  $\epsilon$ -poles of the type  $f(x)/\epsilon$ , where  $f(x)$  can be either an  $x$ -integrable function for  $x = 1$  or  $(1-x)^{-1}$ . In order to get a series in  $\epsilon$  whose coefficients are well-defined  $x$ -distributions, we write:

$$\mathcal{V}^{a,ai}(x; \epsilon) = \frac{1}{\epsilon} \left\{ \frac{1}{x} \left[ \epsilon x \mathcal{V}^{a,ai}(x; \epsilon) \right]_+ + \epsilon \delta(1-x) \int_0^1 dz z \mathcal{V}^{a,ai}(z; \epsilon) \right\} . \quad (5.75)$$

The two terms in the curly bracket can separately be expanded in  $\epsilon$  and lead to contributions with the following structure

$$P^{a,\tilde{ai}}(x) + \text{const.} \left( \frac{1}{\epsilon} + \mathcal{O}(1) \right) \delta(1-x) + \mathcal{O}(\epsilon) , \quad (5.76)$$

where  $P^{a,\tilde{a}i}(x)$  is the *four-dimensional* regularized (real + virtual) Altarelli-Parisi probability for the splitting process  $a(p_a) \rightarrow \tilde{a}i(xp_a) + i((1-x)p_a)$ .

In order to obtain this result, we first compute the spin-averaged splitting functions  $\langle \mathbf{V}_k^{ai}(x; u) \rangle$ :

$$\frac{n_s(\tilde{q})}{n_s(q)} \frac{\langle \mathbf{V}_k^{qg}(x; u) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = C_F \left[ \frac{2}{1-x+u} - (1+x) - \epsilon(1-x) \right], \quad (5.77)$$

$$\frac{n_s(\tilde{q})}{n_s(g)} \frac{\langle \mathbf{V}_k^{g\bar{q}}(x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = T_R \left[ 1 - \frac{2x(1-x)}{1-\epsilon} \right], \quad (5.78)$$

$$\frac{n_s(\tilde{g})}{n_s(g)} \frac{\langle \mathbf{V}_k^{gg}(x; u) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = 2C_A \left[ \frac{1}{1-x+u} + \frac{1-x}{x} - 1 + x(1-x) \right]. \quad (5.79)$$

$$\frac{n_s(\tilde{g})}{n_s(q)} \frac{\langle \mathbf{V}_k^{qq}(x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = C_F \left[ (1-\epsilon)x + 2\frac{1-x}{x} \right]. \quad (5.80)$$

Then, inserting Eqs. (5.77–5.80) into Eqs. (5.74,5.75), we find:

$$\mathcal{V}^{q,g}(x; \epsilon) = -\frac{1}{\epsilon} P^{qg}(x) + P^{qg}(x) \ln(1-x) + C_F x + \mathcal{O}(\epsilon), \quad (5.81)$$

$$\mathcal{V}^{g,q}(x; \epsilon) = -\frac{1}{\epsilon} P^{gq}(x) + P^{gq}(x) \ln(1-x) + T_R 2x(1-x) + \mathcal{O}(\epsilon), \quad (5.82)$$

$$\begin{aligned} \mathcal{V}^{q,q}(x; \epsilon) &= -\frac{1}{\epsilon} P^{qq}(x) + \delta(1-x) \left[ \mathcal{V}_{qg}(\epsilon) + \left( \frac{2}{3}\pi^2 - 5 \right) C_F \right] \\ &+ C_F \left[ -\left( \frac{4}{1-x} \ln \frac{1}{1-x} \right)_+ - \frac{2}{1-x} \ln(2-x) \right. \\ &\left. + (1-x) - (1+x) \ln(1-x) \right] + \mathcal{O}(\epsilon), \end{aligned} \quad (5.83)$$

$$\begin{aligned} \mathcal{V}^{g,g}(x; \epsilon) &= -\frac{1}{\epsilon} P^{gg}(x) + \delta(1-x) \left[ \frac{1}{2} \mathcal{V}_{gg}(\epsilon) + N_f \mathcal{V}_{q\bar{q}}(\epsilon) + \left( \frac{2}{3}\pi^2 - \frac{50}{9} \right) C_A \right. \\ &+ \left. \frac{16}{9} N_f T_R \right] + C_A \left[ -\left( \frac{4}{1-x} \ln \frac{1}{1-x} \right)_+ - \frac{2}{1-x} \ln(2-x) \right. \\ &\left. + 2 \left( -1 + x(1-x) + \frac{1-x}{x} \right) \ln(1-x) \right] + \mathcal{O}(\epsilon), \end{aligned} \quad (5.84)$$

where the functions  $\mathcal{V}_{ij}(\epsilon)$  are given in Eqs. (5.32–5.34) and the (customary) regularized Altarelli-Parisi probabilities are

$$P^{qg}(x) = C_F \frac{1 + (1-x)^2}{x}, \quad (5.85)$$

$$P^{gq}(x) = T_R \left[ x^2 + (1-x)^2 \right], \quad (5.86)$$

$$P^{qq}(x) = C_F \left( \frac{1+x^2}{1-x} \right)_+ , \quad (5.87)$$

$$P^{gg}(x) = 2C_A \left[ \left( \frac{1}{1-x} \right)_+ + \frac{1-x}{x} - 1 + x(1-x) \right] + \delta(1-x) \left( \frac{11}{6}C_A - \frac{2}{3}N_f T_R \right) . \quad (5.88)$$

To simplify our notation in the following Subsections, it is also convenient to introduce the regular (i.e. not singular for  $x \rightarrow 1$ ) parts  $P_{\text{reg}}^{ab}(x)$  of the Altarelli-Parisi probabilities as follows

$$P_{\text{reg}}^{ab}(x) \equiv P^{ab}(x) - \delta^{ab} \left[ 2 \mathbf{T}_a^2 \left( \frac{1}{1-x} \right)_+ + \gamma_a \delta(1-x) \right] , \quad (5.89)$$

$$\gamma_{a=q,\bar{q}} = \frac{3}{2}C_F , \quad \gamma_{a=g} = \frac{11}{6}C_A - \frac{2}{3}T_R N_f , \quad (5.90)$$

that is,

$$\begin{aligned} P_{\text{reg}}^{ab}(x) &= P^{ab}(x) && \text{if } a \neq b , \\ P_{\text{reg}}^{qq}(x) &= -C_F(1+x) , \quad P_{\text{reg}}^{gg}(x) = 2C_A \left[ \frac{1-x}{x} - 1 + x(1-x) \right] . \end{aligned} \quad (5.91)$$

## 5.4 Dipole formulae with one identified parton in the final state

The perturbative QCD treatment of cross sections that, in addition to jets, have identified particles in the final state requires the introduction of fragmentation functions. In order to deal with the collinear singularities related to the fragmentation process, it is convenient to consider dipole factorization formulae that treat the final-state partons in an unsymmetric manner. Thus, the factorization formulae we are going to present below rescale the momenta of identified and non-identified partons in a different way.

Let us consider the matrix element  $|a, 1, \dots, m+1 \rangle_{m+1+a}$ , where  $a$  denotes a parton whose momentum  $p_a$  is identified in the final state. In this case we can get singularities from the regions where  $p_i \cdot p_j \rightarrow 0$  or  $p_i \cdot p_a \rightarrow 0$ . In the first region we write the following dipole formula

$$\begin{aligned} & {}_{m+1+a} \langle a, 1, \dots, m+1 | |a, 1, \dots, m+1 \rangle_{m+1+a} \\ &= \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_a, p_1, \dots, p_{m+1}) + \mathcal{D}_{ij,a}(p_a, p_1, \dots, p_{m+1}) + \dots \end{aligned} \quad (5.92)$$

where  $\dots$  stands for terms that are not singular in the limit  $p_i \cdot p_j \rightarrow 0$ .

The first term on the right-hand side of Eq. (5.92) is analogous to that in Eq. (5.1), i.e. the dipole  $\mathcal{D}_{ij,k}(p_a, p_1, \dots, p_{m+1})$  is:

$$\begin{aligned} \mathcal{D}_{ij,k}(p_a, p_1, \dots, p_{m+1}) &= -\frac{1}{2p_i \cdot p_j} \\ & {}_{m+a} \langle a, 1, \dots, \tilde{i}j, \dots, \tilde{k}, \dots, m+1 | \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} \mathbf{V}_{ij,k} |a, 1, \dots, \tilde{i}j, \dots, \tilde{k}, \dots, m+1 \rangle_{m+a} \end{aligned} \quad (5.93)$$

The second term is similar to the first one, that is,

$$\mathcal{D}_{ij,a}(p_a, p_1, \dots, p_{m+1}) = -\frac{1}{2p_i \cdot p_j} \\ {}_{m+a} \langle \tilde{a}, 1, \dots, \tilde{i}, \dots, m+1 | \frac{\mathbf{T}_a \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} \mathbf{V}_{ij,a} | \tilde{a}, 1, \dots, \tilde{i}, \dots, m+1 \rangle_{m+a} , \quad (5.94)$$

but now the spectator is the identified particle  $\tilde{a}$ . It is thus convenient to rewrite Eqs. (5.3)–(5.9) in terms of the following variable

$$z_{ij,a} = 1 - y_{ij,a} = \frac{(p_i + p_j)p_a}{p_i p_j + p_j p_a + p_a p_i} . \quad (5.95)$$

The dipole momenta  $\tilde{p}_a$  and  $\tilde{p}_{ij}$  are:

$$\tilde{p}_a^\mu = \frac{1}{z_{ij,a}} p_a^\mu , \quad \tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - \frac{1 - z_{ij,a}}{z_{ij,a}} p_a^\mu , \quad (5.96)$$

and the splitting functions are given in terms of  $z_{ij,a}$  and the variables  $\tilde{z}_i, \tilde{z}_j$ ,

$$\tilde{z}_i = \frac{p_i p_a}{(p_j + p_i)p_a} = \frac{p_i \tilde{p}_a}{\tilde{p}_{ij} \tilde{p}_a} , \quad \tilde{z}_j = \frac{p_j p_a}{(p_j + p_i)p_a} = \frac{p_j \tilde{p}_a}{\tilde{p}_{ij} \tilde{p}_a} = 1 - \tilde{z}_i , \quad (5.97)$$

by the following expressions

$$\langle s | \mathbf{V}_{q_i g_j, a}(\tilde{z}_i; 1 - z_{ij,a}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S C_F \left[ \frac{2}{1 - \tilde{z}_i z_{ij,a}} - (1 + \tilde{z}_i) - \epsilon(1 - \tilde{z}_i) \right] \delta_{ss'} , \quad (5.98)$$

$$\langle \mu | \mathbf{V}_{q_i \bar{q}_j, a}(\tilde{z}_i) | \nu \rangle = 8\pi\mu^{2\epsilon} \alpha_S T_R \left[ -g^{\mu\nu} - \frac{2}{p_i p_j} (\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu) (\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu) \right] , \quad (5.99)$$

$$\langle \mu | \mathbf{V}_{g_i g_j, a}(\tilde{z}_i; 1 - z_{ij,a}) | \nu \rangle = 16\pi\mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( \frac{1}{1 - \tilde{z}_i z_{ij,a}} \right. \right. \\ \left. \left. + \frac{1}{1 - \tilde{z}_j z_{ij,a}} - 2 \right) + (1 - \epsilon) \frac{1}{p_i p_j} (\tilde{z}_i p_i^\mu - \tilde{z}_j p_j^\mu) (\tilde{z}_i p_i^\nu - \tilde{z}_j p_j^\nu) \right] . \quad (5.100)$$

The only difference in the treatment of the two dipole contributions in Eq. (5.92) is that  $a$  is identified in the final state and, hence, in the physical cross section we do not have to integrate over its momentum  $p_a$ . Thus we are led to consider the following convolution formula for the phase space (the factor  $1/z^{2-2\epsilon}$  is introduced for later convenience: see Eq. (6.17) and Sect. 9)

$$d\phi(p_i, p_j; Q - p_a) = \int_0^1 \frac{dz}{z^{2-2\epsilon}} d\phi(\tilde{p}_{ij}; Q - p_a/z) [dp_i(\tilde{p}_{ij}, p_a; z)] , \quad (5.101)$$

where

$$[dp_i(\tilde{p}_{ij}, p_a; z)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \Theta(z) \Theta(1-z) \delta(z - z_{ij,a}) \frac{z^{2-2\epsilon}}{1 - \tilde{z}_i} . \quad (5.102)$$

In terms of the kinematic variables in Eqs. (5.95,5.97), Eq. (5.102) is

$$[dp_i(\tilde{p}_{ij}, p_a; z)] = \frac{(2\tilde{p}_{ij}p_a)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} d\tilde{z}_i dz_{ij,a} \Theta(\tilde{z}_i(1-\tilde{z}_i)) \Theta(z(1-z)) \cdot (\tilde{z}_i(1-\tilde{z}_i))^{-\epsilon} \delta(z-z_{ij,a}) (z(1-z))^{-\epsilon}, \quad (5.103)$$

where  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to  $\tilde{p}_{ij}$  and  $p_a$ .

The integration of the splitting functions  $\mathbf{V}_{ij,a}$  over the phase space in Eq. (5.103) defines the functions  $\bar{\mathcal{V}}_{ij}(z; \epsilon)$ :

$$\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_{ij}p_a} \right)^\epsilon \bar{\mathcal{V}}_{ij}(z; \epsilon) \equiv \int [dp_i(\tilde{p}_{ij}, p_a; z)] \frac{1}{2p_i p_j} \langle \mathbf{V}_{ij,a}(\tilde{z}_i; 1-z_{ij,a}) \rangle, \quad (5.104)$$

that is,

$$\bar{\mathcal{V}}_{ij}(z; \epsilon) = \Theta(z)\Theta(1-z) \frac{z^{1-\epsilon}}{(1-z)^{1+\epsilon}} \int_0^1 d\tilde{z}_i (\tilde{z}_i(1-\tilde{z}_i))^{-\epsilon} \frac{\langle \mathbf{V}_{ij,a}(\tilde{z}_i; 1-z) \rangle}{8\pi\alpha_S\mu^{2\epsilon}}. \quad (5.105)$$

As in the case of the function  $\mathcal{V}_{ij}(x; \epsilon)$  in Eq. (5.50),  $\bar{\mathcal{V}}_{ij}(z; \epsilon)$  is a  $z$ -distribution with  $\epsilon$ -dependent coefficients:

$$\bar{\mathcal{V}}_{ij}(z; \epsilon) = \left[ \bar{\mathcal{V}}_{ij}(z; \epsilon = 0) \right]_+ + \delta(1-z) \int_0^1 dz' \bar{\mathcal{V}}_{ij}(z'; \epsilon) + \mathcal{O}(\epsilon). \quad (5.106)$$

The explicit form of the spin-averaged splitting functions  $\langle \mathbf{V}_{ij,a} \rangle$  is the same as in Eqs. (5.29–5.31), apart from the replacements  $k \rightarrow a$ ,  $1-y \rightarrow z$ :

$$\frac{\langle \mathbf{V}_{qg,a}(\tilde{z}; 1-z) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = C_F \left[ \frac{2}{1-\tilde{z}z} - (1+\tilde{z}) - \epsilon(1-\tilde{z}) \right], \quad (5.107)$$

$$\frac{\langle \mathbf{V}_{q\bar{q},a}(\tilde{z}) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = T_R \left[ 1 - \frac{2\tilde{z}(1-\tilde{z})}{1-\epsilon} \right], \quad (5.108)$$

$$\frac{\langle \mathbf{V}_{gg,a}(\tilde{z}; 1-z) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = 2C_A \left[ \frac{1}{1-\tilde{z}z} + \frac{1}{1-(1-\tilde{z})z} - 2 + \tilde{z}(1-\tilde{z}) \right]. \quad (5.109)$$

Inserting Eqs. (5.107)–(5.109) into Eq. (5.105), performing the  $\tilde{z}$ -integration and expanding in  $\epsilon$  as in Eq. (5.106), we find

$$\bar{\mathcal{V}}_{qg}(z; \epsilon) = C_F \left[ \left( \frac{2}{1-z} \ln \frac{1}{1-z} \right)_+ - \frac{3}{2} \left( \frac{z}{1-z} \right)_+ \right] + \delta(1-z) \mathcal{V}_{qg}(\epsilon) + \mathcal{O}(\epsilon), \quad (5.110)$$

$$\bar{\mathcal{V}}_{q\bar{q}}(z; \epsilon) = \frac{2}{3} T_R \left( \frac{z}{1-z} \right)_+ + \delta(1-z) \mathcal{V}_{q\bar{q}}(\epsilon) + \mathcal{O}(\epsilon), \quad (5.111)$$

$$\bar{\mathcal{V}}_{gg}(z; \epsilon) = 2C_A \left[ \left( \frac{2}{1-z} \ln \frac{1}{1-z} \right)_+ - \frac{11}{6} \left( \frac{z}{1-z} \right)_+ \right] + \delta(1-z) \mathcal{V}_{gg}(\epsilon) + \mathcal{O}(\epsilon), \quad (5.112)$$

where  $\mathcal{V}_{ij}(\epsilon)$  are the functions in Eqs. (5.32–5.34).

Let us now consider the singularities from the region where  $p_i \cdot p_a \rightarrow 0$ . In this case we write:

$${}_{m+1+a} \langle a, 1, \dots, m+1 | | a, 1, \dots, m+1 \rangle_{m+1+a} = \sum_{k \neq i} \mathcal{D}_{ai,k}(p_a, p_1, \dots, p_{m+1}) + \dots \quad (5.113)$$

$$\begin{aligned} \mathcal{D}_{ai,k}(p_a, p_1, \dots, p_{m+1}) &= -\frac{1}{2p_i \cdot p_a} \\ \cdot {}_{m+a} \langle \tilde{a}i, 1, \dots, \tilde{k}, \dots, m+1 | \frac{\mathbf{T}_k \cdot \mathbf{T}_{ai}}{\mathbf{T}_{ai}^2} \mathbf{V}_{ai,k} | \tilde{a}i, 1, \dots, \tilde{k}, \dots, m+1 \rangle_{m+a} \quad (5.114) \end{aligned}$$

The parton momenta in the dipole (5.114) are:

$$\tilde{p}_{ai}^\mu = \frac{1}{z_{ik,a}} p_a^\mu, \quad \tilde{p}_k^\mu = p_i^\mu + p_k^\mu - \frac{1 - z_{ik,a}}{z_{ik,a}} p_a^\mu, \quad (5.115)$$

where

$$z_{ik,a} = \frac{(p_i + p_k) p_a}{p_i p_k + p_k p_a + p_a p_i}. \quad (5.116)$$

Note that, unlike the case of the final-state dipole  $\{\tilde{i}, \tilde{k}\}$  with no identified partons, here the momentum of  $\tilde{a}i$  ( $\tilde{k}$ ) is (is not) parallel to  $a$  ( $k$ ).

The splitting functions  $\mathbf{V}_{ai,k}$  are:

$$\langle s | \mathbf{V}_{q_a \bar{q}_i, k}(z_{ik,a}; u_i) | s' \rangle = 8\pi \mu^{2\epsilon} \alpha_S C_F \delta_{ss'} \left[ \frac{2}{1 - z_{ik,a}(1 - u_i)} - (1 + z_{ik,a}) - \epsilon(1 - z_{ik,a}) \right], \quad (5.117)$$

$$\langle s | \mathbf{V}_{g_a \bar{q}_i, k}(z_{ik,a}) | s' \rangle = 8\pi \mu^{2\epsilon} \alpha_S C_F \delta_{ss'} \left[ \frac{1 + (1 - z_{ik,a})^2}{z_{ik,a}} - \epsilon z_{ik,a} \right], \quad (5.118)$$

$$\begin{aligned} \langle \mu | \mathbf{V}_{q_a \bar{q}_i, k}(z_{ik,a}; u_i) | \nu \rangle &= 8\pi \mu^{2\epsilon} \alpha_S T_R \quad (5.119) \\ \cdot \left[ -g^{\mu\nu} - 2z_{ik,a}(1 - z_{ik,a}) \frac{u_i(1 - u_i)}{p_i p_k} \left( \frac{p_i^\mu}{u_i} - \frac{p_k^\mu}{1 - u_i} \right) \left( \frac{p_i^\nu}{u_i} - \frac{p_k^\nu}{1 - u_i} \right) \right], \end{aligned}$$

$$\begin{aligned} \langle \mu | \mathbf{V}_{g_a \bar{q}_i, k}(z_{ik,a}; u_i) | \nu \rangle &= 16\pi \mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( \frac{1}{1 - z_{ik,a}(1 - u_i)} - 2 + \frac{1}{z_{ik,a}} \right) \right. \\ &+ (1 - \epsilon)(1 - z_{ik,a}) z_{ik,a} \frac{u_i(1 - u_i)}{p_i p_k} \left. \left( \frac{p_i^\mu}{u_i} - \frac{p_k^\mu}{1 - u_i} \right) \left( \frac{p_i^\nu}{u_i} - \frac{p_k^\nu}{1 - u_i} \right) \right], \quad (5.120) \end{aligned}$$

where the variable  $u_i$  is defined as follows

$$u_i = \frac{p_i p_a}{(p_i + p_k) p_a}. \quad (5.121)$$

The phase-space for the dipole  $\{ai, k\}$  can be written in terms of the rescaled momenta  $\tilde{p}_k, \tilde{p}_{ai}$  by using the following convolution formula

$$d\phi(p_i, p_k; Q - p_a) = \int_0^1 \frac{dz}{z^{2-2\epsilon}} d\phi(\tilde{p}_k; Q - p_a/z) [dp_i(\tilde{p}_k, p_a; z)] , \quad (5.122)$$

where

$$[dp_i(\tilde{p}_k, p_a; z)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \Theta(z)\Theta(1-z) \delta(z - z_{ik,a}) \frac{z^{2-2\epsilon}}{1-u_i} . \quad (5.123)$$

Using the kinematic variables in Eqs. (5.116,5.121), Eq. (5.123) can be written as

$$\begin{aligned} [dp_i(\tilde{p}_k, p_a; z)] &= \frac{(2\tilde{p}_k p_a)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} du_i dz_{ik,a} \Theta(u_i(1-u_i)) \Theta(z(1-z)) \\ &\cdot (u_i(1-u_i))^{-\epsilon} \delta(z - z_{ik,a}) (z(1-z))^{-\epsilon} , \end{aligned} \quad (5.124)$$

where  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to  $\tilde{p}_k$  and  $p_a$ .

The integration of the splitting functions  $\mathbf{V}_{ai,k}$  over the phase space in Eq. (5.124) defines the functions  $\mathcal{V}_{ai,a}(z; \epsilon)$ :

$$\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_k p_a} \right)^\epsilon \mathcal{V}_{ai,a}(z; \epsilon) \equiv \int [dp_i(\tilde{p}_k, p_a; z)] \frac{1}{2p_a p_i} \langle \mathbf{V}_{ai,k}(z_{ik,a}; u_i) \rangle , \quad (5.125)$$

that is,

$$\mathcal{V}_{ai,a}(z; \epsilon) = \Theta(z)\Theta(1-z) [z(1-z)]^{-\epsilon} \int_0^1 \frac{du_i}{u_i} (u_i(1-u_i))^{-\epsilon} \frac{\langle \mathbf{V}_{ai,k}(z; u_i) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} . \quad (5.126)$$

The spin-averaged splitting functions  $\langle \mathbf{V}_{ai,k}(z; u) \rangle$  are:

$$\frac{\langle \mathbf{V}_{qag_i,k}(z; u) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = C_F \left[ \frac{2}{1-z(1-u)} - (1+z) - \epsilon(1-z) \right] , \quad (5.127)$$

$$\frac{\langle \mathbf{V}_{gaq_i,k}(z) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = C_F \left[ \frac{1+(1-z)^2}{z} - \epsilon z \right] , \quad (5.128)$$

$$\frac{\langle \mathbf{V}_{ga\bar{q}_i,k}(z) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = T_R \left[ 1 - \frac{2z(1-z)}{1-\epsilon} \right] , \quad (5.129)$$

$$\frac{\langle \mathbf{V}_{gag_i,k}(z; u) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = 2C_A \left[ \frac{1}{1-z(1-u)} - 2 + \frac{1}{z} + z(1-z) \right] . \quad (5.130)$$

Inserting Eqs. (5.127–5.130) into Eq. (5.126) and expanding in  $\epsilon$ , we obtain the following  $z$ -distributions

$$\mathcal{V}_{q,g}(z; \epsilon) = -\frac{1}{\epsilon} P_{qg}(z) + P_{qg}(z) \ln[z(1-z)] + C_F z + \mathcal{O}(\epsilon) , \quad (5.131)$$



$$\mathcal{V}_{g,q}(z; \epsilon) = -\frac{1}{\epsilon} P_{gq}(z) + P_{gq}(z) \ln[z(1-z)] + T_R 2z(1-z) + \mathcal{O}(\epsilon) , \quad (5.132)$$

$$\begin{aligned} \mathcal{V}_{q,q}(z; \epsilon) &= -\frac{1}{\epsilon} P_{qq}(z) + \delta(1-z) \left[ \mathcal{V}_{qq}(\epsilon) + \left( \frac{2}{3}\pi^2 - 5 \right) C_F \right] \\ &+ C_F \left[ -\left( \frac{4}{1-z} \ln \frac{1}{1-z} \right)_+ + \frac{2}{1-z} \ln z \right. \\ &\left. + (1-z) - (1+z) \ln[z(1-z)] \right] + \mathcal{O}(\epsilon) , \end{aligned} \quad (5.133)$$

$$\begin{aligned} \mathcal{V}_{g,g}(z; \epsilon) &= -\frac{1}{\epsilon} P_{gg}(z) + \delta(1-z) \left[ \frac{1}{2} \mathcal{V}_{gg}(\epsilon) + N_f \mathcal{V}_{q\bar{q}}(\epsilon) + \left( \frac{2}{3}\pi^2 - \frac{50}{9} \right) C_A \right. \\ &+ \left. \frac{16}{9} N_f T_R \right] + C_A \left[ -\left( \frac{4}{1-z} \ln \frac{1}{1-z} \right)_+ + \frac{2}{1-z} \ln z \right. \\ &\left. + 2 \left( -2 + z(1-z) + \frac{1}{z} \right) \ln[z(1-z)] \right] + \mathcal{O}(\epsilon) , \end{aligned} \quad (5.134)$$

where the functions  $\mathcal{V}_{ij}(\epsilon)$  are given in Eqs. (5.32–5.34) and  $P_{ab}(z) = P^{ab}(z)$  are the regularized Altarelli-Parisi probabilities in Eqs. (5.85–5.88).

## 5.5 Initial-state singularities: two initial-state partons

Let us come back to the treatment of initial-state singularities in the case with two partons  $a$  and  $b$  in the initial state. In the limit  $p_a \cdot p_i \rightarrow 0$ , the dipole factorization formula is

$$\begin{aligned} &_{m+1,ab} \langle 1, \dots, m+1; a, b | 1, \dots, m+1; a, b \rangle_{m+1,ab} \\ &= \sum_{k \neq i} \mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}; p_a, p_b) + \mathcal{D}^{ai,b}(p_1, \dots, p_{m+1}; p_a, p_b) + \dots . \end{aligned} \quad (5.135)$$

This equation generalizes Eq. (5.60).

The second term on the right-hand side represents a new dipole contribution in which the emitter is the initial-state parton  $\widetilde{ai}$  and the spectator is the other initial-state parton  $b$ :

$$\begin{aligned} \mathcal{D}^{ai,b}(p_1, \dots, p_{m+1}; p_a, p_b) &= -\frac{1}{2p_a \cdot p_i} \frac{1}{x_{i,ab}} \\ &_{m,ab} \langle \widetilde{1}, \dots, \widetilde{m+1}; \widetilde{ai}, b | \frac{\mathbf{T}_b \cdot \mathbf{T}_{ai}}{\mathbf{T}_{ai}^2} \mathbf{V}^{ai,b} | \widetilde{1}, \dots, \widetilde{m+1}; \widetilde{ai}, b \rangle_{m,ab} . \end{aligned} \quad (5.136)$$

This dipole contribution differs from that in which the spectator is a final-state parton because, when computing the cross section (see Sect. 10), it is convenient to leave the momentum  $p_b$  *unchanged*. Thus the matrix element  $|\widetilde{1}, \dots, \widetilde{m+1}; \widetilde{ai}, b \rangle_{m,ab}$  involves an initial-state parton  $\widetilde{ai}$  with momentum parallel to  $p_a$ :

$$\widetilde{p}_{ai}^\mu = x_{i,ab} p_a^\mu , \quad (5.137)$$

$$x_{i,ab} = \frac{p_a p_b - p_i p_a - p_i p_b}{p_a p_b} , \quad (5.138)$$

and *all* other final-state momenta  $k_j$  (and *not only* the momenta  $p_j$  of the QCD partons!) rescaled as follows

$$\tilde{k}_j^\mu = k_j^\mu - \frac{2k_j \cdot (K + \tilde{K})}{(K + \tilde{K})^2} (K + \tilde{K})^\mu + \frac{2k_j \cdot K}{K^2} \tilde{K}^\mu , \quad (5.139)$$

where the momenta  $K^\mu$  and  $\tilde{K}^\mu$  are defined by

$$\begin{aligned} K^\mu &= p_a^\mu + p_b^\mu - p_i^\mu , \\ \tilde{K}^\mu &= \tilde{p}_{ai}^\mu + p_b^\mu . \end{aligned} \quad (5.140)$$

Note that the momentum conservation constraint in the  $m + 1$ -parton matrix element is

$$p_a^\mu + p_b^\mu - \sum_j k_j^\mu - p_i^\mu = 0 . \quad (5.141)$$

Therefore we have  $2 \sum_j k_j \cdot K = 2K^2$  and  $2 \sum_j k_j \cdot (K + \tilde{K}) = 2K^2 + 2K \cdot \tilde{K} = (K + \tilde{K})^2$  and it is straightforward to check that momentum conservation is exactly implemented in the  $m$ -parton matrix element on the right-hand side of Eq. (5.136), that is,

$$\tilde{p}_{ai}^\mu + p_b^\mu - \sum_j \tilde{k}_j^\mu = 0 . \quad (5.142)$$

Note also that Eq. (5.139) can be rewritten in the following way:

$$\tilde{k}_j^\mu = \Lambda^\mu{}_\nu(K, \tilde{K}) k_j^\nu , \quad (5.143)$$

$$\Lambda^\mu{}_\nu(K, \tilde{K}) = g^\mu{}_\nu - \frac{2(K + \tilde{K})^\mu (K + \tilde{K})_\nu}{(K + \tilde{K})^2} + \frac{2\tilde{K}^\mu K_\nu}{K^2} , \quad (5.144)$$

and thus the matrix  $\Lambda^\mu{}_\nu(K, \tilde{K})$  generates a Lorentz transformation (actually, a *proper* Lorentz transformation) on all the final-state momenta.

The splitting functions  $\mathbf{V}^{ai,b}$  in Eq. (5.135) are as follows

$$\langle s | \mathbf{V}^{qag_i,b}(x_{i,ab}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S C_F \delta_{ss'} \left[ \frac{2}{1 - x_{i,ab}} - (1 + x_{i,ab}) - \epsilon(1 - x_{i,ab}) \right] , \quad (5.145)$$

$$\langle s | \mathbf{V}^{gag_i,b}(x_{i,ab}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S T_R [1 - \epsilon - 2x_{i,ab}(1 - x_{i,ab})] \delta_{ss'} , \quad (5.146)$$

$$\begin{aligned} \langle \mu | \mathbf{V}^{qag_i,b}(x_{i,ab}) | \nu \rangle &= 8\pi\mu^{2\epsilon} \alpha_S C_F \left[ -g^{\mu\nu} x_{i,ab} \right. \\ &\quad \left. + \frac{1 - x_{i,ab}}{x_{i,ab}} \frac{2p_a \cdot p_b}{p_i \cdot p_a p_i \cdot p_b} \left( p_i^\mu - \frac{p_i p_a}{p_b p_a} p_b^\mu \right) \left( p_i^\nu - \frac{p_i p_a}{p_b p_a} p_b^\nu \right) \right] , \end{aligned} \quad (5.147)$$

$$\begin{aligned} \langle \mu | \mathbf{V}^{gag_i,b}(x_{i,ab}) | \nu \rangle &= 16\pi\mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( \frac{x_{i,ab}}{1 - x_{i,ab}} + x_{i,ab}(1 - x_{i,ab}) \right) \right. \\ &\quad \left. + (1 - \epsilon) \frac{1 - x_{i,ab}}{x_{i,ab}} \frac{p_a \cdot p_b}{p_i \cdot p_a p_i \cdot p_b} \left( p_i^\mu - \frac{p_i p_a}{p_b p_a} p_b^\mu \right) \left( p_i^\nu - \frac{p_i p_a}{p_b p_a} p_b^\nu \right) \right] . \end{aligned} \quad (5.148)$$

The Jacobian factor associated with the Lorentz transformation (5.143) acting on the final-state momenta is equal to unity. Therefore the phase space for the dipole  $\{\tilde{a}i, b\}$  has a trivial convolution structure:

$$d\phi(p_i, k_1, \dots; p_a + p_b) = \int_0^1 dx d\phi(\tilde{k}_1, \dots; xp_a + p_b) [dp_i(p_a, p_b, x)] , \quad (5.149)$$

where

$$[dp_i(p_a, p_b, x)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \Theta(x)\Theta(1-x) \delta(x - x_{i,ab}) . \quad (5.150)$$

The phase space in Eq. (5.150) can be written as follows

$$\begin{aligned} [dp_i(p_a, p_b, x)] &= \frac{(2p_a p_b)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} d\tilde{v}_i dx_{i,ab} \Theta(x(1-x)) \Theta(\tilde{v}_i) \Theta\left(1 - \frac{\tilde{v}_i}{1-x}\right) \\ &\cdot (1-x)^{-2\epsilon} \delta(x - x_{i,ab}) \left[ \frac{\tilde{v}_i}{1-x} \left(1 - \frac{\tilde{v}_i}{1-x}\right) \right]^{-\epsilon} , \end{aligned} \quad (5.151)$$

where  $x_{i,ab}$  is defined in Eq. (5.138),  $\tilde{v}_i = p_a p_i / p_a p_b$  and  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to  $p_a$  and  $p_b$ .

The following integral of the splitting function  $\mathbf{V}^{ai,b}$  defines the  $x$ -distribution  $\tilde{\mathcal{V}}^{a,ai}$  :

$$\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_a p_b} \right)^\epsilon \tilde{\mathcal{V}}^{a,ai}(x; \epsilon) \equiv \int [dp_i(p_a, p_b, x)] \frac{1}{2p_a p_i} \frac{n_s(\tilde{a}i)}{n_s(a)} \langle \mathbf{V}^{ai,b}(x_{i,ab}) \rangle . \quad (5.152)$$

Using the phase space in Eq. (5.151), we thus have:

$$\tilde{\mathcal{V}}^{a,ai}(x; \epsilon) = -\frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \Theta(x)\Theta(1-x) (1-x)^{-2\epsilon} \frac{n_s(\tilde{a}i)}{n_s(a)} \frac{\langle \mathbf{V}^{ai,b}(x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} , \quad (5.153)$$

where the spin averages  $\langle \mathbf{V}^{ai,b}(x) \rangle$  are exactly proportional to the spin average of the  $d$ -dimensional Altarelli-Parisi splitting functions in Eqs. (4.18–4.21):

$$\frac{n_s(\tilde{a}i)}{n_s(a)} \frac{\langle \mathbf{V}^{ai,b}(x) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = \langle \hat{P}_{a,\tilde{a}i}(x; \epsilon) \rangle . \quad (5.154)$$

Performing the  $\epsilon$ -expansion in Eqs.(5.153,5.154) according to the procedure in Eqs. (5.51) and (5.52), we find

$$\tilde{\mathcal{V}}^{a,b}(x; \epsilon) = \mathcal{V}^{a,b}(x; \epsilon) + \delta^{ab} \mathbf{T}_a^2 \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] + \tilde{K}^{ab}(x) + \mathcal{O}(\epsilon) , \quad (5.155)$$

where  $\mathcal{V}^{a,b}(x; \epsilon)$  are the functions defined in Eqs. (5.81–5.84) and  $\tilde{K}^{ab}(x)$  are given in terms of the regular part (see Eq. (5.89)) of the Altarelli-Parisi probabilities as follows

$$\begin{aligned} \tilde{K}^{ab}(x) &= P_{\text{reg}}^{ab}(x) \ln(1-x) \\ &+ \delta^{ab} \mathbf{T}_a^2 \left[ \left( \frac{2}{1-x} \ln(1-x) \right)_+ - \frac{\pi^2}{3} \delta(1-x) \right] . \end{aligned} \quad (5.156)$$

## 5.6 Factorization formulae with many identified partons

In the most general case one deals with QCD cross sections involving initial-state partons and many identified partons in the final state. Here the NLO matrix element has collinear singularities when  $p_i \cdot p_j \rightarrow 0$  ( $i$  and  $j$  being unidentified final-state partons) and when  $p_i \cdot p_a \rightarrow 0$  ( $a$  being either an initial-state parton or an identified final-state parton). The former singularities can be factorized in terms of the dipoles  $\mathcal{D}_{ij,k}$ ,  $\mathcal{D}_{ij}^a$ ,  $\mathcal{D}_{ij,a}$  in Eqs. (5.2,5.36,5.94). As for the latter singularities, one should consider two different possibilities. If the spectator is an unidentified final-state parton  $k$ , one can factorize in terms of the dipoles  $\mathcal{D}_k^{ai}$ ,  $\mathcal{D}_{ai,k}$  in Eqs. (5.61,5.114). If the spectator  $b$  is an initial-state parton or an identified final-state parton, it is convenient to introduce new dipoles  $\mathcal{D}_{ai,b}^{(n)}$ ,  $\mathcal{D}_{ai}^{(n)b}$ ,  $\mathcal{D}_b^{(n)ai}$ ,  $\mathcal{D}^{(n)ai,b}$  in which the momentum of the spectator is left *unchanged*. Actually, with respect to the momentum dependence, these objects are ‘pseudo-dipoles’ rather than dipoles, in the sense that they depend on the momentum  $\tilde{p}_{ai}$  of the emitter, on the momentum  $p_b$  of the spectator and on an additional momentum  $n$ . This momentum  $n$  is:

$$n^\mu = p_{\text{in}}^\mu - \sum_{a \in \text{final state}} p_a^\mu, \quad (5.157)$$

where  $p_{\text{in}}^\mu$  is the total incoming momentum and the second term on the right-hand side is the sum of all the momenta of the identified partons in the final state<sup>‡</sup>. Note that, by momentum conservation,  $n^\mu$  is equal to the sum of the momenta of the final-state unidentified particles (QCD partons or not). Therefore the momentum  $n^\mu$  is time-like (and with positive definite energy). Furthermore, since we only consider non-trivial quantities in which the lowest order process has at least one unidentified parton, it cannot be light-like.

As we shall see, the dipole  $\mathcal{D}^{ai,b}$  considered in the previous Subsection corresponds to  $\mathcal{D}^{(n)ai,b}$  for the particular case with no identified partons in the final-state (i.e. when  $n = p_a + p_b$ ).

Let us first consider the singularities  $p_i \cdot p_a \rightarrow 0$  when  $a$  is an initial-state parton. In this case the factorization formula is:

$$\begin{aligned} & m_{+1,\dots,a,\dots} < 1, \dots, m+1, \dots; a.. || 1, \dots, m+1, \dots; a.. >_{m+1,\dots,a,\dots} \\ & = \sum_{k \neq i} \mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}, \dots; p_a, \dots) + \sum_{b \neq a} \mathcal{D}_b^{(n)ai}(p_1, \dots, p_{m+1}, \dots; p_a, \dots) + \dots \quad (5.158) \end{aligned}$$

The first term on the right-hand side of Eq. (5.158) is the same as that in Eq. (5.135).

The new dipole contributions are given by the second term on the right-hand side of Eq. (5.158). Note that the sum over the spectators  $b$  refers to all initial-state partons as well as to all identified partons in the final state and no distinction is made between these two cases. The explicit expression for these dipole terms is:

$$\begin{aligned} \mathcal{D}_b^{(n)ai}(p_1, \dots, p_{m+1}, \dots; p_a, \dots) &= -\frac{1}{2p_a \cdot p_i} \frac{1}{x_{ain}} \\ & m_{\dots,a,\dots} < \tilde{1}, \dots, \widetilde{m+1}, \dots; \widetilde{ai}.. | \frac{\mathbf{T}_b \cdot \mathbf{T}_{ai}}{\mathbf{T}_{ai}^2} \mathbf{V}_b^{(n)ai} | \tilde{1}, \dots, \widetilde{m+1}, \dots; \widetilde{ai}.. >_{m_{\dots,a,\dots}} \quad (5.159) \end{aligned}$$

<sup>‡</sup>Note that, in general, one can change  $p_{\text{in}}^\mu$  by adding some momentum transfer that does not involve QCD partons. For instance, in deep-inelastic lepton-hadron collisions  $l(k) + h(p) \rightarrow l'(k') + \dots$  one can replace  $p_{\text{in}}^\mu = k^\mu + p^\mu$  with  $p_{\text{in}}^\mu = Q^\mu + p^\mu$  where  $Q_\mu = k_\mu - k'_\mu$ .

The assignment of momenta in the matrix element  $|\widetilde{1}, \dots, \widetilde{m+1}, \dots; \widetilde{a}\widetilde{i} \dots \rangle_{m\dots a\dots}$  is the following. The momentum of the emitter is parallel to  $p_a$ :

$$\widetilde{p}_{ai}^\mu = x_{ain} p_a^\mu, \quad (5.160)$$

$$x_{ain} = \frac{(p_a - p_i) \cdot n}{p_a \cdot n}, \quad (5.161)$$

and all other initial-state and identified final-state momenta are left *unchanged*. All other final-state momenta  $k_j$  (QCD partons or not) are transformed according to the following Lorentz transformation

$$\widetilde{k}_j^\mu = \Lambda^\mu{}_\nu(K, \widetilde{K}) k_j^\nu, \quad (5.162)$$

$$\Lambda^\mu{}_\nu(K, \widetilde{K}) = g^\mu{}_\nu - \frac{2(K + \widetilde{K})^\mu (K + \widetilde{K})_\nu}{(K + \widetilde{K})^2} + \frac{2\widetilde{K}^\mu K_\nu}{K^2}, \quad (5.163)$$

where the momenta  $K^\mu$  and  $\widetilde{K}^\mu$  are defined by

$$\begin{aligned} K^\mu &= n^\mu - p_i^\mu, \\ \widetilde{K}^\mu &= n^\mu - (1 - x_{ain})p_a^\mu. \end{aligned} \quad (5.164)$$

Note that Eq. (5.162) actually defines a proper Lorentz transformation.

The splitting functions  $\mathbf{V}_b^{(n)ai}$  in Eq. (5.158) are as follows

$$\langle s | \mathbf{V}_b^{(n)q_a g_i}(x_{ain}; v_{i,ab}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S C_F \delta_{ss'} [2v_{i,ab} - (1 + x_{ain}) - \epsilon(1 - x_{ain})], \quad (5.165)$$

$$\langle s | \mathbf{V}_b^{(n)g_a \bar{q}_i}(x_{ain}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S T_R [1 - \epsilon - 2x_{ain}(1 - x_{ain})] \delta_{ss'}, \quad (5.166)$$

$$\begin{aligned} \langle \mu | \mathbf{V}_b^{(n)q_a g_i}(x_{ain}) | \nu \rangle &= 8\pi\mu^{2\epsilon} \alpha_S C_F \left[ -g^{\mu\nu} x_{ain} + \frac{1 - x_{ain}}{x_{ain}} \right. \\ &\cdot \left. \frac{4p_i \cdot p_a}{2(p_a \cdot n)(p_i \cdot n) - n^2 p_i \cdot p_a} \left( \frac{np_a}{p_i p_a} p_i^\mu - n^\mu \right) \left( \frac{np_a}{p_i p_a} p_i^\nu - n^\nu \right) \right], \end{aligned} \quad (5.167)$$

$$\begin{aligned} \langle \mu | \mathbf{V}_b^{(n)g_a g_i}(x_{ain}; v_{i,ab}) | \nu \rangle &= 16\pi\mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( v_{i,ab} - 1 + x_{ain}(1 - x_{ain}) \right) \right. \\ &+ \left. (1 - \epsilon) \frac{1 - x_{ain}}{x_{ain}} \frac{2p_i \cdot p_a}{2(p_a \cdot n)(p_i \cdot n) - n^2 p_i \cdot p_a} \left( \frac{np_a}{p_i p_a} p_i^\mu - n^\mu \right) \left( \frac{np_a}{p_i p_a} p_i^\nu - n^\nu \right) \right], \end{aligned} \quad (5.168)$$

where we have defined

$$v_{i,ab} = \frac{p_a p_b}{p_i(p_a + p_b)}. \quad (5.169)$$

Since all the momenta  $\widetilde{k}_j$  are obtained by means of the Lorentz transformation (5.162), the ‘pseudo-dipole’ phase space has a trivial convolution structure (note that in the physical region of interest,  $(\sum_j k_j)^2 - Q^2 \geq 0$ ), namely

$$d\phi(p_i, k_1, \dots; p_a + Q) = \int_0^1 dx d\phi(\widetilde{k}_1, \dots; xp_a + Q) [dp_i(n = Q + p_a, p_a, x)], \quad (5.170)$$

where

$$[dp_i(n, p_a, x)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \Theta(x) \Theta(1-x) \delta(x - x_{ain}) . \quad (5.171)$$

The phase space in Eq. (5.171) can explicitly be written in terms of the kinematic variables  $x_{ain}$  in Eq. (5.161) and  $\tilde{v}_i = p_a p_i / p_a n$  :

$$[dp_i(n, p_a, x)] = \frac{(2p_a n)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} d\tilde{v}_i dx_{ain} \Theta(x(1-x)) \Theta(\tilde{v}_i) \Theta\left(1 - \frac{n^2 \tilde{v}_i}{2(1-x)p_a n}\right) \cdot (1-x)^{-2\epsilon} \delta(x - x_{ain}) \left[ \frac{\tilde{v}_i}{1-x} \left(1 - \frac{n^2 \tilde{v}_i}{2(1-x)p_a n}\right) \right]^{-\epsilon} , \quad (5.172)$$

where  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to the light-like momenta  $\bar{n}^\mu = n^\mu - n^2 p_a^\mu / (2p_a n)$  and  $p_a^\mu$ .

Performing the integration of the splitting kernels  $\mathbf{V}_b^{(n)ai}$  over the phase space in Eq. (5.172), we introduce the functions  $\tilde{\mathcal{V}}^{a,ai}(x; \epsilon; p_a, p_b, n)$  :

$$\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_a p_b} \right)^\epsilon \tilde{\mathcal{V}}^{a,ai}(x; \epsilon; p_a, p_b, n) \equiv \int [dp_i(n, p_a, x)] \frac{1}{2p_a p_i} \frac{n_s(\tilde{ai})}{n_s(a)} \cdot < \mathbf{V}_b^{(n)ai}(x_{ain}; v_{i,ab}) > . \quad (5.173)$$

The spin averages of the kernels  $\mathbf{V}_b^{(n)ai}$  are related in a simple way to the corresponding averages of the ( $d$ -dimensional) Altarelli-Parisi splitting functions, that is,

$$\frac{n_s(\tilde{ai})}{n_s(a)} \frac{< \mathbf{V}_b^{(n)ai}(x; v_{i,ab}) >}{8\pi\alpha_S\mu^{2\epsilon}} = < \hat{P}_{a,\tilde{ai}}(x; \epsilon) > + 2 \delta^{a,\tilde{ai}} \mathbf{T}_a^2 \left[ v_{i,ab} - \frac{1}{1-x} \right] . \quad (5.174)$$

Therefore, the only non-trivial integration in Eq. (5.173) is that which involves the term  $v_{i,ab}$  (see Appendix B) and leads to the following result

$$\tilde{\mathcal{V}}^{a,b}(x; \epsilon; p_a, p_b, n) = \tilde{\mathcal{V}}^{a,b}(x; \epsilon) + \mathcal{L}^{a,b}(x; p_a, p_b, n) + \mathcal{O}(\epsilon) . \quad (5.175)$$

The first term on the right-hand side is given in Eq. (5.155) and the second term is defined by

$$\mathcal{L}^{a,b}(x; p_a, p_b, n) = \delta^{ab} \delta(1-x) 2 \mathbf{T}_a^2 \left[ \text{Li}_2 \left( 1 - \frac{(1+v)(p_a + p_b) \cdot n}{2 p_a \cdot n} \right) + \text{Li}_2 \left( 1 - \frac{(1-v)(p_a + p_b) \cdot n}{2 p_a \cdot n} \right) \right] - P_{\text{reg}}^{ab}(x) \ln \frac{n^2(p_a \cdot p_b)}{2(p_a \cdot n)^2} , \quad (5.176)$$

$$v = \sqrt{1 - \frac{n^2(p_a + p_b)^2}{[(p_a + p_b) \cdot n]^2}} , \quad (5.177)$$

$P_{\text{reg}}^{ab}(x)$  being the regular part of the Altarelli-Parisi probabilities in Eq. (5.89) and  $\text{Li}_2(x)$  is the dilogarithm function:

$$\text{Li}_2(x) = - \int_0^x \frac{dz}{z} \ln(1-z) . \quad (5.178)$$

Note that unlike the  $\mathcal{V}$ -functions considered in the previous Subsections, Eq. (5.175) depends not only on the momentum fraction  $x$ , but also on the momenta  $p_a, p_b, n$ . This momentum dependence is entirely accounted for by the  $\mathcal{L}^{a,b}$  function on the right-hand side of Eq. (5.175). In the case with no final-state identified partons,  $p_b$  is the momentum of an incoming parton,  $n = p_a + p_b$ , so  $\mathcal{L}^{a,b}$  vanishes, thus recovering the results already discussed in Sect. 5.5.

In order to deal with the singularities for  $p_i \cdot p_a \rightarrow 0$ , when  $a$  is an identified parton in the final state, we introduce the following factorization formula

$$\begin{aligned} & \langle m+1a, \dots | \dots, m+1, a, \dots | \dots \rangle_{m+1a, \dots} \\ &= \sum_{k \neq i} \mathcal{D}_{ai,k}(p_1, \dots, p_{m+1}, p_a, \dots) + \sum_{b \neq a} \mathcal{D}_{ai,b}^{(n)}(p_1, \dots, p_{m+1}, p_a, \dots) + \dots \quad , \quad (5.179) \end{aligned}$$

where  $\mathcal{D}_{ai,k}$  is the dipole in Eq. (5.114) and the new dipole contribution is given by

$$\begin{aligned} \mathcal{D}_{ai,b}^{(n)}(p_1, \dots, p_{m+1}, p_a, \dots) &= -\frac{1}{2p_a \cdot p_i} \\ & \cdot \langle m+1, \dots, m+1, \widetilde{ai}, \dots | \dots \rangle_{m+1, \dots} \frac{\mathbf{T}_b \cdot \mathbf{T}_{ai}}{\mathbf{T}_{ai}^2} \mathbf{V}_{ai,b}^{(n)} | \widetilde{1}, \dots, m+1, \widetilde{ai}, \dots \rangle_{m+1, \dots} \quad . \quad (5.180) \end{aligned}$$

As in Eq. (5.158), the sum over the spectators  $b$  in Eq. (5.179) refers to all initial-state partons as well as to all identified partons in the final state with no distinction between these two cases.

The assignment of momenta in the matrix element  $|\widetilde{1}, \dots, m+1, \widetilde{ai}, \dots \rangle_{m+1, \dots}$  is the following. The momentum of the emitter is parallel to  $p_a$ :

$$\widetilde{p}_{ai}^\mu = \frac{1}{z_{ain}} p_a^\mu \quad , \quad (5.181)$$

$$z_{ain} = \frac{p_a \cdot n}{(p_a + p_i) \cdot n} \quad , \quad (5.182)$$

and all other initial-state and identified final-state momenta are left unchanged. All other final-state momenta  $k_j$  (QCD partons or not) are transformed according to the Lorentz transformation in Eqs. (5.162–5.164).

The splitting functions  $\mathbf{V}_{ai,b}^{(n)}$  in Eq. (5.179) are given in terms of the variable  $v_{i,ab}$  in Eq. (5.169) as follows

$$\langle s | \mathbf{V}_{qa\bar{q}_i,b}^{(n)}(z_{ain}; v_{i,ab}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S C_F \delta_{ss'} \left[ 2 \frac{v_{i,ab}}{z_{ain}} - (1 + z_{ain}) - \epsilon (1 - z_{ain}) \right] \quad , \quad (5.183)$$

$$\langle s | \mathbf{V}_{ga\bar{q}_i,b}^{(n)}(z_{ain}) | s' \rangle = 8\pi\mu^{2\epsilon} \alpha_S C_F \left[ \frac{1 + (1 - z_{ain})^2}{z_{ain}} - \epsilon z_{ain} \right] \delta_{ss'} \quad , \quad (5.184)$$

$$\begin{aligned} \langle \mu | \mathbf{V}_{qa\bar{q}_i,b}^{(n)}(z_{ain}) | \nu \rangle &= 8\pi\mu^{2\epsilon} \alpha_S T_R \left[ -g^{\mu\nu} - 4z_{ain}(1 - z_{ain}) \right. \\ & \cdot \left. \frac{p_i \cdot p_a}{2(p_a \cdot n)(p_i \cdot n) - n^2} \frac{p_i \cdot p_a}{p_i \cdot p_a} \left( \frac{np_a}{p_i p_a} p_i^\mu - n^\mu \right) \left( \frac{np_a}{p_i p_a} p_i^\nu - n^\nu \right) \right] \quad , \quad (5.185) \end{aligned}$$

$$\begin{aligned}
\langle \mu | \mathbf{V}_{g_a g_i, b}^{(n)}(z_{ain}; v_{i,ab}) | \nu \rangle &= 16\pi \mu^{2\epsilon} \alpha_S C_A \left[ -g^{\mu\nu} \left( \frac{v_{i,ab}}{z_{ain}} - 1 + \frac{1 - z_{ain}}{z_{ain}} \right) \right. \\
&+ \left. 2(1 - \epsilon) z_{ain} (1 - z_{ain}) \frac{p_i \cdot p_a}{2(p_a \cdot n)(p_i \cdot n) - n^2 p_i \cdot p_a} \left( \frac{np_a}{p_i p_a} p_i^\mu - n^\mu \right) \left( \frac{np_a}{p_i p_a} p_i^\nu - n^\nu \right) \right] .
\end{aligned} \tag{5.186}$$

The phase space for the dipole (5.180) has the following convolution form

$$d\phi(p_i, k_1, \dots; Q - p_a) = \int_0^1 \frac{dz}{z^{2-2\epsilon}} d\phi(\tilde{k}_1, \dots; Q - p_a/z) [dp_i(n = Q - p_a; p_a, z)] , \tag{5.187}$$

where

$$[dp_i(n; p_a, z)] = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \Theta(z) \Theta(1 - z) \delta(z - z_{ain}) z^{2-2\epsilon} . \tag{5.188}$$

Introducing the kinematic variables  $z_{ain}$  in Eq. (5.182) and  $\tilde{v}_i = p_a p_i / p_a n$ , Eq. (5.188) can be written as follows

$$\begin{aligned}
[dp_i(n; p_a, z)] &= \frac{(2p_a n)^{1-\epsilon}}{16\pi^2} \frac{d\Omega^{(d-3)}}{(2\pi)^{1-2\epsilon}} d\tilde{v}_i dz_{ain} \Theta(z(1 - z)) \Theta(\tilde{v}_i) \Theta\left(1 - \frac{n^2 \tilde{v}_i z}{2(1 - z)p_a n}\right) \\
&\cdot (1 - z)^{-2\epsilon} \delta(z - z_{ain}) \left[ \frac{\tilde{v}_i z}{1 - z} \left(1 - \frac{n^2 \tilde{v}_i z}{2(1 - z)p_a n}\right) \right]^{-\epsilon} ,
\end{aligned} \tag{5.189}$$

where  $d\Omega^{(d-3)}$  is an element of solid angle perpendicular to the light-like momenta  $\bar{n}^\mu = n^\mu - n^2 p_a^\mu / (2p_a n)$  and  $p_a^\mu$ .

Thus, the integration of the splitting kernels  $\mathbf{V}_{ai,b}^{(n)}$  defines the functions  $\bar{\mathcal{V}}_{ai,a}(z; \epsilon; p_a, p_b, n)$ :

$$\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{2p_a p_b} \right)^\epsilon \bar{\mathcal{V}}_{ai,a}(z; \epsilon; p_a, p_b, n) \equiv \int [dp_i(n; p_a, z)] \frac{1}{2p_a p_i} \langle \mathbf{V}_{ai,b}^{(n)}(z_{ain}; v_{i,ab}) \rangle . \tag{5.190}$$

As in the case of initial-state singularities, the spin averages  $\langle \mathbf{V}_{ai,b}^{(n)} \rangle$  are simply related to the ( $d$ -dimensional) Altarelli-Parisi splitting functions:

$$\frac{\langle \mathbf{V}_{ai,b}^{(n)}(z; v_{i,ab}) \rangle}{8\pi\alpha_S\mu^{2\epsilon}} = \langle \hat{P}_{ai,a}^\sim(z; \epsilon) \rangle + 2 \delta^{\tilde{a}i,a} \mathbf{T}_a^2 \left[ \frac{v_{i,ab}}{z} - \frac{1}{1 - z} \right] , \tag{5.191}$$

and, explicitly performing the integration of the  $v_{i,ab}$  contributions in Eq. (5.191) (see Appendix B), we find that the functions  $\bar{\mathcal{V}}_{a,b}$  are equal to the analogous functions  $\tilde{\mathcal{V}}^{a,b}$  in Eq. (5.175):

$$\bar{\mathcal{V}}_{a,b}(z; \epsilon; p_a, p_b, n) = \tilde{\mathcal{V}}^{a,b}(z; \epsilon; p_a, p_b, n) . \tag{5.192}$$



## 6 QCD cross sections at NLO

In the following Sections we describe in detail our subtraction method for evaluating QCD cross sections. To this end, it is useful to recall the general and precise definitions of the NLO cross sections.

In the case of processes with no initial-state hadrons, for instance in  $e^+e^-$  annihilation, the partonic cross section is<sup>§</sup>

$$\sigma = \sigma^{LO} + \sigma^{NLO} , \quad (6.1)$$

$$\sigma^{LO} = \int_m d\sigma^B , \quad \sigma^{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V , \quad (6.2)$$

where  $d\sigma^B, d\sigma^R, d\sigma^V$  are the cross sections in the Born approximation and to one-loop order ( $R$ : real emission;  $V$ : virtual correction). If  $\sigma$  is a jet cross section (no final-state hadrons observed), hadron-level and parton-level cross sections coincide.

In the case of processes with one initial-state hadron carrying momentum  $p^\mu$  (for instance, deep inelastic lepton-hadron scattering), the calculation of the QCD cross section requires the introduction of parton distributions. If we denote by  $f_a(\eta, \mu_F^2)$  the density of partons of type  $a$  in the incoming hadron, the hadronic cross section is given by

$$\sigma(p) = \sum_a \int_0^1 d\eta f_a(\eta, \mu_F^2) \left[ \sigma_a^{LO}(\eta p) + \sigma_a^{NLO}(\eta p; \mu_F^2) \right] , \quad (6.3)$$

and the corresponding parton-level cross sections are:

$$\sigma_a^{LO}(p) = \int_m d\sigma_a^B(p) , \quad (6.4)$$

$$\sigma_a^{NLO}(p; \mu_F^2) = \int_{m+1} d\sigma_a^R(p) + \int_m d\sigma_a^V(p) + \int_m d\sigma_a^C(p; \mu_F^2) . \quad (6.5)$$

The notation  $B, R, V$  is as in Eq. (6.2). The contribution  $d\sigma_a^C$  represents the collinear-subtraction counterterm and is explicitly given by the following expression

$$d\sigma_a^C(p; \mu_F^2) = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_b \int_0^1 dz \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{ab}(z) + K_{\text{F.S.}}^{ab}(z) \right] d\sigma_b^B(zp) . \quad (6.6)$$

The partonic contributions on the right-hand side of Eq. (6.5) are separately divergent for  $\epsilon \rightarrow 0$ . Their sum  $\sigma_a^{NLO}$  is finite for  $\epsilon \rightarrow 0$  but depends on the *factorization scale* and on the *factorization scheme* of collinear singularities. Both dependences are contained in the definition of  $d\sigma_a^C$ :  $\mu_F$  is the factorization scale and the actual form of the kernel  $K_{\text{F.S.}}^{ab}(z)$  specifies the factorization scheme. Setting  $K_{\text{F.S.}}^{ab}(z) = 0$  defines the  $\overline{\text{MS}}$  subtraction scheme. The functions  $P^{ab}(z)$  in Eq. (6.6) are the *four dimensional* Altarelli Parisi probabilities in Eqs. (5.85–5.88). The parton densities  $f_a(\eta, \mu_F^2)$  are also scale/scheme dependent, so that this dependence cancels in the hadronic cross section of Eq. (6.3).

Note that

$$\sum_b \int_0^1 dx x P^{ab}(x) = 1 , \quad (6.7)$$

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<sup>§</sup>We are using the same notation as in Sect. 2. Thus,  $m$  is the number of unobserved final-state partons for the leading-order process.

and that, in the  $\overline{\text{MS}}$  scheme, momentum conservation reads as follows

$$\sum_a \int_0^1 dx x f_a(x, \mu_F^2) = 1 \quad . \quad (6.8)$$

In other factorization schemes the generalization of Eq. (6.8) is:

$$\sum_{a,b} \int_0^1 dx x f_b(x, \mu_F^2) \left[ \delta^{ba} - \frac{\alpha_S}{2\pi} \int_0^1 dz z K_{\text{F.S.}}^{ba}(z) \right] = 1 \quad . \quad (6.9)$$

The extension of Eq. (6.3) to processes with two initial-state hadrons is straightforward. Denoting by  $f_a$  and  $\bar{f}_b$  the parton densities of the two incoming hadrons, we have

$$\sigma(p, \bar{p}) = \sum_{a,b} \int_0^1 d\eta f_a(\eta, \mu_F^2) \int_0^1 d\bar{\eta} \bar{f}_b(\bar{\eta}, \mu_F^2) \left[ \sigma_{ab}^{LO}(\eta p, \bar{\eta} \bar{p}) + \sigma_{ab}^{NLO}(\eta p, \bar{\eta} \bar{p}; \mu_F^2) \right] \quad , \quad (6.10)$$

$$\sigma_{ab}^{LO}(p, \bar{p}) = \int_m d\sigma_{ab}^B(p, \bar{p}) \quad , \quad (6.11)$$

$$\sigma_{ab}^{NLO}(p, \bar{p}; \mu_F^2) = \int_{m+1} d\sigma_{ab}^R(p, \bar{p}) + \int_m d\sigma_{ab}^V(p, \bar{p}) + \int_m d\sigma_{ab}^C(p, \bar{p}; \mu_F^2) \quad , \quad (6.12)$$

where the collinear counterterm is:

$$\begin{aligned} d\sigma_{ab}^C(p, \bar{p}; \mu_F^2) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_{cd} \int_0^1 dz \int_0^1 d\bar{z} d\sigma_{cd}^B(zp, \bar{z}\bar{p}) \\ &\cdot \left\{ \delta_{bd} \delta(1-\bar{z}) \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{ac}(z) + K_{\text{F.S.}}^{ac}(z) \right] \right. \\ &\left. + \delta_{ac} \delta(1-z) \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{bd}(\bar{z}) + K_{\text{F.S.}}^{bd}(\bar{z}) \right] \right\} \quad . \quad (6.13) \end{aligned}$$

It is completely trivial to generalize the resulting formulae to the case in which one introduces different factorization scales for the two hadrons, as one might in photoproduction for example. The replacement  $\bar{f}_b(\bar{\eta}, \mu_F^2) \rightarrow \bar{f}_b(\bar{\eta}, \bar{\mu}_F^2)$  in Eq. (6.10) is simply accompanied by  $\mu_F \rightarrow \bar{\mu}_F$  in the second term in the curly bracket on the right-hand side of Eq. (6.13).

Let us now consider fragmentation processes. The one-hadron inclusive cross section  $\sigma_{(incl)}(p)$  in the case with no initial-state hadrons is:

$$\sigma_{(incl)}(p) = \sum_a \int_0^1 \frac{d\eta}{\eta^2} d^a(\eta, \mu_F^2) \left[ \sigma_{(incl)a}^{LO}(p/\eta) + \sigma_{(incl)a}^{NLO}(p/\eta; \mu_F^2) \right] \quad , \quad (6.14)$$

where  $d^a(\eta, \mu_F^2)$  is the fragmentation function of the parton  $a$  into the observed hadron and the partonic cross sections are:

$$\sigma_{(incl)a}^{LO}(p) = \int_m d\sigma_{(incl)a}^B(p) \quad , \quad (6.15)$$

$$\sigma_{(incl)a}^{NLO}(p) = \int_{m+1} d\sigma_{(incl)a}^R(p) + \int_m d\sigma_{(incl)a}^V(p) + \int_m d\sigma_{(incl)a}^C(p; \mu_F^2) \quad , \quad (6.16)$$

$$d\sigma_{(incl)a}^C(p; \mu_F^2) = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_b \int_0^1 \frac{dz}{z^{2-2\epsilon}} \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P_{ba}(z) + H_{ba}^{\text{FS}}(z) \right] d\sigma_{(incl)b}^B(p/z), \quad (6.17)$$

where the Altarelli-Parisi probability  $P_{ab}(z)$  describes the time-like splitting process  $a(p) \rightarrow b(zp)$ . Note that  $P_{ab}(z) = P^{ab}(z)$ . The kernel  $H_{ba}^{\text{FS}}(z)$  in Eq. (6.17) defines the factorization scheme ( $H_{ba}^{\text{FS}}(z) = 0$  in the  $\overline{\text{MS}}$  subtraction scheme).

Note that the one-particle inclusive cross section  $\sigma_{(incl)}(p)$  in Eq. (6.14) is defined without integrating over any component of the momentum  $p^\mu$  of the observed hadron (the unusual convolution measures  $d\eta/\eta^2$  and  $dz/z^{2-2\epsilon}$  in Eqs. (6.14) and (6.17) follow from that). Thus the following integral

$$\int \frac{d^4p}{(2\pi)^3} \delta_+(p^2) \sigma_{(incl)}(p) \quad (6.18)$$

is equal to the associated multiplicity times the total cross section. The corresponding associated multiplicities at partonic level (i.e. in  $d$  dimensions) are ( $I = B, R, V, C, \dots$ ):

$$\int \frac{d^d p}{(2\pi)^{d-1}} \delta_+(p^2) d\sigma_{(incl)a}^I(p) \quad (6.19)$$

In the most general case, one should consider multi-particle correlations, that is, one deals with processes of the type

$$p + \bar{p} \rightarrow q_1 + \dots + q_n + X, \quad (6.20)$$

where  $p, \bar{p}$  are the momenta of two incoming hadrons,  $q_1, \dots, q_n$  are the momenta of  $n$  hadrons detected in the final state and  $X$  stands for unobserved final-state particles or jets. Note that, by definition, the momenta  $q_1, \dots, q_n$  are supposed not to be parallel to each other or to the incoming momenta  $p$  and  $\bar{p}$ . The hadronic cross section is:

$$\begin{aligned} \sigma_{(incl)}(p, \bar{p}; q_1, \dots, q_n) &= \sum_{a,b} \int_0^1 d\eta f_a(\eta, \mu_F^2) \int_0^1 d\bar{\eta} \bar{f}_b(\bar{\eta}, \mu_F^2) \\ &\cdot \sum_{a_1, \dots, a_n} \int_0^1 \frac{d\eta_1}{\eta_1^2} \dots \int_0^1 \frac{d\eta_n}{\eta_n^2} d^{a_1}(\eta_1, \mu_1^2) \dots d^{a_n}(\eta_n, \mu_n^2) \\ &\cdot \left[ \sigma_{ab, (incl) a_1, \dots, a_n}^{LO}(\eta p, \bar{\eta} \bar{p}; q_1/\eta_1, \dots, q_n/\eta_n) \right. \\ &\left. + \sigma_{ab, (incl) a_1, \dots, a_n}^{NLO}(\eta p, \bar{\eta} \bar{p}; q_1/\eta_1, \dots, q_n/\eta_n; \mu_F^2, \mu_1^2, \dots, \mu_n^2) \right], \quad (6.21) \end{aligned}$$

where we have introduced a single factorization scale  $\mu_F^2$  for the parton densities and  $n$  different factorization scales  $\mu_1^2, \dots, \mu_n^2$  for the fragmentation functions.

The corresponding cross sections at parton level are the following

$$\sigma_{ab, (incl) a_1, \dots, a_n}^{LO}(p, \bar{p}; q_1, \dots, q_n) = \int_m d\sigma_{ab, (incl) a_1, \dots, a_n}^B(p, \bar{p}; q_1, \dots, q_n), \quad (6.22)$$

$$\begin{aligned} \sigma_{ab, (incl) a_1, \dots, a_n}^{NLO}(p, \bar{p}; q_1, \dots, q_n; \mu_F^2, \mu_1^2, \dots, \mu_n^2) &= \int_{m+1} d\sigma_{ab, (incl) a_1, \dots, a_n}^R(p, \bar{p}; q_1, \dots, q_n) \quad (6.23) \\ &+ \int_m d\sigma_{ab, (incl) a_1, \dots, a_n}^V(p, \bar{p}; q_1, \dots, q_n) + \int_m d\sigma_{ab, (incl) a_1, \dots, a_n}^C(p, \bar{p}; q_1, \dots, q_n; \mu_F^2, \mu_1^2, \dots, \mu_n^2), \end{aligned}$$

where the collinear counterterm  $d\sigma^C$  is given by

$$\begin{aligned}
d\sigma_{ab,(incl) a_1,\dots,a_n}^C(p, \bar{p}; q_1, \dots, q_n; \mu_F^2, \mu_1^2, \dots, \mu_n^2) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \\
&\cdot \left\{ \sum_{a'} \int_0^1 dz \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{aa'}(z) + K_{\text{F.S.}}^{aa'}(z) \right] d\sigma_{a'b,(incl) a_1,\dots,a_n}^B(zp, \bar{p}; q_1, \dots, q_n) \right. \\
&+ \sum_{b'} \int_0^1 dz \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{bb'}(z) + K_{\text{F.S.}}^{bb'}(z) \right] d\sigma_{ab',(incl) a_1,\dots,a_n}^B(p, z\bar{p}; q_1, \dots, q_n) \\
&+ \sum_{i=1}^n \sum_{a'_i} \int_0^1 \frac{dz}{z^{2-2\epsilon}} d\sigma_{ab,(incl) a_1,\dots,a'_i,\dots,a_n}^B(p, \bar{p}; q_1, \dots, q_i/z, \dots, q_n) \\
&\cdot \left. \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_i^2} \right)^\epsilon P_{a'_i a_i}(z) + H_{a'_i a_i}^{\text{F.S.}}(z) \right] \right\} . \tag{6.24}
\end{aligned}$$

Note that, computing the hadronic cross sections in Eqs. (6.3,6.10,6.14,6.21), the factorization-scale evolution of the parton distribution functions has to be consistently carried out at NLO. For the parton densities  $f_a(\eta, \mu_F^2)$  and the fragmentation functions  $d^a(\eta, \mu_F^2)$  we have

$$\frac{d f_a(\eta, \mu_F^2)}{d \ln \mu_F^2} = \sum_b \int_\eta^1 \frac{dz}{z} f_b(\eta/z, \mu_F^2) \frac{\alpha_S(\mu_F^2)}{2\pi} \left[ P^{ba}(z) + \frac{\alpha_S(\mu_F^2)}{2\pi} P^{(1)ba}(z) \right] , \tag{6.25}$$

$$\frac{d d^a(\eta, \mu_F^2)}{d \ln \mu_F^2} = \sum_b \int_\eta^1 \frac{dz}{z} \frac{\alpha_S(\mu_F^2)}{2\pi} \left[ P_{ab}(z) + \frac{\alpha_S(\mu_F^2)}{2\pi} P_{ab}^{(1)}(z) \right] d^b(\eta/z, \mu_F^2) , \tag{6.26}$$

where  $P^{(1)ab}(x)$  and  $P_{ab}^{(1)}(x)$  respectively are the space-like and time-like NLO Altarelli-Parisi probabilities. They depend on the factorization scheme and this dependence is given in terms of the flavour kernels  $K_{\text{F.S.}}^{ab}$  and  $H_{ba}^{\text{F.S.}}$  as follows

$$P^{(1)ab}(x) = P_{\overline{\text{MS}}}^{(1)ab}(x) + \sum_c \int_x^1 \frac{dz}{z} \left[ P^{ac}(z) K_{\text{F.S.}}^{cb}(x/z) - K_{\text{F.S.}}^{ac}(x/z) P^{cb}(z) \right] - 2\pi\beta_0 K_{\text{F.S.}}^{ab}(x) , \tag{6.27}$$

$$P_{ab}^{(1)}(x) = P_{ab}^{(1)\overline{\text{MS}}}(x) + \sum_c \int_x^1 \frac{dz}{z} \left[ H_{ac}^{\text{F.S.}}(x/z) P_{cb}(z) - P_{ac}(z) H_{cb}^{\text{F.S.}}(x/z) \right] - 2\pi\beta_0 H_{ab}^{\text{F.S.}}(x) , \tag{6.28}$$

where  $P_{\overline{\text{MS}}}^{(1)ab}(x)$  and  $P_{ab}^{(1)\overline{\text{MS}}}(x)$  are the corresponding probabilities evaluated in the  $\overline{\text{MS}}$  subtraction scheme [26].

## 7 Jet cross sections with no initial-state hadrons

In processes with no initial-state hadrons, the QCD cross section for jet observables is given by Eqs. (6.1,6.2). In terms of the QCD matrix elements, the Born-level cross section in  $d$  dimensions is the following

$$d\sigma^B = \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; Q) \frac{1}{S_{\{m\}}} |\mathcal{M}_m(p_1, \dots, p_m)|^2 F_J^{(m)}(p_1, \dots, p_m) \ , \quad (7.1)$$

where  $\mathcal{N}_{in}$  includes all the factors that are QCD independent,  $\sum_{\{m\}}$  denotes the sum over all the configurations with  $m$  partons,  $d\phi_m$  is the partonic phase space in Eq. (3.1),  $S_{\{m\}}$  is the Bose symmetry factor for identical partons in the final state and  $\mathcal{M}_m$  is the tree-level matrix element.

The phase space function  $F_J^{(m)}(p_1, \dots, p_m)$  defines the jet observable in terms of the momenta of the  $m$  final-state partons. In general  $F_J$  may contain  $\theta$ -functions (thus, Eq. (7.1) defines precisely a cross section),  $\delta$ -functions (Eq. (7.1) defines a differential cross section), numerical and kinematic factors (Eq. (7.1) refers to an inclusive observable), or any combination of these. The essential property of  $F_J^{(m)}$  is that the jet observable we are interested in has to be infrared and collinear safe. From a formal viewpoint this implies that  $F_J$  fulfils the following properties

$$F_J^{(n+1)}(p_1, \dots, p_j = \lambda q, \dots, p_{n+1}) \rightarrow F_J^{(n)}(p_1, \dots, p_{n+1}) \quad \text{if } \lambda \rightarrow 0 \ , \quad (7.2)$$

$$F_J^{(n+1)}(p_1, \dots, p_i, \dots, p_j, \dots, p_{n+1}) \rightarrow F_J^{(n)}(p_1, \dots, p, \dots, p_{n+1}) \quad \text{if } p_i \rightarrow zp \ , \ p_j \rightarrow (1-z)p \quad (7.3)$$

for all  $n \geq m$ , and

$$F_J^{(m)}(p_1, \dots, p_m) \rightarrow 0 \quad \text{if } p_i \cdot p_j \rightarrow 0 \ . \quad (7.4)$$

Equations (7.2) and (7.3) respectively guarantee that the jet observable is infrared and collinear safe for *any* number  $n$  of final-state partons, i.e. to *any* order in QCD perturbation theory. The  $n$ -parton jet function  $F_J^{(n)}$  on the right-hand side of Eq. (7.2) is obtained from the original  $F_J^{(n+1)}$  by removing the soft parton  $p_j$ , and that on the right-hand side of Eq. (7.3) by replacing the collinear partons  $\{p_i, p_j\}$  by  $p = p_i + p_j$ .

Equation (7.4) defines the leading-order cross section, that is, it ensures that the Born-level cross section  $d\sigma^B$  in Eq. (7.1) is well-defined (i.e. finite after integration) in  $d = 4$  dimensions.

The cross section  $d\sigma^R$  has the same expression as  $d\sigma^B$ , apart from the replacement  $m \rightarrow m + 1$ .

### 7.1 Subtraction term

In order to compute  $\sigma^{NLO}$  we write the following identity

$$\sigma^{NLO} = \int_{m+1} (d\sigma^R - d\sigma^A) + \left[ \int_{m+1} d\sigma^A + \int_m d\sigma^V \right] \ , \quad (7.5)$$

where  $d\sigma^A$  is a local counterterm for  $d\sigma^R$ , i.e.  $d\sigma^A$  has the same (*unintegrated*) singular behaviour as  $d\sigma^R$ . An explicit and general form of  $d\sigma^A$  is provided by the dipole factorization formula introduced in Sect. 5.1. Thus we can define:

$$d\sigma^A = \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1, \dots, p_{m+1}; Q) \frac{1}{S_{\{m+1\}}} \cdot \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}) , \quad (7.6)$$

where  $\mathcal{D}_{ij,k}(p_1, \dots, p_{m+1})$  is the dipole contribution in Eq. (5.2), namely

$$\mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) = -\frac{1}{2p_i \cdot p_j} \cdot \quad m < 1, \dots, \tilde{i}_j, \dots, \tilde{k}, \dots, m+1 \mid \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} \mathbf{V}_{ij,k} \mid 1, \dots, \tilde{i}_j, \dots, \tilde{k}, \dots, m+1 >_m , \quad (7.7)$$

and  $F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1})$  is the jet function for the corresponding  $m$ -parton state  $\{p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}\}$ .

We can check that the definition (7.6) makes the difference ( $d\sigma^R - d\sigma^A$ ) integrable in  $d = 4$  dimensions. Indeed, its explicit expression is

$$d\sigma^R - d\sigma^A = \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1, \dots, p_{m+1}; Q) \frac{1}{S_{\{m+1\}}} \cdot \left\{ |\mathcal{M}_{m+1}(p_1, \dots, p_{m+1})|^2 F_J^{(m+1)}(p_1, \dots, p_{m+1}) - \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}) \right\} . \quad (7.8)$$

Each term in the curly bracket is separately singular in the soft and collinear regions. However, as discussed in Sect. 5.1, in each of these regions both the matrix element  $\mathcal{M}_{m+1}$  and the phase space for the  $m+1$ -parton configuration behave as the corresponding dipole contribution and dipole phase space:

$$|\mathcal{M}_{m+1}(p_1, \dots, p_{m+1})|^2 \rightarrow \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) , \quad (7.9)$$

$$\{p_1, \dots, p_i, \dots, p_j, \dots, p_k, \dots, p_{m+1}\} \rightarrow \{p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}\} . \quad (7.10)$$

Thus, because of Eqs. (7.2) and (7.3), the singularities of the first term in the curly bracket are cancelled by similar singularities due to the second term. On the other hand, each dipole  $\mathcal{D}_{ij,k}$  in Eq. (7.7) has no other singularities but those due to the  $m$ -parton matrix element  $|1, \dots, \tilde{i}_j, \dots, \tilde{k}, \dots, m+1 >_m$ . However, because of Eq. (7.4), these singularities are screened (regularized) by the jet function  $F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1})$  in the curly bracket of Eq. (7.8).

Note that this cancellation mechanism is completely independent of the actual form of the jet defining function but it is essential that  $d\sigma^R$  and  $d\sigma^A$  are proportional to  $F_J^{(m+1)}$  and  $F_J^{(m)}$  respectively. Nonetheless, both  $d\sigma^R$  and  $d\sigma^A$  live on the same  $m+1$ -parton phase space  $d\phi_{m+1}$ . Thus the numerical integration (in  $d = 4$  dimensions) of Eq. (7.8) via Monte

Carlo techniques is straightforward. One simply generates an  $m + 1$ -parton configuration and gives it a positive ( $+ |\mathcal{M}_{m+1}|^2$ ) or negative ( $-\sum_{k \neq i,j} \mathcal{D}_{ij,k}$ ) weight. The role of the two different jet functions  $F_J^{(m+1)}$  and  $F_J^{(m)}$  is that of binning these weighted events into different bins of the jet observable. Any time that the generated configuration approaches a singular region, these two bins coincide and the cancellation of the large positive and negative weights takes place.

## 7.2 Integral of the subtraction term

Having discussed the four-dimensional integrability of  $(d\sigma^R - d\sigma^A)$ , the only other step we have to consider is the  $d$ -dimensional analytical integrability of  $d\sigma^A$  over the one-parton subspace leading to soft and collinear divergences.

We start by noting that the dipole contribution in Eq. (7.7) can be written as follows

$$\mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) = - \left[ \frac{\mathbf{V}_{ij,k}}{2p_i \cdot p_j} \frac{1}{\mathbf{T}_{ij}^2} |\mathcal{M}_m^{ij,k}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1})|^2 \right]_h \quad (7.11)$$

where  $\mathcal{M}_m^{ij,k}$  is a colour-correlated  $m$ -parton amplitude (see Eq. (3.9)) depending only on  $p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}$  while  $\mathbf{V}_{ij,k}/p_i \cdot p_j$  depends only on  $p_i, p_j, p_k$  or, equivalently,  $p_i, \tilde{p}_{ij}, \tilde{p}_k$  (the subscript  $h$  in the square bracket of Eq. (7.11) means that  $\mathbf{V}_{ij,k}$  and  $\mathcal{M}_m^{ij,k}$  are still coupled in helicity space). Using the phase space factorization property in Eq. (5.17), we can thus completely factorize the  $p_i$  dependence in the following form

$$\begin{aligned} \int_{m+1} d\sigma^A &= - \int_m \mathcal{N}_{in} \sum_{\{m+1\}} \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} d\phi_m(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}; Q) \frac{1}{S_{\{m+1\}}} \\ &\cdot F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}) \\ &\cdot \left[ \frac{1}{\mathbf{T}_{ij}^2} |\mathcal{M}_m^{ij,k}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1})|^2 \int_1 \frac{\mathbf{V}_{ij,k}}{2p_i \cdot p_j} \right]_h [dp_i(\tilde{p}_{ij}, \tilde{p}_k)] \quad , \end{aligned} \quad (7.12)$$

and we can perform the integration over  $p_i$ . In particular, according to the discussion in Sect. 5.1, the azimuthal correlation between  $\mathbf{V}_{ij,k}$  and  $|\mathcal{M}_m^{ij,k}|^2$  vanishes after integration over  $[dp_i(\tilde{p}_{ij}, \tilde{p}_k)]$  and we get (see Eq. (5.27))

$$\begin{aligned} \int_{m+1} d\sigma^A &= - \int_m \mathcal{N}_{in} \sum_{\{m+1\}} \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} d\phi_m(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}; Q) \frac{1}{S_{\{m+1\}}} \\ &\cdot F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}) |\mathcal{M}_m^{ij,k}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1})|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_{ij}\tilde{p}_k} \right)^\epsilon \frac{1}{\mathbf{T}_{ij}^2} \mathcal{V}_{ij}(\epsilon) \quad . \end{aligned} \quad (7.13)$$

In order to rewrite Eq. (7.13) in terms of an  $m$ -parton contribution times a factor, we have to perform the counting of the symmetry factors for going from  $m$  partons to  $m + 1$  partons.

Consider a Born-level  $m$ -parton configuration with  $m_f$  quarks of flavour  $f$ ,  $\bar{m}_f$  anti-quarks of flavour  $f$ , and  $m_g$  gluons. From this parton configuration we can obtain an  $(m+1)$ -parton configuration by changing

$$\text{a) } m_g \rightarrow m_g + 1 \quad \text{or} \quad \text{b) } m_f \rightarrow m_f + 1, \bar{m}_f \rightarrow \bar{m}_f + 1, m_g \rightarrow m_g - 1 \quad . \quad (7.14)$$

The corresponding ratios of the symmetry factors for identical particles are

$$\begin{aligned} \frac{S_{\{m\}}^{(a)}}{S_{\{m+1\}}} &= \frac{\dots m_g!}{\dots (m_g + 1)!} = \frac{1}{m_g + 1} \quad , \quad (7.15) \\ \frac{S_{\{m\}}^{(b)}}{S_{\{m+1\}}} &= \frac{\dots m_f! \bar{m}_f! m_g!}{\dots (m_f + 1)! (\bar{m}_f + 1)! (m_g - 1)!} = \frac{m_g}{(m_f + 1)(\bar{m}_f + 1)} \quad . \end{aligned}$$

Thus we write

$$\begin{aligned} \sum_{\{m+1\}} \frac{1}{S_{\{m+1\}}} \sum_{\substack{\text{pairs} \\ i,j}} \dots &= \sum_{\{m+1\}}^{(a)} \frac{1}{S_{\{m\}}} \frac{1}{m_g + 1} \left( \sum_{\substack{\text{pairs} \\ i,j=q_f,g}} \dots + \sum_{\substack{\text{pairs} \\ i,j=\bar{q}_f,g}} \dots + \sum_{\substack{\text{pairs} \\ i,j=g,g}} \dots \right) \\ &+ \sum_{\{m+1\}}^{(b)} \frac{1}{S_{\{m\}}} \frac{m_g}{(m_f + 1)(\bar{m}_f + 1)} \sum_{\substack{\text{pairs} \\ i,j=q_f,\bar{q}_f}} \dots \quad , \quad (7.16) \end{aligned}$$

and then

$$\sum_{\substack{\text{pairs} \\ i,j}} \dots = \frac{\#(i,j)_{m+1}}{\#(\tilde{i}\tilde{j})_m} \sum_{\tilde{i}\tilde{j}} \dots \quad , \quad (7.17)$$

where  $\#(i,j)_{m+1}$  denotes the number of pairs  $i,j$  in the configuration with  $m+1$  partons and  $\#(\tilde{i}\tilde{j})_m$  denotes the number of partons with flavour  $\tilde{i}\tilde{j}$  in the corresponding  $m$ -parton configuration. Since we have

$$\begin{aligned} \frac{\#(q_f, g)_{m+1}}{\#(q_f)_m} &= \frac{m_f(m_g + 1)}{m_f} \quad , \quad \frac{\#(\bar{q}_f, g)_{m+1}}{\#(\bar{q}_f)_m} = \frac{\bar{m}_f(m_g + 1)}{\bar{m}_f} \quad , \quad (7.18) \\ \frac{\#(g, g)_{m+1}}{\#(g)_m} &= \frac{m_g(m_g + 1)/2}{m_g} \quad , \quad \frac{\#(q_f, \bar{q}_f)_{m+1}}{\#(g)_m} = \frac{(m_f + 1)(\bar{m}_f + 1)}{m_g} \quad , \end{aligned}$$

we end up with

$$\begin{aligned} \sum_{\{m+1\}} \frac{1}{S_{\{m+1\}}} \sum_{\substack{\text{pairs} \\ i,j}} \dots &= \sum_{\{m\}}^{(a)} \frac{1}{S_{\{m\}}} \left( \sum_{\tilde{i}\tilde{j}=q_f} \dots + \sum_{\tilde{i}\tilde{j}=\bar{q}_f} \dots + \frac{1}{2} \sum_{\tilde{i}\tilde{j}=g} \dots \right) \\ &+ \sum_{\{m\}}^{(b)} \frac{1}{S_{\{m\}}} N_f \sum_{\tilde{i}\tilde{j}=g} \dots \quad . \quad (7.19) \end{aligned}$$

Inserting Eq. (7.19) into Eq. (7.13), we obtain:

$$\begin{aligned} \int_{m+1} d\sigma^A &= - \int_m \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; Q) \frac{1}{S_{\{m\}}} F_J^{(m)}(p_1, \dots, p_m) \\ &\cdot \sum_i \sum_{k \neq i} |\mathcal{M}_m^{i,k}(p_1, \dots, p_m)|^2 \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \quad , \quad (7.20) \end{aligned}$$



where we have defined

$$\mathcal{V}_i(\epsilon) \equiv \mathcal{V}_{qg}(\epsilon) , \quad \text{if } i = q, \bar{q} , \quad (7.21)$$

$$\mathcal{V}_i(\epsilon) \equiv \frac{1}{2}\mathcal{V}_{gg}(\epsilon) + N_f \mathcal{V}_{q\bar{q}}(\epsilon) , \quad \text{if } i = g . \quad (7.22)$$

Equation (7.20) explicitly shows that the subtraction contribution  $d\sigma^A$  can be integrated in closed analytic form over the subspace leading to soft and collinear divergences. These divergences are indeed collected in terms of  $\epsilon$  poles into the factors  $\mathcal{V}_i(\epsilon)$ .

The final result for  $\int_{m+1} d\sigma^A$  in Eq. (7.20) can be written as follows

$$\int_{m+1} d\sigma^A = \int_m [d\sigma^B \cdot \mathbf{I}(\epsilon)] . \quad (7.23)$$

Comparing Eq. (7.20) with Eq. (7.1), we see that the notation  $[d\sigma^B \cdot \mathbf{I}(\epsilon)]$  on the right-hand side means that one has to write down the expression for  $d\sigma^B$  and then replace the corresponding matrix element squared at the Born level

$$|\mathcal{M}_m|^2 = {}_{m < 1, \dots, m} |1, \dots, m >_m , \quad (7.24)$$

by

$${}_{m < 1, \dots, m} | \mathbf{I}(\epsilon) | 1, \dots, m >_m , \quad (7.25)$$

where the insertion operator  $\mathbf{I}(\epsilon)$  depends on the colour charges and momenta of the  $m$  final-state partons in  $d\sigma^B$ :

$$\mathbf{I}(p_1, \dots, p_m; \epsilon) = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_i \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \sum_{k \neq i} \mathbf{T}_i \cdot \mathbf{T}_k \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon . \quad (7.26)$$

The singular factors  $\mathcal{V}_i(\epsilon)$ , defined in Eqs. (7.21,7.22), are given by (see Eqs. (5.32–5.34))

$$\mathcal{V}_i(\epsilon) = \mathbf{T}_i^2 \left( \frac{1}{\epsilon^2} - \frac{\pi^2}{3} \right) + \gamma_i \frac{1}{\epsilon} + \gamma_i + K_i + \mathcal{O}(\epsilon) , \quad (7.27)$$

where  $\gamma_i$  is defined in Eq. (5.90) and we have introduced the following constants

$$K_{i=q,\bar{q}} = \left( \frac{7}{2} - \frac{\pi^2}{6} \right) C_F , \quad K_{i=g} = \left( \frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R N_f . \quad (7.28)$$

Note that all the terms in  $\mathcal{V}_i(\epsilon)$  have a simple interpretation. The coefficient of the double pole  $1/\epsilon^2$  is the square of the colour charge of the parton  $i$ , that of the single pole  $1/\epsilon$  is related to the integral of the non-soft part of its four-dimensional Altarelli-Parisi splitting function, and the  $\pi^2$ -term is a customary phase space factor. The constant  $K_g$  typically appears in the resummation program of higher-order logarithmic corrections of Sudakov type [27] if one uses dimensional regularization and the  $\overline{\text{MS}}$  renormalization scheme (we do not know of an analogous role for the constant  $K_q$ ). Actually, in the context of these calculations the constant  $K_g$  can be absorbed by a redefinition of the renormalized coupling, thus introducing a ‘more physical’ renormalization scheme, accidentally called the ‘Monte Carlo’ scheme in Ref. [28].

### 7.3 One-loop corrections

The NLO QCD cross section in Eq. (7.5) is finite by definition, i.e. because of the infrared and collinear safety of the jet observable. Since the first term on the right-hand side is finite, the second term

$$\int_{m+1} d\sigma^A + \int_m d\sigma^V = \int_m \left\{ d\sigma^V + \left[ d\sigma^B \cdot \mathbf{I}(\epsilon) \right] \right\} \quad (7.29)$$

has to be finite as well. Thus all the  $\epsilon$  poles in  $d\sigma^V$  must be cancelled by those in  $d\sigma^B \cdot \mathbf{I}(\epsilon)$ . The virtual contribution  $d\sigma^V$  has the following expression in terms of the (renormalized) one-loop matrix element

$$d\sigma^V = \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; Q) \frac{1}{S_{\{m\}}} |\mathcal{M}_m(p_1, \dots, p_m)|_{(1-loop)}^2 F_J^{(m)}(p_1, \dots, p_m) . \quad (7.30)$$

Comparing Eq. (7.30) with Eqs. (7.20,7.23), we find that the singular terms of the one-loop matrix element have the following universal structure

$$\begin{aligned} |\mathcal{M}_m(p_1, \dots, p_m)|_{(1-loop)}^2 &= - \, m < 1, \dots, m \, | \mathbf{I}(\epsilon) \, | 1, \dots, m >_m + \dots \\ &= \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_i \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \sum_{k \neq i} \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon |\mathcal{M}_m^{i,k}(p_1, \dots, p_m)|^2 + \dots , \end{aligned} \quad (7.31)$$

where  $|\mathcal{M}_m^{i,k}|^2 = \, m < 1, \dots, m \, | \mathbf{T}_i \cdot \mathbf{T}_k \, | 1, \dots, m >_m$  is the square of the colour-correlated tree-amplitude in Eq. (3.9) and the dots stand for contributions that do not contain any  $\epsilon$  poles. Thus, using the finiteness property of the NLO cross section, we have obtained as by-product of our algorithm the general expression (7.31) for the singular terms of the one-loop QCD amplitudes.

Alternatively, we can use the results of Ref. [29] (see also Ref. [8]) on the singular behaviour of the one-loop amplitudes to prove that our algorithm correctly produces the cancellation of all the  $\epsilon$ -poles in Eq. (7.29), thus leading to a finite NLO cross section. As a matter of fact, using Eqs. (7.27) and keeping only the  $\epsilon$  poles, we can rewrite Eq. (7.31) as follows

$$\begin{aligned} |\mathcal{M}_m(p_1, \dots, p_m)|_{(1-loop)}^2 &= \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_i \sum_{k \neq i} \left[ \frac{1}{\epsilon^2} \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon + \frac{1}{\epsilon} \frac{\gamma_i}{\mathbf{T}_i^2} \right] \\ &\cdot \, m < 1, \dots, m \, | \mathbf{T}_i \cdot \mathbf{T}_k \, | 1, \dots, m >_m + \dots \\ &= \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left[ \sum_{\substack{i,k \\ i \neq k}} \frac{1}{\epsilon^2} \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \, m < 1, \dots, m \, | \mathbf{T}_i \cdot \mathbf{T}_k \, | 1, \dots, m >_m \right. \\ &\quad \left. - \left( \sum_i \gamma_i \right) \frac{1}{\epsilon} \, m < 1, \dots, m \, | 1, \dots, m >_m \right] + \dots , \end{aligned} \quad (7.32)$$

where in the last expression we have used colour-charge conservation ( $\sum_{k \neq i} \mathbf{T}_i \cdot \mathbf{T}_k = -\mathbf{T}_i^2$ ). Equation (7.32) is completely equivalent to Eqs. (2.3) and (2.9) in Ref. [29]<sup>¶</sup>.

<sup>¶</sup>Note a misprint in Eq. (2.9) of Ref. [29]: the overall sign on the right-hand side should be plus instead of minus.

## 7.4 Final formulae

The final results of the application of our algorithm to the calculation of jet cross sections with no hadrons in the initial state are summarized below.

The full QCD cross section in Eq. (6.1) contains a LO and a NLO component. Assuming that the LO calculation involves  $m$  final-state partons, the LO cross section is given by

$$\sigma^{LO} = \int_m d\sigma^B = \int d\Phi^{(m)} |\mathcal{M}_m(p_1, \dots, p_m)|^2 F_J^{(m)}(p_1, \dots, p_m) \quad , \quad (7.33)$$

where  $\mathcal{M}_m$  is the tree-level QCD matrix element to produce  $m$  partons in the final state and the function  $F_J^{(m)}$  defines the particular jet observable we are interested in (see Eqs. (7.2–7.4) for the general properties that  $F_J^{(m)}$  has to fulfil). The factor  $d\Phi^{(m)}$  collects all the relevant phase space factors, i.e. all the remaining terms on the right-hand side of Eq. (7.1). The whole calculation (phase space integration and evaluation of the matrix element) has to be carried out in four space-time dimensions.

According to the notation in Eq. (2.13), the NLO cross section is split into two terms with  $m + 1$ -parton and  $m$ -parton kinematics, respectively:

$$\sigma^{NLO} = \sigma^{NLO\{m+1\}} + \sigma^{NLO\{m\}} \quad . \quad (7.34)$$

The contribution with  $m + 1$ -parton kinematics is the following

$$\begin{aligned} \sigma^{NLO\{m+1\}} &= \int_{m+1} \left[ (d\sigma^R)_{\epsilon=0} - \left( \sum_{\text{dipoles}} d\sigma^B \otimes dV_{\text{dipole}} \right)_{\epsilon=0} \right] \\ &= \int d\Phi^{(m+1)} \left\{ |\mathcal{M}_{m+1}(p_1, \dots, p_{m+1})|^2 F_J^{(m+1)}(p_1, \dots, p_{m+1}) \right. \\ &\quad \left. - \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}) F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}) \right\} \quad , \end{aligned} \quad (7.35)$$

where the term in the curly bracket is exactly the same as that in Eq. (7.8):  $\mathcal{M}_{m+1}$  is the tree-level matrix element,  $\mathcal{D}_{ij,k}$  is the dipole factor in Eq. (5.2) and  $F_J^{(m)}$  is the jet defining function for the corresponding  $m$ -parton state (note, again, the difference between the two jet functions  $F_J^{(m+1)}$  and  $F_J^{(m)}$  in the curly bracket). Despite their original  $d$ -dimensional definition, at this stage the full calculation is carried out in four dimensions.

The NLO contribution with  $m$ -parton kinematics is given by

$$\begin{aligned} \sigma^{NLO\{m\}} &= \int_m [d\sigma^V + d\sigma^B \otimes \mathbf{I}]_{\epsilon=0} \\ &= \int d\Phi^{(m)} \left\{ |\mathcal{M}_m(p_1, \dots, p_m)|_{(1-loop)}^2 + \sum_{m < 1, \dots, m} \mathbf{I}(\epsilon) |1, \dots, m >_m \right\}_{\epsilon=0} F_J^{(m)}(p_1, \dots, p_m) \quad . \end{aligned} \quad (7.36)$$

The first term in the curly bracket is the one-loop<sup>||</sup> *renormalized* matrix element square to produce  $m$  final-state partons. The second term is obtained by inserting the colour-charge operator of Eq. (7.26) into the tree-level matrix element to produce  $m$  partons as in

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<sup>||</sup>Remember that, according to our calculation of the insertion operator  $\mathbf{I}$ , the one-loop matrix element in Eq. (7.36) has to be evaluated by using conventional dimensional regularization. We refer to the discussion in Sect. 3.3 for the use of different regularization schemes.

Eq. (7.25) (see also Appendix C). These two terms have to be first evaluated in  $d = 4 - 2\epsilon$  dimensions. Then one has to carry out their expansion in  $\epsilon$ -poles (the expansion for the singular factors  $\mathcal{V}_i(\epsilon)$  is recalled in Appendix C), cancel analytically (by trivial addition) the poles and perform the limit  $\epsilon \rightarrow 0$ . At this point the phase-space integration is carried out in four space-time dimensions.

## 8 Jet cross sections with one initial-state hadron

Sections 8–11 are devoted to the generalization of the results of the previous Section to processes with identified hadrons (cfr. Sect. 2.3). In each of these Sections, we first describe the implementation of our subtraction procedure by following closely (although with less detail) the steps of Sect. 7. We start by defining the jet cross sections for each class of processes, then we introduce the explicit expression for our subtraction term  $d\sigma^A$  and, finally, we perform its integration, calculate the appropriate combinatorial factors, and show how the ensuing contribution can be combined with the virtual term  $d\sigma^V$  and the collinear counter-term  $d\sigma^C$  to provide a finite NLO partonic cross section. The final formulae of our algorithm are summarized at the end of each Section.

Let us start by considering hard-scattering processes with a single incoming hadron (cfr. Eqs. (6.3–6.6)) like, for instance, deep-inelastic lepton-hadron scattering. In the case of unpolarized scattering, the Born-level partonic cross section with one parton of flavour  $a$  and momentum  $p_a$  in the initial state has the following expression in terms of the QCD matrix elements

$$d\sigma_a^B(p_a) = \mathcal{N}_{in} \frac{1}{n_s(a)n_c(a)\Phi(p_a)} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; p_a + Q) \frac{1}{S_{\{m\}}} \cdot |\mathcal{M}_{m,a}(p_1, \dots, p_m; p_a)|^2 F_J^{(m)}(p_1, \dots, p_m; p_a) , \quad (8.1)$$

Here the factor  $1/(n_s(a)n_c(a))$  accounts for the average over the number of initial-state polarizations and colours and  $\Phi(p_a)$  is the flux factor. Since  $p_a^2 = 0$ , the flux factor fulfils the following scaling property

$$\Phi(\eta p_a) = \eta \Phi(p_a) . \quad (8.2)$$

The jet function  $F_J^{(m)}(p_1, \dots, p_m; p_a)$  is infrared and collinear safe (see Eqs. (7.2–7.4)) and, moreover, it fulfils the property of factorizability of initial-state collinear singularities. From a formal viewpoint this implies that

$$F_J^{(n+1)}(p_1, \dots, p_i, \dots, p_{n+1}; p_a) \rightarrow F_J^{(n)}(p_1, \dots, p_{n+1}; x p_a) , \quad \text{if } p_i \rightarrow (1-x)p_a \quad (8.3)$$

for any number  $n$  of final-state partons (the  $n$ -parton jet function on the right-hand side is obtained from the  $n+1$ -parton function on the left-hand side by removing the parton  $i$ ) and

$$F_J^{(m)}(p_1, \dots, p_m; p_a) \rightarrow 0 , \quad \text{if } p_i \cdot p_a \rightarrow 0 \quad (8.4)$$

for the leading-order process (i.e.  $n = m$ ).

All the other factors in Eq. (8.1) are analogous to those in Eq. (7.1).

### 8.1 Implementation of the subtraction procedure

In order to compute the NLO cross section in Eq. (6.5), we write the following identity

$$\begin{aligned} \sigma_a^{NLO}(p_a; \mu_F^2) &= \int_{m+1} \left( d\sigma_a^R(p_a) - d\sigma_a^A(p_a) \right) \\ &+ \left[ \int_{m+1} d\sigma_a^A(p_a) + \int_m d\sigma_a^V(p_a) + \int_m d\sigma_a^C(p_a; \mu_F^2) \right] , \quad (8.5) \end{aligned}$$

where, according to the dipole formulae in Sects. 5.1–5.3, a local counterterm  $d\sigma_a^A(p_a)$  is provided by:

$$\begin{aligned}
d\sigma_a^A(p_a) &= \mathcal{N}_{in} \frac{1}{n_s(a)\Phi(p_a)} \sum_{\{m+1\}} d\phi_{m+1}(p_1, \dots, p_{m+1}; p_a + Q) \frac{1}{S_{\{m+1\}}} \\
&\cdot \left\{ \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}; p_a) F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}; p_a) \right. \\
&+ \sum_{\substack{\text{pairs} \\ i,j}} \mathcal{D}_{ij}^a(p_1, \dots, p_{m+1}; p_a) F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; \tilde{p}_a) \\
&\left. + \sum_i \sum_{k \neq i} \mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}; p_a) F_J^{(m)}(p_1, \dots, \tilde{p}_k, \dots, p_{m+1}; \tilde{p}_{ai}) \right\} . \quad (8.6)
\end{aligned}$$

Here  $\mathcal{D}_{ij,k}(p_1, \dots, p_{m+1})$ ,  $\mathcal{D}_{ij}^a(p_1, \dots, p_{m+1}; p_a)$  and  $\mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}; p_a)$  are respectively the dipoles in Eqs. (5.2), (5.36) and (5.61) and  $F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}; p_a)$ ,  $F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; \tilde{p}_a)$  and  $F_J^{(m)}(p_1, \dots, \tilde{p}_k, \dots, p_{m+1}; \tilde{p}_{ai})$  are the jet defining functions for the corresponding  $m$ -parton states  $\{p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}; p_a\}$ ,  $\{p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; \tilde{p}_a\}$  and  $\{p_1, \dots, \tilde{p}_k, \dots, p_{m+1}; \tilde{p}_{ai}\}$ .

Since the dipole contributions exactly match the soft and collinear divergences of the square of the matrix element  $|\mathcal{M}_{m+1,a}|^2$ , the subtracted expression  $(d\sigma_a^R(p_a) - d\sigma_a^A(p_a))$  in Eq. (8.5) is integrable in  $d = 4$  dimensions.

In order to compute the contribution in the square bracket of Eq. (8.5), we write  $d\sigma_a^A(p_a)$  as follows

$$d\sigma_a^A(p_a) = d\sigma_a^{A'}(p_a) + d\sigma_a^{A''}(p_a) + d\sigma_a^{A'''}(p_a) , \quad (8.7)$$

where the three terms on the right-hand side of Eq. (8.7) are in one-to-one correspondence with those in the curly bracket on the right-hand side of Eq. (8.6).

The integration of  $d\sigma_a^{A'}(p_a)$  can be performed analogously to that of  $d\sigma^A$  in the previous Section, thus leading to the following result

$$\begin{aligned}
\int_{m+1} d\sigma_a^{A'}(p_a) &= - \int_m \mathcal{N}_{in} \frac{1}{n_s(a)\Phi(p_a)} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; p_a + Q) \frac{1}{S_{\{m\}}} \\
&\cdot F_J^{(m)}(p_1, \dots, p_m; p_a) \sum_i \sum_{k \neq i} |\mathcal{M}_{m,a}^{i,k}(p_1, \dots, p_m; p_a)|^2 \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) , \quad (8.8)
\end{aligned}$$

where the functions  $\mathcal{V}_i(\epsilon)$  are defined in Eqs. (7.21,7.22).

Let us now consider the integration of  $d\sigma_a^{A''}(p_a)$ . We first rewrite Eq. (5.36) as follows

$$\mathcal{D}_{ij}^a(p_1, \dots, p_{m+1}; p_a) = - \left[ \frac{\mathbf{V}_{ij}^a}{2x_{ij,a} p_i \cdot p_j} \frac{1}{\mathbf{T}_{ij}^2} |\mathcal{M}_{m,a}^{ij,a}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; \tilde{p}_a)|^2 \right]_h , \quad (8.9)$$

where the notation is similar to that in Eq. (7.11), and then, using the phase space convolution properties in Eqs. (5.46,5.47), we obtain:

$$\int_{m+1} d\sigma_a^{A''}(p_a) = - \int_m \mathcal{N}_{in} \sum_{\{m+1\}} \sum_{\substack{\text{pairs} \\ i,j}} \int_0^1 dx d\phi_m(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; xp_a + Q) \frac{1}{S_{\{m+1\}}}$$

$$\begin{aligned}
& \cdot F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; xp_a) \frac{1}{n_s(a)\Phi(p_a)} \\
& \cdot \frac{1}{x} \left[ \frac{1}{\mathbf{T}_{ij}^2} |\mathcal{M}_{m,a}^{ij,a}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; xp_a)|^2 \int_1 \frac{\mathbf{V}_{ij}^a}{2p_i \cdot p_j} \right]_h [dp_i(\tilde{p}_{ij}; p_a, x)] \ .
\end{aligned} \tag{8.10}$$

As in the case of the splitting functions  $\mathbf{V}_{ij,k}$ , the azimuthal correlations due to  $\mathbf{V}_{ij}^a$  vanish after integration over  $p_i$  (keeping  $\tilde{p}_{ij}$  and  $x$  fixed) and we find

$$\begin{aligned}
& \int_{m+1} d\sigma_a^{A''}(p_a) = - \int_m \mathcal{N}_{in} \sum_{\{m+1\}} \sum_{\substack{\text{pairs} \\ i,j}} \int_0^1 dx \frac{1}{n_s(a)\Phi(xp_a)} \\
& \cdot d\phi_m(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; xp_a + Q) \frac{1}{S_{\{m+1\}}} F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; xp_a) \\
& \cdot |\mathcal{M}_{m,a}^{ij,a}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; xp_a)|^2 \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_{ij} \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_{ij}^2} \mathcal{V}_{ij}(x; \epsilon) \ , \tag{8.11}
\end{aligned}$$

where the functions  $\mathcal{V}_{ij}(x; \epsilon)$  are given in Eqs. (5.57–5.59) and we have used Eq. (8.2) to replace  $x\Phi(p_a)$  with  $\Phi(xp_a)$ .

Equation (8.11) is similar to Eq. (7.13) in Sect. 7, apart from the replacement  $k \rightarrow a$  as spectator parton. In order to rewrite Eq. (8.11) in terms of an  $m$ -parton contribution times a factor, we have to perform the counting of the symmetry factors for going from  $m$  partons to  $m+1$  partons in the final state. Since, as shown in Sect. 7, this counting is independent of the spectator parton, we immediately get

$$\begin{aligned}
& \int_{m+1} d\sigma_a^{A''}(p_a) = - \int_m \mathcal{N}_{in} \int_0^1 dx \frac{1}{n_s(a)\Phi(xp_a)} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; xp_a + Q) \\
& \cdot \frac{1}{S_{\{m\}}} F_J^{(m)}(p_1, \dots, p_m; xp_a) \sum_i |\mathcal{M}_{m,a}^{i,a}(p_1, \dots, p_m; xp_a)|^2 \\
& \cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(x; \epsilon) \ , \tag{8.12}
\end{aligned}$$

where we have defined

$$\mathcal{V}_i(x; \epsilon) \equiv \mathcal{V}_{q\bar{q}}(x; \epsilon) \ , \quad \text{if } i = q, \bar{q} \ , \tag{8.13}$$

$$\mathcal{V}_i(x; \epsilon) \equiv \frac{1}{2} \mathcal{V}_{gg}(x; \epsilon) + N_f \mathcal{V}_{q\bar{q}}(x; \epsilon) \ , \quad \text{if } i = g \ . \tag{8.14}$$

Recall that all the  $\epsilon$ -poles of  $\mathcal{V}_i(x; \epsilon)$  are accounted for by  $\delta(1-x)\mathcal{V}_i(\epsilon)$  terms (see Eqs. (5.57–5.59)), where  $\mathcal{V}_i(\epsilon)$  are the functions defined in Eqs. (7.21,7.22).

Let us now consider the integration of  $d\sigma_a^{A''}(p_a)$ . We first rewrite the corresponding dipole contribution (see Eq. (5.61)) as follows

$$\mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}; p_a) = - \left[ \frac{\mathbf{V}_k^{ai}}{2x_{ik,a} p_i \cdot p_a} \frac{1}{\mathbf{T}_{ai}^2} |\mathcal{M}_{m,\tilde{a}i}^{k,ai}(p_1, \dots, \tilde{p}_k, \dots, p_{m+1}; \tilde{p}_{ai})|^2 \right]_h \ . \tag{8.15}$$

Thus, using the phase space convolution in Eqs. (5.70,5.71) and performing the integration over  $p_i$  (keeping  $\tilde{p}_k$  and  $x$  fixed), the azimuthal correlation due to  $\mathbf{V}_k^{ai}$  vanishes and we

obtain

$$\begin{aligned}
\int_{m+1} d\sigma_a^{Aim}(p_a) &= - \int_m \mathcal{N}_{in} \int_0^1 dx \sum_{\{m+1\}} \sum_i \sum_{k \neq i} \frac{1}{n_s(\tilde{a}i) \Phi(xp_a)} \\
&\cdot d\phi_m(p_1, \dots, \tilde{p}_k, \dots, p_m; xp_a + Q) \frac{1}{S_{\{m+1\}}} F_J^{(m)}(p_1, \dots, \tilde{p}_k, \dots, p_m; xp_a) \\
&\cdot |\mathcal{M}_{m, \tilde{a}i}^{ai,k}(p_1, \dots, \tilde{p}_k, \dots, p_m; xp_a)|^2 \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2\tilde{p}_k \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_{\tilde{a}i}^2} \mathcal{V}^{a,ai}(x; \epsilon) \quad , \quad (8.16)
\end{aligned}$$

where the functions  $\mathcal{V}^{a,ai}(x; \epsilon)$  are given in Eqs. (5.81–5.84).

The right-hand side of Eq. (8.16) can easily be rewritten in terms of a sum over  $m$ -parton configurations. To this end, we have to perform the corresponding counting of symmetry factors, which, nonetheless, is trivial in this case. If the Born-level  $m$ -parton configuration has  $m_i$  partons of type  $i$ , the corresponding  $m+1$ -parton configuration has  $m_i+1$  partons of the same type, so that:

$$\sum_{\{m+1\}} \frac{1}{S_{\{m+1\}}} \sum_i \dots = \sum_{\{m+1\}} \sum_i \frac{\dots m_i!}{\dots (m_i+1)!} \frac{1}{S_{\{m\}}} \dots \quad (8.17)$$

However, there are  $m_i+1$  possible ways of choosing the parton  $i$  in the  $m+1$ -parton configuration and, hence, we obtain

$$\begin{aligned}
\sum_{\{m+1\}} \frac{1}{S_{\{m+1\}}} \sum_i \dots &= \sum_i \frac{\dots m_i!}{\dots (m_i+1)!} (m_i+1) \sum_{\{m\}} \frac{1}{S_{\{m\}}} \dots \\
&= \sum_i \sum_{\{m\}} \frac{1}{S_{\{m\}}} \dots \quad , \quad (8.18)
\end{aligned}$$

where the  $\sum_i$  in the last line of Eq. (8.18) simply denotes the sum over the flavours  $i$  in the  $m$ -parton configuration. We can thus rewrite this sum as a sum over the flavours  $b = \tilde{a}i$  in the initial state and Eq. (8.16) becomes:

$$\begin{aligned}
\int_{m+1} d\sigma_a^{Aim}(p_a) &= - \int_m \mathcal{N}_{in} \sum_{\{m\}} \int_0^1 dx d\phi_m(p_1, \dots, p_m; xp_a + Q) \frac{1}{S_{\{m\}}} \\
&\cdot F_J^{(m)}(p_1, \dots, p_m; xp_a) \sum_k \sum_b \frac{1}{n_s(b) \Phi(xp_a)} |\mathcal{M}_{m,b}^{b,k}(p_1, \dots, p_m; xp_a)|^2 \\
&\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_k \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \mathcal{V}^{a,b}(x; \epsilon) \quad . \quad (8.19)
\end{aligned}$$

Note that, in addition to  $\delta^{ab} \delta(1-x) \mathcal{V}_b(\epsilon)$  terms like those in Eqs. (8.13,8.14),  $\mathcal{V}^{a,b}(x; \epsilon)$  contains  $\epsilon$ -poles in terms of the form  $P^{ab}(x)/\epsilon$  (see Eqs. (5.81–5.84)).

Collecting Eqs. (8.8,8.12,8.19) and adding Eq. (6.6), we find

$$\begin{aligned}
\int_{m+1} d\sigma_a^A(p_a) + \int_m d\sigma_a^C(p_a) &= - \sum_b \int_0^1 dx \int_m \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; xp_a + Q) \\
&\cdot \frac{1}{S_{\{m\}}} F_J^{(m)}(p_1, \dots, p_m; xp_a) \frac{1}{n_s(b) \Phi(xp_a)} \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)}
\end{aligned}$$



$$\begin{aligned}
& \cdot \left\{ \delta^{ab} \delta(1-x) \sum_i \sum_{k \neq i} |\mathcal{M}_{m,b}^{i,k}(p_1, \dots, p_m; xp_a)|^2 \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \right. \\
& + \delta^{ab} \sum_i |\mathcal{M}_{m,b}^{i,b}(p_1, \dots, p_m; xp_a)|^2 \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(x; \epsilon) \\
& + \sum_i |\mathcal{M}_{m,b}^{i,b}(p_1, \dots, p_m; xp_a)|^2 \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \mathcal{V}^{a,b}(x; \epsilon) \\
& \left. + \frac{1}{n_c(b)} |\mathcal{M}_{m,b}(p_1, \dots, p_m; xp_a)|^2 \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{ab}(x) + K_{\text{FS}}^{ab}(x) \right] \right\} . \quad (8.20)
\end{aligned}$$

We see that  $\int_{m+1} d\sigma_a^A(p_a) + \int_m d\sigma_a^C(p_a)$  is obtained from the leading-order expression  $\int_m d\sigma_a^B(xp_a)$  by replacing the Born-level matrix element squared

$$\frac{1}{n_s(a)} \quad m, a < \dots; p_a | \dots; p_a >_{m, a} , \quad (8.21)$$

by

$$\sum_b \frac{1}{n_s(b)} \quad m, b < \dots; xp_a | \mathbf{I}^{a,b}(x; \epsilon) | \dots; xp_a >_{m, b} , \quad (8.22)$$

and performing the  $x$ -integration. Here the insertion operator  $\mathbf{I}(x; \epsilon)$  depends on the colour charges, momenta and flavours of the QCD partons

$$\begin{aligned}
\mathbf{I}^{a,b}(p_1, \dots, p_m; p_a, x; \epsilon; \mu_F^2) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \\
& \cdot \left\{ \delta^{ab} \delta(1-x) \sum_i \sum_{k \neq i} \mathbf{T}_i \cdot \mathbf{T}_k \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) + \delta^{ab} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(x; \epsilon) \right. \\
& \left. + \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \mathcal{V}^{a,b}(x; \epsilon) - \frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{ab}(x) + K_{\text{FS}}^{ab}(x) \right\} . \quad (8.23)
\end{aligned}$$

This form of the insertion operator can be simplified in the limit  $\epsilon \rightarrow 0$ . We start by rewriting Eq. (8.23) as follows

$$\begin{aligned}
\mathbf{I}^{a,b}(p_1, \dots, p_m; p_a, x; \epsilon; \mu_F^2) &= \delta^{ab} \delta(1-x) \mathbf{I}(p_1, \dots, p_m; p_a; \epsilon) \\
& + \delta^{ab} \mathbf{I}_{(1)}(p_1, \dots, p_m; p_a, x; \epsilon) + \mathbf{I}_{(2)}^{a,b}(p_1, \dots, p_m; p_a, x; \epsilon; \mu_F^2) , \quad (8.24)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{I}(p_1, \dots, p_m; p_a; \epsilon) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \sum_i \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \left[ \sum_{k \neq i} \mathbf{T}_i \cdot \mathbf{T}_k \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \right. \right. \\
& \left. \left. + \mathbf{T}_i \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \right] + \frac{1}{\mathbf{T}_a^2} \mathcal{V}_a(\epsilon) \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \right\} , \quad (8.25)
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_{(1)}(p_1, \dots, p_m; p_a, x; \epsilon) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \\
& \cdot \left[ \mathcal{V}_i(x; \epsilon) - \delta(1-x) \mathcal{V}_i(\epsilon) \right] , \quad (8.26)
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_{(2)}^{a,b}(p_1, \dots, p_m; p_a, x; \epsilon; \mu_F^2) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \right. \\
&\cdot \left[ \mathcal{V}^{a,b}(x; \epsilon) - \delta^{ab} \delta(1-x) \mathcal{V}_a(\epsilon) \right] - \frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{ab}(x) + K_{\text{F.S.}}^{ab}(x) \left. \right\} . \quad (8.27)
\end{aligned}$$

The operator  $\mathbf{I}(p_1, \dots, p_m, p_a; \epsilon)$  in Eq. (8.25) is fully symmetric with respect to the dependence on colour charges and momenta of the  $m+1$  partons  $\{p_1, \dots, p_m, p_a\}$ . In other words, this operator is identical (apart from depending on the additional initial-state parton  $a$ ) to that in Eq. (7.26). Since crossing the momentum of partons from the final to the initial state does not change the singular terms in the one-loop QCD amplitudes, it follows that the insertion operator (8.25) cancels all the singularities in the virtual contribution  $\int_m d\sigma_a^V(p_a)$ . As a consequence, the two other operators  $\mathbf{I}_{(1)}$  and  $\mathbf{I}_{(2)}$  should contribute as finite counterterms. Actually,  $\mathbf{I}_{(1)}$  and  $\mathbf{I}_{(2)}$  are separately finite in the limit  $\epsilon \rightarrow 0$ .

In order to show that, we can use Eqs. (5.57–5.59), (8.13,8.14) and (7.21,7.22) and thus obtain

$$\begin{aligned}
\mathcal{V}_i(x; \epsilon) - \delta(1-x) \mathcal{V}_i(\epsilon) &= \mathbf{T}_i^2 \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] \\
&- \gamma_i \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] + \mathcal{O}(\epsilon) , \quad (8.28)
\end{aligned}$$

where  $\gamma_i$  are given in Eq. (5.90). Then we can write:

$$\begin{aligned}
\mathbf{I}_{(1)}(p_1, \dots, p_m; p_a, x; \epsilon) &= -\frac{\alpha_S}{2\pi} \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \left\{ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ \right. \\
&+ \frac{2}{1-x} \ln(2-x) - \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] \left. \right\} + \mathcal{O}(\epsilon) . \quad (8.29)
\end{aligned}$$

Coming to  $\mathbf{I}_{(2)}$ , let us rewrite Eq. (8.27) in the following form

$$\begin{aligned}
\mathbf{I}_{(2)}^{a,b}(p_1, \dots, p_m; p_a, x; \epsilon; \mu_F^2) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \right. \\
&\cdot \left[ \mathcal{V}^{a,b}(x; \epsilon) + \frac{1}{\epsilon} P^{ab}(x) - \delta^{ab} \delta(1-x) \mathcal{V}_a(\epsilon) \right] \\
&- \left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} + \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon \right] \frac{1}{\epsilon} P^{ab}(x) + K_{\text{F.S.}}^{ab}(x) \left. \right\} . \quad (8.30)
\end{aligned}$$

Using Eqs. (5.81–5.84), we see that the first square bracket contribution on the right-hand side of Eq. (8.30) is finite for  $\epsilon \rightarrow 0$  and given by

$$\begin{aligned}
\mathcal{V}^{a,b}(x; \epsilon) + \frac{1}{\epsilon} P^{ab}(x) - \delta^{ab} \delta(1-x) \mathcal{V}_a(\epsilon) &= \overline{K}^{ab}(x) + P^{ab}(x) \ln x \\
&- \delta^{ab} \mathbf{T}_a^2 \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] + \mathcal{O}(\epsilon) , \quad (8.31)
\end{aligned}$$

where the  $\overline{K}^{ab}(x)$  functions are defined so as to simplify the final formulae,

$$\overline{K}^{qg}(x) = P^{qg}(x) \ln \frac{1-x}{x} + C_F x , \quad (8.32)$$

$$\overline{K}^{gq}(x) = P^{gq}(x) \ln \frac{1-x}{x} + T_R 2x(1-x) , \quad (8.33)$$

$$\begin{aligned} \overline{K}^{qq}(x) &= C_F \left[ \left( \frac{2}{1-x} \ln \frac{1-x}{x} \right)_+ - (1+x) \ln \frac{1-x}{x} + (1-x) \right] \\ &\quad - \delta(1-x) (5 - \pi^2) C_F , \end{aligned} \quad (8.34)$$

$$\begin{aligned} \overline{K}^{gg}(x) &= 2C_A \left[ \left( \frac{1}{1-x} \ln \frac{1-x}{x} \right)_+ + \left( \frac{1-x}{x} - 1 + x(1-x) \right) \ln \frac{1-x}{x} \right] \\ &\quad - \delta(1-x) \left[ \left( \frac{50}{9} - \pi^2 \right) C_A - \frac{16}{9} T_R N_f \right] . \end{aligned} \quad (8.35)$$

As for the second square bracket contribution on the right-hand side of Eq. (8.30), using colour conservation ( $\sum_i \mathbf{T}_i = -\mathbf{T}_b$ ) and expanding in  $\epsilon$ , we get an  $\mathcal{O}(\epsilon)$  term:

$$\begin{aligned} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} + \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon &= \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \frac{1}{\mathbf{T}_b^2} \left[ \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon - \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon \right] \\ &= \epsilon \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \frac{1}{\mathbf{T}_b^2} \ln \frac{\mu_F^2}{2p_i \cdot p_a} + \mathcal{O}(\epsilon^2) . \end{aligned} \quad (8.36)$$

Inserting Eqs. (8.31,8.36) into Eq. (8.30), adding the  $\mathbf{I}_{(1)}$  contribution in Eq. (8.29) and again using colour charge conservation, we end up with

$$\begin{aligned} &\delta^{ab} \mathbf{I}_{(1)}(p_1, \dots, p_m; p_a, x; \epsilon) + \mathbf{I}_{(2)}^{a,b}(p_1, \dots, p_m; p_a, x; \epsilon; \mu_F^2) \\ &= \mathbf{K}^{a,b}(x) + \mathbf{P}^{a,b}(p_1, \dots, p_m; xp_a, x; \mu_F^2) + \mathcal{O}(\epsilon) , \end{aligned} \quad (8.37)$$

where we have defined

$$\mathbf{K}^{a,b}(x) = \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{ab}(x) - K_{\text{F.S.}}^{ab}(x) + \delta^{ab} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] \right\} , \quad (8.38)$$

$$\mathbf{P}^{a,b}(p_1, \dots, p_m; xp_a, x; \mu_F^2) = \frac{\alpha_S}{2\pi} P^{ab}(x) \frac{1}{\mathbf{T}_b^2} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \ln \frac{\mu_F^2}{2xp_a \cdot p_i} . \quad (8.39)$$

The final result for  $\mathbf{I}^{a,b}(x; \epsilon)$  is the following

$$\begin{aligned} \mathbf{I}^{a,b}(p_1, \dots, p_m; p_a, x; \epsilon; \mu_F^2) &= \delta^{ab} \delta(1-x) \mathbf{I}(p_1, \dots, p_m, p_a; \epsilon) \\ &\quad + \mathbf{K}^{a,b}(x) + \mathbf{P}^{a,b}(p_1, \dots, p_m; xp_a, x; \mu_F^2) + \mathcal{O}(\epsilon) , \end{aligned} \quad (8.40)$$

where the insertion operators  $\mathbf{I}$ ,  $\mathbf{K}^{a,b}$  and  $\mathbf{P}^{a,b}$  are given in Eqs. (8.25), (8.38) and (8.39). Therefore, using a notation similar to that in Eq.(7.23), we can write

$$\begin{aligned} \int_{m+1} d\sigma_a^A(p) + \int_m d\sigma_a^C(p; \mu_F^2) &= \int_m [d\sigma_a^B(p) \cdot \mathbf{I}(\epsilon)] \\ &\quad + \sum_b \int_0^1 dx \int_m [\mathbf{K}^{a,b}(x) \cdot d\sigma_b^B(xp)] + \sum_b \int_0^1 dx \int_m [\mathbf{P}^{a,b}(xp, x; \mu_F^2) \cdot d\sigma_b^B(xp)] . \end{aligned} \quad (8.41)$$

Note that all the insertion operators  $\mathbf{I}(\epsilon)$ ,  $\mathbf{K}^{a,b}(x)$ ,  $\mathbf{P}^{a,b}(xp, x; \mu_F^2)$  depend on the colour charges and flavours of the QCD partons. However, while this dependence is fully symmetric in  $\mathbf{I}(\epsilon)$ , the operators  $\mathbf{K}^{a,b}(x)$  and  $\mathbf{P}^{a,b}(xp, x; \mu_F^2)$  do depend asymmetrically on the flavour and colour charge of the incoming parton  $p$ . In addition,  $\mathbf{I}(\epsilon)$  depends on the parton momenta,  $\mathbf{K}^{a,b}(x)$  depends on the momentum fraction  $x$  (but not on the parton momenta) and  $\mathbf{P}^{a,b}(xp, x; \mu_F^2)$  depends on  $x$ , parton momenta and factorization scale.

As in the case of processes with no initial-state hadrons, the term  $d\sigma_a^B(p) \cdot \mathbf{I}(\epsilon)$  in Eq. (8.41) cancels all the  $\epsilon$ -poles in the virtual contribution  $d\sigma_a^V(p)$ , thus making the NLO cross section in Eq. (8.5) finite. The other two terms on the right-hand side of Eq. (8.41) are finite remainders that are left after factorization of the initial-state collinear singularities into the parton densities.

The operator  $\mathbf{P}^{a,b}(xp, x; \mu_F^2)$  is directly related to the Altarelli-Parisi probabilities. In particular, from Eq. (8.39) we can check that it fulfils the relation:

$$\frac{\partial \mathbf{P}^{a,b}(p_1, \dots, p_m; xp_a, x; \mu_F^2)}{\partial \ln \mu_F^2} = \frac{\alpha_S}{2\pi} P^{ab}(x) \frac{1}{\mathbf{T}_b^2} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b = -\frac{\alpha_S}{2\pi} P^{ab}(x) \quad , \quad (8.42)$$

where we have used colour-charge conservation ( $\sum_i \mathbf{T}_i = -\mathbf{T}_b$ ). It follows that it cancels the similar (and with opposite sign) factorization-scale dependence of the parton densities  $f_a(x, \mu_F^2)$  thus making the hadronic cross section (6.3)  $\mu_F$ -independent to NLO accuracy.

The operator  $\mathbf{K}^{a,b}(x)$  contains  $(\ )_+$  and  $\delta$  distributions with coefficients  $\gamma_i$  of the same type as those appearing in the singular operator  $\mathbf{I}(\epsilon)$  (see Eqs. (7.27,8.25)). These terms, due to colour correlations between the incoming parton and the final-state partons, are the heritage of the initial-state collinear divergences originally present in the real contribution  $d\sigma_a^R(p)$ . Moreover,  $\mathbf{K}^{a,b}(x)$  depends on the flavour (and colour diagonal) kernels  $K_{\text{F.S.}}^{ab}(x)$  and  $\overline{K}^{ab}(x)$ . The kernel  $K_{\text{F.S.}}^{ab}(x)$  is related to the definition of the factorization scheme (cfr. Eq. (6.6)) while  $\overline{K}^{ab}(x)$  has a close relationship with the parton splitting functions. As a matter of fact, if we define  $\hat{P}'_{ab}(x)$  by expanding in  $\epsilon$  the  $d$ -dimensional Altarelli-Parisi splitting functions in Eqs. (4.18–4.21):

$$\langle \hat{P}_{ab}(x; \epsilon) \rangle = \langle \hat{P}_{ab}(x; \epsilon = 0) \rangle - \epsilon \hat{P}'_{ab}(x) + \mathcal{O}(\epsilon^2) \quad , \quad (8.43)$$

we can rewrite Eqs. (8.32–8.35) as follows

$$\begin{aligned} \overline{K}^{ab}(x) &= \hat{P}'_{ab}(x) + P_{\text{reg}}^{ab}(x) \ln \frac{1-x}{x} \\ &+ \delta^{ab} \left[ \mathbf{T}_a^2 \left( \frac{2}{1-x} \ln \frac{1-x}{x} \right)_+ - \delta(1-x) \left( \gamma_a + K_a - \frac{5}{6} \pi^2 \mathbf{T}_a^2 \right) \right] \quad , \quad (8.44) \end{aligned}$$

where  $P_{\text{reg}}^{ab}(x)$  is the non-singular part of the (four-dimensional) Altarelli-Parisi probabilities (see Eq. (5.89)) and the coefficients  $K_a$  are defined in Eq. (7.28). The contribution  $\hat{P}'_{ab}(x)$  on the right-hand side of Eq. (8.44), is directly related to the dimensional regularization of the initial-state collinear singularities. Indeed, it comes from the interference of the  $1/\epsilon$  collinear pole and the  $\mathcal{O}(\epsilon)$ -contribution to the  $d$ -dimensional splitting functions. Also the other terms on the right-hand side of Eq. (8.44) have a simple interpretation. After having factorized the initial-state collinear divergences, the finite remainder  $\overline{K}^{ab}(x)$  is proportional

to the phase space available for the emission from the incoming parton. The factor  $\ln(1 - x)/x$  multiplying  $P_{\text{reg}}^{ab}(x)$  has this kinematic origin. The same factor controls radiation that is both collinear and soft and thus it enters into the  $( )_+$ -distribution in the square bracket.

## 8.2 Final formulae

Summarizing the results derived in the previous Subsection, we obtain the following final formulae for jet cross sections involving one initial-state hadron.

The partonic cross section on the right-hand side of Eq. (6.3) consists of a LO and a NLO component. In the case of an incoming parton of flavour  $a$  and momentum  $p_a$ , the explicit expression for the LO component is:

$$\sigma_a^{LO}(p_a) = \int_m d\sigma_a^B(p_a) = \int d\Phi^{(m)}(p_a) \frac{1}{n_c(a)} |\mathcal{M}_{m,a}(p_1, \dots, p_m; p_a)|^2 F_J^{(m)}(p_1, \dots, p_m; p_a) , \quad (8.45)$$

where  $\mathcal{M}_{m,a}$  is the tree-level matrix element to produce  $m$  final-state partons,  $F_J^{(m)}$  is the jet defining function (it fulfils the properties in Eqs. (8.3,8.4) in addition to those in Eqs. (7.2–7.4)), the factor  $1/n_c(a)$  comes from the average over the colours of the initial-state parton and  $d\Phi^{(m)}(p_a)$  collects all the remaining factors (phase space, flux, spin average) on the right-hand side of Eq. (8.1). The evaluation of the LO cross section (8.45) is carried out in four space-time dimensions.

The NLO partonic cross section is split into three terms, as in Eq. (2.15):

$$\sigma_a^{NLO}(p_a; \mu_F^2) = \sigma_a^{NLO\{m+1\}}(p_a) + \sigma_a^{NLO\{m\}}(p_a) + \int_0^1 dx \hat{\sigma}_a^{NLO\{m\}}(x; xp_a, \mu_F^2) . \quad (8.46)$$

The term with  $m + 1$ -parton kinematics is given by

$$\begin{aligned} \sigma_a^{NLO\{m+1\}}(p_a) &= \int_{m+1} \left[ \left( d\sigma_a^R(p_a) \right)_{\epsilon=0} - \left( \sum_{\text{dipoles}} d\sigma_a^B(p_a) \otimes \left( dV_{\text{dipole}} + dV'_{\text{dipole}} \right) \right)_{\epsilon=0} \right] \\ &= \int d\Phi^{(m+1)}(p_a) \left\{ \frac{1}{n_c(a)} |\mathcal{M}_{m+1,a}(p_1, \dots, p_{m+1}; p_a)|^2 F_J^{(m+1)}(p_1, \dots, p_{m+1}; p_a) \right. \\ &\quad \left. - \sum_{\text{dipoles}} \left( \mathcal{D} \cdot F^{(m)} \right) (p_1, \dots, p_{m+1}; p_a) \right\} , \quad (8.47) \end{aligned}$$

where  $\mathcal{M}_{m+1,a}$  is the tree-level matrix element with  $m + 1$  partons in the final state and  $\sum_{\text{dipoles}} \left( \mathcal{D} \cdot F^{(m)} \right) (p_1, \dots, p_{m+1}; p_a)$  is the sum of the dipole factors contained into the curly bracket on the right-hand side of Eq. (8.6). Note that the  $m + 1$ -parton matrix element is multiplied by  $F_J^{(m+1)}$ , the jet function for  $m + 1$  final-state partons, while the dipole contributions involve the  $m$ -parton jet function  $F_J^{(m)}$ . All the terms in Eq. (8.47) are evaluated and integrated in four dimensions.

The NLO contribution with  $m$ -parton kinematics is exactly like that in Eq. (7.36) for  $e^+e^-$ -type processes, apart from the additional dependence on the colour and momentum

of the incoming parton. Indeed, combining the virtual cross section with the first term on the right-hand side of Eq. (8.41), we obtain

$$\begin{aligned}
\sigma_a^{NLO\{m\}}(p_a) &= \int_m \left[ d\sigma_a^V(p_a) + d\sigma_a^B(p_a) \otimes \mathbf{I} \right]_{\epsilon=0} \\
&= \int d\Phi^{(m)}(p_a) \left\{ \frac{1}{n_c(a)} |\mathcal{M}_{m,a}(p_1, \dots, p_m; p_a)|_{(1-loop)}^2 \right. \\
&\quad \left. + \left. \left. \left. m, a < 1, \dots, m; a \right| \mathbf{I}(\epsilon) \left| 1, \dots, m; a >_{m,a} \right. \right\}_{\epsilon=0} F_J^{(m)}(p_1, \dots, p_m; p_a) \ , \quad (8.48)
\end{aligned}$$

where  $|\mathcal{M}_{m,a}|_{(1-loop)}^2$  is the one-loop matrix element square and the colour charge operator  $\mathbf{I}(\epsilon)$  is given in Eq. (8.25) (see also Appendix C). The two terms in the curly bracket have to be separately computed in  $d = 4 - 2\epsilon$  dimensions and the limit  $\epsilon \rightarrow 0$  performed after having cancelled analytically the  $\epsilon$  poles. At this point the phase space integration is carried out in four dimensions.

The NLO component involving the one-dimensional convolution with respect to the longitudinal-momentum fraction  $x$  is given by the last two terms on the right-hand side of Eq. (8.41):

$$\begin{aligned}
\int_0^1 dx \hat{\sigma}_a^{NLO\{m\}}(x; xp_a, \mu_F^2) &= \sum_b \int_0^1 dx \int_m \left[ d\sigma_b^B(xp_a) \otimes (\mathbf{K} + \mathbf{P})^{a,b}(x) \right]_{\epsilon=0} \\
&= \sum_b \int_0^1 dx \int d\Phi^{(m)}(xp_a) F_J^{(m)}(p_1, \dots, p_m; xp_a) \\
&\quad \cdot \left. \left. \left. m, b < 1, \dots, m; xp_a \right| \left( \mathbf{K}^{a,b}(x) + \mathbf{P}^{a,b}(xp_a, x; \mu_F^2) \right) \left| 1, \dots, m; xp_a >_{m,b} \right. \right. \ . \quad (8.49)
\end{aligned}$$

The colour-charge operators  $\mathbf{K}$  and  $\mathbf{P}$  are respectively defined in Eqs. (8.38) and (8.39). Their explicit expressions, as well as those of the related flavour kernels  $P^{ab}(x)$ ,  $\overline{K}^{ab}(x)$  and  $K_{\text{F.S.}}^{ab}(x)$ , are recalled in Appendix C. The calculation of Eq. (8.49) is directly performed in four space-time dimensions.

The partonic cross sections in Eqs. (8.45–8.49) have to be convoluted with the parton densities as in Eq. (6.3), in order to compute the corresponding hadronic cross sections.

Note that, because of the  $x$ -dependence of the operators  $\mathbf{K}$  and  $\mathbf{P}$  in Eqs. (8.38,8.39), the cross section component in Eq. (8.49) is boost-invariant with respect to the direction of the incoming momentum  $p_a$ . Therefore, in the evaluation of the hadronic cross section, this contribution enters in the form of multiple convolution of a Born-type partonic cross section with the kernel  $\mathbf{K}$  or  $\mathbf{P}$  and with the parton densities.

## 9 Jet cross sections with one final-state identified hadron

Let us now consider fragmentation processes, starting with the simplest case, which does not involve initial-state hadrons. According to the definition in Sect. 6, the inclusive cross section to produce a parton of flavour  $a$  and momentum  $p_a$  has the following expression at the Born level

$$d\sigma_{(incl)a}^B(p_a) = \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; Q - p_a) \frac{1}{S_{\{m\}}} \cdot |\mathcal{M}_{m+a}(p_a, p_1, \dots, p_m)|^2 F_J^{(m)}([p_a], p_1, \dots, p_m) . \quad (9.1)$$

The notation in Eq. (9.1) is similar to that in Eqs. (7.1) and (8.1). The only relevant difference regards the jet defining function  $F_J^{(m)}([p_a], p_1, \dots, p_m)$ . Besides being infrared and collinear safe (see Eqs. (7.2–7.4)), it should guarantee the factorizability of final-state collinear singularities. This implies the following general (i.e. for any number  $n$  of partons) property

$$F_J^{(n+1)}([p_a], p_1, \dots, p_i, \dots, p_{n+1}) \rightarrow F_J^{(n)}([p_a/z], p_1, \dots, p_{n+1}) , \quad \text{if } p_i \rightarrow (1/z - 1)p_a , \quad (9.2)$$

and the leading-order constraint

$$F_J^{(m)}([p_a], p_1, \dots, p_m) \rightarrow 0 , \quad \text{if } p_i \cdot p_a \rightarrow 0 . \quad (9.3)$$

### 9.1 Implementation of the subtraction procedure

As usual, in order to evaluate the NLO partonic cross section, we rewrite Eq. (6.16) in the following form

$$\begin{aligned} \sigma_{(incl)a}^{NLO}(p_a) &= \int_{m+1} \left( d\sigma_{(incl)a}^R(p_a) - d\sigma_{(incl)a}^A(p_a) \right) \\ &+ \left[ \int_{m+1} d\sigma_{(incl)a}^A(p_a) + \int_m d\sigma_{(incl)a}^V(p_a) + \int_m d\sigma_{(incl)a}^C(p_a; \mu_F^2) \right] , \end{aligned} \quad (9.4)$$

where, using the dipole formulae in Eqs. (5.92) and (5.113), we define the local counterterm:

$$\begin{aligned} d\sigma_{(incl)a}^A(p_a) &= \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1, \dots, p_{m+1}; Q - p_a) \frac{1}{S_{\{m+1\}}} \\ &\cdot \left\{ \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_a, p_1, \dots, p_{m+1}) F_J^{(m)}([p_a], p_1, \dots, \tilde{p}_{ij}, \dots, \tilde{p}_k, \dots, p_{m+1}) \right. \\ &+ \sum_{\substack{\text{pairs} \\ i,j}} \mathcal{D}_{ij,a}(p_a, p_1, \dots, p_{m+1}) F_J^{(m)}([\tilde{p}_a], p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}) \\ &\left. + \sum_i \sum_{k \neq i} \mathcal{D}_{ai,k}(p_a, p_1, \dots, p_{m+1}) F_J^{(m)}([\tilde{p}_{ai}], p_1, \dots, \tilde{p}_k, \dots, p_{m+1}) \right\} . \end{aligned} \quad (9.5)$$

The four-dimensional integrability of  $(d\sigma_{(incl)a}^R(p_a) - d\sigma_{(incl)a}^A(p_a))$  follows in exactly the same way as for Eqs. (7.5) and (8.5).

To compute the term in the square bracket of Eq. (9.4), we write  $d\sigma_{(incl)a}^A(p_a)$  as follows

$$d\sigma_{(incl)a}^A(p_a) = d\sigma_{(incl)a}^{A'}(p_a) + d\sigma_{(incl)a}^{A''}(p_a) + d\sigma_{(incl)a}^{A'''}(p_a) , \quad (9.6)$$

where the three terms on the right-hand side are in one-to-one correspondence with those in the curly bracket on the right-hand side of Eq. (9.5).

The integration of  $d\sigma_{(incl)a}^{A'}(p_a)$  can be carried out analogously to that of  $d\sigma^A$  in Sect. 7 and, thus, we obtain

$$\begin{aligned} \int_{m+1} d\sigma_{(incl)a}^{A'}(p_a) &= - \int_m \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; Q - p_a) \frac{1}{S_{\{m\}}} \\ &\cdot F_J^{(m)}([p_a], p_1, \dots, p_m) \sum_i \sum_{k \neq i} |\mathcal{M}_{m+a}^{i,k}(p_a, p_1, \dots, p_m)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) , \end{aligned} \quad (9.7)$$

where the functions  $\mathcal{V}_i(\epsilon)$  are defined in Eqs. (7.21,7.22).

The integration of  $d\sigma_{(incl)a}^{A''}(p_a)$  is similar to that of  $d\sigma_a^{A''}(p_a)$  in Sect. 8. The main difference is that the phase space convolution in Eq. (5.101) replaces that in Eq. (5.46). Thus, we find:

$$\begin{aligned} \int_{m+1} d\sigma_{(incl)a}^{A''}(p_a) &= - \int_m \mathcal{N}_{in} \int_0^1 \frac{dz}{z^{2-2\epsilon}} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; Q - p_a/z) \\ &\cdot \frac{1}{S_{\{m\}}} F_J^{(m)}([p_a/z], p_1, \dots, p_m) \sum_i |\mathcal{M}_{m+a}^{i,a}(p_a/z, p_1, \dots, p_m)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \bar{\mathcal{V}}_i(z; \epsilon) , \end{aligned} \quad (9.8)$$

where we have defined

$$\bar{\mathcal{V}}_i(z; \epsilon) \equiv \bar{\mathcal{V}}_{q\bar{q}}(z; \epsilon) , \quad \text{if } i = q, \bar{q} , \quad (9.9)$$

$$\bar{\mathcal{V}}_i(z; \epsilon) \equiv \frac{1}{2} \bar{\mathcal{V}}_{g\bar{g}}(z; \epsilon) + N_f \bar{\mathcal{V}}_{q\bar{q}}(z; \epsilon) , \quad \text{if } i = g . \quad (9.10)$$

The integration of  $d\sigma_{(incl)a}^{A'''}(p_a)$  is again analogous to that of  $d\sigma_a^{A'''}(p_a)$  in Sect. 8, apart from the different phase space convolution in Eq. (5.122) for the dipole  $\{ai, k\}$ . We find:

$$\begin{aligned} \int_{m+1} d\sigma_{(incl)a}^{A'''}(p_a) &= - \int_m \mathcal{N}_{in} \sum_{\{m\}} \int_0^1 \frac{dz}{z^{2-2\epsilon}} d\phi_m(p_1, \dots, p_m; Q - p_a/z) \frac{1}{S_{\{m\}}} \\ &\cdot F_J^{(m)}([p_a/z], p_1, \dots, p_m) \sum_k \sum_b |\mathcal{M}_{m+b}^{b,k}(p_a/z, p_1, \dots, p_m)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_k \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \mathcal{V}_{b,a}(z; \epsilon) , \end{aligned} \quad (9.11)$$

where the functions  $\mathcal{V}_{a,ai}(z; \epsilon)$  are given in Eqs. (5.131–5.134).



Combining the results in Eqs. (9.7,9.8,9.11) and adding the collinear counterterm in Eq. (6.17), we have

$$\begin{aligned}
& \int_{m+1} d\sigma_{(incl)a}^A(p_a) + \int_m d\sigma_{(incl)a}^C(p_a) = - \sum_b \int_0^1 \frac{dz}{z^{2-2\epsilon}} \int_m \mathcal{N}_{in} \\
& \cdot \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; Q - p_a/z) \frac{1}{S_{\{m\}}} F_J^{(m)}([p_a/z], p_1, \dots, p_m) \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \\
& \cdot \left\{ \delta_{ab} \delta(1-z) \sum_i \sum_{k \neq i} |\mathcal{M}_{m+b}^{i,k}(p_a/z, p_1, \dots, p_m)|^2 \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \right. \\
& + \delta_{ab} \sum_i |\mathcal{M}_{m+b}^{i,b}(p_a/z, p_1, \dots, p_m)|^2 \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \bar{\mathcal{V}}_i(z; \epsilon) \\
& + \sum_i |\mathcal{M}_{m+b}^{i,b}(p_a/z, p_1, \dots, p_m)|^2 \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \mathcal{V}_{b,a}(z; \epsilon) \\
& \left. + |\mathcal{M}_{m+b}(p_a/z, p_1, \dots, p_m)|^2 \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P_{ba}(z) + H_{ba}^{\text{F.S.}}(z) \right] \right\} .
\end{aligned} \tag{9.12}$$

Hence,  $\int_{m+1} d\sigma_{(incl)a}^A(p_a) + \int_m d\sigma_{(incl)a}^C(p_a)$  is obtained from the leading order expression  $\int_m d\sigma_{(incl)a}^B(p_a)$  by replacing the Born-level matrix element squared

$${}_{m+a} \langle p_a, \dots | p_a, \dots \rangle_{m+a} , \tag{9.13}$$

by

$$\sum_b {}_{m+b} \langle p_a/z, \dots | \mathbf{I}_{b,a}(z; \epsilon) | p_a/z, \dots \rangle_{m+b} , \tag{9.14}$$

where the insertion operator  $\mathbf{I}(z; \epsilon)$  depends on the colour charges, momenta and flavours of the QCD partons:

$$\begin{aligned}
\mathbf{I}_{b,a}(p_a, p_1, \dots, p_m; z; \epsilon) &= -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \\
& \cdot \left\{ \delta_{ab} \delta(1-z) \sum_i \sum_{k \neq i} \mathbf{T}_i \cdot \mathbf{T}_k \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) + \delta_{ab} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \bar{\mathcal{V}}_i(z; \epsilon) \right. \\
& \left. + \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_b^2} \mathcal{V}_{b,a}(z; \epsilon) - \frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P_{ba}(z) + H_{ba}^{\text{F.S.}}(z) \right\} .
\end{aligned} \tag{9.15}$$

This insertion operator is similar to that in Eq. (8.23) for the cross section with a single incoming parton, apart from the replacements  $\mathcal{V}_i(x; \epsilon) \rightarrow \bar{\mathcal{V}}_i(z; \epsilon)$ ,  $\mathcal{V}^{a,b}(x; \epsilon) \rightarrow \mathcal{V}_{b,a}(z; \epsilon)$ ,  $P_{ab}(x) \rightarrow P_{ba}(z)$ ,  $K_{\text{F.S.}}^{ab}(x) \rightarrow H_{ba}^{\text{F.S.}}(z)$  (note, in particular, the transposition of the flavour indices that is involved in these replacements). Therefore, using the analogues of Eqs. (8.28,8.31), namely

$$\begin{aligned}
\bar{\mathcal{V}}_i(z; \epsilon) - \delta(1-z) \mathcal{V}_i(\epsilon) &= \gamma_i + \mathbf{T}_i^2 \left( \frac{2}{1-z} \ln \frac{1}{1-z} \right)_+ \\
& - \gamma_i \left[ \left( \frac{1}{1-z} \right)_+ + \delta(1-z) \right] + \mathcal{O}(\epsilon) ,
\end{aligned} \tag{9.16}$$

$$\begin{aligned}
& \mathcal{V}_{b,a}(z; \epsilon) + \frac{1}{\epsilon} P_{ba}(z) - \delta_{ba} \delta(1-z) \mathcal{V}_a(\epsilon) = \overline{K}^{ba}(z) + 2P_{ba}(z) \ln z \\
& - \delta^{ba} \mathbf{T}_a^2 \left( \frac{2}{1-z} \ln \frac{1}{1-z} \right)_+ + \mathcal{O}(\epsilon) \ , \tag{9.17}
\end{aligned}$$

and performing the same algebraic manipulations as in Sect. 8, we end up with the final result:

$$\begin{aligned}
& \int_{m+1} d\sigma_{(incl)a}^A(p) + \int_m d\sigma_{(incl)a}^C(p; \mu_F^2) = \int_m \left[ d\sigma_{(incl)a}^B(p) \cdot \mathbf{I}(\epsilon) \right] \\
& + \sum_b \int_0^1 \frac{dz}{z^2} \int_m \left[ d\sigma_{(incl)b}^B(p/z) \cdot \mathbf{H}_{b,a}(z) \right] + \sum_b \int_0^1 \frac{dz}{z^2} \int_m \left[ d\sigma_{(incl)b}^B(p/z) \cdot \mathbf{P}_{b,a}(p/z, z; \mu_F^2) \right] \ , \tag{9.18}
\end{aligned}$$

where the insertion operator  $\mathbf{I}(\epsilon)$  is exactly the same as in Eq. (8.25) and the insertion operators  $\mathbf{H}_{b,a}(z)$  and  $\mathbf{P}_{b,a}(p/z, z; \mu_F^2)$  are defined as follows

$$\begin{aligned}
& \mathbf{H}_{b,a}(z) = \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{ba}(z) + 3P_{ba}(z) \ln z - H_{ba}^{\text{F.S.}}(z) \right. \\
& \left. + \delta_{ab} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-z} \right)_+ + \delta(1-z) - 1 \right] \right\} \ , \tag{9.19}
\end{aligned}$$

$$\mathbf{P}_{b,a}(p_1, \dots, p_m; p_a/z, z; \mu_F^2) = \frac{\alpha_S}{2\pi} P_{ba}(z) \frac{1}{\mathbf{T}_b^2} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \ln \frac{z\mu_F^2}{2p_a \cdot p_i} \ . \tag{9.20}$$

Equation (9.18) is the time-like (a single identified parton in the final state) analogue of Eq. (8.41) for the space-like (a single parton in the initial state) case. The contribution  $d\sigma_{(incl)a}^B(p) \cdot \mathbf{I}(\epsilon)$  cancels all the  $\epsilon$ -poles in  $d\sigma_{(incl)a}^V(p)$  thus making the NLO cross section (9.4) finite in the four-dimensional limit.

The operators  $\mathbf{H}_{b,a}(z)$  and  $\mathbf{P}_{b,a}(p/z, z; \mu_F^2)$  are instead finite for  $\epsilon \rightarrow 0$  (for this reason, in Eq. (9.18) we have replaced the phase space factor  $dz/z^{2-2\epsilon}$  of Eq. (9.12) with  $dz/z^2$ ) and are similar to the operators  $\mathbf{K}^{a,b}(x)$  and  $\mathbf{P}^{a,b}(xp, x; \mu_F^2)$  in Eq. (8.41). Actually, apart from the momentum rescaling  $xp \rightarrow p/z$ , the only other difference between the two  $\mathbf{P}$  operators is in the transposition of the flavour indices  $a$  and  $b$ . Therefore, in spite of the identity  $P_{ab}(z) = P^{ab}(z)$  of the Altarelli-Parisi probabilities for time-like and space-like splittings, in the case of the operators  $\mathbf{P}$  we have  $\mathbf{P}_{b,a}(p, z; \mu_F^2) \neq \mathbf{P}^{b,a}(p, z; \mu_F^2)$ . This difference is due to the colour correlations  $\mathbf{T}_i \cdot \mathbf{T}_b$  or, more precisely, to the fact that momentum fraction and colour flow in opposite direction in the time-like and space-like cases (Fig. 4).

Comparing Eq. (8.38) and (9.19), we see that this transposition of the flavour indices also affects the difference between  $\mathbf{K}$  and  $\mathbf{H}$ . Moreover,  $\mathbf{H}$  differs from  $\mathbf{K}$  by an extra term  $3P_{ba}(z) \ln z - \delta_{ab} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \gamma_i / \mathbf{T}_i^2$ , which can be attributed to the kinematic crossing from initial- to final-state partons. The appearance of the two different kernels  $K_{\text{FS}}^{ab}$  and  $H_{ba}^{\text{F.S.}}$  is trivially related to the choice of the factorization scheme of collinear singularities (cfr. Eqs.(6.6) and (6.17)).

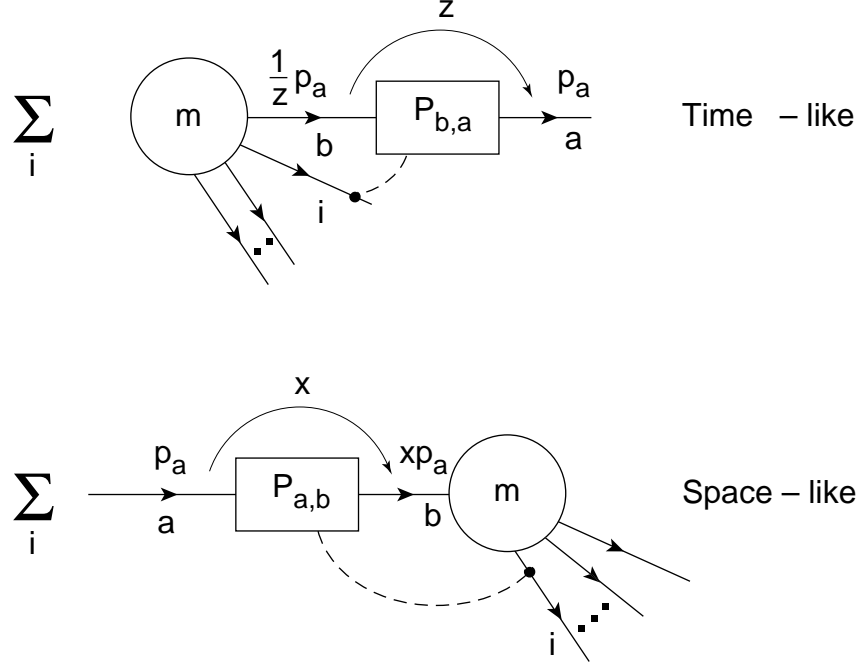


Figure 4: Altarelli-Parisi insertion operators for the time-like (Eq. (9.20)) and the space-like (Eq. (8.39)) cases.

## 9.2 Final formulae

The calculations carried out in Subsection 9.1 for jet cross sections with a single identified hadron in the final state lead to results that are very similar to those described in Subsection 8.2 for the kinematically-crossed process with a single hadron in the initial state.

In summary, the LO parton-level cross section (6.15), to be convoluted with the non-perturbative fragmentation function as in Eq. (6.14), can be written as follows

$$\sigma_{(incl) a}^{LO}(p_a) = \int_m d\sigma_{(incl) a}^B(p_a) = \int d\Phi^{(m)}(p_a) |\mathcal{M}_{m+a}(p_a, p_1, \dots, p_m)|^2 F_J^{(m)}([p_a], p_1, \dots, p_m) . \quad (9.21)$$

Here  $a$  and  $p_a$  respectively denote the flavour and momentum of the identified parton,  $\mathcal{M}_{m+a}$  is the tree-level matrix element to produce  $m$  unidentified partons in the final state,  $F_J^{(m)}$  is the jet defining function (its general properties are listed in Eqs. (9.2,9.3) and Eqs. (7.2–7.4)) and  $d\Phi^{(m)}(p_a)$  stands for all the remaining phase-space factors on the right-hand side of Eq. (9.1).

The NLO partonic cross section is decomposed into three terms. Following the symbolic notation in Eq. (2.15), we write:

$$\sigma_{(incl) a}^{NLO}(p_a; \mu_F^2) = \sigma_{(incl) a}^{NLO\{m+1\}}(p_a) + \sigma_{(incl) a}^{NLO\{m\}}(p_a) + \int_0^1 \frac{dz}{z^2} \hat{\sigma}_{(incl) a}^{NLO\{m\}}(z; p_a/z, \mu_F^2) . \quad (9.22)$$

The NLO contribution with  $m+1$ -parton kinematics has the following explicit expression

$$\sigma_{(incl) a}^{NLO\{m+1\}}(p_a) = \int_{m+1} \left[ \left( d\sigma_{(incl) a}^R(p_a) \right)_{\epsilon=0} \right]$$

$$\begin{aligned}
& - \left( \sum_{\text{dipoles}} d\sigma_{(incl)a}^B(p_a) \otimes (dV_{\text{dipole}} + dV'_{\text{dipole}}) \right)_{\epsilon=0} \Big] \\
& = \int d\Phi^{(m+1)}(p_a) \left\{ |\mathcal{M}_{m+1+a}(p_a, p_1, \dots, p_{m+1})|^2 F_J^{(m+1)}([p_a], p_1, \dots, p_{m+1}) \right. \\
& \left. - \sum_{\text{dipoles}} (\mathcal{D} \cdot F^{(m)})(p_a, p_1, \dots, p_{m+1}) \right\} , \tag{9.23}
\end{aligned}$$

where  $\mathcal{M}_{m+1+a}$  is the tree-level matrix element with  $m+1$  unidentified partons in the final state and  $\sum_{\text{dipoles}} (\mathcal{D} \cdot F^{(m)})(p_a, p_1, \dots, p_{m+1})$  is the sum of the dipole factors contained into the curly bracket on the right-hand side of Eq. (9.5).

The NLO term with  $m$ -parton kinematics is obtained by adding the virtual cross section and the first contribution on the right-hand side of Eq. (9.18). This term is completely analogous to that in Eq. (8.48) and is given by:

$$\begin{aligned}
\sigma_{(incl)a}^{NLO\{m\}}(p_a) & = \int_m \left[ d\sigma_{(incl)a}^V(p_a) + d\sigma_{(incl)a}^B(p_a) \otimes \mathbf{I} \right]_{\epsilon=0} \\
& = \int d\Phi^{(m)}(p_a) \left\{ |\mathcal{M}_{m+a}(p_a, p_1, \dots, p_m)|_{(1-loop)}^2 \right. \\
& \left. + \sum_{m+a < a, 1, \dots, m} \mathbf{I}(\epsilon) |a, 1, \dots, m \rangle_{m+a} \right\}_{\epsilon=0} F_J^{(m)}([p_a], p_1, \dots, p_m) . \tag{9.24}
\end{aligned}$$

where the colour-charge operator  $\mathbf{I}(\epsilon)$  is defined in Eq. (8.25) (see also Appendix C).

The third contribution on the right-hand side of Eq. (9.22) involves the integration of an  $m$ -parton cross section with respect to the fraction  $z$  of the longitudinal momentum carried by the identified parton. This contribution is given by the last two terms on the right-hand side of Eq. (9.18):

$$\begin{aligned}
\int_0^1 \frac{dz}{z^2} \hat{\sigma}_{(incl)a}^{NLO\{m\}}(z; p_a/z, \mu_F^2) & = \sum_b \int_0^1 \frac{dz}{z^2} \int_m \left[ d\sigma_{(incl)b}^B(p_a/z) \otimes (\mathbf{H} + \mathbf{P})_{b,a}(z) \right]_{\epsilon=0} \\
& = \sum_b \int_0^1 \frac{dz}{z^2} \int d\Phi^{(m)}(p_a/z) F_J^{(m)}([p_a/z], p_1, \dots, p_m) \tag{9.25} \\
& \cdot \sum_{m+b < p_a/z, 1, \dots, m} (\mathbf{H}_{b,a}(z) + \mathbf{P}_{b,a}(p_a/z, z; \mu_F^2)) |p_a/z, 1, \dots, m \rangle_{m+b} ,
\end{aligned}$$

where the colour-charge operators  $\mathbf{H}$  and  $\mathbf{P}$  are respectively defined in Eqs. (9.19) and (9.20) (see also Appendix C).

The actual evaluation of Eqs. (9.21), (9.23) and (9.25) is directly performed in four space-time dimensions. As for Eq. (9.24), one should first cancel analytically the  $\epsilon$  poles of the one-loop matrix element with those of the insertion operator  $\mathbf{I}$ , perform the limit  $\epsilon \rightarrow 0$  and then carry out the phase-space integration in four space-time dimensions.

## 10 Jet cross sections with two initial-state hadrons

In the case of unpolarized scattering, the Born-level cross section with two incoming partons of flavours  $a$  and  $b$  and momenta  $p_a$  and  $p_b$  is the following

$$d\sigma_{ab}^B(p_a, p_b) = \mathcal{N}_{in} \frac{1}{n_s(a)n_s(b)n_c(a)n_c(b)\Phi(p_a \cdot p_b)} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; p_a + p_b + Q) \cdot \frac{1}{S_{\{m\}}} |\mathcal{M}_{m,ab}(p_1, \dots, p_m; p_a, p_b)|^2 F_J^{(m)}(p_1, \dots, p_m; p_a, p_b) . \quad (10.1)$$

Here the factor  $1/(n_s(a)n_s(b)n_c(a)n_c(b))$  accounts for the average over the number of initial-state polarizations and colours and  $\Phi(p_a \cdot p_b)$  is the flux factor. The flux factor fulfils the following scaling property

$$\Phi(\eta p_a \cdot p_b) = \eta \Phi(p_a \cdot p_b) . \quad (10.2)$$

The function  $F_J^{(m)}(p_1, \dots, p_m; p_a, p_b)$  defines the jet observable and has the same properties as the function  $F_J^{(m)}(p_1, \dots, p_m; p_a)$  in Sect. 8 (more precisely, the factorizability of initial-state collinear singularities has to be valid with respect to both  $p_a$  and  $p_b$ ). All the other factors in Eq. (10.1) are analogous to those in Eq. (8.1).

### 10.1 Implementation of the subtraction procedure

In order to compute the NLO cross section, we can write it as follows

$$\sigma_{ab}^{NLO}(p_a, p_b; \mu_F^2) = \int_{m+1} \left( d\sigma_{ab}^R(p_a, p_b) - d\sigma_{ab}^A(p_a, p_b) \right) + \left[ \int_{m+1} d\sigma_{ab}^A(p_a, p_b) + \int_m d\sigma_{ab}^V(p_a, p_b) + \int_m d\sigma_{ab}^C(p_a, p_b; \mu_F^2) \right] , \quad (10.3)$$

where the local counterterm  $d\sigma_{ab}^A(p_a, p_b)$  is given by

$$d\sigma_{ab}^A(p_a, p_b) = \mathcal{N}_{in} \frac{1}{n_s(a)n_s(b)\Phi(p_a p_b)} \sum_{\{m+1\}} d\phi_{m+1}(p_1, \dots, p_{m+1}; p_a + p_b + Q) \frac{1}{S_{\{m+1\}}} \cdot \left\{ \sum_{\substack{\text{pairs} \\ i,j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, \dots, p_{m+1}; p_a, p_b) F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}; p_a, p_b) + \sum_{\substack{\text{pairs} \\ i,j}} \left[ \mathcal{D}_{ij}^a(p_1, \dots, p_{m+1}; p_a, p_b) F_J^{(m)}(p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; \tilde{p}_a, p_b) + (a \leftrightarrow b) \right] + \sum_i \sum_{k \neq i} \left[ \mathcal{D}_k^{ai}(p_1, \dots, p_{m+1}; p_a, p_b) F_J^{(m)}(p_1, \dots, \tilde{p}_k, \dots, p_{m+1}; \tilde{p}_{ai}, p_b) + (a \leftrightarrow b) \right] + \sum_i \left[ \mathcal{D}^{ai,b}(p_1, \dots, p_{m+1}; p_a, p_b) F_J^{(m)}(\tilde{p}_1, \dots, \tilde{p}_{m+1}; \tilde{p}_{ai}, p_b) + (a \leftrightarrow b) \right] \right\} . \quad (10.4)$$

While the first three terms in the curly bracket exactly correspond to those in the curly bracket of Eq. (8.6), the last term is the new dipole contribution introduced in Eq. (5.135).

In order to compute the integral of  $d\sigma_{ab}^A(p_a, p_b)$ , we write it as follows

$$d\sigma_{ab}^A(p_a, p_b) = d\sigma_{ab}^{A'}(p_a, p_b) + d\sigma_{ab}^{A''}(p_a, p_b) + d\sigma_{ab}^{A'''}(p_a, p_b) + d\sigma_{ab}^{A''''}(p_a, p_b) \ , \quad (10.5)$$

where the four terms on the right-hand side are in one-to-one correspondence with those in the curly bracket of Eq. (10.4).

The integration over  $p_i$  in the first three terms on the right-hand side of Eq. (10.5) is completely analogous to that carried out in the Sect. 8. Thus we obtain:

$$\begin{aligned} \int_{m+1} d\sigma_{ab}^{A'}(p_a, p_b) &= - \int_m \mathcal{N}_{in} \frac{1}{n_s(a)n_s(b)\Phi(p_a p_b)} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; p_a + p_b + Q) \\ &\cdot \frac{1}{S_{\{m\}}} F_J^{(m)}(p_1, \dots, p_m; p_a, p_b) \sum_i \sum_{k \neq i} |\mathcal{M}_{m,ab}^{i,k}(p_1, \dots, p_m; p_a, p_b)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \ , \end{aligned} \quad (10.6)$$

$$\begin{aligned} \int_{m+1} d\sigma_{ab}^{A''}(p_a, p_b) &= - \int_m \mathcal{N}_{in} \int_0^1 dx \frac{1}{n_s(a)n_s(b)\Phi(xp_a p_b)} \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; xp_a + p_b + Q) \\ &\cdot \frac{1}{S_{\{m\}}} F_J^{(m)}(p_1, \dots, p_m; xp_a, p_b) \sum_i |\mathcal{M}_{m,ab}^{i,a}(p_1, \dots, p_m; xp_a, p_b)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(x; \epsilon) + (a \leftrightarrow b) \ , \end{aligned} \quad (10.7)$$

$$\begin{aligned} \int_{m+1} d\sigma_{ab}^{A'''}(p_a, p_b) &= - \int_m \mathcal{N}_{in} \sum_{\{m\}} \int_0^1 dx \ d\phi_m(p_1, \dots, p_m; xp_a + p_b + Q) \\ &\cdot \frac{1}{S_{\{m\}}} F_J^{(m)}(p_1, \dots, p_m; xp_a, p_b) \\ &\cdot \sum_k \sum_c \frac{1}{n_s(c)n_s(b)\Phi(xp_a p_b)} |\mathcal{M}_{m,cb}^{c,k}(p_1, \dots, p_m; xp_a, p_b)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_k \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_c^2} \mathcal{V}^{a,c}(x; \epsilon) + (a \leftrightarrow b) \ , \end{aligned} \quad (10.8)$$

where the functions  $\mathcal{V}_i(\epsilon)$ ,  $\mathcal{V}_i(x; \epsilon)$ , and  $\mathcal{V}^{a,b}(x; \epsilon)$  are respectively defined in Eqs. (7.21,7.22), Eqs. (8.13,8.14) and Eqs. (5.81–5.84).

Let us now consider the  $p_i$ -integration of  $d\sigma_{ab}^{A''''}(p_a, p_b)$ . We first use the phase space convolution in Eqs. (5.149,5.150) in order to factorize the  $p_i$  integration. Then we can integrate the splitting function  $\mathbf{V}^{ai,c}$  over  $p_i$  and we find:

$$\begin{aligned} \int_{m+1} d\sigma_{ab}^{A''''}(p_a, p_b) &= - \int_m \mathcal{N}_{in} \int_0^1 dx \sum_{\{m+1\}} \sum_i \frac{1}{n_s(\tilde{a}i)n_s(b)\Phi(xp_a p_b)} \\ &\cdot d\phi_m(p_1, \dots, p_m; xp_a + p_b + Q) \frac{1}{S_{\{m+1\}}} \\ &\cdot F_J^{(m)}(p_1, \dots, p_m; xp_a, p_b) |\mathcal{M}_{m,\tilde{a}ib}^{ai,b}(p_1, \dots, p_m; xp_a, p_b)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_a \cdot p_b} \right)^\epsilon \frac{1}{\mathbf{T}_{ai}^2} \tilde{\mathcal{V}}^{a,ai}(x; \epsilon) + (a \leftrightarrow b) \ , \end{aligned} \quad (10.9)$$

where the functions  $\tilde{\mathcal{V}}^{a,ai}(x; \epsilon)$  are given in Eq. (5.155). In order to rewrite Eq. (10.9) in terms of a sum over  $m$ -parton configurations, we have to perform the corresponding counting of symmetry factors. However this counting is exactly the same as that already considered in Sect. 8 (see Eqs. (8.17,8.18)) for the case in which the spectator is a final-state parton  $k$ . Thus we obtain:

$$\begin{aligned}
\int_{m+1} d\sigma_{ab}^{A''''}(p_a, p_b) &= - \int_m \mathcal{N}_{in} \sum_{\{m\}} \int_0^1 dx \, d\phi_m(p_1, \dots, p_m; xp_a + p_b + Q) \frac{1}{S_{\{m\}}} \\
&\cdot F_J^{(m)}(p_1, \dots, p_m; xp_a, p_b) \\
&\cdot \sum_c \frac{1}{n_s(c)n_s(b)\Phi(xp_a p_b)} |\mathcal{M}_{m,cb}^{c,b}(p_1, \dots, p_m; xp_a, p_b)|^2 \\
&\cdot \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{2p_a \cdot p_b} \right)^\epsilon \frac{1}{\mathbf{T}_c^2} \tilde{\mathcal{V}}^{a,c}(x; \epsilon) + (a \leftrightarrow b) \quad . \quad (10.10)
\end{aligned}$$

Collecting Eqs. (10.6,10.7,10.8,10.10) and adding Eq. (6.13), we can write the following expression

$$\begin{aligned}
\int_{m+1} d\sigma_{ab}^A(p_a, p_b) + \int_m d\sigma_{ab}^C(p_a, p_b; \mu_F^2) &= \sum_{c,d} \int_0^1 dx \int_0^1 dy \int_m \mathcal{N}_{in} \frac{1}{n_s(c)n_s(d)\Phi(xyp_a p_b)} \\
&\cdot \sum_{\{m\}} d\phi_m(p_1, \dots, p_m; xp_a + yp_b + Q) \frac{1}{S_{\{m\}}} F_J^{(m)}(p_1, \dots, p_m; xp_a, yp_b) \\
&\cdot {}_{m,cd < 1, \dots, m; a, b} | \mathbf{I}^{ab,cd}(x, y; \epsilon) | {}_{1, \dots, m; a, b} >_{m,cd} \quad . \quad (10.11)
\end{aligned}$$

Here  $\int_{m+1} d\sigma_{ab}^A(p_a, p_b) + \int_m d\sigma_{ab}^C(p_a, p_b; \mu_F^2)$  is obtained from the leading-order contribution  $d\sigma_{ab}^B(xp_a, yp_b)$  by replacing the corresponding Born-level matrix element squared

$$\frac{1}{n_s(a)n_s(b)} {}_{m,ab < \dots} | \dots >_{m,ab} \quad , \quad (10.12)$$

by

$$\sum_{c,d} \frac{1}{n_s(c)n_s(d)} {}_{m,cd < \dots} | \mathbf{I}^{ab,cd}(x, y; \epsilon) | \dots >_{m,cd} \quad , \quad (10.13)$$

and performing the  $x$  and  $y$  integrations. The insertion operator  $\mathbf{I}(x, y; \epsilon)$  depends on the colour charges, momenta and flavours of the QCD partons. Its explicit expression can be written as follows

$$\begin{aligned}
\mathbf{I}^{ab,cd}(p_1, \dots, p_m; p_a, x; p_b, y; \epsilon; \mu_F^2) &= \delta^{ac} \delta^{bd} \delta(1-x) \delta(1-y) \mathbf{I}(p_1, \dots, p_m, p_a, p_b; \epsilon) \\
&+ \delta^{ac} \delta^{bd} \left[ \delta(1-y) \mathbf{I}_{(1)}(p_1, \dots, p_m; p_a, x; \epsilon) + \delta(1-x) \mathbf{I}_{(1)}(p_1, \dots, p_m; p_b, y; \epsilon) \right] \\
&+ \left[ \delta^{bd} \delta(1-y) \mathbf{I}_{(2)}^{a,c}(p_1, \dots, p_m; p_a, x; p_b, y; \epsilon; \mu_F^2) + \delta^{ac} \delta(1-x) \mathbf{I}_{(2)}^{b,d}(p_1, \dots, p_m; p_a, p_b, y; \epsilon; \mu_F^2) \right] \quad , \quad (10.14)
\end{aligned}$$

where

$$\mathbf{I}(p_1, \dots, p_m, p_a, p_b; \epsilon) = - \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \sum_i \frac{1}{\mathbf{T}_i^2} \mathcal{V}_i(\epsilon) \left[ \sum_{k \neq i} \mathbf{T}_i \cdot \mathbf{T}_k \left( \frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^\epsilon \right. \right.$$

$$\begin{aligned}
& + \mathbf{T}_i \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon + \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_b} \right)^\epsilon \Big] \\
& + \frac{1}{\mathbf{T}_a^2} \mathcal{V}_a(\epsilon) \left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon + \mathbf{T}_b \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_b \cdot p_a} \right)^\epsilon \right] \\
& + \frac{1}{\mathbf{T}_b^2} \mathcal{V}_b(\epsilon) \left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \left( \frac{4\pi\mu^2}{2p_i \cdot p_b} \right)^\epsilon + \mathbf{T}_b \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_b \cdot p_a} \right)^\epsilon \right] \Big\} , \tag{10.15}
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_{(1)}(p_1, \dots, p_m; p_a, x; \epsilon) & = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_i^2} \\
& \cdot \left[ \mathcal{V}_i(x; \epsilon) - \delta(1-x) \mathcal{V}_i(\epsilon) \right] , \tag{10.16}
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_{(2)}^{a,c}(p_1, \dots, p_m; p_a, x; p_b; \epsilon; \mu_F^2) & = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \sum_i \mathbf{T}_i \cdot \mathbf{T}_c \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_c^2} \right. \\
& \cdot \left[ \mathcal{V}^{a,c}(x; \epsilon) - \delta^{ac} \delta(1-x) \mathcal{V}_a(\epsilon) \right] + \mathbf{T}_b \cdot \mathbf{T}_c \left( \frac{4\pi\mu^2}{2p_b \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_c^2} \\
& \cdot \left[ \tilde{\mathcal{V}}^{a,c}(x; \epsilon) - \delta^{ac} \delta(1-x) \mathcal{V}_a(\epsilon) \right] + \left[ -\frac{1}{\epsilon} \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{ac}(x) + K_{\text{F.S.}}^{ac}(x) \right] \Big\} . \tag{10.17}
\end{aligned}$$

All the contributions that are not proportional to  $\delta(1-x)$  in Eqs. (10.16) and (10.17) respectively come from Eqs. (10.7) and (6.13,10.8,10.10). The operator  $\mathbf{I}$  in Eq. (10.15) instead contains all the terms coming from Eq. (10.6) plus those proportional to  $\delta(1-x)$  that have been subtracted in Eqs. (10.16) and (10.17).

We see that  $\mathbf{I}(p_1, \dots, p_m, p_a, p_b; \epsilon)$  in Eq. (10.15) is exactly like that in Eq. (7.26), apart from depending on the additional initial-state partons  $a$  and  $b$ . Therefore it cancels all the singularities in the virtual contribution  $\int_m d\sigma_{ab}^V(p_a, p_b)$ .

The other two operators  $\mathbf{I}_{(1)}$  and  $\mathbf{I}_{(2)}$  contribute as finite counterterms. As a matter of fact, the insertion operator  $\mathbf{I}_{(1)}$  in Eq. (10.16) is exactly the same as the insertion operator in Eqs. (8.26, 8.29) for the case of cross sections with a single incoming parton (note, however, that in Eq. (10.16) colour-charge conservation reads  $\sum_i \mathbf{T}_i = -(\mathbf{T}_a + \mathbf{T}_b)$ ). As for the operator  $\mathbf{I}_{(2)}$ , in order to show that it is finite itself for  $\epsilon \rightarrow 0$ , we rewrite Eq. (10.17) as follows

$$\begin{aligned}
\mathbf{I}_{(2)}^{a,c}(p_1, \dots, p_m; p_a, x; p_b; \epsilon; \mu_F^2) & = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_c \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \right. \right. \\
& + \left. \left. \mathbf{T}_b \cdot \mathbf{T}_c \left( \frac{4\pi\mu^2}{2p_b \cdot p_a} \right)^\epsilon \right] \frac{1}{\mathbf{T}_c^2} \left[ \mathcal{V}^{a,c}(x; \epsilon) + \frac{1}{\epsilon} P^{ac}(x) - \delta^{ac} \delta(1-x) \mathcal{V}_a(\epsilon) \right] \right. \\
& - \left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_c \left( \frac{4\pi\mu^2}{2p_i \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_c^2} + \mathbf{T}_b \cdot \mathbf{T}_c \left( \frac{4\pi\mu^2}{2p_b \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_c^2} + \left( \frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon \right] \frac{1}{\epsilon} P^{ac}(x) \\
& + \left. \left. \mathbf{T}_b \cdot \mathbf{T}_c \left( \frac{4\pi\mu^2}{2p_b \cdot p_a} \right)^\epsilon \frac{1}{\mathbf{T}_c^2} \left[ \tilde{\mathcal{V}}^{a,c}(x; \epsilon) - \mathcal{V}^{a,c}(x; \epsilon) \right] + K_{\text{F.S.}}^{ab}(x) \right\} . \tag{10.18}
\end{aligned}$$



Then, the first term in the curly bracket of Eq. (10.18) gives:

$$-\bar{K}^{ac}(x) - P^{ac}(x) \ln x + \delta^{ac} \mathbf{T}_a^2 \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] + \mathcal{O}(\epsilon) \quad (10.19)$$

where we have used Eq. (8.31) and charge conservation ( $\sum_i \mathbf{T}_i + \mathbf{T}_b = -\mathbf{T}_c$ ). From the second term in the curly bracket of Eq. (10.18) we obtain:

$$-\left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_c \frac{1}{\mathbf{T}_c^2} \ln \frac{\mu_F^2}{2p_i \cdot p_a} + \mathbf{T}_b \cdot \mathbf{T}_c \frac{1}{\mathbf{T}_c^2} \ln \frac{\mu_F^2}{2p_b \cdot p_a} \right] P^{ac}(x) + \mathcal{O}(\epsilon) \quad (10.20)$$

where we have used charge conservation and performed the  $\epsilon$  expansion as in Eq. (8.36). Finally, the third term in the curly bracket of Eq. (10.18) gives:

$$\mathbf{T}_b \cdot \mathbf{T}_c \frac{1}{\mathbf{T}_c^2} \widetilde{K}^{ac}(x) + \delta^{ac} \mathbf{T}_b \cdot \mathbf{T}_a \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] + \mathcal{O}(\epsilon) \quad (10.21)$$

where we have used Eq. (5.155). Collecting these results we find:

$$\begin{aligned} \mathbf{I}_{(2)}^{a,c}(p_1, \dots, p_m; p_a, x; p_b; \epsilon; \mu_F^2) &= -\frac{\alpha_S}{2\pi} \left\{ -\bar{K}^{ac}(x) - P^{ac}(x) \ln x + K_{\text{F.S.}}^{ac}(x) \right. \\ &+ \delta^{ac} \mathbf{T}_a^2 \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] \\ &- \left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_c \frac{1}{\mathbf{T}_c^2} \ln \frac{\mu_F^2}{2p_i \cdot p_a} + \mathbf{T}_b \cdot \mathbf{T}_c \frac{1}{\mathbf{T}_c^2} \ln \frac{\mu_F^2}{2p_b \cdot p_a} \right] P^{ac}(x) \\ &\left. + \mathbf{T}_b \cdot \mathbf{T}_c \frac{1}{\mathbf{T}_c^2} \widetilde{K}^{ac}(x) + \delta^{ac} \mathbf{T}_b \cdot \mathbf{T}_a \left[ \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right)_+ + \frac{2}{1-x} \ln(2-x) \right] \right\} + \mathcal{O}(\epsilon) \quad (10.22) \end{aligned}$$

and, adding  $\mathbf{I}_{(1)}$ , we can write the following final expression

$$\begin{aligned} \int_{m+1} d\sigma_{ab}^A(p, \bar{p}) + \int_m d\sigma_{ab}^C(p, \bar{p}; \mu_F^2) &= \int_m [d\sigma_{ab}^B(p, \bar{p}) \cdot \mathbf{I}(\epsilon)] \\ &+ \sum_{a'} \int_0^1 dx \int_m [\mathbf{K}^{a,a'}(x) \cdot d\sigma_{a'b}^B(xp, \bar{p})] + \sum_{a'} \int_0^1 dx \int_m [\mathbf{P}^{a,a'}(xp, x; \mu_F^2) \cdot d\sigma_{a'b}^B(xp, \bar{p})] \\ &+ \sum_{b'} \int_0^1 dx \int_m [\mathbf{K}^{b,b'}(x) \cdot d\sigma_{ab'}^B(p, x\bar{p})] + \sum_{b'} \int_0^1 dx \int_m [\mathbf{P}^{b,b'}(x\bar{p}, x; \mu_F^2) \cdot d\sigma_{ab'}^B(p, x\bar{p})] \quad , \end{aligned} \quad (10.23)$$

where  $\mathbf{I}(\epsilon)$  is given by Eq. (10.15) and the insertion operators  $\mathbf{K}^{a,b}(x)$  and  $\mathbf{P}^{a,b}(xp, x; \mu_F^2)$  are:

$$\begin{aligned} \mathbf{K}^{a,a'}(x) &= \frac{\alpha_S}{2\pi} \left\{ \bar{K}^{aa'}(x) - K_{\text{F.S.}}^{aa'}(x) \right. \\ &\left. + \delta^{aa'} \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] \right\} - \frac{\alpha_S}{2\pi} \mathbf{T}_b \cdot \mathbf{T}_{a'} \frac{1}{\mathbf{T}_{a'}^2} \widetilde{K}^{aa'}(x) \quad , \end{aligned} \quad (10.24)$$

$$\begin{aligned} \mathbf{P}^{a,a'}(p_1, \dots, p_m, p_b; xp_a, x; \mu_F^2) &= \frac{\alpha_S}{2\pi} P^{aa'}(x) \frac{1}{\mathbf{T}_{a'}^2} \left[ \sum_i \mathbf{T}_i \cdot \mathbf{T}_{a'} \ln \frac{\mu_F^2}{2xp_a \cdot p_i} + \mathbf{T}_b \cdot \mathbf{T}_{a'} \ln \frac{\mu_F^2}{2xp_a \cdot p_b} \right] \quad (10.25) \end{aligned}$$

The operators  $\mathbf{I}(\epsilon)$  and  $\mathbf{P}^{a,a'}(xp, x; \mu_F^2)$  are completely analogous to those in Eqs. (8.25) and (8.39) for the case with a single incoming parton, apart from the trivial dependence on the additional initial-state parton. Note, instead, a new feature of the operator  $\mathbf{K}$ . While the term in the curly bracket on the right-hand side of Eq. (10.24) is equal to that in Eq. (8.38), in the present case there is an additional contribution to  $\mathbf{K}$ , namely  $\widetilde{K}^{aa'}(x)$ , due to *parton-parton correlations* in the initial state.

## 10.2 Final formulae

The results of the previous Subsection can be combined into the following final expressions for the jet cross sections in hadron-hadron scattering processes.

The hadron-level cross section is obtained by convoluting the partonic cross sections on the right-hand side of Eq. (6.10) with the parton densities. The LO parton-level cross section is given by

$$\begin{aligned} \sigma_{ab}^{LO}(p_a, p_b) &= \int_m d\sigma_{ab}^B(p_a, p_b) \\ &= \int d\Phi^{(m)}(p_a, p_b) \frac{1}{n_c(a)n_c(b)} |\mathcal{M}_{m,ab}(p_1, \dots, p_m; p_a, p_b)|^2 F_J^{(m)}(p_1, \dots, p_m; p_a, p_b) \ , \end{aligned} \quad (10.26)$$

where  $a$  and  $b$  denote the flavours of the incoming partons,  $p_a$  and  $p_b$  are their momenta and  $n_c(a), n_c(b)$  are their number of colours. The matrix element  $|\mathcal{M}_{m,ab}|^2$  is the square of the tree-level amplitude to produce  $m$  final-state partons and  $F_J^{(m)}$  is the function that defines the jet observable we want to compute (the properties that  $F_J$  has to fulfil are given in Eqs. (7.2–7.4)) and Eqs. (8.3,8.4)). All the other phase space factors on the right-hand side of Eq. (10.1) are collected into the factor  $d\Phi^{(m)}(p_a, p_b)$ .

According to the general notation in Eq. (2.15), the full NLO partonic cross section is obtained by adding three different types of contribution:

$$\begin{aligned} \sigma_{ab}^{NLO}(p_a, p_b; \mu_F^2) &= \sigma_{ab}^{NLO\{m+1\}}(p_a, p_b) + \sigma_{ab}^{NLO\{m\}}(p_a, p_b) \\ &+ \int_0^1 dx \left[ \hat{\sigma}_{ab}^{NLO\{m\}}(x; xp_a, p_b, \mu_F^2) + \hat{\sigma}_{ab}^{NLO\{m\}}(x; p_a, xp_b, \mu_F^2) \right] \ . \end{aligned} \quad (10.27)$$

The first contribution has  $m + 1$ -parton kinematics and is given by the following expression

$$\begin{aligned} \sigma_{ab}^{NLO\{m+1\}}(p_a, p_b) &= \int_{m+1} \left[ \left( d\sigma_{ab}^R(p_a, p_b) \right)_{\epsilon=0} \right. \\ &- \left( \sum_{\text{dipoles}} d\sigma_{ab}^B(p_a, p_b) \otimes \left( dV_{\text{dipole}} + dV'_{\text{dipole}} \right) \right)_{\epsilon=0} \left. \right] \\ &= \int d\Phi^{(m+1)}(p_a, p_b) \left\{ \frac{1}{n_c(a)n_c(b)} |\mathcal{M}_{m+1,ab}(p_1, \dots, p_{m+1}; p_a, p_b)|^2 F_J^{(m+1)}(p_1, \dots, p_{m+1}; p_a, p_b) \right. \\ &- \left. \sum_{\text{dipoles}} \left( \mathcal{D} \cdot F^{(m)} \right) (p_1, \dots, p_{m+1}; p_a, p_b) \right\} \ , \end{aligned} \quad (10.28)$$

where  $\mathcal{M}_{m+1,ab}$  is the tree-level matrix element with  $m + 1$  partons in the final state and  $\sum_{\text{dipoles}} \left( \mathcal{D} \cdot F^{(m)} \right) (p_1, \dots, p_{m+1}; p_a, p_b)$  is the sum of the dipole factors contained into the curly bracket on the right-hand side of Eq. (10.4).

The NLO contribution with  $m$ -parton kinematics is obtained by adding the virtual cross section and the first term on the right-hand side of Eq. (10.23). As in all the other scattering processes, its explicit expression is given in terms of the square of the one-loop matrix element  $|\mathcal{M}_{m,ab}|_{(1-loop)}^2$  and of the insertion operator  $\mathbf{I}(\epsilon)$ :

$$\begin{aligned} \sigma_{ab}^{NLO\{m\}}(p_a, p_b) &= \int_m \left[ d\sigma_{ab}^V(p_a, p_b) + d\sigma_{ab}^B(p_a, p_b) \otimes \mathbf{I} \right]_{\epsilon=0} \\ &= \int d\Phi^{(m)}(p_a, p_b) \left\{ \frac{1}{n_c(a)n_c(b)} |\mathcal{M}_{m,ab}(p_1, \dots, p_m; p_a, p_b)|_{(1-loop)}^2 \right. \\ &\quad \left. + \left. \sum_{m,ab < 1, \dots, m; a, b} \mathbf{I}(\epsilon) \right| 1, \dots, m; a, b \right. \left. \sum_{\epsilon=0} F_J^{(m)}(p_1, \dots, p_m; p_a, p_b) \right\} . \end{aligned} \quad (10.29)$$

In the present case, the colour-charge operator  $\mathbf{I}(\epsilon)$  is explicitly written down in Eq. (10.15) (see also Appendix C).

The third term on the right-hand side of Eq. (10.27) comes from the second and third lines on the right-hand side of Eq. (10.23) and contains two contributions that are similar to that involved in the processes with a single incoming hadron (cfr. Eq. (8.46)). Each of these contributions is obtained by integrating a cross section with  $m$ -parton kinematics with respect to the fraction  $x$  of the longitudinal momentum carried by one of the incoming partons. When this parton is the parton  $a$ , we explicitly have:

$$\begin{aligned} \int_0^1 dx \hat{\sigma}_{ab}^{NLO\{m\}}(x; xp_a, p_b, \mu_F^2) &= \sum_{a'} \int_0^1 dx \int_m \left[ d\sigma_{a'b}^B(xp_a, p_b) \otimes (\mathbf{K} + \mathbf{P})^{a,a'}(x) \right]_{\epsilon=0} \\ &= \sum_{a'} \int_0^1 dx \int d\Phi^{(m)}(xp_a, p_b) F_J^{(m)}(p_1, \dots, p_m; xp_a, p_b) \\ &\quad \cdot \left. \sum_{m, a'b < 1, \dots, m; xp_a, p_b} \left( \mathbf{K}^{a,a'}(x) + \mathbf{P}^{a,a'}(xp_a, x; \mu_F^2) \right) \right| 1, \dots, m; xp_a, p_b \right. \left. \sum_{\epsilon=0} \right\} , \end{aligned} \quad (10.30)$$

where the colour-charge operators  $\mathbf{K}$  and  $\mathbf{P}$  are respectively defined in Eqs. (10.24) and (10.25) (see also Appendix C). The expression for  $\hat{\sigma}_{ab}^{NLO\{m\}}(x; p_a, xp_b, \mu_F^2)$  is completely analogous to Eq. (10.30), apart from the replacements  $xp_a \rightarrow p_a, p_b \rightarrow xp_b$  and  $\sum_{a'} \rightarrow \sum_{b'}$  (as in Eq. (10.23)). Note that the right-hand side of Eq. (10.30) has exactly the same structure as in Eq. (8.49) for the case with a single incoming parton. However, we should recall that the colour-charge operator  $\mathbf{K}^{a,a'}$  entering into Eq. (10.30) differs from that appearing into Eq. (8.49) by the additional correlation term  $\widetilde{K}^{a,a'}$  (see Eq. (10.24)), which is due to the presence of the other incoming parton  $b$ .

Equations (10.26), (10.28) and (10.30) are directly evaluated in four space-time dimensions. As for Eq. (10.29), one should first cancel analytically the  $\epsilon$  poles of the one-loop matrix element with those of the insertion operator  $\mathbf{I}$ , perform the limit  $\epsilon \rightarrow 0$  and then carry out the phase-space integration in four space-time dimensions.

# 11 Multi-particle correlations

In the case of processes involving multi-particle correlations (see Eq. (6.20)), the partonic cross section at the Born level is given by

$$\begin{aligned}
d\sigma_{ab,(incl)a_1,\dots,a_n}^B(p,\bar{p};q_1,\dots,q_n) &= \mathcal{N}_{in} \frac{1}{n_s(a)n_s(b)n_c(a)n_c(b)\Phi(p\cdot\bar{p})} \\
&\cdot \sum_{\{m\}} d\phi_m(p_1,\dots,p_m;p+\bar{p}+Q-q_1-\dots-q_n) \frac{1}{S_{\{m\}}} \\
&\cdot |\mathcal{M}_{m+a_1+a_2,\dots,ab}(q_1,\dots,q_n,p_1,\dots,p_m;p,\bar{p})|^2 F_J^{(m)}([q_1],\dots,[q_n],p_1,\dots,p_m;p,\bar{p}) \quad . \quad (11.1)
\end{aligned}$$

Here, we denote by  $a$  and  $b$  the flavour indices of the two incoming partons with momenta  $p$  and  $\bar{p}$ , while  $a_1, \dots, a_n$  are the flavour indices of the final-state identified partons with momenta  $q_1, \dots, q_n$ . In addition, the leading-order cross section in Eq. (11.1) has  $m$  final-state unidentified partons with momenta  $p_1, \dots, p_m$  (non-QCD partons are understood). The jet defining function  $F_J$  has the properties already discussed in Sects. 7–10, namely, infrared and collinear safety and factorizability of initial- and final-state collinear singularities. Remember that, by definition, the momenta  $p, \bar{p}, q_1, \dots, q_n$  are supposed not to be parallel to each other.

## 11.1 Implementation of the subtraction procedure

According to our general procedure, we write the NLO partonic cross section (6.23) in the following form

$$\begin{aligned}
&\sigma_{ab,(incl)a_1,\dots,a_n}^{NLO}(p,\bar{p};q_1,\dots,q_n;\mu_F^2,\mu_1^2,\dots,\mu_n^2) \\
&= \int_{m+1} \left( d\sigma_{ab,(incl)a_1,\dots,a_n}^R(p,\bar{p};q_1,\dots,q_n) - d\sigma_{ab,(incl)a_1,\dots,a_n}^A(p,\bar{p};q_1,\dots,q_n) \right) \\
&+ \left[ \int_{m+1} d\sigma_{ab,(incl)a_1,\dots,a_n}^A(p,\bar{p};q_1,\dots,q_n) + \int_m d\sigma_{ab,(incl)a_1,\dots,a_n}^V(p,\bar{p};q_1,\dots,q_n) \right. \\
&\left. + \int_m d\sigma_{ab,(incl)a_1,\dots,a_n}^C(p,\bar{p};q_1,\dots,q_n;\mu_F^2,\mu_1^2,\dots,\mu_n^2) \right] \quad , \quad (11.2)
\end{aligned}$$

where the subtraction term is defined by

$$\begin{aligned}
d\sigma_{ab,(incl)a_1,\dots,a_n}^A(p,\bar{p};q_1,\dots,q_n) &= \mathcal{N}_{in} \frac{1}{n_s(a)n_s(b)\Phi(p\cdot\bar{p})} \\
&\cdot \sum_{\{m+1\}} d\phi_{m+1}(p_1,\dots,p_{m+1};p+\bar{p}+Q-q_1-\dots-q_n) \frac{1}{S_{\{m+1\}}} \\
&\cdot \left\{ \sum_{\substack{\text{pairs} \\ i,j}} (\mathcal{D} \cdot F_J^{(m)})_{ij} + \sum_{l=1}^n \sum_i (\mathcal{D} \cdot F_J^{(m)})_{ai} + \sum_i \left[ (\mathcal{D} \cdot F_J^{(m)})^{ai} + (\mathcal{D} \cdot F_J^{(m)})^{bi} \right] \right\} \quad . \quad (11.3)
\end{aligned}$$

Here we have introduced the shorthand notation  $\mathcal{D} \cdot F_J^{(m)}$  to denote the different dipole contributions in which the emitter is a final-state unidentified parton, a final-state identified parton or an initial-state parton. Their explicit expressions, according to the dipole

formulae in Sect. 5, are respectively the following

$$\begin{aligned}
(\mathcal{D} \cdot F_J^{(m)})_{ij} &= \sum_{k \neq i, j} \mathcal{D}_{ij,k}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) F_J^{(m)}([q_1], \dots, [q_n], p_1, \dots, \tilde{p}_{ij}, \tilde{p}_k, \dots, p_{m+1}; p, \bar{p}) \\
&+ \sum_{l=1}^n \mathcal{D}_{ij,a_l}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) F_J^{(m)}([q_1], \dots, [\tilde{q}_l], \dots, [q_n], p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; p, \bar{p}) \\
&+ \mathcal{D}_{ij}^a(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) F_J^{(m)}([q_1], \dots, [q_n], p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; \tilde{p}, \bar{p}) \\
&+ \mathcal{D}_{ij}^b(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) F_J^{(m)}([q_1], \dots, [q_n], p_1, \dots, \tilde{p}_{ij}, \dots, p_{m+1}; p, \tilde{p}) \quad , \quad (11.4)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{D} \cdot F_J^{(m)})_{a_i} &= \sum_{k \neq i} \mathcal{D}_{a_i,k}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) \\
&\cdot F_J^{(m)}([q_1], \dots, [\tilde{q}_l], \dots, [q_n], p_1, \dots, \tilde{p}_k, \dots, p_{m+1}; p, \bar{p}) \\
&+ \left[ \sum_{\substack{r=1 \\ r \neq l}}^n \mathcal{D}_{a_i,a_r}^{(n)}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) + \mathcal{D}_{a_i}^{(n)a}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) \right. \\
&\left. + \mathcal{D}_{a_i}^{(n)b}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) \right] F_J^{(m)}([q_1], \dots, [\tilde{q}_l], \dots, [q_n], \tilde{p}_1, \dots, \tilde{p}_{m+1}; p, \bar{p}) \quad , \quad (11.5)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{D} \cdot F_J^{(m)})^{ai} + (a \leftrightarrow b) &= \left\{ \sum_{k \neq i} \mathcal{D}_k^{ai}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) \right. \\
&\cdot F_J^{(m)}([q_1], \dots, [q_n], p_1, \dots, \tilde{p}_k, \dots, p_{m+1}; \tilde{p}, \bar{p}) \\
&+ \left[ \sum_{l=1}^n \mathcal{D}_{a_i}^{(n)ai}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) + \mathcal{D}^{(n)ai,b}(q_1, \dots, q_n, p_1, \dots, p_{m+1}; p, \bar{p}) \right] \\
&\left. \cdot F_J^{(m)}([q_1], \dots, [q_n], \tilde{p}_1, \dots, \tilde{p}_{m+1}; \tilde{p}, \bar{p}) \right\} + (a \leftrightarrow b) \quad . \quad (11.6)
\end{aligned}$$

Comparing Eqs. (11.3–11.6) with the form of the subtraction terms introduced in Sects. 7–10, we see that the only new feature of the present case is due to the ‘pseudodipoles’  $\mathcal{D}_{a_i,a_r}^{(n)}$ ,  $\mathcal{D}_{a_i}^{(n)a}$ ,  $\mathcal{D}_{a_i}^{(n)b}$ ,  $\mathcal{D}_{a_i}^{(n)ai}$ ,  $\mathcal{D}^{(n)ai,b}$  where both the emitter and the spectator are incoming partons or identified final-state partons. These dipoles are defined in Sect. 5.6. Note, in particular, that no distinction is made between initial- and final-state spectator (i.e.  $\mathcal{D}_{a_i}^{(n)a} = \mathcal{D}_{a_i,a}^{(n)}$  in Eq. (11.5) and  $\mathcal{D}^{(n)ai,b} = \mathcal{D}_b^{(n)ai}$  in Eq. (11.6)) and that the dipole partonic states  $\{q_1, \dots, \tilde{q}_l, \dots, q_n, \tilde{p}_1, \dots, \tilde{p}_{m+1}, p, \bar{p}\}$ ,  $\{q_1, \dots, q_n, \tilde{p}_1, \dots, \tilde{p}_{m+1}, \tilde{p}, \bar{p}\}$ ,  $\{q_1, \dots, q_n, \tilde{p}_1, \dots, \tilde{p}_{m+1}, p, \tilde{p}\}$  depend only on the emitter (the corresponding jet functions  $F_J$  appear as common factors in Eqs. (11.5,11.6)).

The subtracted contribution ( $d\sigma^R - d\sigma^A$ ) in Eq. (11.2) is integrable in four dimensions by explicit construction.

In order to evaluate the  $d$ -dimensional integral of  $d\sigma^A$ , we decompose it as follows

$$\int_{m+1} d\sigma^A = \int_{m+1} (d\sigma^{A'} + d\sigma^{A''} + d\sigma^{A'''} + d\sigma^{A''''}) \quad . \quad (11.7)$$

The first three terms on the right-hand side of Eq. (11.7) respectively correspond to the integral of  $(\mathcal{D} \cdot F_J)_{ij}$ ,  $\mathcal{D}_{a_i,k}$  and  $\mathcal{D}_k^{ai}$ . Their treatment has been already considered in Sects. 7–9. The fourth term contains the integral of all the pseudodipoles and can be handled as

$d\sigma^{A''''}$  in Sect. 10. As a result of this integration procedure we find:

$$\begin{aligned}
& \int_{m+1} d\sigma_{ab,(incl)a_1,\dots,a_n}^A(p, \bar{p}; q_1, \dots, q_n) + \int_m d\sigma_{ab,(incl)a_1,\dots,a_n}^C(p, \bar{p}; q_1, \dots, q_n; \mu_F^2, \mu_1^2, \dots, \mu_n^2) \\
&= \int_m \left[ d\sigma_{ab,(incl)a_1,\dots,a_n}^B(p, \bar{p}; q_1, \dots, q_n) \cdot \mathbf{I}(\epsilon) \right] \\
&+ \sum_{a'} \int_0^1 dx \int_m \left[ \left( \mathbf{K}^{a,a'}(x) + \mathbf{P}^{a,a'}(xp, x; \mu_F^2) \right) \cdot d\sigma_{a'b,(incl)a_1,\dots,a_n}^B(xp, \bar{p}; q_1, \dots, q_n) \right] \\
&+ \sum_{b'} \int_0^1 dx \int_m \left[ \left( \mathbf{K}^{b,b'}(x) + \mathbf{P}^{b,b'}(x\bar{p}, x; \mu_F^2) \right) \cdot d\sigma_{ab',(incl)a_1,\dots,a_n}^B(p, x\bar{p}; q_1, \dots, q_n) \right] \\
&+ \sum_{l=1}^n \sum_{a'_l} \int_0^1 \frac{dz}{z^2} \int_m \left[ d\sigma_{ab,(incl)a_1,\dots,a'_l,\dots,a_n}^B(p, \bar{p}; q_1, \dots, q_l/z, \dots, q_n) \cdot \left( \mathbf{H}_{a'_l,a_l}(z) + \mathbf{P}_{a'_l,a_l}(q_l/z, z; \mu_l^2) \right) \right].
\end{aligned} \tag{11.8}$$

The factor  $\mathbf{I}(\epsilon)$  comes from the integration of  $\mathcal{D}_{ij,k}$  and, in addition, collects all the  $\epsilon$ -poles due to the other dipole factors. The finite parts of the dipoles  $\mathcal{D}_{ij}^a$ ,  $\mathcal{D}_{ij}^b$ ,  $\mathcal{D}_k^{ai}$ ,  $\mathcal{D}_{a_i}^{(n)ai}$ ,  $\mathcal{D}^{(n)ai,b}$ ,  $\mathcal{D}^{(n)bi,a}$  contribute to the initial-state operators  $\mathbf{K}(x) + \mathbf{P}(xp, x; \mu_F^2)$ , and those of the dipoles  $\mathcal{D}_{ij,a_i}$ ,  $\mathcal{D}_{a_i i,k}$ ,  $\mathcal{D}_{a_i i,a_r}^{(n)}$ ,  $\mathcal{D}_{a_i i}^{(n)a}$ ,  $\mathcal{D}_{a_i i}^{(n)b}$  contribute to the final-state operators  $\mathbf{H}(z) + \mathbf{P}(q_l/z, z; \mu_l^2)$ .

The insertion operator  $\mathbf{I}(\epsilon)$  in Eq. (11.8) is fully symmetric with respect to all the QCD partons. Therefore, denoting by  $I$  a generic QCD parton ( $I = \{i, a_l, a, b\}$ ), we can write (the singular factors  $\mathcal{V}_I(\epsilon)$  are given in Eq. (7.27)):

$$\mathbf{I}(q_1, \dots, q_n, p_1, \dots, p_m, p, \bar{p}; \epsilon) = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_I \frac{1}{\mathbf{T}_I^2} \mathcal{V}_I(\epsilon) \sum_{J \neq I} \mathbf{T}_I \cdot \mathbf{T}_J \left( \frac{4\pi\mu^2}{2p_I \cdot p_J} \right)^\epsilon. \tag{11.9}$$

The initial-state operator  $\mathbf{P}^{a,a'}(xp, x; \mu_F^2)$  is instead symmetric with respect to all the partons except  $p$ . Its explicit expression is:

$$\mathbf{P}^{a,a'}(q_1, \dots, q_n, p_1, \dots, p_m, \bar{p}; xp, x; \mu_F^2) = \frac{\alpha_S}{2\pi} P^{aa'}(x) \frac{1}{\mathbf{T}_{a'}^2} \sum_{I \neq a'} \mathbf{T}_I \cdot \mathbf{T}_{a'} \ln \frac{\mu_F^2}{2xp \cdot p_I}. \tag{11.10}$$

Similarly, for the final-state operator  $\mathbf{P}_{a'_l,a_l}(q_l/z, z; \mu_l^2)$  we find

$$\mathbf{P}_{a'_l,a_l}(q_1, \dots, q_n, p_1, \dots, p_m, p, \bar{p}; q_l/z, z; \mu_l^2) = \frac{\alpha_S}{2\pi} P_{a'_l,a_l}(z) \frac{1}{\mathbf{T}_{a'_l}^2} \sum_{I \neq a'_l} \mathbf{T}_I \cdot \mathbf{T}_{a'_l} \ln \frac{z\mu_l^2}{2q_l \cdot p_I}. \tag{11.11}$$

The insertion operators  $\mathbf{K}^{a,a'}(x)$  and  $\mathbf{H}_{a'_l,a_l}(z)$  are separately symmetric with respect to the sets of the unidentified and identified (or initial-state) partons. They are given by the following equations

$$\begin{aligned}
\mathbf{K}^{a,a'}(x) &= \frac{\alpha_S}{2\pi} \left\{ \bar{K}^{aa'}(x) - K_{\text{F.S.}}^{aa'}(x) \right. \\
&+ \delta^{aa'} \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] - \frac{1}{\mathbf{T}_{a'}^2} \left( \sum_{l=1}^n \mathbf{T}_{a_l} \cdot \mathbf{T}_{a'} + \mathbf{T}_b \cdot \mathbf{T}_{a'} \right) \bar{K}^{aa'}(x) \\
&\left. - \frac{1}{\mathbf{T}_{a'}^2} \left[ \sum_{l=1}^n \mathbf{T}_{a_l} \cdot \mathbf{T}_{a'} \mathcal{L}^{a,a'}(x; p, q_l, n) + \mathbf{T}_b \cdot \mathbf{T}_{a'} \mathcal{L}^{a,a'}(x; p, \bar{p}, n) \right] \right\}, \tag{11.12}
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}_{a'_i, a_i}(z) = & \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{a'_i a_i}(z) + 3P_{a'_i a_i}(z) \ln z - H_{a'_i a_i}^{\text{F.S.}}(z) \right. \\
& + \delta_{a'_i a_i} \sum_i \mathbf{T}_i \cdot \mathbf{T}_{a'_i} \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-z} \right)_+ + \delta(1-z) - 1 \right] \\
& + \frac{1}{\mathbf{T}_{a'_i}^2} \left( \sum_{\substack{r=1 \\ r \neq l}}^n \mathbf{T}_{a_r} \cdot \mathbf{T}_{a'_i} + \mathbf{T}_a \cdot \mathbf{T}_{a'_i} + \mathbf{T}_b \cdot \mathbf{T}_{a'_i} \right) \left[ P_{a'_i a_i}(z) \ln z - \widetilde{K}^{a'_i a_i}(z) \right] \\
& - \frac{1}{\mathbf{T}_{a'_i}^2} \left[ \sum_{\substack{r=1 \\ r \neq l}}^n \mathbf{T}_{a_r} \cdot \mathbf{T}_{a'_i, a_i} \mathcal{L}^{a'_i, a_i}(z; q_l, q_r, n) + \mathbf{T}_a \cdot \mathbf{T}_{a'_i} \mathcal{L}^{a'_i, a_i}(z; q_l, p, n) \right. \\
& \left. + \mathbf{T}_b \cdot \mathbf{T}_{a'_i} \mathcal{L}^{a'_i, a_i}(z; q_l, \bar{p}, n) \right] \left. \right\} , \tag{11.13}
\end{aligned}$$

where the flavour kernels  $\overline{K}^{ab}$ ,  $K_{\text{F.S.}}^{ab}$ ,  $\widetilde{K}^{ab}$ ,  $\mathcal{L}^{a,b}$ ,  $H_{ba}^{\text{F.S.}}$  are defined respectively in Eqs. (8.32–8.35), (6.6), (5.156), (5.176) and (6.17).

Note that the operators  $\mathbf{I}$  and  $\mathbf{P}$  in Eqs. (11.9–11.11) are completely analogous to the corresponding operators defined in Sects. 7–10. Some new features instead appear in Eqs. (11.12) and (11.13). Comparing Eq. (11.12) with Eq. (10.24), we find a new contribution (that in the last square bracket of (11.12)) due to *correlations* between the incoming parton  $a$  and the identified final-state partons. As a result, the insertion operator  $\mathbf{K}$  explicitly depends on the momenta of the identified partons unlike the previous cases. This contribution is indeed non-vanishing only if there are final-state identified partons. A similar term is also present in the expression (11.13) for  $\mathbf{H}$ . However, in this case also the term in the second square bracket on the right-hand side of Eq. (11.13) has no analogue in the corresponding Eq. (9.19). This term is due to *final-state* parton correlations and is similar to that proportional to  $\widetilde{K}^{aa'}(x)$  on the right-hand side of Eq. (11.12). In this respect the difference between initial- and final-state parton correlations simply amounts to the replacement  $\widetilde{K}^{a',a}(z) \rightarrow \widetilde{K}^{a',a}(z) - P^{a',a}(z) \ln z$ .

## 11.2 Final formulae

The results of the previous Subsection can be summarized by the following final formulae.

The LO parton-level cross section, which enters into the calculation of the multi-particle hadronic cross section of Eq. (6.21), is obtained as follows

$$\begin{aligned}
\sigma_{ab,(\text{incl})a_1, \dots, a_n}^{LO}(p, \bar{p}; q_1, \dots, q_n) &= \int_m d\sigma_{ab,(\text{incl})a_1, \dots, a_n}^B(p, \bar{p}; q_1, \dots, q_n) \\
&= \int d\Phi^{(m)}(p, \bar{p}; q_1, \dots, q_n) \frac{1}{n_c(a)n_c(b)} |\mathcal{M}_{m+a_1+a_2, \dots, ab}(q_1, \dots, q_n, p_1, \dots, p_m; p, \bar{p})|^2 \\
&\cdot F_J^{(m)}([q_1], \dots, [q_n], p_1, \dots, p_m; p, \bar{p}) . \tag{11.14}
\end{aligned}$$

Here  $n_c(a)$  and  $n_c(b)$  are the number of colours of the incoming partons,  $\mathcal{M}_{m+a_1+a_2, \dots, ab}$  is the tree-level matrix element to produce  $m$  final-state partons in addition to the identified partons, and  $F_J^{(m)}$  is the most general jet defining function (it fulfils Eqs. (7.2–7.4), Eqs. (8.3,8.4) and Eqs. (9.2,9.3) with respect to the dependence on the momenta of the





The last three terms on the right-hand side of Eq. (11.15) are in one-to-one correspondence with the last three terms on the right-hand side of Eq. (11.8). Their expressions as a function of the QCD matrix elements are the following

$$\begin{aligned}
& \int_0^1 dx \hat{\sigma}_{ab,(incl)a_1,\dots,a_n}^{NLO\{m\}}(x; xp, \bar{p}; q_1, \dots, q_n; \mu_F^2) \\
&= \sum_{a'} \int_0^1 dx \int_m \left[ d\sigma_{a'b,(incl)a_1,\dots,a_n}^B(x; xp, \bar{p}; q_1, \dots, q_n) \otimes (\mathbf{K} + \mathbf{P})^{a,a'}(x) \right]_{\epsilon=0} \\
&= \sum_{a'} \int_0^1 dx \int d\Phi^{(m)}(xp, \bar{p}; q_1, \dots, q_n) F_J^{(m)}([q_1], \dots, [q_n], p_1, \dots, p_m; xp, \bar{p}) \quad (11.18) \\
&\cdot m+a_1\dots a'b < q_1, \dots, q_n, 1, \dots, m; xp, \bar{p} \mid \left( \mathbf{K}^{a,a'}(x) \right. \\
&\left. + \mathbf{P}^{a,a'}(xp, x; \mu_F^2) \right) \mid q_1, \dots, q_n, 1, \dots, m; xp, \bar{p} >_{m+a_1\dots a'b} \cdot,
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \frac{dz}{z^2} \hat{\sigma}_{ab,(incl)a_1,\dots,a_n}^{NLO\{m\}}(z; p, \bar{p}; q_1, \dots, q_l/z, \dots, q_n; \mu_l^2) \\
&= \sum_{a'_i} \int_0^1 \frac{dz}{z^2} \int_m \left[ d\sigma_{ab,(incl)a_1,\dots,a'_i,\dots,a_n}^B(z; p, \bar{p}; q_1, \dots, q_l/z, \dots, q_n) \otimes (\mathbf{H} + \mathbf{P})_{a'_i,a_l}(z) \right]_{\epsilon=0} \\
&= \sum_{a'_i} \int_0^1 \frac{dz}{z^2} \int d\Phi^{(m)}(p, \bar{p}; q_1, \dots, q_l/z, \dots, q_n) F_J^{(m)}([q_1], \dots, [q_l/z], \dots, [q_n], p_1, \dots, p_m; p, \bar{p}) \quad (11.19) \\
&\cdot m+\dots+a'_i+\dots,ab < q_1, \dots, q_l/z, \dots, q_n, 1, \dots, m; p, \bar{p} \mid \left( \mathbf{H}_{a'_i,a_l}(z) \right. \\
&\left. + \mathbf{P}_{a'_i,a_l}(q_l/z, z; \mu_l^2) \right) \mid q_1, \dots, q_l/z, \dots, q_n, 1, \dots, m; p, \bar{p} >_{m+\dots+a'_i+\dots,ab} \cdot.
\end{aligned}$$

The contribution  $\hat{\sigma}_{ab,(incl)a_1,\dots,a_n}^{NLO\{m\}}(x; p, x\bar{p}; q_1, \dots, q_n; \mu_F^2)$  is obtained from Eq. (11.18) by the replacements  $xp \rightarrow p$ ,  $\bar{p} \rightarrow x\bar{p}$ ,  $\sum_{a'} \rightarrow \sum_{b'}$  (see Eq. (11.8)). Note that each of the contributions  $\hat{\sigma}_{ab,(incl)a_1,\dots,a_n}^{NLO\{m\}}$  in Eqs. (11.15,11.18,11.19) depends on a single factorization scale.

The colour-charge operators  $\mathbf{K}$  and  $\mathbf{P}$  of Eq. (11.18) are defined in Eqs. (11.12) and (11.10) respectively. The colour-charge operators  $\mathbf{H}$  and  $\mathbf{P}$  of Eq. (11.19) are respectively given in Eqs. (11.13) and (11.11). The definitions of the related flavour kernels  $P^{aa'}(x)$ ,  $\bar{K}^{aa'}(x)$ ,  $K_{\text{FS}}^{aa'}(x)$ ,  $\widetilde{K}^{aa'}(x)$ ,  $H_{a'a}^{\text{FS}}(x)$  and of the functions  $\mathcal{L}^{a,a'}$  are also recalled in Appendix C.

The actual evaluation of Eqs. (11.14), (11.16), (11.18) and (11.19) is directly performed in four space-time dimensions. As for Eq. (11.17), one should first cancel analytically the  $\epsilon$  poles of the one-loop matrix element with those of the insertion operator  $\mathbf{I}$ , perform the limit  $\epsilon \rightarrow 0$  and then carry out the phase-space integration in four space-time dimensions.

Note that the formulae presented in this Subsection can be applied also to the simplified cases of multi-particle correlations with a single incoming hadron or with no hadrons in the initial state. For this purpose, it is sufficient to remove one or both of the contributions in the square bracket on the right-hand side of Eq. (11.15) and to set equal to zero the colour charge of the removed incoming parton both in the dipole factors of Eq. (11.16) and in the colour-charge operators of Eqs. (11.17–11.19).

The results obtained in this Section show that, in the most general case of multi-particle correlations, the NLO partonic cross section in Eq. (11.15) is finite, i.e. free from soft and collinear singularities. We should point out that its finiteness directly follows from the primitive definition in Eq. (6.23), which is nothing other than the formal restatement, to the lowest non-trivial order, of the factorization theorem of mass singularities [30]. Doubts have been raised in the past about the validity of the theorem for processes involving more than one identified parton, for instance, the Drell-Yan process. Although those doubts have been proved to be unfounded, a Cartesian (and fully general) proof of the factorization theorem in the context of QCD is still missing. Our check, by means of the explicit calculations in this Section, that the partonic cross section in Eq. (6.23) is finite is, to our knowledge, the first proof (at NLO) of the factorization theorem in the most general case of QCD jet cross sections.

## 12 Summary and discussion

### 12.1 Summary

In this paper, we have introduced a new general algorithm for calculating arbitrary jet cross sections to NLO in arbitrary scattering processes, based on the subtraction method. The key ingredients are the dipole factorization formulae, which implement both the usual soft and collinear factorization formulae, smoothly interpolating the two. The corresponding dipole phase space obeys exact factorization, so that the dipole contributions to the cross section can be exactly integrated analytically over the whole of phase space.

The steps that are necessary to set up a general method for evaluating NLO QCD cross sections are recalled in Sect. 1. In order to compute a jet quantity to NLO, one must calculate the contributions from one-loop corrections to the Born-level cross section and from the cross section for processes in which one additional parton is present in the final state. Each leads to singularities, which are regularized by working in a number of dimensions  $d = 4 - 2\epsilon$  other than four. Analytical integration must then be used to extract the singular terms as poles in  $\epsilon$  and combine them with one another to yield a finite result. However, almost all experimentally important jet quantities are sufficiently complicated that analytical integration is impossible and one must resort to numerical techniques. Thus one must somehow extract the singular parts of the cross section in a way that is independent of the exact details of the observable and treat them analytically. This leaves a remainder that depends on the full complications of the jet quantity, but which is finite so can be treated numerically.

One way of doing this is provided by the subtraction method, described in Sect. 2.1. This works by introducing an approximate cross section, which is defined in such a way as to match all the singularities of the real cross section to produce one additional parton. Thus the difference between the two is guaranteed to be finite and numerical integration can be used in 4 dimensions. However, the approximate cross section is required to be simple enough to be integrated analytically without knowing the details of the definition of the jet observable.

This is achieved by considering, for every  $m + 1$ -parton configuration, a mapping to a ‘similar’  $m$ -parton configuration (or in general, several of them), defined such that in the singular regions of phase space the two configurations become indistinguishable. The approximate cross section is then defined to be proportional to the jet quantity calculated from this similar  $m$ -parton configuration. Thus, the analytical integration can be performed without knowing the definition of the jet quantity, by holding the  $m$ -parton configuration fixed and integrating over all  $m + 1$ -parton configurations that map onto it. This gives rise to  $\epsilon$  poles, which cancel those from the one-loop cross section to produce that same  $m$ -parton configuration. The result is a finite  $m$ -parton cross section that can then be integrated numerically.

Note that, using the subtraction method, no approximation is actually performed in the evaluation of the NLO cross section. The approximate cross section is subtracted from the real cross section and then added back to the one-loop cross section. Moreover, and most

importantly, no approximation is performed in the analytical integration of the approximate cross section. In that respect, the adjective ‘approximate’ may sound misleading. Rather than approximating the true cross section, the subtracted contribution defines a fake cross section that has the same dynamical singularities as the real cross section and whose kinematics are sufficiently simple to allow its analytic integration.

To define this fake cross section, one must find an approximation (in  $d$  dimensions) to the matrix element (or its square) that matches the real matrix element in all the singular regions of phase space. While this can be done by calculating the full matrix element in  $d$  dimensions and taking all the relevant limits, this is an extremely laborious and ungeneralizable procedure. Instead, one can use an approach based on the soft and collinear factorization theorems (see Sect. 4), which guarantee that the singular terms are process independent. Thus one can define the subtraction cross section as a sum of soft and collinear pieces in a completely process-independent way and integrate it once-and-for-all.

In fact, we go one step further even than this. By introducing a *dipole* factorization theorem (see Sect. 5), we are able to construct a single approximation to the matrix element that matches all of its soft and collinear singularities. This approximation is given in terms of the Born-level matrix element times universal (process independent) dipole factors. Thus we can provide a *single* formula that approximates the real matrix element (squared) for an arbitrary process, in *all* of its singular limits. This avoids problems that can arise from using the separate soft and collinear approximations, where one must either define an explicit arbitrary cutoff to separate the two regions, or add back on another term to compensate for the double-counting of the region of overlapping singularities that gives rise to double poles.

While the soft and collinear factorization theorems dictate the form the dipole factor must take in the exactly singular limits, we are free to choose the extrapolation away from those limits arbitrarily. Clearly we should use this freedom to make the necessary analytical integrals as straightforward as possible. Physical processes in which some external momenta are fixed (by incoming hadrons or by measurement of outgoing hadrons) impose additional constraints on the phase space. In order to ensure that the phase-space integrals are still analytically tractable, we make different choices for the extrapolation away from these limits, depending on which of the participating parton momenta are fixed by external hadrons. Thus we end up with a finite set of different dipole formulae (see Sects. 5.1–5.6), applicable to different physical processes.

Closely related to the choice of dipole formulae is the definition of the ‘similar’  $m$ -parton configuration. Each dipole factor smoothly describes the merging of three partons into two new partons (emitter and spectator) while one of the three partons approaches a singular region. Thus, for each of the different constraints on the  $m + 1$ -parton phase space, we are able to define a *one-to-one* mapping to a set of  $m$  momenta plus a single-parton momentum in such a way that the single-parton subspace obeys *exact* phase space factorization. That is, the  $m + 1$ -parton phase space can be written as the product of a *physical*  $m$ -parton phase space times the dipole phase space. By physical, we mean that all partons are on-shell and energy-momentum conservation is implemented exactly, as are all the phase-space constraints. The fact that each dipole factor depends on more than two parton momenta is essential in order to implement these kinematic features.

Owing to the exact phase-space factorization and the convenient choice of dipole formulae, it is possible to integrate all of the dipole contributions analytically over the full dipole phase space in  $d$  dimensions. These result in a set of  $\epsilon$  poles that cancel those in the one-loop cross section, as well as a set of finite corrections (see Sects. 7.2,7.3). It is worth noting that because of the simple definitions of the dipole formulae, the origin of all the finite terms that arise can be simply traced, and all are well-known constants, allowing a powerful check that the integrals have been performed correctly.

In addition to the poles that cancel with the one-loop contribution, when there are identified external hadrons the integration of the dipole factors leads to additional poles that must be subtracted into the process-independent parton distribution functions. Practically, this means cancellation against universal (but factorization scheme- and scale-dependent) collinear counter-terms (see Sect. 6). The scheme- and scale-dependences resurface in the finite remainder left over after the cancellation (see Sect. 2.3 and Sects. 8–11). Once again, the finite terms can be easily checked, because they have simple physical origins related to different integrals and projections of the Altarelli-Parisi splitting functions.

One feature of our algorithm is that it does not require the convolution with the parton distribution function to be made during Monte Carlo integration. One is free to choose either to calculate a hadron-level cross section, including the convolution, or a parton-level cross section as a function of the partonic momentum fraction. The latter can then be convoluted with the distribution function after Monte Carlo integration. This can be extremely useful in many respects. One can produce cross sections with a wide variety of parton distribution functions, or study the scheme- and scale-dependence of the results without having to reintegrate for each new scheme or scale. Moreover one can check whether, in extreme phase space regions, the NLO partonic cross section contains enhanced (typically, logarithmically-enhanced) contributions that may spoil the convergence of the fixed-order perturbative expansion, thus requiring all-order summations. Finally, the parton-level calculation can also be important for comparing different computations, since all the dependence on non-perturbative input can be removed, and the result is a purely perturbative, parameter-free, quantity.

Starting from the dipole formulae and the integrals of the dipole factors, in Sects. 7–11 we have explicitly carried out all the  $d$ -dimensional analytic work that is necessary for a straightforward numerical implementation of NLO calculations in any scattering process. The results are collected in effective final formulae, which are recalled in the last Subsection of Sects. 7–11 for each different class of scattering processes. Using these final formulae, any NLO calculation requires only the corresponding matrix elements as input. More precisely, the only additional ingredients needed to construct a numerical program to calculate the NLO corrections to arbitrary jet quantities in a given process are:

- a set of independent colour projections of the matrix element squared at the Born level, summed over parton polarizations, in  $d$  dimensions (if the total number of QCD partons involved in the LO matrix element is less than or equal to three this is unnecessary, because the colour structure exactly factorizes, see Appendix A);
- the one-loop matrix element in  $d$  dimensions;
- an additional projection of the Born level matrix element over the helicity of each external gluon in four dimensions;

- the tree-level NLO matrix element in four dimensions.

We should emphasize that, independently of the actual set-up of our algorithm, the dipole formalism is fully general and, hence, highly *flexible*. The main point is that it provides an explicit, universal and simple (in  $d$  dimensions) expression for the approximate cross section  $d\sigma^A$ . Starting from it and having the process-independent dipole splitting functions and their integrals to hand, by direct inspection one can try to modify the subtraction term (for instance, introducing finite weighting factors or cut-offs for the dipole terms) to simplify its treatment in any particular scattering process. This may be useful for improving the convergence of the numerical program and can always be done at the expense of some extra analytic work in four dimensions on the finite difference between the two approximate cross sections. In other words, using the dipole formalism one can set up ones own sub-algorithm.

Generalizing the procedure for constructing NLO Monte Carlo programs for arbitrary quantities has several advantages. These are principally because of the reduction in the number and complexity of ingredients that have to be calculated for each new process, and because the  $d$ -dimensional integrals only need be done once and can be easily checked independently, rather than being buried inside a specific calculation.

Perhaps the single biggest advantage is the fact that the NLO matrix element for additional real emission can be calculated in four dimensions. In calculations involving several partons this can result in great savings in computation time and in size of the final expressions, because helicity amplitude techniques can be used. On the other hand, specific calculations that construct the approximate cross section directly from the real one must work in  $d$  dimensions, producing very cumbersome formulae at intermediate stages even though the final result is simple, since it is just the soft and collinear limits, which factorize.

It is often said that the bottleneck in producing new NLO calculations is the calculation of the necessary one-loop amplitudes. While this is partially true, one can cite many examples where the relevant matrix elements have been available for a long time, yet no Monte Carlo programs for arbitrary jet quantities have been available. Most notorious has been the case of jet production in deep inelastic scattering, where the matrix elements can be simply obtained by crossing the  $e^+e^-$  ones, which have been known for many years, but only recently have any NLO jet calculations been produced [31,21]. This has severely hampered jet studies by the HERA experiments, which have been forced to compare data with parton shower models, or partial calculations. Other examples include the longitudinal fragmentation function in  $e^+e^-$  annihilation, which was, until very recently [32], perhaps the single simplest uncalculated QCD quantity.

It is thus clear that the numerical implementation of NLO calculations forms a second bottleneck. General algorithms such as ours will certainly help to reduce the amount of work needed to make these implementations, so will help to reduce this bottleneck and hence increase the number of processes in which perturbative QCD can be compared with data in a quantitative way.

As a final comment on the Monte Carlo implementation of the subtraction method, we should note that the cancellation of soft and collinear singularities does not completely

solve all the numerical problems. Although the subtracted cross section is finite, integrable square-root singularities may arise in the  $m + 1$  parton integral. Square-root singularities cannot be integrated by naïve Monte Carlo methods because they have infinite variance and the integration procedure would never converge. The presence of these singularities can depend on the behaviour of the particular jet observable, so no subtraction procedure can universally overcome them. However, it is straightforward to control the square-root singularities in a completely general way, without any knowledge of the jet quantity being calculated, using the standard technique of multi-channel Monte Carlo integration [33] during the phase-space generation. In the Monte Carlo implementations of our algorithm a user-supplied routine analyses the generated momenta, and any infrared safe jet observable can be used [20,21].

## 12.2 Comparison with other general approaches

The first general algorithm for constructing NLO Monte Carlo programs for jet cross sections in arbitrary processes was proposed in Refs. [8,9], using the phase-space slicing method. Several subsequent calculations have been based on it. In the simple case in which there are no identified hadrons, discussed in Ref. [8], colour-ordered amplitudes are used to derive the soft and collinear factorization formulae. An arbitrary, but small, parameter is introduced to separate the phase space into ‘resolved’ and ‘unresolved’ regions. In the latter the cross section and jet quantity are really approximated by their soft and collinear limits. The resulting formulae can then be integrated analytically over the unresolved regions, to cancel the poles in the virtual cross section.

Although this is an approximate calculation, it becomes exact in the limit that the cutoff parameter goes to zero. Unfortunately the errors on the numerical integral over the resolved region of phase space then diverge and one must always make a compromise in the choice of cutoff. While rules of thumb have developed to give a rough idea of how small it needs to be, there is no substitute for explicitly checking that there is no dependence on it. However, this is not always feasible owing to constraints on computer time. A particularly poignant example is that of the energy-energy correlation in  $e^+e^-$  annihilation. Since discrepancies exist between dedicated analytical calculations and general-purpose Monte Carlo programs, a very high-statistics comparison between three different Monte Carlo programs was made [34], to check that they really were in agreement. The result was a small, but statistically significant, discrepancy between the program of Ref. [8] and those of Refs. [14,20]. While this was attributed to a residual dependence on the cutoff, it was not possible within the available computer time to make a sufficiently accurate calculation to confirm this, or to extrapolate to zero cutoff.

These problems become increasingly severe as one approaches the edges of phase space because the approximation performed within the slicing procedure can strongly interfere with the actual definition of certain jet observables. Indeed, at the edges of phase space there are physical parameters, namely ratios of physical scales, that become large. These large parameters can produce logarithmic and even power-like (!) enhancement of the cutoff-dependence, thus requiring such small cutoffs that numerical stability is never reached, in practice.

However, we should stress that these problems do not imply that, in some case, the phase-space slicing method is unable to provide reliable QCD predictions. On one hand, in the evaluation of jet quantities far from the phase-space boundary, these problems are all obviated if proper care is taken to explicitly check the cutoff-dependence for every result quoted and to work in a region in which it is negligible. On the other hand, in the case of quantities at the edges of phase space, the kinematic region where one can lose control of the numerical stability is that in which the perturbative expansion is affected by very large coefficients: in this region the convergence of the fixed-order expansion is spoiled, the NLO calculation is insufficient to provide reliable QCD predictions and the latter require, anyhow, analytic summation techniques to all orders.

A possible disadvantage of the phase-space slicing method regards precisely this last point. For most jet observables the only analytical NLO results available are for the coefficients of logarithmically enhanced terms near the kinematic limits. Thus, it is a useful test of a complete NLO numerical calculation to compare it with these analytical results. At the same time, the numerical calculation can be used to check these analytical results, which are the first step in the resummation procedures. The approximation embodied in the phase-space slicing method, and the ensuing problems of numerical stability in extreme phase-space regions, may reduce the amount of information that, otherwise, can be provided by a NLO calculation.

The slicing method of Ref. [8] was extended to include identified hadrons in Ref. [9]. The approach is to first consider the case in which all partons are outgoing and then cross some to the initial state as necessary. As we have seen in our approach, the kinematic constraints imposed by identifying external hadrons do not act symmetrically on initial- and final-state partons. Therefore ‘crossing kernels’ have to be introduced.

These crossing kernels are similar to the insertion operators  $\mathbf{K}$  and  $\mathbf{P}$  (and  $\mathbf{H}$ , in the case of fragmentation processes) that, in our approach, arise as finite remainders left over after the subtraction of the universal collinear counter-terms. The main difference between the crossing kernels of Ref. [9] and our kernels  $\mathbf{K}$  and  $\mathbf{P}$  is that the latter do depend on the colour charges of the partons involved in the scattering process. In order to compute the full parton-level cross sections in NLO, the crossing kernels as well as the insertion operators  $\mathbf{K}$  and  $\mathbf{P}$  must be convoluted with  $m$ -parton cross sections (see Sect. 2.3).

In Ref. [9], the crossing kernels are preconvoluted with the parton density set to provide a new effective set of ‘crossing functions’, which are used in the main integration stage of the Monte Carlo program. This procedure can be a considerable problem in one of the main applications of NLO jet calculations, the extraction of parton distribution functions by fitting to the data, since this process is usually iterative, with the distribution functions gradually converging on the best fit. The use of the crossing functions would require a new Monte Carlo integration for each iteration. In our method on the other hand, it is straightforward to calculate parton-level cross sections. These can then be convoluted with any parton distribution functions after the Monte Carlo integration is completed. Thus each iteration would then only require a simple one-dimensional numerical integral.

This disadvantage of the usual implementation of the method of Ref. [9] can obviously be avoided if the crossing kernels are first convoluted with partonic cross section contributions



rather than with parton densities.

The general properties of soft and collinear radiation were used to construct a subtraction algorithm for the first time in Ref. [4], for one- and two-jet production in hadron-hadron collisions. This has recently been modified to deal with three-jet production [16,17] and, in general, with  $n$ -jet cross sections. Although this formalism is based on the subtraction method, so looks superficially similar to our algorithm, there are in fact a great many differences.

The subtraction procedures used in Refs. [16] and [17] differ in many details but share some common features. Firstly, they select energy and angle variables by working in a definite reference frame and thus breaking Lorentz invariance at intermediate steps of the calculation (although of course it is restored in the final results). The definite choice of a reference frame then unambiguously specifies the integration variables that can lead to singularities in the integration of the real matrix element. Thus, one can introduce a partition of the  $m + 1$ -parton phase space in such a way that in each subregion only one energy or angle variable can kinematically vanish. Having done that, the approximate cross section  $d\sigma^A$  is defined by means of a double (soft and collinear) subtraction procedure as the product of the singular variable times its residue. The residue is evaluated by exactly performing the soft and collinear limits and thus it can be computed in a process independent manner.

The two main differences between this ‘residue approach’ and our formalism regard the treatment of the soft and collinear limits and the related definition of the subtracted  $m$ -parton configuration. While the dipole formulae provide a single and smooth approximation of the real matrix element in all of its singular limits, in the residue approach the soft and collinear regions are treated separately. Correspondingly, while in the dipole formalism the subtracted parton configuration is obtained by a one-to-one mapping from the original  $m + 1$ -parton configuration, in the residue approach the mapping is achieved by a projection procedure. This projection of the  $m + 1$ -parton phase space onto  $m$ -parton phase space does not allow for an exact phase space factorization. For this reason, one has to introduce arbitrary cutoffs to define the upper limits on the integration variables, although the dependence on these cutoffs should cancel, to within the numerical accuracy, in the final result.

The residue approach can be recovered as a particular set-up of the dipole formalism. Starting from our general expression for the subtracted cross section  $d\sigma^A$  one can project it onto any properly defined soft and collinear subspace.

### 12.3 Future outlook

Looking to the future, there are several avenues along which the present work could be continued. The first is obviously to apply it to as many processes as possible, in particular those for which no other calculations exist. We have already constructed Monte Carlo programs for jets in  $e^+e^-$  annihilation [20] and deep inelastic scattering [21] and several more applications are in progress.

In terms of extending the algorithm itself, the most obvious missing feature at present is the treatment of heavy quark effects. It is straightforward to extend the dipole formalism to incorporate massive partons, either in jet calculations or with fragmentation functions [19]. This is clearly an important extension, as many heavy quark processes are good probes of perturbative QCD, but many of these that are being experimentally measured are not yet predicted to NLO.

Another potentially important extension is the generalization to polarized partons, which is also straightforward in the dipole formalism.

Looking further ahead, it is to be hoped that at some stage NNLO calculations of jet observables will be attempted. Even once the necessary two-loop matrix elements are calculated, the amount of work needed to provide a numerical implementation will be enormous. Clearly any progress that can be made in the meantime to set up a general-purpose NNLO subtraction algorithm will speed up the process of bringing the new calculation to the marketplace. The dipole formalism seems particularly suited to this task.

When starting this research project, we had in mind a main final goal: a method for carrying out NNLO QCD calculations. Having set up a completely general NLO algorithm, we are confident that this main goal can indeed be achieved.

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## Appendix A: Colour Algebra

In order to practice with the colour algebra we can consider some simple examples. In the simplest cases with two or three partons (regardless of whether they are final- or initial-state partons), the colour algebra can be performed in factorized (closed) form.

For the case with two partons, using colour conservation, we have:

$$\mathbf{T}_1 \cdot \mathbf{T}_2 |1, 2 \rangle = -\mathbf{T}_1 \cdot \mathbf{T}_1 |1, 2 \rangle = -\mathbf{T}_1^2 |1, 2 \rangle = -\mathbf{T}_2^2 |1, 2 \rangle , \quad (\text{A.1})$$

so that all the charge operators  $\{\mathbf{T}_1^2, \mathbf{T}_2^2, -\mathbf{T}_1 \cdot \mathbf{T}_2\}$  are factorizable in terms of the (scalar) Casimir operator  $C_1 = C_2$ .

Using colour conservation for the three-parton case we have:

$$\begin{aligned} 0 &= \left( \sum_{i=1}^3 \mathbf{T}_i \right)^2 |1, 2, 3 \rangle \\ &= \left( \mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2 + 2\mathbf{T}_1 \cdot \mathbf{T}_2 + 2\mathbf{T}_1 \cdot \mathbf{T}_3 + 2\mathbf{T}_2 \cdot \mathbf{T}_3 \right) |1, 2, 3 \rangle , \end{aligned} \quad (\text{A.2})$$

and

$$(\mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{T}_1 \cdot \mathbf{T}_3) |1, 2, 3 \rangle = -\mathbf{T}_1^2 |1, 2, 3 \rangle . \quad (\text{A.3})$$

Combining these two equations we get:

$$2\mathbf{T}_2 \cdot \mathbf{T}_3 |1, 2, 3 \rangle = (\mathbf{T}_1^2 - \mathbf{T}_2^2 - \mathbf{T}_3^2) |1, 2, 3 \rangle \quad (\text{A.4})$$

and similarly for  $\mathbf{T}_1 \cdot \mathbf{T}_3$  and  $\mathbf{T}_1 \cdot \mathbf{T}_2$ . Therefore, all the charge operators are factorizable in terms of linear combinations of the Casimir invariants  $C_1, C_2, C_3$  of the three partons.

When the total number  $n$  of partons is  $n \geq 4$  the colour algebra does not factorize any longer. For instance, if  $n = 4$  we have four trivial relations, namely

$$\mathbf{T}_i^2 |1, 2, 3, 4 \rangle = C_i |1, 2, 3, 4 \rangle , \quad i = 1, \dots, 4 . \quad (\text{A.5})$$

As for the remaining six charge operators  $\mathbf{T}_i \cdot \mathbf{T}_j (i \neq j)$ , we can use the following four identities (due to charge conservation)

$$\mathbf{T}_i \cdot \sum_{j=1}^4 \mathbf{T}_j |1, 2, 3, 4 \rangle = 0 , \quad i = 1, \dots, 4 , \quad (\text{A.6})$$

in order to single out two independent charge operators. For instance we can write:

$$\begin{aligned} \mathbf{T}_3 \cdot \mathbf{T}_4 |1, 2, 3, 4 \rangle &= \left[ \frac{1}{2} (C_1 + C_2 - C_3 - C_4) + \mathbf{T}_1 \cdot \mathbf{T}_2 \right] |1, 2, 3, 4 \rangle , \\ \mathbf{T}_2 \cdot \mathbf{T}_4 |1, 2, 3, 4 \rangle &= \left[ \frac{1}{2} (C_1 + C_3 - C_2 - C_4) + \mathbf{T}_1 \cdot \mathbf{T}_3 \right] |1, 2, 3, 4 \rangle , \\ \mathbf{T}_2 \cdot \mathbf{T}_3 |1, 2, 3, 4 \rangle &= \left[ \frac{1}{2} (C_4 - C_1 - C_2 - C_3) - \mathbf{T}_1 \cdot \mathbf{T}_2 - \mathbf{T}_1 \cdot \mathbf{T}_3 \right] |1, 2, 3, 4 \rangle , \\ \mathbf{T}_1 \cdot \mathbf{T}_4 |1, 2, 3, 4 \rangle &= -(C_1 + \mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{T}_1 \cdot \mathbf{T}_3) |1, 2, 3, 4 \rangle , \end{aligned} \quad (\text{A.7})$$

and express all the charge operators in terms of Casimir invariants and  $\mathbf{T}_1 \cdot \mathbf{T}_2, \mathbf{T}_1 \cdot \mathbf{T}_3$ . The actual values of the latter depend on the detailed colour configuration of the four-parton state, namely\*\*

$$\begin{aligned}
\langle 1, 2, 3, 4 | \mathbf{T}_1 \cdot \mathbf{T}_2 | 1, 2, 3, 4 \rangle &= \left[ \mathcal{M}_4^{b_1 b_2 a_3 a_4}(p_1, p_2, p_3, p_4) \right]^* T_{b_1 a_1}^c T_{b_2 a_2}^c \mathcal{M}_4^{a_1 a_2 a_3 a_4}(p_1, p_2, p_3, p_4), \\
\langle 1, 2, 3, 4 | \mathbf{T}_1 \cdot \mathbf{T}_3 | 1, 2, 3, 4 \rangle &= \left[ \mathcal{M}_4^{b_1 a_2 b_3 a_4}(p_1, p_2, p_3, p_4) \right]^* T_{b_1 a_1}^c T_{b_3 a_3}^c \mathcal{M}_4^{a_1 a_2 a_3 a_4}(p_1, p_2, p_3, p_4).
\end{aligned}
\tag{A.8}$$

In the general case with  $n > 4$  partons, one should consider  $n(n+1)/2$  charge operators  $\mathbf{T}_i \cdot \mathbf{T}_j$ . Among them there are  $n$  trivial factorizable contributions, i.e.  $\mathbf{T}_i^2 | \dots \rangle_n = C_i | \dots \rangle_n$ . In addition, one can use  $n$  charge conservation constraints (i.e.  $\mathbf{T}_i \cdot \sum_{j=1}^n \mathbf{T}_j | \dots \rangle_n = 0$ ) to end up with  $n(n-3)/2$  independent charge operators whose action onto the matrix elements has to be evaluated explicitly.

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\*\*To be precise, if some of the partons in  $|1, 2, 3, 4 \rangle$  are initial-state partons, the right-hand sides of Eq. (A.8) should contain an additional normalization factor as in Eq. (3.13).

## Appendix B: Soft Integrals

In this Appendix we explicitly perform the integration of the soft terms  $v_{i,ab}$  in Eqs. (5.174) and (5.191) over the phase space volumes in Eqs. (5.171) and (5.188) respectively.

Let us first discuss the space-like case (see Eq. (5.173)). We have to carry out the following integration

$$I_{soft}(x) = 8\pi\alpha_S\mu^{2\epsilon} \int [dp_i(n, p_a, x)] \frac{1}{2p_a p_i} v_{i,ab} . \quad (\text{B.1})$$

Since the vector  $n^\mu$  in Eq. (5.157) is time-like, we can work in its rest frame and consider the integration variables  $E_i, \theta, \phi$  defined as follows

$$\begin{aligned} n^\mu &= E(1, \dots) , & p_a^\mu &= E_a(1, \dots, 1) , & p_a^\mu + p_b^\mu &= E_{ab}(1, \dots, v \sin \chi, v \cos \chi) , \\ p_i^\mu &= E_i(1, \dots, \text{'angles' } \dots, \sin \theta \cos \phi, \cos \theta) . \end{aligned} \quad (\text{B.2})$$

In Eq. (B.2) the dots stand for vanishing components, while the notation ‘angles’ denotes the dependence of  $p_i$  on the  $d - 4$  angular variables that can be trivially integrated in Eq. (B.1).

Using Eqs. (5.161,5.171) for the phase space  $[dp_i(n, p_a, x)]$  and Eq. (5.169) for  $v_{i,ab}$  and introducing the variables in Eq. (B.2), the integral (B.1) becomes:

$$\begin{aligned} I_{soft}(x) &= \Theta(x) \Theta(1-x) \frac{\alpha_S}{2\pi} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_0^\infty \frac{dE_i}{E_i} \left( \frac{\pi\mu^2}{E_i^2} \right)^\epsilon \delta\left(x - \frac{E_a - E_i}{E_a}\right) \\ &\cdot \int_0^\pi d\theta (\sin \theta)^{1-2\epsilon} \int_0^\pi \frac{d\phi}{\pi} (\sin \phi)^{-2\epsilon} \frac{1 - v \cos \chi}{(1 - \cos \theta)(1 - v \sin \chi \sin \theta \cos \phi - v \cos \chi \cos \theta)} . \end{aligned} \quad (\text{B.3})$$

The integration over the energy  $E_i$  in Eq. (B.3) is straightforward:

$$\int_0^\infty \frac{dE_i}{E_i} \left( \frac{\pi\mu^2}{E_i^2} \right)^\epsilon \delta\left(x - \frac{E_a - E_i}{E_a}\right) = \frac{1}{1-x} \left( \frac{\pi\mu^2}{(1-x)^2 E_a^2} \right)^\epsilon , \quad (\text{B.4})$$

while for the angular integrals we can use the result of Ref. [35], namely

$$\begin{aligned} &\int_0^\pi d\theta (\sin \theta)^{1-2\epsilon} \int_0^\pi \frac{d\phi}{\pi} (\sin \phi)^{-2\epsilon} \frac{1 - v \cos \chi}{(1 - \cos \theta)(1 - v \sin \chi \sin \theta \cos \phi - v \cos \chi \cos \theta)} \\ &= -\frac{1}{\epsilon} \left[ \frac{1 - v^2}{(1 - v \cos \chi)^2} \right]^\epsilon - \epsilon \left[ 2\text{Li}_2\left(-\frac{v(1 - \cos \chi)}{1 - v}\right) - 2\text{Li}_2\left(-\frac{v(1 + \cos \chi)}{1 - v \cos \chi}\right) \right. \\ &\left. + \ln^2 \frac{1 - v}{1 - v \cos \chi} - \frac{1}{2} \ln^2 \frac{1 + v}{1 - v} - \frac{1}{2} \ln^2 \frac{1 - v^2}{(1 - v \cos \chi)^2} \right] + \mathcal{O}(\epsilon^2) , \end{aligned} \quad (\text{B.5})$$

which, by means of the identity  $\text{Li}_2(-v(1 + \cos \chi)/(1 - v \cos \chi)) = -\text{Li}_2(v(1 + \cos \chi)/(1 + v)) - (1/2) \ln^2((1 - v \cos \chi)/(1 + v))$  (i.e.  $\text{Li}_2(1 - 1/x) = -\text{Li}_2(1 - x) - (1/2) \ln^2 x$ , for  $0 \leq x \leq 1$ ), can be rewritten as follows

$$\begin{aligned} &\int_0^\pi d\theta (\sin \theta)^{1-2\epsilon} \int_0^\pi \frac{d\phi}{\pi} (\sin \phi)^{-2\epsilon} \frac{1 - v \cos \chi}{(1 - \cos \theta)(1 - v \sin \chi \sin \theta \cos \phi - v \cos \chi \cos \theta)} \\ &= - \left[ \frac{1 - v^2}{(1 - v \cos \chi)^2} \right]^\epsilon \left\{ \frac{1}{\epsilon} + \epsilon \left[ 2\text{Li}_2\left(-\frac{v(1 - \cos \chi)}{1 - v}\right) + 2\text{Li}_2\left(\frac{v(1 + \cos \chi)}{1 + v}\right) \right] \right\} + \mathcal{O}(\epsilon^2) . \end{aligned} \quad (\text{B.6})$$

Thus, we find

$$I_{soft}(x) = \Theta(x) \Theta(1-x) \frac{\alpha_S}{2\pi} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} (1-x)^{-1-2\epsilon} \left[ \frac{(1-v^2)\pi\mu^2}{E_a^2(1-v\cos\chi)^2} \right]^\epsilon \cdot \left\{ -\frac{1}{\epsilon} - \epsilon \left[ 2\text{Li}_2\left(-\frac{v(1-\cos\chi)}{1-v}\right) + 2\text{Li}_2\left(\frac{v(1+\cos\chi)}{1+v}\right) \right] + \mathcal{O}(\epsilon^2) \right\} . \quad (\text{B.7})$$

We can then perform the  $\epsilon$ -expansion of the factor  $(1-x)^{-1-2\epsilon}$  according to Eq. (5.51):

$$(1-x)^{-1-2\epsilon} = -\frac{1}{2\epsilon}\delta(1-x) + \left(\frac{1}{1-x}\right)_+ + 2\epsilon \left(\frac{1}{1-x} \ln \frac{1}{1-x}\right)_+ + \mathcal{O}(\epsilon^2) , \quad (\text{B.8})$$

and, relating the variables  $v, \cos\chi, E_a$  in Eq. (B.2) to the Lorentz invariants:

$$\frac{4E_a^2(1-v\cos\chi)^2}{1-v^2} = 2p_a \cdot p_b , \quad \frac{2(1-v\cos\chi)}{1-v^2} = \frac{(p_a + p_b) \cdot n}{p_a \cdot n} ,$$

$$v = \sqrt{1 - \frac{n^2(p_a + p_b)^2}{[(p_a + p_b) \cdot n]^2}} , \quad (\text{B.9})$$

we end up with the final result (note,  $\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon) = 1 - \epsilon^2\pi^2/6 + \mathcal{O}(\epsilon^3)$ )

$$I_{soft}(x) = \Theta(x) \Theta(1-x) \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{2p_a p_b}\right)^\epsilon \left\{ \frac{1}{2\epsilon^2}\delta(1-x) - \frac{1}{\epsilon} \left(\frac{1}{1-x}\right)_+ - \left(\frac{2}{1-x} \ln \frac{1}{1-x}\right)_+ + \delta(1-x) \left[ 2\text{Li}_2\left(1 - \frac{(1+v)(p_a + p_b) \cdot n}{2 p_a \cdot n}\right) + 2\text{Li}_2\left(1 - \frac{(1-v)(p_a + p_b) \cdot n}{2 p_a \cdot n}\right) - \frac{\pi^2}{12} \right] + \mathcal{O}(\epsilon) \right\} . \quad (\text{B.10})$$

Let us now consider the time-like case (see Eq. (5.190)) and compute the integral

$$J_{soft}(z) = 8\pi\alpha_S\mu^{2\epsilon} \int [dp_i(n; p_a, z)] \frac{1}{2p_a p_i} \frac{v_{i,ab}}{z} . \quad (\text{B.11})$$

As in the previous case we work in the rest frame of  $n^\mu$ . Using Eqs. (5.182,5.188) for  $[dp_i(n; p_a, z)]$  and Eq. (5.169) for  $v_{i,ab}$  and introducing the kinematic variables in Eq. (B.2), we obtain an expression for  $J_{soft}(z)$  that is equal to that in Eq. (B.3) apart from the replacement:

$$\delta\left(x - \frac{E_a - E_i}{E_a}\right) \rightarrow z^{1-2\epsilon} \delta\left(z - \frac{E_a}{E_a + E_i}\right) . \quad (\text{B.12})$$

Therefore the  $E_i$  integration in  $J_{soft}(z)$ , namely

$$\int_0^\infty \frac{dE_i}{E_i} \left(\frac{\pi\mu^2}{E_i^2}\right)^\epsilon z^{1-2\epsilon} \delta\left(z - \frac{E_a}{E_a + E_i}\right) = \frac{1}{1-z} \left(\frac{\pi\mu^2}{(1-z)^2 E_a^2}\right)^\epsilon , \quad (\text{B.13})$$

gives exactly the same factor as in Eq. (B.4) and we immediately obtain

$$J_{soft}(z) = I_{soft}(z) . \quad (\text{B.14})$$

## Appendix C: Collection of Main Formulae

In this Appendix we collect together the main formulae that are needed to implement our algorithm for calculating jet cross sections.

According to our method, the final expressions for the NLO cross sections are given in terms of contributions with  $m+1$ -parton and  $m$ -parton kinematics, denoted as  $\sigma^{NLO\{m+1\}}$ ,  $\sigma^{NLO\{m\}}$ ,  $\hat{\sigma}^{NLO\{m\}}$ . The cross section  $\sigma^{NLO\{m+1\}}$  is obtained by subtracting the counter-term  $d\sigma^A$  from the real contribution  $d\sigma^R$ . The counter-term  $d\sigma^A$  is constructed by using the dipole factors introduced in Sect. 5. These dipole factors are collected in Table 1, where we list the Equations with the dipole definition, the corresponding kinematics and the related splitting functions. Remember that, when computing  $\sigma^{NLO\{m+1\}}$ , all the dipole factors have to be directly evaluated in four space-time dimensions.

The cross sections  $\sigma^{NLO\{m\}}$  and  $\hat{\sigma}^{NLO\{m\}}$  are obtained by adding the virtual contribution  $d\sigma^V$  and the collinear counter-term  $d\sigma^C$  to the integral of the subtraction counter-term  $d\sigma^A$ . In the rest of this Appendix we concentrate on the formulae needed to construct the final integral of the subtraction counter-term.

dipole factor	definition	dipole kinematics	phase space	dipole splitting functions $\mathbf{V}$
$\mathcal{D}_{ij,k}$	(5.2)	(5.3), (5.4), (5.6)	(5.20)	(5.7–5.9)
$\mathcal{D}_{ij}^a$	(5.36)	(5.37), (5.38), (5.42)	(5.48)	(5.39–5.41)
$\mathcal{D}_k^{ai}$	(5.61)	(5.62–5.64), (5.69)	(5.72)	(5.65–5.68)
$\mathcal{D}_{ij,a}$	(5.94)	(5.95–5.97),	(5.103)	(5.98–5.100)
$\mathcal{D}_{ai,k}$	(5.114)	(5.115), (5.116), (5.121)	(5.124)	(5.117–5.120)
$\mathcal{D}^{ai,b}$	(5.136)	(5.137–5.140),	(5.151)	(5.145–5.148)
$\mathcal{D}_b^{(n)ai}$ , $\mathcal{D}^{(n)ai,b}$	(5.159)	(5.157), (5.160–5.164), (5.169)	(5.172)	(5.165–5.168)
$\mathcal{D}_{ai,b}^{(n)}$ , $\mathcal{D}_{ai}^{(n)b}$	(5.180)	(5.157), (5.181), (5.182), (5.162–5.164), (5.169)	(5.189)	(5.183–5.186)

Table 1: List of the dipole factorization formulae used to construct the universal subtraction term  $d\sigma^A$ . The numbering refers to the Equations in Sect. 5.

## Flavour kernels

The majority of the functions and constants we encounter are derived from the spin-averaged Altarelli-Parisi splitting functions in  $d = 4 - 2\epsilon$  dimensions:

$$\langle \hat{P}_{qq}(x; \epsilon) \rangle = \langle \hat{P}_{\bar{q}\bar{q}}(x; \epsilon) \rangle = C_F \left[ \frac{1+x^2}{1-x} - \epsilon(1-x) \right] , \quad (\text{C.1})$$

$$\langle \hat{P}_{qg}(x; \epsilon) \rangle = \langle \hat{P}_{\bar{q}g}(x; \epsilon) \rangle = C_F \left[ \frac{1+(1-x)^2}{x} - \epsilon x \right] , \quad (\text{C.2})$$

$$\langle \hat{P}_{gg}(x; \epsilon) \rangle = \langle \hat{P}_{g\bar{q}}(x; \epsilon) \rangle = T_R \left[ 1 - \frac{2x(1-x)}{1-\epsilon} \right] , \quad (\text{C.3})$$

$$\langle \hat{P}_{gg}(x; \epsilon) \rangle = 2C_A \left[ \frac{x}{1-x} + \frac{1-x}{x} + x(1-x) \right] , \quad (\text{C.4})$$

and  $\langle \hat{P}_{q\bar{q}}(x; \epsilon) \rangle = \langle \hat{P}_{\bar{q}q}(x; \epsilon) \rangle = 0$ . It is sometimes useful to express these as a power series in  $\epsilon$ ,

$$\langle \hat{P}_{ab}(x; \epsilon) \rangle = \langle \hat{P}_{ab}(x; \epsilon = 0) \rangle - \epsilon \hat{P}'_{ab}(x) + \mathcal{O}(\epsilon^2) . \quad (\text{C.5})$$

The regularized Altarelli-Parisi probabilities are the sum of these and the virtual corrections, evaluated in four dimensions,

$$P^{qq}(x) = P^{\bar{q}\bar{q}}(x) = C_F \frac{1+(1-x)^2}{x} , \quad (\text{C.6})$$

$$P^{gq}(x) = P^{g\bar{q}}(x) = T_R [x^2 + (1-x)^2] , \quad (\text{C.7})$$

$$P^{qq}(x) = P^{\bar{q}\bar{q}}(x) = C_F \left( \frac{1+x^2}{1-x} \right)_+ , \quad (\text{C.8})$$

$$P^{gg}(x) = 2C_A \left[ \left( \frac{1}{1-x} \right)_+ + \frac{1-x}{x} - 1 + x(1-x) \right] + \delta(1-x) \left( \frac{11}{6}C_A - \frac{2}{3}N_f T_R \right) . \quad (\text{C.9})$$

Note that the final-state Altarelli-Parisi probabilities  $P_{ab}(x)$  are identical to these initial-state ones,  $P_{ab}(x) = P^{ab}(x)$ , so we do not usually make any distinction. Their regular parts are denoted as follows

$$\begin{aligned} P_{\text{reg}}^{ab}(x) &= P^{ab}(x) && \text{if } a \neq b , \\ P_{\text{reg}}^{qq}(x) &= -C_F(1+x) , && P_{\text{reg}}^{gg}(x) = 2C_A \left[ \frac{1-x}{x} - 1 + x(1-x) \right] . \end{aligned} \quad (\text{C.10})$$

After integrating the dipole formulae for final state emission, we obtain the constants

$$\gamma_q = \gamma_{\bar{q}} = \frac{3}{2}C_F , \quad \gamma_g = \frac{11}{6}C_A - \frac{2}{3}T_R N_f , \quad (\text{C.11})$$

and

$$K_q = K_{\bar{q}} = \left( \frac{7}{2} - \frac{\pi^2}{6} \right) C_F , \quad K_g = \left( \frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9}T_R N_f , \quad (\text{C.12})$$

which are related to various integrals of the Altarelli-Parisi splitting functions. As a matter of fact, we have

$$-\frac{1}{2} \sum_b \int_0^1 dz (z(1-z))^{-\epsilon} \langle \hat{P}_{ab}(z; \epsilon) \rangle = 2\mathbf{T}_a^2 \frac{1}{\epsilon} + \gamma_a + \left( K_a - \frac{\pi^2}{6}\mathbf{T}_a^2 \right) \epsilon + \mathcal{O}(\epsilon^2) . \quad (\text{C.13})$$



After integrating the expressions for emission from a dipole in which one identified hadron participates, we obtain the functions:

$$\begin{aligned} \overline{K}^{ab}(x) &= \hat{P}'_{ab}(x) + P_{\text{reg}}^{ab}(x) \ln \frac{1-x}{x} \\ &+ \delta^{ab} \left[ \mathbf{T}_a^2 \left( \frac{2}{1-x} \ln \frac{1-x}{x} \right)_+ - \delta(1-x) \left( \gamma_a + K_a - \frac{5}{6} \pi^2 \mathbf{T}_a^2 \right) \right] , \end{aligned} \quad (\text{C.14})$$

or, explicitly,

$$\overline{K}^{qg}(x) = \overline{K}^{\bar{q}g}(x) = P^{qg}(x) \ln \frac{1-x}{x} + C_F x , \quad (\text{C.15})$$

$$\overline{K}^{gq}(x) = \overline{K}^{g\bar{q}}(x) = P^{gq}(x) \ln \frac{1-x}{x} + T_R 2x(1-x) , \quad (\text{C.16})$$

$$\begin{aligned} \overline{K}^{qq}(x) = \overline{K}^{\bar{q}\bar{q}}(x) &= C_F \left[ \left( \frac{2}{1-x} \ln \frac{1-x}{x} \right)_+ - (1+x) \ln \frac{1-x}{x} + (1-x) \right] \\ &- \delta(1-x) (5 - \pi^2) C_F , \end{aligned} \quad (\text{C.17})$$

$$\begin{aligned} \overline{K}^{gg}(x) &= 2C_A \left[ \left( \frac{1}{1-x} \ln \frac{1-x}{x} \right)_+ + \left( \frac{1-x}{x} - 1 + x(1-x) \right) \ln \frac{1-x}{x} \right] \\ &- \delta(1-x) \left[ \left( \frac{50}{9} - \pi^2 \right) C_A - \frac{16}{9} T_R N_f \right] , \end{aligned} \quad (\text{C.18})$$

and  $\overline{K}^{\bar{q}q}(x) = \overline{K}^{q\bar{q}}(x) = 0$ .

Finally, when two identified hadrons participate, we obtain the following additional functions

$$\widetilde{K}^{ab}(x) = P_{\text{reg}}^{ab}(x) \ln(1-x) + \delta^{ab} \mathbf{T}_a^2 \left[ \left( \frac{2}{1-x} \ln(1-x) \right)_+ - \frac{\pi^2}{3} \delta(1-x) \right] . \quad (\text{C.19})$$

Although it is not as closely related to the Altarelli-Parisi splitting functions, we include in this Section the integral of the *pseudodipole* splitting function encountered in multiparticle correlations,

$$\begin{aligned} \mathcal{L}^{a,b}(x; p_a, p_b, n) &= \delta^{ab} \delta(1-x) 2 \mathbf{T}_a^2 \left[ \text{Li}_2 \left( 1 - \frac{(1+v)(p_a + p_b) \cdot n}{2 p_a \cdot n} \right) \right. \\ &+ \left. \text{Li}_2 \left( 1 - \frac{(1-v)(p_a + p_b) \cdot n}{2 p_a \cdot n} \right) \right] - P_{\text{reg}}^{ab}(x) \ln \frac{n^2(p_a \cdot p_b)}{2(p_a \cdot n)^2} , \end{aligned} \quad (\text{C.20})$$

$$v = \sqrt{1 - \frac{n^2(p_a + p_b)^2}{[(p_a + p_b) \cdot n]^2}} , \quad (\text{C.21})$$

$$\text{Li}_2(x) = - \int_0^x \frac{dz}{z} \ln(1-z) . \quad (\text{C.22})$$

Remember that the four-momentum  $n^\mu$  is defined as follows

$$n^\mu = p_{\text{in}}^\mu - \sum_{a \in \text{final state}} p_a^\mu , \quad (\text{C.23})$$

where  $p_{\text{in}}^\mu$  is the total incoming momentum in the scattering process and the second term on the right-hand side is the sum of all the momenta of the identified partons in the final-state.

In addition, partonic cross sections for processes involving identified hadrons also depend on the scheme-dependent flavour functions  $K_{\text{F.S.}}^{ab}(x)$  and  $H_{ba}^{\text{F.S.}}(x)$ . In the  $\overline{\text{MS}}$  scheme, all are zero,

$$K_{\overline{\text{MS}}}^{ab} = H_{ba}^{\overline{\text{MS}}} = 0. \quad (\text{C.24})$$

In the DIS scheme, the initial-state functions are given by (see [36] and Appendix D),

$$K_{\text{DIS}}^{qq}(x) = K_{\text{DIS}}^{\bar{q}\bar{q}}(x) = C_F \left[ \frac{1+x^2}{1-x} \left( \log \frac{1-x}{x} - \frac{3}{4} \right) + \frac{1}{4}(9+5x) \right]_+, \quad (\text{C.25})$$

$$K_{\text{DIS}}^{gq}(x) = K_{\text{DIS}}^{g\bar{q}}(x) = T_R \left[ (x^2 + (1-x)^2) \log \frac{1-x}{x} + 8x(1-x) - 1 \right], \quad (\text{C.26})$$

and  $K_{\text{DIS}}^{qq}(x) = K_{\text{DIS}}^{\bar{q}\bar{q}}(x) = -K_{\text{DIS}}^{qq}(x)$ ,  $K_{\text{DIS}}^{gg}(x) = -2N_f K_{\text{DIS}}^{gq}(x)$ ,  $K_{\text{DIS}}^{\bar{q}\bar{q}}(x) = K_{\text{DIS}}^{q\bar{q}}(x) = 0$ .

In the case of fragmentation processes, a factorization scheme conceptually analogous to the DIS scheme has been introduced in Ref. [37].

## Insertion Operators

The final result for singular part of the integral of the dipole splitting functions is the same in all processes, and depends on the universal insertion operator  $\mathbf{I}$ . In order to write the result in a uniform way for all dipole contributions, we use the notation  $\{p\}$  to denote a set of parton momenta, without specifying which are identified and whether they are in the initial or final state.  $I$  and  $J$  are indices over all these momenta, and we obtain

$$\mathbf{I}(\{p\}; \epsilon) = -\frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_I \frac{1}{\mathbf{T}_I^2} \mathcal{V}_I(\epsilon) \sum_{J \neq I} \mathbf{T}_I \cdot \mathbf{T}_J \left( \frac{4\pi\mu^2}{2p_I \cdot p_J} \right)^\epsilon. \quad (\text{C.27})$$

Note that the scalar products  $p_I \cdot p_J$  are always positive. If either momentum is crossed between the initial and final states, our uniform notation ensures that  $p_I \cdot p_J$  retains the same sign. The universal singular function  $\mathcal{V}_I(\epsilon)$  in Eq. (C.27) depends only on the parton flavour and has the following  $\epsilon$ -expansion

$$\mathcal{V}_I(\epsilon) = \mathbf{T}_I^2 \left( \frac{1}{\epsilon^2} - \frac{\pi^2}{3} \right) + \gamma_I \frac{1}{\epsilon} + \gamma_I + K_I + \mathcal{O}(\epsilon), \quad (\text{C.28})$$

where the constants  $\gamma_I$  and  $K_I$  are given in Eqs. (C.11,C.12). The insertion operator  $\mathbf{I}$  enters into the calculation of the cross sections  $\sigma^{NLO\{m\}}$  with  $m$ -parton kinematics.

When there are identified external hadrons, we obtain finite remainders from the infinite subtraction into the parton distribution functions. These finite remainders, which contribute to the one-dimensional convolution of the cross sections  $\hat{\sigma}^{NLO\{m\}}$  with  $m$ -parton kinematics, contain two terms which respectively depend on the factorization scale and on the factorization scheme.

The factorization-scale dependent term is proportional to the insertion operator  $\mathbf{P}$ , which is again universal. Using the  $\{p\}$  and  $I$  notation again, it is given by

$$\mathbf{P}^{a,b}(\{p\}; xp_a, x; \mu_F^2) = \frac{\alpha_S}{2\pi} P^{ab}(x) \frac{1}{\mathbf{T}_b^2} \sum_{I \neq b} \mathbf{T}_I \cdot \mathbf{T}_b \ln \frac{\mu_F^2}{2xp_a \cdot p_I} , \quad (\text{C.29})$$

for initial-state partons and

$$\mathbf{P}_{b,a}(\{p\}; p_a/z, z; \mu_F^2) = \frac{\alpha_S}{2\pi} P_{ba}(z) \frac{1}{\mathbf{T}_b^2} \sum_{I \neq b} \mathbf{T}_I \cdot \mathbf{T}_b \ln \frac{z\mu_F^2}{2p_a \cdot p_I} , \quad (\text{C.30})$$

for final-state partons. Here  $P^{ab} = P_{ab}$  are the Altarelli-Parisi probabilities in Eqs. (C.6–C.9). Note that if we denote by  $p_b$  the rescaled momentum in the insertion operator  $\mathbf{P}$  (i.e.  $p_b = xp_a$  in Eq. (C.29) and  $p_b = p_a/z$  in Eq. (C.30)), the right-hand sides of Eqs. (C.29) and (C.30) only differ by the transposition  $ab \rightarrow ba$  of the flavour indices in the Altarelli-Parisi probabilities. Nonetheless, as discussed at the end of Sect. 9.1, we have  $\mathbf{P}^{a,b}(\{p\}; p_b, x; \mu_F^2) \neq \mathbf{P}_{a,b}(\{p\}; p_b, x; \mu_F^2)$ .

As for the term that depends on the factorization scheme, it is proportional to the initial-state insertion operator  $\mathbf{K}$  and/or to the final-state insertion operator  $\mathbf{H}$ . When there is only one initial-state hadron, the insertion operator  $\mathbf{K}$  is:

$$\mathbf{K}^{a,b}(x) = \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{ab}(x) - K_{\text{F.S.}}^{ab}(x) + \delta^{ab} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] \right\} . \quad (\text{C.31})$$

Similarly, when there is only one final-state identified hadron, the insertion operator  $\mathbf{H}$  is:

$$\begin{aligned} \mathbf{H}_{b,a}(z) = & \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{ba}(z) + 3P_{ba}(z) \ln z - H_{ba}^{\text{F.S.}}(z) \right. \\ & \left. + \delta_{ab} \sum_i \mathbf{T}_i \cdot \mathbf{T}_b \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-z} \right)_+ + \delta(1-z) - 1 \right] \right\} . \end{aligned} \quad (\text{C.32})$$

Likewise, when there are two initial-state hadrons, we obtain

$$\begin{aligned} \mathbf{K}^{a,a'}(x) = & \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{aa'}(x) - K_{\text{F.S.}}^{aa'}(x) \right. \\ & \left. + \delta^{aa'} \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] \right\} - \frac{\alpha_S}{2\pi} \mathbf{T}_b \cdot \mathbf{T}_{a'} \frac{1}{\mathbf{T}_{a'}^2} \widetilde{K}^{aa'}(x) . \end{aligned} \quad (\text{C.33})$$

Finally, in the most general case of multiparton correlations, we obtain

$$\begin{aligned} \mathbf{K}^{a,a'}(x) = & \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{aa'}(x) - K_{\text{F.S.}}^{aa'}(x) \right. \\ & + \delta^{aa'} \sum_i \mathbf{T}_i \cdot \mathbf{T}_a \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-x} \right)_+ + \delta(1-x) \right] - \frac{1}{\mathbf{T}_{a'}^2} \left( \sum_{l=1}^n \mathbf{T}_{a_l} \cdot \mathbf{T}_{a'} + \mathbf{T}_b \cdot \mathbf{T}_{a'} \right) \widetilde{K}^{aa'}(x) \\ & \left. - \frac{1}{\mathbf{T}_{a'}^2} \left[ \sum_{l=1}^n \mathbf{T}_{a_l} \cdot \mathbf{T}_{a'} \mathcal{L}^{a,a'}(x; p, q_l, n) + \mathbf{T}_b \cdot \mathbf{T}_{a'} \mathcal{L}^{a,a'}(x; p, \bar{p}, n) \right] \right\} , \end{aligned} \quad (\text{C.34})$$

and

$$\begin{aligned}
\mathbf{H}_{a'_l, a_l}(z) = & \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{a'_l a_l}(z) + 3P_{a'_l a_l}(z) \ln z - H_{a'_l a_l}^{\text{F.S.}}(z) \right. \\
& + \delta_{a'_l a_l} \sum_i \mathbf{T}_i \cdot \mathbf{T}_{a'_l} \frac{\gamma_i}{\mathbf{T}_i^2} \left[ \left( \frac{1}{1-z} \right)_+ + \delta(1-z) - 1 \right] \\
& + \frac{1}{\mathbf{T}_{a'_l}^2} \left( \sum_{\substack{r=1 \\ r \neq l}}^n \mathbf{T}_{a_r} \cdot \mathbf{T}_{a'_l} + \mathbf{T}_a \cdot \mathbf{T}_{a'_l} + \mathbf{T}_b \cdot \mathbf{T}_{a'_l} \right) \left[ P_{a'_l a_l}(z) \ln z - \widetilde{K}^{a'_l a_l}(z) \right] \\
& - \frac{1}{\mathbf{T}_{a'_l}^2} \left[ \sum_{\substack{r=1 \\ r \neq l}}^n \mathbf{T}_{a_r} \cdot \mathbf{T}_{a'_l, a_l} \mathcal{L}^{a'_l, a_l}(z; q_l, q_r, n) + \mathbf{T}_a \cdot \mathbf{T}_{a'_l} \mathcal{L}^{a'_l, a_l}(z; q_l, p, n) \right. \\
& \left. + \mathbf{T}_b \cdot \mathbf{T}_{a'_l} \mathcal{L}^{a'_l, a_l}(z; q_l, \bar{p}, n) \right] \left. \right\} . \tag{C.35}
\end{aligned}$$

Note that setting  $\mathbf{T}_b = 0$  in Eqs. (C.34) and (C.35) we respectively obtain the operators  $\mathbf{K}$  and  $\mathbf{H}$  for the case of multiparton correlations with a single incoming parton. Likewise, setting  $\mathbf{T}_a = \mathbf{T}_b = 0$  in Eq. (C.35) we obtain the operator  $\mathbf{H}$  for multiparton correlations in processes with no hadrons in the initial state.

The definition of the flavour kernels  $\overline{K}^{ab}(x)$ ,  $\widetilde{K}^{ab}(x)$ ,  $\mathcal{L}^{a,b}$ ,  $K_{\text{F.S.}}^{ab}(x)$  and  $H_{ba}^{\text{F.S.}}(x)$  that appear in Eqs. (C.31–C.35) is recalled in Eqs. (C.14), (C.19), (C.20) and (C.24–C.26).

In order to evaluate the NLO cross sections with  $m$ -parton kinematics, the colour-charge operators  $\mathbf{I}$ ,  $\mathbf{P}$ ,  $\mathbf{K}$  and  $\mathbf{H}$  have to be inserted into the tree-level matrix elements. This leads to the computation of colour-correlated tree-amplitudes. We conclude this Appendix by recalling their definition. As above we denote by  $\{p\}$  a generic set of  $N$  parton momenta. The square  $|\mathcal{M}^{I,J}|^2$  of the colour-correlated amplitude has the following expression in terms of the coloured tree-level amplitude  $\mathcal{M}^{a_1 \dots a_N}(\{p\})$ :

$$\begin{aligned}
|\mathcal{M}^{I,J}(\{p\})|^2 & \equiv \langle \{p\} | \mathbf{T}_I \cdot \mathbf{T}_J | \{p\} \rangle \tag{C.36} \\
& = \frac{1}{n_c(a)n_c(b)} \left[ \mathcal{M}^{a_1 \dots b_I \dots b_J \dots a_N}(\{p\}) \right]^* T_{b_I a_I}^c T_{b_J a_J}^c \mathcal{M}^{a_1 \dots a_I \dots a_J \dots a_N}(\{p\}) .
\end{aligned}$$

Here the labels  $a$  and  $b$  refer to the initial-state partons in  $|\{p\}\rangle$  and  $n_c(a)$  and  $n_c(b)$  is their number of colours. The factor  $1/(n_c(a)n_c(b))$  on the right-hand side of Eq. (C.36) comes from the definition of the state vector  $|\{p\}\rangle$  (see Eqs. (3.3,3.11)) used throughout this paper. If there is only one initial state parton, or none, then this factor becomes  $1/n_c(a)$  or 1 respectively, as in Eqs. (3.13) and (3.9).

## Appendix D: Examples

In this appendix we give some simple applications of our method.

### $e^+e^- \rightarrow 2$ jets

We begin with a trivial example: two-jet observables in  $e^+e^-$  annihilation. We use customary notation for the kinematic variables:  $Q^2$  is the square of the centre-of-mass energy,  $y_{ij} = 2p_i \cdot p_j/Q^2$  and  $x_i = 2p_i \cdot Q/Q^2$ , where  $p_i$  is the momentum of any QCD parton in the final state.

The LO contribution is the parton model process  $e^+e^- \rightarrow q(p_1) + \bar{q}(p_2)$ , with matrix element  $\mathcal{M}_2$ . We average over event orientation, so  $\mathcal{M}_2$  has no dependence on the parton momenta. Moreover, we choose the overall normalization of  $\mathcal{M}_2$  such that the two-parton phase space is:

$$d\Phi^{(2)} = dy_{12} \delta(1 - y_{12}) \ , \quad (\text{D.1})$$

and the LO cross section in Eq. (7.33) is given by

$$\sigma^{LO} = |\mathcal{M}_2|^2 \int dy_{12} \delta(1 - y_{12}) F_J^{(2)}(p_1, p_2) \ . \quad (\text{D.2})$$

The NLO real-emission process is  $e^+e^- \rightarrow q(p_1) + \bar{q}(p_2) + g(p_3)$ , with matrix element  $\mathcal{M}_3(p_1, p_2, p_3)$ . In four dimensions, the matrix element is:

$$|\mathcal{M}_3(p_1, p_2, p_3)|^2 = C_F \frac{8\pi\alpha_S}{Q^2} \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} |\mathcal{M}_2|^2 \ , \quad (\text{D.3})$$

and the phase-space is given by

$$d\Phi^{(3)} = \frac{Q^2}{16\pi^2} dx_1 dx_2 \Theta(1 - x_1)\Theta(1 - x_2)\Theta(x_1 + x_2 - 1). \quad (\text{D.4})$$

The calculation of the subtracted cross section involves the evaluation of two dipole contributions:  $\mathcal{D}_{13,2}$  and  $\mathcal{D}_{23,1}$ . Their definition is given in Eqs. (5.2,5.7). The associated colour algebra is trivial, as shown in Eq. (A.1), and we find

$$\mathcal{D}_{13,2}(p_1, p_2, p_3) = \frac{1}{2p_1 p_3} V_{q_1 g_3, 2} |\mathcal{M}_2|^2 \ , \quad (\text{D.5})$$

with the following dipole kinematics

$$\tilde{p}_2^\mu = \frac{1}{x_2} p_2^\mu \ , \quad \tilde{p}_{13}^\mu = Q^\mu - \frac{1}{x_2} p_2^\mu \ . \quad (\text{D.6})$$

The dipole contribution  $\mathcal{D}_{23,1}$  is obtained from Eqs. (D.5,D.6) by the replacement  $p_1 \leftrightarrow p_2$ . Inserting the (four-dimensional) definitions of  $V_{q_1 g_3, 2}$  and  $x_i$ , we obtain the final expression

for the three-parton cross section in Eq. (7.35):

$$\begin{aligned}
\sigma^{NLO\{3\}} &= \int_3 [d\sigma_{\epsilon=0}^R - d\sigma_{\epsilon=0}^A] \\
&= |\mathcal{M}_2|^2 \frac{C_{F\alpha_S}}{2\pi} \int_0^1 dx_1 dx_2 \Theta(x_1 + x_2 - 1) \left\{ \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} F_J^{(3)}(p_1, p_2, p_3) \right. \\
&\quad - \left[ \frac{1}{1-x_2} \left( \frac{2}{2-x_1-x_2} - (1+x_1) \right) + \frac{1-x_1}{x_2} \right] F_J^{(2)}(\tilde{p}_{13}, \tilde{p}_2) \\
&\quad \left. - \left[ \frac{1}{1-x_1} \left( \frac{2}{2-x_1-x_2} - (1+x_2) \right) + \frac{1-x_2}{x_1} \right] F_J^{(2)}(\tilde{p}_{23}, \tilde{p}_1) \right\} . \quad (D.7)
\end{aligned}$$

Since the three-parton matrix element can be written as follows

$$\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} = \frac{1}{1-x_2} \left( \frac{2}{2-x_1-x_2} - (1+x_1) \right) + (x_1 \leftrightarrow x_2) , \quad (D.8)$$

it is straightforward to see that for any infrared safe observable (implying that  $F_J^{(3)} \rightarrow F_J^{(2)}$  as  $x_i \rightarrow 1$ ), Eq. (D.7) is finite.

Next we have to evaluate the insertion operator  $\mathbf{I}(\epsilon)$  and combine it with the virtual cross section. The one-loop matrix element in the  $\overline{\text{MS}}$  renormalization scheme is given by

$$|\mathcal{M}_2|_{(1-loop)}^2 = |\mathcal{M}_2|^2 \frac{C_{F\alpha_S}}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right\} , \quad (D.9)$$

while the insertion operator in Eqs. (C.27,C.28) gives

$${}_{2<1,2} \langle \mathbf{I}(\epsilon) | 1, 2 \rangle_2 = |\mathcal{M}_2|^2 \frac{C_{F\alpha_S}}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left\{ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 10 - \pi^2 + \mathcal{O}(\epsilon) \right\} . \quad (D.10)$$

Note that in Eqs. (D.9,D.10) the two-parton matrix element  $\mathcal{M}_2$  is consistently evaluated in  $d = 4 - 2\epsilon$  dimensions. Combining these two contributions as in Eq. (7.36), we obtain a finite (for  $\epsilon \rightarrow 0$ ) final expression for the two-parton cross section:

$$\sigma^{NLO\{2\}} = \int_2 [d\sigma^V + \int_1 d\sigma^A]_{\epsilon=0} = |\mathcal{M}_2|^2 \frac{C_{F\alpha_S}}{\pi} \int dy_{12} \delta(1-y_{12}) F_J^{(2)}(p_1, p_2) . \quad (D.11)$$

It is straightforward to check that the total cross section ( $F_J^{(3)} = F_J^{(2)} = 1$ ) agrees with the well-known result,  $\sigma_{\text{tot}}^{NLO} = \frac{3}{4} C_F \frac{\alpha_S}{\pi} \sigma_{\text{tot}}^{LO}$ .

## $e^+e^- \rightarrow 3$ jets

In Ref. [20] we presented the simplest non-trivial case: three-jet production in  $e^+e^-$  annihilation. For completeness we repeat it here. We again average over event orientation, so our formalism can be directly compared with that in Ref. [6].

The LO partonic process to be considered is  $e^+e^- \rightarrow q + \bar{q} + g$ , with matrix element  $\mathcal{M}_3$  and kinematic variables as defined above for the case of  $e^+e^- \rightarrow 2$  jets.

At NLO, two different real-emission subprocesses contribute: a)  $e^+e^- \rightarrow q(p_1) + \bar{q}(p_2) + g(p_3) + g(p_4)$ ; b)  $e^+e^- \rightarrow q(p_1) + \bar{q}(p_2) + q(p_3) + \bar{q}(p_4)$ . The calculation of the subtracted cross section (7.35) for the subprocess a) involves the evaluation of the following dipole contributions:  $\mathcal{D}_{13,2}, \mathcal{D}_{13,4}, \mathcal{D}_{14,2}, \mathcal{D}_{14,3}, \mathcal{D}_{23,1}, \mathcal{D}_{23,4}, \mathcal{D}_{24,1}, \mathcal{D}_{24,3}, \mathcal{D}_{34,1}, \mathcal{D}_{34,2}$ . The associated colour algebra can again be easily performed because the several colour projections of the three-parton matrix element fully factorize (see Eq.(A.4)). Thus we do not need to compute any colour-correlated tree amplitudes and we find

$$\begin{aligned}\mathcal{D}_{13,2}(p_1, p_2, p_3, p_4) &= \frac{1}{2p_1 p_3} \left(1 - \frac{C_A}{2C_F}\right) V_{q_1 g_3, 2} |\mathcal{M}_3(\tilde{p}_{13}, \tilde{p}_2, p_4)|^2, \\ \mathcal{D}_{13,4}(p_1, p_2, p_3, p_4) &= \frac{1}{2p_1 p_3} \frac{C_A}{2C_F} V_{q_1 g_3, 4} |\mathcal{M}_3(\tilde{p}_{13}, p_2, \tilde{p}_4)|^2, \\ \mathcal{D}_{34,1}(p_1, p_2, p_3, p_4) &= \frac{1}{2p_3 p_4} \frac{1}{2} V_{g_3 g_4, 1}^{\mu\nu} \mathcal{T}_{\mu\nu}(\tilde{p}_1, p_2, \tilde{p}_{34}).\end{aligned}\tag{D.12}$$

The dipole contributions  $\mathcal{D}_{23,1}, \mathcal{D}_{23,4}, \mathcal{D}_{34,2}$  are obtained respectively from  $\mathcal{D}_{13,2}, \mathcal{D}_{13,4}, \mathcal{D}_{34,1}$  by the replacement  $p_1 \leftrightarrow p_2$ . Analogously, one can obtain  $\mathcal{D}_{14,2}$  and  $\mathcal{D}_{14,3}$  respectively from  $\mathcal{D}_{13,2}$  and  $\mathcal{D}_{13,4}$  by the replacement  $p_3 \leftrightarrow p_4$ , and  $\mathcal{D}_{24,1}$  and  $\mathcal{D}_{24,3}$  respectively from  $\mathcal{D}_{13,2}$  and  $\mathcal{D}_{13,4}$  by the replacement  $p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4$ .

In the case of the subprocess b) we have to consider the following dipole contributions:  $\mathcal{D}_{12,3}, \mathcal{D}_{12,4}, \mathcal{D}_{14,2}, \mathcal{D}_{14,3}, \mathcal{D}_{23,1}, \mathcal{D}_{23,4}, \mathcal{D}_{34,1}, \mathcal{D}_{34,2}$ . Performing the colour algebra we get

$$\mathcal{D}_{34,1}(p_1, p_2, p_3, p_4) = \frac{1}{2p_3 p_4} \frac{1}{2} V_{q_3 \bar{q}_4, 1}^{\mu\nu} \mathcal{T}_{\mu\nu}(\tilde{p}_1, p_2, \tilde{p}_{34}),\tag{D.13}$$

and all the other dipoles are obtained by the corresponding permutation of the parton momenta.

The splitting functions  $V_{ij,k}$  of Eqs. (D.12,D.13) are explicitly given in Eqs. (5.7–5.9). The tensor  $\mathcal{T}_{\mu\nu}$  is the squared amplitude for the LO process  $e^+e^- \rightarrow q\bar{q}g$  not summed over the gluon polarizations ( $\mu$  and  $\nu$  are the gluon spin indices and  $-g^{\mu\nu}\mathcal{T}_{\mu\nu} = |\mathcal{M}_3|^2$ ). This can be easily calculated. In the case of jet observables averaged over the directions of the incoming leptons (un-oriented events) we find (in  $d = 4$  dimensions)

$$\mathcal{T}^{\mu\nu}(p_1, p_2, p_3) = -\frac{1}{x_1^2 + x_2^2} |\mathcal{M}_3(p_1, p_2, p_3)|^2 T^{\mu\nu},\tag{D.14}$$

where

$$\begin{aligned}T^{\mu\nu} &= +2\frac{p_1^\mu p_2^\nu}{Q^2} + 2\frac{p_2^\mu p_1^\nu}{Q^2} - 2\frac{1-x_1}{1-x_2} \frac{p_1^\mu p_1^\nu}{Q^2} - 2\frac{1-x_2}{1-x_1} \frac{p_2^\mu p_2^\nu}{Q^2} \\ &\quad - \frac{1-x_1-x_2+x_2^2}{1-x_2} \left[ \frac{p_1^\mu p_3^\nu}{Q^2} + \frac{p_3^\mu p_1^\nu}{Q^2} \right] - \frac{1-x_2-x_1+x_1^2}{1-x_1} \left[ \frac{p_2^\mu p_3^\nu}{Q^2} + \frac{p_3^\mu p_2^\nu}{Q^2} \right] \\ &\quad + \left(1 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 - x_2\right) g^{\mu\nu}.\end{aligned}\tag{D.15}$$

The few ingredients listed in Eqs. (D.12–D.15) have to be combined with the four-parton matrix elements  $\mathcal{M}_4$  for evaluating the four-parton cross section  $\sigma^{NLO\{4\}}$  in Eq. (7.35). Obviously, due to the very long expressions for the matrix elements [6], we do not report here the explicit formula for  $\sigma^{NLO\{4\}}$ .

To complete the NLO calculation we also need the virtual cross section. In the case of un-oriented events, we take the one-loop matrix element in the  $\overline{\text{MS}}$  renormalization scheme from Ref. [6] (we use slightly different notation):

$$\begin{aligned}
|\mathcal{M}_3(p_1, p_2, p_3)|_{(1-loop)}^2 &= |\mathcal{M}_3(p_1, p_2, p_3)|^2 \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \\
&\cdot \left\{ -\frac{1}{\epsilon^2} \left[ (2C_F - C_A)y_{12}^{-\epsilon} + C_A (y_{13}^{-\epsilon} + y_{23}^{-\epsilon}) \right] - \frac{1}{\epsilon} \left( 3C_F + \frac{11}{6}C_A - \frac{2}{3}T_R N_f \right) \right. \\
&\left. + \frac{\pi^2}{2}(2C_F + C_A) - 8C_F \right\} + \frac{\alpha_S}{2\pi} [F(y_{12}, y_{13}, y_{23}) + \mathcal{O}(\epsilon)] \quad , \quad (D.16)
\end{aligned}$$

where  $F(y_{12}, y_{13}, y_{23})$  is defined in Eq. (2.21) of Ref. [6].

The explicit evaluation of the insertion operator  $\mathbf{I}(\epsilon)$  in Eqs. (C.27,C.28) gives:

$$\begin{aligned}
{}_{3<} \langle 1, 2, 3 | \mathbf{I}(\epsilon) | 1, 2, 3 \rangle {}_{3>} &= |\mathcal{M}_3(p_1, p_2, p_3)|^2 \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \\
&\left\{ \frac{1}{\epsilon^2} \left[ (2C_F - C_A)y_{12}^{-\epsilon} + C_A (y_{13}^{-\epsilon} + y_{23}^{-\epsilon}) \right] + \frac{1}{\epsilon} (2\gamma_q + \gamma_g) \right. \\
&- \gamma_q \frac{1}{C_F} \left[ (2C_F - C_A) \ln y_{12} + \frac{1}{2}C_A \ln(y_{13}y_{23}) \right] - \frac{1}{2}\gamma_g \ln(y_{13}y_{23}) \\
&\left. - \frac{\pi^2}{3}(2C_F + C_A) + 2(\gamma_q + K_q) + \gamma_g + K_g + \mathcal{O}(\epsilon) \right\} \quad . \quad (D.17)
\end{aligned}$$

Combining the one-loop matrix element (D.16) with the result (D.17), and using the explicit expressions (C.11,C.12) for  $\gamma_I$  and  $K_I$ , all the pole terms cancel. Note that as well as the pole terms, the closely related  $\pi^2$  and  $\ln^2$  terms cancel:

$$\begin{aligned}
|\mathcal{M}_3(p_1, p_2, p_3)|_{(1-loop)}^2 + {}_{3<} \langle 1, 2, 3 | \mathbf{I}(\epsilon) | 1, 2, 3 \rangle {}_{3>} &= |\mathcal{M}_3(p_1, p_2, p_3)|^2 \\
&\cdot \frac{\alpha_S}{2\pi} \left[ -\frac{3}{2}(2C_F - C_A) \ln y_{12} - \frac{1}{3}(5C_A - T_R N_f) \ln(y_{13}y_{23}) \right. \\
&\left. + 2C_F + \frac{50}{9}C_A - \frac{16}{9}T_R N_f \right] + \frac{\alpha_S}{2\pi} [F(y_{12}, y_{13}, y_{23}) + \mathcal{O}(\epsilon)] \quad . \quad (D.18)
\end{aligned}$$

The integration of the expression (D.18) (with the matrix element  $\mathcal{M}_3$  given in Eq. (D.3)) over the phase space in Eq. (D.4) provides the three-parton cross section  $\sigma^{NLO\{3\}}$  in Eq. (7.36).

The results presented here form the basis for a Monte Carlo program [20] that can calculate the NLO corrections to arbitrary three-jet-like observables in  $e^+e^-$  annihilation: it will be described in more detail elsewhere.

## 1 + 1 jets in DIS and the Structure Function $F_2$

We next discuss the simplest case in which there is an incoming hadron: 1+1-jet observables in deep inelastic lepton-hadron scattering (DIS). We limit ourselves to considering the case of virtual-photon exchange and, in particular, we compute  $\sigma = \sigma_T + \sigma_L$ ,  $\sigma_T$  and  $\sigma_L$



respectively being the scattering cross sections off transversely and longitudinally polarized photons. In the fully inclusive limit (i.e. when no final-state jet observable is defined), our cross section is simply proportional to the customary structure function  $F_2$ . We use standard notation for the kinematic variables:  $p$  is the incoming momentum,  $q$  is the off-shell photon momentum ( $q^2 = -Q^2 < 0$ ),  $x = Q^2/(2pq)$  is the Bjorken variable and  $z_i = p_i p/pq$ ,  $p_i$  being any parton (hadron) momentum in the final state. The relevant matrix elements are obtained from the hadronic tensor  $W_{\mu\nu}$  by applying the following ( $d$ -dimensional) projection operator

$$P_2^{\mu\nu} = \frac{Q^2}{2pq} \left[ -g^{\mu\nu} + \frac{(d-1)Q^2}{(pq)^2} p^\mu p^\nu \right]. \quad (\text{D.19})$$

Note that to implement the final formulae of our algorithm, the  $d$ -dimensional definition of  $P_2^{\mu\nu}$  is relevant only for a consistent calculation of the one-loop matrix element.

The hadronic cross section  $\sigma(p)$  in Eq. (6.3) is obtained by convoluting partonic cross sections  $\sigma_a = \sigma_q, \sigma_{\bar{q}}, \sigma_g$  with parton densities  $f_a$ . Since we are considering only photon exchange, by charge conjugation invariance we have  $\sigma_{\bar{q}} = \sigma_q$ . Thus, we explicitly compute only  $\sigma_q$  and  $\sigma_g$ . At LO,  $\sigma_g^{LO} = 0$  while  $\sigma_q^{LO}$  is obtained by the parton model process  $q(p) + \gamma^*(q) \rightarrow q(p_1)$ . By momentum conservation we have  $p_1 = q + p$ , so the LO matrix element  $\mathcal{M}_{1,q}(q+p;p)$  has, in practice, no dependence on final-state parton momenta. We choose the overall normalization of  $\mathcal{M}_{1,q}(q+p;p)$  such that the single-parton phase space is:

$$d\Phi^{(1)}(p) = \frac{Q^2}{2pq} \delta\left(\frac{2pq}{Q^2} - 1\right) = \delta(1-x) \quad , \quad (\text{D.20})$$

and the LO cross section in Eq. (8.45) is given by

$$\sigma_q^{LO}(p) = \frac{1}{N_c} |\mathcal{M}_{1,q}(q+p;p)|^2 \delta(1-x) F_J^{(1)}(q+p;p) \quad . \quad (\text{D.21})$$

At NLO, we have to consider real emission processes with two final-state partons with momenta  $p_1$  and  $p_2$ . By momentum conservation we have  $2p_1 p_2 = (1-x)Q^2/x$  and the two-parton phase space is given by

$$d\Phi^{(2)}(p) = \frac{Q^2 x}{16\pi^2} dz_1 dz_2 \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \Theta(z_1)\Theta(z_2) \delta(1-z_1-z_2) 2\pi \delta(\pi + \phi_1 - \phi_2) \quad . \quad (\text{D.22})$$

In the following, we consider jet quantities that are averaged over the azimuthal angles  $\phi_{1,2}$ , so they can be trivially integrated and  $\int d\phi_1 d\phi_2 \delta(\pi + \phi_1 - \phi_2) \rightarrow 2\pi$ . The NLO cross sections  $\sigma_q^{NLO}(p)$ ,  $\sigma_g^{NLO}(p)$  respectively receive contributions from the real emission processes  $q(p) + \gamma^*(q) \rightarrow q(p_1) + g(p_2)$  and  $g(p) + \gamma^*(q) \rightarrow q(p_1) + \bar{q}(p_2)$ . The corresponding matrix elements in  $d = 4$  dimensions are:

$$\begin{aligned} \frac{1}{N_c} |\mathcal{M}_{2,q}(p_1, p_2; p)|^2 &= C_F \frac{8\pi\alpha_S}{Q^2} \left[ \frac{x^2 + z_1^2}{(1-x)(1-z_1)} + 2(1+3xz_1) \right] \\ &\cdot \frac{1}{N_c} |\mathcal{M}_{1,q}(q+xp; xp)|^2 \quad , \\ \frac{1}{N_c^2 - 1} |\mathcal{M}_{2,g}(p_1, p_2; p)|^2 &= T_R \frac{8\pi\alpha_S}{Q^2} \left[ \frac{(z_1^2 + (1-z_1)^2)(x^2 + (1-x)^2)}{z_1(1-z_1)} + 8x(1-x) \right] \\ &\cdot \frac{1}{N_c} |\mathcal{M}_{1,q}(q+xp; xp)|^2 \quad . \end{aligned} \quad (\text{D.23})$$

According to Eq. (8.47), the calculation of the two-parton cross sections  $\sigma_q^{NLO\{2\}}$  and  $\sigma_g^{NLO\{2\}}$  is performed by subtracting, in the first case, the final-state dipole  $\mathcal{D}_{12}^q$  and the initial-state dipole  $\mathcal{D}_1^{q2}$  and, in the second case, the two initial-state dipoles  $\mathcal{D}_2^{g1}$ ,  $\mathcal{D}_1^{g2}$ . The associated colour algebra is trivial (as in the example of  $e^+e^- \rightarrow 2$  jets) and we find

$$\begin{aligned}\mathcal{D}_{12}^q(p_1, p_2; p) &= \frac{1}{2p_1 p_2} \frac{1}{x} V_{q_1 g_2}^q \frac{1}{N_c} |\mathcal{M}_{1,q}(q + \tilde{p}; \tilde{p})|^2, \\ \mathcal{D}_1^{q2}(p_1, p_2; p) &= \frac{1}{2pp_2} \frac{1}{x} V_1^{qq_2} \frac{1}{N_c} |\mathcal{M}_{1,q}(q + \tilde{p}; \tilde{p})|^2, \\ \mathcal{D}_2^{g1}(p_1, p_2; p) &= \frac{1}{2pp_1} \frac{1}{x} V_2^{gq_1} \frac{1}{N_c} |\mathcal{M}_{1,q}(q + \tilde{p}; \tilde{p})|^2,\end{aligned}\tag{D.24}$$

with  $\mathcal{D}_1^{g2}$  being obtained from  $\mathcal{D}_2^{g1}$  by the replacement  $p_1 \leftrightarrow p_2$ . Note that the dipole kinematics turns out to be the same for all these dipole contributions: the incoming and outgoing parton momenta in each dipole respectively are  $\tilde{p}^\mu = xp^\mu$  and (by momentum conservation)  $q^\mu + xp^\mu$ .

Inserting into Eq. (D.24) the definitions (5.39,5.65,5.66) of the splitting functions  $V$  and combining Eqs. (D.22,D.23,D.24) as in Eq. (8.47), we obtain the following expressions for the NLO cross sections contributions  $\sigma_a^{NLO\{2\}}$

$$\begin{aligned}\sigma_q^{NLO\{2\}}(p) &= \int_0^1 dz_1 dz_2 \delta(1 - z_1 - z_2) \frac{1}{N_c} |\mathcal{M}_{1,q}(q + xp; xp)|^2 \frac{C_F \alpha_S}{2\pi} x \\ &\cdot \left\{ \left[ \frac{x^2 + z_1^2}{(1-x)(1-z_1)} + 2(1 + 3xz_1) \right] F_J^{(2)}(p_1, p_2; p) \right. \\ &\quad \left. - \frac{x^2 + z_1^2}{(1-x)(1-z_1)} F_J^{(1)}(q + xp; xp) \right\},\end{aligned}\tag{D.25}$$

$$\begin{aligned}\sigma_g^{NLO\{2\}}(p) &= \int_0^1 dz_1 dz_2 \delta(1 - z_1 - z_2) \frac{1}{N_c} |\mathcal{M}_{1,q}(q + xp; xp)|^2 \frac{T_R \alpha_S}{2\pi} x \\ &\cdot \left\{ \left[ \frac{(z_1^2 + (1-z_1)^2)(x^2 + (1-x)^2)}{z_1(1-z_1)} + 8x(1-x) \right] F_J^{(2)}(p_1, p_2; p) \right. \\ &\quad \left. - \frac{x^2 + (1-x)^2}{z_1(1-z_1)} F_J^{(1)}(q + xp; xp) \right\}.\end{aligned}\tag{D.26}$$

Clearly, for any jet observable (implying that  $F_J^{(2)} \rightarrow F_J^{(1)}$  as  $z_1 \rightarrow 1, 0$  or  $x \rightarrow 1$ ) the integrals in Eqs. (D.25,D.26) are finite.

In order to compute the NLO cross sections with  $1 \rightarrow 1$  parton kinematics, we have to evaluate the insertion operators  $\mathbf{I}$ ,  $\mathbf{P}$  and  $\mathbf{K}$  in Eqs. (C.27,C.29,C.31). We find

$$\begin{aligned}_{1,q < 1; p} \mathbf{I}(\epsilon) |1; p >_{1,q} &= \frac{1}{N_c} |\mathcal{M}_{1,q}(q + p; p)|^2 \\ &\cdot \frac{C_F \alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left\{ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 10 - \pi^2 + \mathcal{O}(\epsilon) \right\},\end{aligned}\tag{D.27}$$

$$\sum_b {}_{1,b < 1; zp} \mathbf{P}^{qb}(zp, z; \mu_F^2) |1; zp >_{1,b} = -\frac{1}{N_c} |\mathcal{M}_{1,q}(q + zp; zp)|^2 \frac{\alpha_S}{2\pi} P^{qq}(z) \ln \frac{x\mu_F^2}{zQ^2},\tag{D.28}$$

$$\sum_b \sum_{1,b < 1; zp} |\mathbf{P}^{g,b}(zp, z; \mu_F^2) |1; zp >_{1,b} = -\frac{1}{N_c} |\mathcal{M}_{1,q}(q + zp; zp)|^2 2 \frac{\alpha_S}{2\pi} P^{gq}(z) \ln \frac{x\mu_F^2}{zQ^2} , \quad (\text{D.29})$$

$$\begin{aligned} \sum_b \sum_{1,b < 1; zp} |\mathbf{K}^{q,b}(z) |1; zp >_{1,b} &= \frac{1}{N_c} |\mathcal{M}_{1,q}(q + zp; zp)|^2 \frac{\alpha_S}{2\pi} \\ &\cdot \left\{ \overline{K}^{qq}(z) - K_{\text{F.S.}}^{qq}(z) - \frac{3}{2} C_F \left[ \left( \frac{1}{1-z} \right)_+ + \delta(1-z) \right] \right\} , \end{aligned} \quad (\text{D.30})$$

$$\sum_b \sum_{1,b < 1; zp} |\mathbf{K}^{g,b}(z) |1; zp >_{1,b} = \frac{1}{N_c} |\mathcal{M}_{1,q}(q + zp; zp)|^2 2 \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{gq}(z) - K_{\text{F.S.}}^{gq}(z) \right\} . \quad (\text{D.31})$$

According to Eq. (8.48), the result in Eq. (D.27) has to be combined with the following one-loop matrix element

$$\begin{aligned} \frac{1}{N_c} |\mathcal{M}_{1,q}(q + p; p)|_{(1\text{-loop})}^2 &= \frac{1}{N_c} |\mathcal{M}_{1,q}(q + p; p)|^2 \\ &\cdot \frac{C_F \alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right\} \end{aligned} \quad (\text{D.32})$$

Obviously, the  $\epsilon$  poles cancel and we obtain

$$\sigma_q^{NLO\{1\}}(p) = \frac{1}{N_c} |\mathcal{M}_{1,q}(q + p; p)|^2 \frac{C_F \alpha_S}{2\pi} (2 - \pi^2) \delta(1-x) F_J^{(1)}(q + p; p) . \quad (\text{D.33})$$

From Eqs. (D.28–D.31) we can evaluate the NLO cross section contributions in Eq. (8.49). Using the phase space factor in Eq. (D.20), we find

$$\begin{aligned} \int_0^1 dz \hat{\sigma}_q^{NLO\{1\}}(z; zp, \mu_F^2) &= \int_0^1 dz \delta(z/x - 1) F_J^{(1)}(q + zp; zp) \frac{1}{N_c} |\mathcal{M}_{1,q}(q + zp; zp)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} \left\{ \overline{K}^{qq}(z) - K_{\text{F.S.}}^{qq}(z) - \frac{3}{2} C_F \left[ \left( \frac{1}{1-z} \right)_+ + \delta(1-z) \right] - P^{qq}(z) \ln \frac{x\mu_F^2}{zQ^2} \right\} \\ &= \frac{1}{N_c} |\mathcal{M}_{1,q}(q + xp; xp)|^2 F_J^{(1)}(q + xp; xp) \frac{\alpha_S}{2\pi} C_F x \\ &\cdot \left\{ \left[ \frac{1+x^2}{1-x} \left( \ln \frac{(1-x)Q^2}{x\mu_F^2} - \frac{3}{4} \right) \right]_+ + \frac{1-7x}{4} + \delta(1-x) \left( \pi^2 - \frac{39}{8} \right) - \frac{1}{C_F} K_{\text{F.S.}}^{qq}(x) \right\} , \end{aligned} \quad (\text{D.34})$$

$$\begin{aligned} \int_0^1 dz \hat{\sigma}_g^{NLO\{1\}}(z; zp, \mu_F^2) &= \int_0^1 dz \delta(z/x - 1) F_J^{(1)}(q + zp; zp) \frac{1}{N_c} |\mathcal{M}_{1,q}(q + zp; zp)|^2 \\ &\cdot \frac{\alpha_S}{2\pi} 2 \left\{ \overline{K}^{gq}(z) - K_{\text{F.S.}}^{gq}(z) - P^{gq}(z) \ln \frac{x\mu_F^2}{zQ^2} \right\} \\ &= \frac{1}{N_c} |\mathcal{M}_{1,q}(q + xp; xp)|^2 F_J^{(1)}(q + xp; xp) \frac{\alpha_S}{2\pi} T_R 2x \\ &\cdot \left\{ [x^2 + (1-x)^2] \ln \frac{(1-x)Q^2}{x\mu_F^2} + 2x(1-x) - \frac{1}{T_R} K_{\text{F.S.}}^{gq}(x) \right\} , \end{aligned} \quad (\text{D.35})$$

where, in the last expression on the right-hand side of Eqs. (D.34,D.35) we have performed the  $z$ -integration and introduced the explicit formulae (C.16,C.17) for the kernels  $\overline{K}^{ab}$  and (C.7,C.8) for the splitting probabilities  $P^{ab}$ .

We can explicitly check that our calculation correctly reproduces the known NLO results [36] for the structure function  $F_2$ . The partonic coefficient functions  $F_{2a}^{NLO}$  for  $F_2$  are obtained from our cross sections by integrating over all possible final states (i.e. by setting  $F_J^{(1)} = F_J^{(2)} = 1$ ) and by fixing the overall normalization with  $|\mathcal{M}_{1,q}(q+xp; xp)|^2 = N_c$ . Gathering together all the terms in Eqs. (D.25,D.33,D.34) and those in Eqs. (D.26,D.35), we obtain

$$F_{2q}^{NLO}(p, \mu_F^2) = \frac{\alpha_S}{2\pi} C_F x \left\{ \left[ \frac{1+x^2}{1-x} \left( \ln \frac{(1-x)Q^2}{x \mu_F^2} - \frac{3}{4} \right) + \frac{1}{4}(9+5x) \right]_+ - \frac{1}{C_F} K_{\text{F.S.}}^{qq}(x) \right\} , \quad (\text{D.36})$$

$$F_{2g}^{NLO}(p, \mu_F^2) = \frac{\alpha_S}{2\pi} T_R 2x \left\{ [x^2 + (1-x)^2] \ln \frac{(1-x)Q^2}{x \mu_F^2} + 8x(1-x) - 1 - \frac{1}{T_R} K_{\text{F.S.}}^{gg}(x) \right\} . \quad (\text{D.37})$$

The NLO expressions in Eqs. (D.36,D.37) are factorization-scheme dependent and this dependence is accounted for by the kernels  $K_{\text{F.S.}}^{qq}$  and  $K_{\text{F.S.}}^{gg}$ . The DIS factorization scheme is defined in such a way that  $F_{2q}^{NLO}(p, Q^2) = F_{2g}^{NLO}(p, Q^2) = 0$ : thus Eqs. (C.25,C.26) follow. The definition of the gluon kernels  $K_{\text{DIS}}^{qq}$  and  $K_{\text{DIS}}^{gg}$  is chosen in order to define parton densities that fulfil momentum conservation [36].

The next simplest process involving an incoming hadron is the 2+1-jet rate in DIS. Conceptually, there are no additional problems that are not dealt with either in the example of 1+1 jets in DIS or in  $e^+e^- \rightarrow 3$  jets, and it is straightforward to implement a general purpose Monte Carlo program for arbitrary 2+1-jet-like quantities in DIS. Specific details of the algorithm and numerical results will be presented elsewhere.

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Thus, for our purposes, the main feature of dimensional reduction is that, provided the one-loop matrix elements are computed in this regularization scheme, the shopping list (see Sects. 2.2 and 12) to construct a numerical program to calculate the NLO corrections to arbitrary jet quantities in a given process can be shortened as follows

- the one-loop matrix element in the dimensional-reduction scheme in  $d$  dimensions;
- an additional projection of the Born level matrix element over the helicity of each external gluon in four dimensions;
- the tree-level NLO matrix element in four dimensions.

The computation of one-loop matrix elements in the dimensional-reduction scheme is greatly simplified by directly evaluating helicity amplitudes [3]. Using these calculations, the above shopping list can be further shortened by eliminating the second item.