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N=2 Supergravity and N=2 Super Yang-Mills Theory on General Scalar Manifolds:

Symplectic Covariance, Gaugings and the Momentum Map. *

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Abstract

The general form of $N=2$ supergravity coupled to an arbitrary number of vector multiplets and hypermultiplets, with a generic gauging of the scalar manifold isometries is given. This extends the results already available in the literature in that we use a coordinate independent and manifestly symplectic covariant formalism which allows to cover theories difficult to formulate within superspace or tensor calculus approach. We provide the complete lagrangian and supersymmetry variations with all fermionic terms, and the form of the scalar potential for arbitrary quaternionic manifolds and special geometry, not necessarily in special coordinates. Lagrangians for rigid theories are also written in this general setting and the connection with local theories elucidated. The derivation of these results using geometrical techniques is briefly summarized.

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1 Introduction

Impressive results over the last year on non perturbative properties of $N=2$ supersymmetric Yang-Mills theories[1, 2] and their extension to string theory[3]-[6] through the notion of string-string duality[7, 8], have used the deep underlying mathematical structure of these theories and its relation to algebraic geometry [9]- [18].

In the case of $N=2$ vector multiplets, describing the effective interactions in the Abelian (Coulomb) phase of a spontaneously broken gauge theory, Seiberg and Witten [1] have shown that positivity of the metric on the underlying moduli space identifies the geometrical data of the effective $N=2$ rigid theory with the periods of a particular torus.

In the coupling to gravity it was conjectured by some of the present authors [3, 4] and later confirmed by heterotic-Type II duality [11, 12, 18, 19], that the very same argument based on positivity of the vector multiplet kinetic metric identifies the corresponding geometrical data of the effective $N=2$ supergravity with the periods of Calabi-Yau threefolds.

On the other hand, when matter is added, the underlying geometrical structure is much richer, since $N=2$ matter hypermultiplets are associated with quaternionic geometry[21, 22, 23], and charged hypermultiplets are naturally associated with the gauging of triholomorphic isometries of these quaternionic manifolds [24, 25].

It is the aim of this paper to complete the general form of the $N=2$ supergravity lagrangian coupled to an arbitrary number of vector multiplets and hypermultiplets in presence of a general gauging of the isometries of both the vector multiplets and hypermultiplets scalar manifolds. Actually this extends results already obtained years ago by some of us [24], that in turn extended previous work by Bagger and Witten on ungauged general quaternionic manifolds coupled to $N=2$ supergravity[21], by de Wit, Lauwers and Van Proeyen on gauged special geometry and gauged quaternionic manifolds obtained by quaternionic quotient in the tensor calculus framework [26], and by Castellani, D'Auria and Ferrara on covariant formulation of special geometry for matter coupled supergravity [27].

This paper firstly provides in a geometrical setting the full lagrangian with all the fermionic terms and the supersymmetry variations. Secondly, it uses a coordinate independent and manifestly symplectic covariant formalism which in particular does not require the use of a prepotential function $F(X)$. Whether a prepotential $F(X)$ exists or not depends on the choice of a symplectic gauge[4]. Moreover, some physically interesting cases are precisely instances where $F(X)$ does not exist[4].

Of particular relevance is the fact that we exhibit a scalar potential for arbitrary quaternionic geometries and for special geometry not necessarily in special coordiantes. This allows us to go beyond what is obtainable with the tensor calculus (or superspace) approach. Among many applications, our results allow the study of general conditions for spontaneous supersymmetry breaking in a manner analogous to what was done for $N=1$ matter coupled supergravity [28]. Many examples of supersymmetry breaking studied in the past are then reproduced in a unified framework.

Recently the power of using simple geometrical formulae for the scalar potential was exploited while studying the breaking of half supersymmetries in a particular simple model, using a symplectic basis where $F(X)$ is not defined[29]. The method has potential applications in string theory to study non perturbative phenomena such as conifold transitions [10], p-forms condensation [30] and Fayet-Iliopoulos terms [29, 32].

$N = 2$ supergravity displays a high degree of complexity in its structure, based however on the simplicity of few principles. The supersymmetric Lagrangian and the transformation rules are indeed quite involved but all the couplings, the mass matrices and the vacuum energy are completely fixed and organized in terms of three geometrical data:

1. *The choice of a special Kähler manifold $\mathcal{SM}$ describing the self-interactions of the vector multiplets*

2. *The choice of a quaternionic manifold $\mathcal{HM}$ describing the self-interaction of the hypermultiplets*

3. *The choice of a gauge group $\mathcal{G}$, that in the non abelian case must be a subgroup of the isometry group of the scalar manifold $\mathcal{M}_{scalar} \equiv \mathcal{SM} \otimes \mathcal{HM}$ with a block diagonal immersion in the symplectic group $Sp(2\overline{n}+2, \mathrm{I\!R})$ of electric–magnetic duality rotations* (see eq. 6.4).

For this reason we devote the first and largest part of the paper (sections 2-7) to review and discuss, in a way independent from supersymmetric Lagrangians and supersymmetry algebras, the geometrical ingredients of the construction that we listed above. This part of the paper can be read as an independent essay and should be quite accessible to mathematicians as well as to readers who have no background or interest in supersymmetry.

The second part of the paper (sections 8-9) presents instead the Lagrangian and supersymmetry transformation rules for both $N = 2$ supergravity and $N = 2$ matter coupled rigid Yang–Mills theory that is retrieved from supergravity in the infinite Planck mass limit $\mu \to \infty$. The theory is presented in a completely explicit component formalism, and no formulae employ or require the use of superfields, superspace or conformal tensor calculus. All items entering such formulae are rather geometrical objects whose nature and properties were described and explained in previous sections.

The reader interested in applications of $N = 2$ supergravity or Yang–Mills theory can directly jump to sections 8-9, that are self–contained, and insert, in the ready-to-use formulae the specific geometrical data corresponding to the problem considered. References to formulae in previous sections are given to fix normalizations.

The derivation of the results presented in sections 8-9 was obtained by means of the geometric ("rheonomic") approach (for a general review see the book by some of us [31]). The details of the derivation are given in the Appendices for the interested reader, while the results are presented in the main text. It is indeed one of the main advantages of the geometrical approach to supersymmetry that the final outcome of the construction is directly written in space–time component formalism.

As emphasized our results are general and apply to generic choice of the scalar manifold. As an illustration of our formulae in the appendix we specialize them to the case of the manifolds 1.1. More specifically, our paper is organized as follows:

1. **Section 2** reviews duality rotations and symplectic covariance in field theory.

2. **Section 3** describes the symplectic embedding of the homogeneous spaces,in particular the special symmetric spaces which appear at tree level in heterotic string theory.

3. **Section 4** reviews Special Kähler geometry, both for rigid and local supersymmetry.

4. **Section 5** describes the geometry of hypermultiplets, their associated quaternionic and hyperKähler manifolds in local and rigid supersymmetry.

5. **Section 6** faces the gauging of special and quaternionic manifolds.

6. **Section 7** deals with the so called momentum map on Special Kähler and quaternionic manifolds giving rise to the introduction of prepotential functions which enter in the construction of the scalar potential.

7. **Section 8** reports the full $N = 2$ Lagrangian in a symplectic covariant form

8. **Section 9** contains the rigid limit and reports the general form of a matter coupled $N = 2$ super Yang–Mills theory on a generic rigid special manifold and a generic rigid hyperKähler manifold.

9. **Appendices A, B** give a detailed derivation of the Lagrangian and transformation rules using the geometrical approach.

10. **Appendix C** deals with the relevant formulas for $N = 2$ supergravity based on the manifolds

$$\begin{aligned} \text{special manifold} &= ST[2,n] \equiv \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2) \times SO(n)} \\ \text{quaternionic manifold} &= HQ[m] \equiv \frac{SO(4,m)}{SO(4) \times SO(m)} \end{aligned} \tag{1.1}$$

    This is done as an exemplification of the general formulae for the potential, mass matrices and kinetic period matrices and for its intrinsic interest in applications to tree level string theory

11. **Appendix D** contains a list of conventions and normalizations that we have employed.

An expanded version of this paper, with particular attention to the geometrical properties of the scalar manifolds, the rigidly supersymmetric version and further related issues is given in [33].

2 Duality Rotations and Symplectic Covariance

In this section, both for completeness and in order to fix our conventions and notations, we review the general structure of an abelian theory of vectors and scalars displaying covariance under a group of duality rotations. The basic reference is the 1981 paper by Gaillard and Zumino [46]. A general presentation in $D = 2p$ dimensions was recently given in [47]. Here we fix $D = 4$.

We consider a theory of $\overline{n}$ gauge fields A^Λ_μ, in a $D = 4$ space–time with Lorentz signature. They correspond to a set of $\overline{n}$ differential 1–forms

$$A^\Lambda \equiv A^\Lambda_\mu \, dx^\mu \qquad (\Lambda = 1, \ldots, \overline{n}) \tag{2.1}$$

The corresponding field strengths and their Hodge duals are defined by

$$
\begin{aligned}
F^{\Lambda} &\equiv d\,A^{\Lambda} \equiv \mathcal{F}^{\Lambda}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \\
\mathcal{F}^{\Lambda}_{\mu\nu} &\equiv \frac{1}{2}\left(\partial_{\mu} A^{\Lambda}_{\nu} - \partial_{\nu} A^{\Lambda}_{\mu}\right) \\
{}^{\star}F^{\Lambda} &\equiv \widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \\
\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu} &\equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda|\rho\sigma}
\end{aligned}
\tag{2.2}
$$

Defining the space–time integration volume as

$$
\mathrm{d}^4 x \equiv -\frac{1}{4!}\,\varepsilon_{\mu_1\ldots\mu_4}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_4} \ , \tag{2.3}
$$

we obtain

$$
F^{\Lambda} \wedge F^{\Sigma} = \varepsilon^{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda}_{\mu\nu}\, \mathcal{F}^{\Sigma}_{\rho\sigma}\, d^4x \qquad ; \qquad F^{\Lambda} \wedge {}^{\star}F^{\Sigma} = -2\, \mathcal{F}^{\Lambda}_{\mu\nu}\, \mathcal{F}^{\Sigma|\mu\nu} d^4 x \ . \tag{2.4}
$$

In addition to the gauge fields let us also introduce a set of real scalar fields ϕ^I ($I = 1, \ldots, \overline{m}$) spanning an $\overline{m}$–dimensional manifold $\mathcal{M}_{scalar}$ [1] endowed with a metric $g_{IJ}(\phi)$. Utilizing the above field content we can write the following action functional:

$$
\mathcal{S} = \frac{1}{2} \int \left\{ \left[\gamma_{\Lambda\Sigma}(\phi)\, F^{\Lambda} \wedge \star F^{\Sigma} + \theta_{\Lambda\Sigma}(\phi)\, F^{\Lambda} \wedge F^{\Sigma} \right] + g_{IJ}(\phi)\, \partial_{\mu}\phi^I\, \partial^{\mu}\phi^J\, \mathrm{d}^4x \right\} \ , \tag{2.5}
$$

where the scalar fields dependent $\overline{n} \times \overline{n}$ matrix $\gamma_{\Lambda\Sigma}(\phi)$ generalizes the inverse of the squared coupling constant $\frac{1}{g^2}$ appearing in ordinary gauge theories. The field dependent matrix $\theta_{\Lambda\Sigma}(\phi)$ is instead a generalization of the *theta*–angle of quantum chromodynamics. Both γ and θ are symmetric matrices. Introducing a formal operator j that maps a field strength into its Hodge dual

$$
\left(j\, \mathcal{F}^{\Lambda}\right)_{\mu\nu} \equiv \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda|\rho\sigma} \tag{2.6}
$$

and a formal scalar product

$$
(G\,,\, K) \equiv G^T K \equiv \sum_{\Lambda=1}^{\overline{n}} G^{\Lambda}_{\mu\nu} K^{\Lambda|\mu\nu} \tag{2.7}
$$

the total Lagrangian of eq. 2.5 can be rewritten as

$$
\mathcal{L}^{(tot)} = \mathcal{F}^T \left(-\gamma \otimes \mathbb{1} + \theta \otimes j\right) \mathcal{F} + \frac{1}{2}\, g_{IJ}(\phi)\, \partial_{\mu}\phi^I\, \partial^{\mu}\phi^J \tag{2.8}
$$

The operator j satisfies $j^2 = -\mathbb{1}$ so that its eigenvalues are $\pm\mathrm{i}$. Introducing self–dual and antiself–dual combinations

$$
\begin{aligned}
\mathcal{F}^{\pm} &= \frac{1}{2}\left(\mathcal{F} \pm \mathrm{i}\, j\mathcal{F}\right) \\
j\, \mathcal{F}^{\pm} &= \mp \mathrm{i} \mathcal{F}^{\pm}
\end{aligned}
\tag{2.9}
$$

[1] Whether the ϕ^I can be arranged into complex fields is not relevant at this level of the discussion.

and the field–dependent symmetric matrices

$$
\begin{aligned}
\mathcal{N} &= \theta - \mathrm{i}\gamma \\
\overline{\mathcal{N}} &= \theta + \mathrm{i}\gamma \,,
\end{aligned} \tag{2.10}
$$

the vector part of the Lagrangian 2.8 can be rewritten as

$$
\mathcal{L}_{vec} = \mathrm{i}\left[\mathcal{F}^{-T}\overline{\mathcal{N}}\mathcal{F}^{-} - \mathcal{F}^{+T}\mathcal{N}\mathcal{F}^{+}\right] \tag{2.11}
$$

Introducing the new tensors

$$
\widetilde{\mathcal{G}}^{\Lambda}_{\mu\nu} \equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{\Lambda}_{\mu\nu}} \leftrightarrow \mathcal{G}^{\mp\Lambda}_{\mu\nu} \equiv \mp\frac{\mathrm{i}}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{\mp\Lambda}_{\mu\nu}} \tag{2.12}
$$

which, in matrix notation, corresponds to

$$
j\,\mathcal{G} \equiv \frac{1}{2}\,\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{T}} = -\,\left(\gamma\otimes \mathbb{1} - \theta\otimes j\right)\,\mathcal{F} \tag{2.13}
$$

the Bianchi identities and field equations associated with the Lagrangian 2.5 can be written as

$$
\partial^{\mu}\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu} = 0 \tag{2.14}
$$

$$
\partial^{\mu}\widetilde{\mathcal{G}}^{\Lambda}_{\mu\nu} = 0 \tag{2.15}
$$

or equivalently

$$
\partial^{\mu} Im\mathcal{F}^{\pm\Lambda}_{\mu\nu} = 0 \tag{2.16}
$$

$$
\partial^{\mu} Im\mathcal{G}^{\pm\Lambda}_{\mu\nu} = 0\,. \tag{2.17}
$$

This suggests that we introduce the $2\overline{n}$ column vector

$$
\mathbf{V} \equiv \begin{pmatrix} j\,\mathcal{F} \\ j\,\mathcal{G} \end{pmatrix} \tag{2.18}
$$

and that we consider general linear transformations on such a vector

$$
\begin{pmatrix} j\,\mathcal{F} \\ j\,\mathcal{G} \end{pmatrix}^{\prime} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} j\,\mathcal{F} \\ j\,\mathcal{G} \end{pmatrix} \tag{2.19}
$$

For any matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2\overline{n}, \mathbb{R})$ the new vector $\mathbf{V}'$ of *magnetic and electric* field–strengths satisfies the same equations 2.15 as the old one. In a condensed notation we can write

$$
\partial\,\mathbf{V} = 0 \quad \Longleftrightarrow \quad \partial\,\mathbf{V}' = 0 \tag{2.20}
$$

Separating the self–dual and anti–self–dual parts

$$
\mathcal{F} = \left(\mathcal{F}^{+} + \mathcal{F}^{-}\right) \qquad ; \qquad \mathcal{G} = \left(\mathcal{G}^{+} + \mathcal{G}^{-}\right) \tag{2.21}
$$

and taking into account that we have

$$
\mathcal{G}^{+} = \mathcal{N}\mathcal{F}^{+} \quad \mathcal{G}^{-} = \overline{\mathcal{N}}\mathcal{F}^{-} \tag{2.22}
$$

the duality rotation of eq. 2.19 can be rewritten as

$$\begin{pmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ \mathcal{N}\mathcal{F}^+ \end{pmatrix} \quad ; \quad \begin{pmatrix} \mathcal{F}^- \\ \mathcal{G}^- \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^- \\ \overline{\mathcal{N}}\mathcal{F}^- \end{pmatrix} \tag{2.23}$$

The problem is that the transformation rule 2.23 of $\mathcal{G}^\pm$ must be consistent with the definition of the latter as variation of the Lagrangian with respect to $\mathcal{F}^\pm$ (see eq. 2.12). This request restricts the form of the matrix $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. As we are going to show, Λ must belong to the symplectic subgroup of the general linear group

$$\Lambda \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2\overline{n}, \mathrm{I\!R}) \subset GL(2\overline{n}, \mathrm{I\!R}) \tag{2.24}$$

the subgroup $Sp(2\overline{n}, \mathrm{I\!R})$ being defined as the set of $2\overline{n} \times 2\overline{n}$ matrices that satisfy the condition

$$\Lambda \in Sp(2\overline{n}, \mathrm{I\!R}) \longrightarrow \Lambda^T \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix} \Lambda = \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix} \tag{2.25}$$

that is, using $n \otimes n$ block components

$$A^T C - C^T A = B^T D - D^T B = 0 \qquad A^T D - C^T B = 1 \tag{2.26}$$

To prove the statement we just made, we calculate the transformed Lagrangian $\mathcal{L}'$ and then we compare its variation $\frac{\partial \mathcal{L}'}{\partial \mathcal{F}'^T}$ with $\mathcal{G}^{\pm\prime}$ as it follows from the postulated transformation rule 2.23. To perform such a calculation we rely on the following basic idea. While the duality rotation 2.23 is performed on the field strengths and on their duals, also the scalar fields are transformed by the action of some diffeomorphism $\xi \in \mathrm{Diff}\,(\mathcal{M}_{scalar})$ of the scalar manifold and, as a consequence of that, also the matrix $\mathcal{N}$ changes. In other words given the scalar manifold $\mathcal{M}_{scalar}$ we assume that there exists a homomorphism of the form

$$\iota_\delta : \mathrm{Diff}\,(\mathcal{M}_{scalar}) \longrightarrow GL(2\overline{n}, \mathrm{I\!R}) \tag{2.27}$$

so that

$$\begin{aligned} \forall \quad \xi \quad &\in \mathrm{Diff}\,(\mathcal{M}_{scalar}) : \phi^I \xrightarrow{\xi} \phi^{I\prime} \\ \exists \quad \iota_\delta(\xi) \quad &= \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \in GL(2\overline{n}, \mathrm{I\!R}) \end{aligned} \tag{2.28}$$

(In the sequel the subfix ξ will be omitted when no confusion can arise and be reinstalled when necessary for clarity.)

Using such a homomorphism we can define the simultaneous action of ξ on all the fields of our theory by setting

$$\xi : \begin{cases} \phi \longrightarrow \xi(\phi) \\ \mathbf{V} \longrightarrow \iota_\delta(\xi)\,\mathbf{V} \\ \mathcal{N}(\phi) \longrightarrow \mathcal{N}'(\xi(\phi)) \end{cases} \tag{2.29}$$

where the notation 2.18 has been utilized. In the gauge sector the transformed Lagrangian is

$$\mathcal{L}'_{vec} = \mathrm{i}\left[\mathcal{F}^{-T}\,(A + B\overline{\mathcal{N}})^T \overline{\mathcal{N}}'(A + B\overline{\mathcal{N}})\mathcal{F}^- - \mathcal{F}^{+T}\,(A + B\mathcal{N})^T \mathcal{N}'(A + B\mathcal{N})\mathcal{F}^+\right] \tag{2.30}$$

Consistency with the definition of $\mathcal{G}^+$ requires that

$$\mathcal{N}' \equiv \mathcal{N}'(\xi(\phi)) = (C + D\mathcal{N}(\phi))\,(A + B\mathcal{N}(\phi))^{-1} \tag{2.31}$$

while consistency with the definition of $\mathcal{G}^-$ imposes the transformation rule

$$\overline{\mathcal{N}}' \equiv \overline{\mathcal{N}}'(\xi(\phi)) = \left(C + D\overline{\mathcal{N}}(\phi)\right)\left(A + B\overline{\mathcal{N}}(\phi)\right)^{-1} \tag{2.32}$$

It is from the transformation rules 2.31 and 2.32 that we derive a restriction on the form of the duality rotation matrix $\Lambda \equiv \iota_\delta(\xi)$. Indeed by requiring that the transformed matrix $\mathcal{N}'$ be again symmetric one easily finds that Λ must obey eq. 2.25, namely $\Lambda \in Sp(2\overline{n}, \mathrm{I\!R})$. Consequently the homomorphism of eq. 2.27 specializes as

$$\iota_\delta : \mathrm{Diff}\,(\mathcal{M}_{scalar}) \longrightarrow Sp(2\overline{n}, \mathrm{I\!R}) \tag{2.33}$$

Clearly, since $Sp(2\overline{n}, \mathrm{I\!R})$ is a finite dimensional Lie group, while Diff $(\mathcal{M}_{scalar})$ is infinite–dimensional, the homomorphism ι_δ can never be an isomorphism. Defining the Torelli group of the scalar manifold as

$$\mathrm{Diff}\,(\mathcal{M}_{scalar}) \supset \mathrm{Tor}\,(\mathcal{M}_{scalar}) \equiv \ker \iota_\delta \tag{2.34}$$

we always have

$$\dim \mathrm{Tor}\,(\mathcal{M}_{scalar}) = \infty \tag{2.35}$$

The reason why we have given the name of Torelli to the group defined by eq. 2.34 is because of its similarity with the Torelli group that occurs in algebraic geometry.

What should be clear from the above discussion is that a family of Lagrangians as in eq. 2.5 will admit a group of duality–rotations/field–redefinitions that will map elements of the family into each other, as long as a *kinetic matrix* $\mathcal{N}_{\Lambda\Sigma}$ can be constructed that transforms as in eq. 2.31. A way to obtain such an object is to identify it with the *period matrix* occurring in problems of algebraic geometry. At the level of the present discussion, however, this identification is by no means essential: any construction of $\mathcal{N}_{\Lambda\Sigma}$ with the appropriate transformation properties is acceptable. Note also that so far we have used the words *duality–rotations/field–redefinitions* and not the word duality symmetry. Indeed the diffeomorphisms of the scalar manifold we have considered were quite general and, as such had no pretension to be symmetries of the action, or of the theory. Indeed the question we have answered is the following: what are the appropriate transformation properties of the tensor gauge fields and of the generalized coupling constants under diffeomorphisms of the scalar manifold? The next question is obviously that of duality symmetries.

As it is the case with the difference between general covariance and isometries in the context of general relativity, duality symmetries correspond to the subset of duality transformations for which we obtain an invariance in form of the theory. In this respect, however, we have to stress that what is invariant in form cannot be the Lagrangian but only the set of field equations plus Bianchi identities. Indeed, while any $\Lambda \in Sp(2\overline{n}, \mathrm{I\!R})$ can, in principle, be an invariance in form of eqs. 2.17, the same is not true for the Lagrangian. One can easily find that the vector kinetic part of this latter transforms as follows:

$$\begin{aligned}
\mathrm{Im}\mathcal{F}^{-\Lambda}\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} \quad \rightarrow \quad & \mathrm{Im}\widetilde{\mathcal{F}}^{-\Lambda}\,\widetilde{\mathcal{G}}^-_\Sigma \\
& = \mathrm{Im}\Big(\mathcal{F}^{-\Lambda}\mathcal{G}^-_\Lambda + 2\mathcal{F}^{-\Lambda}\left(C^TB\right)_\Lambda{}^\Sigma\,\mathcal{G}^-_\Sigma \\
& + \mathcal{F}^{-\Lambda}\left(C^TA\right)_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} + \mathcal{G}^-_\Lambda\left(D^TB\right)^{\Lambda\Sigma}\mathcal{G}^-_\Sigma\Big)
\end{aligned} \tag{2.36}$$

whence we conclude that proper symmetries of the Lagrangian are to be looked for only among matrices with $C = B = 0$. If $C \neq 0$ and $B = 0$, the Lagrangian varies through the addition of a topological density (see below eq. 6.7). Elements of $Sp(2\overline{n}, \mathrm{I\!R})$ with $B \neq 0$, cannot be symmetries of the classical action under any circumstance.

The scalar part of the Lagrangian, on the other hand, is invariant under all those diffeomorphisms of the scalar manifolds that are *isometries* of the scalar metric g_{IJ}. Naming $\xi^\star : T\mathcal{M}_{scalar} \to T\mathcal{M}_{scalar}$ the push–forward of ξ, this means that

$$\begin{aligned} &\forall X, Y \in T\mathcal{M}_{scalar} \\ g(X,Y) &= g(\xi^\star X, \xi^\star Y) \end{aligned} \tag{2.37}$$

and ξ is an exact global symmetry of the scalar part of the Lagrangian in eq. 2.5. In view of our previous discussion these symmetries of the scalar sector are not guaranteed to admit an extension to symmetries of the complete action. Yet we can insist that they extend to symmetries of the field equations plus Bianchi identities, namely to duality symmetries in the sense defined above. This requires that the group of isometries of the scalar metric $\mathcal{I}(\mathcal{M}_{scalar})$ be suitably embedded into the duality group $Sp(2\overline{n}, \mathrm{I\!R})$ and that the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ satisfies the covariance law:

$$\mathcal{N}(\xi(\phi)) = (C_\xi + D_\xi \mathcal{N}(\phi)) (A_\xi + B_\xi \mathcal{N}(\phi))^{-1} \ . \tag{2.38}$$

3 Symplectic embeddings of homogenous spaces

A general construction of the kinetic coupling matrix $\mathcal{N}$ can be derived in the case where the scalar manifold is taken to be a homogeneous space $\mathcal{G}/\mathcal{H}$. This is what happens in all extended supergravities for $N \geq 3$ and also in specific instances of N=2 theories. For this reason we shortly review the construction of the *kinetic period matrix* $\mathcal{N}$ in the case of homogeneous spaces. Although the basic construction was introduced in the literature by Gaillard and Zumino in 1981 [46] and was reviewed by some of us in [31], a derivation of the basic formulae that matches completely with the modern notations of N=2 and N=4 theories, such as they emerge in string compactifications and in the discussion of S–duality, is not available, to our knowledge, in the existing literature. To make the present paper self contained we consider therefore essential to review such a construction in modern gear.

The relevant homomorphism ι_δ (see eq. 2.33) becomes:

$$\iota_\delta : \mathrm{Diff}\left(\frac{\mathcal{G}}{\mathcal{H}}\right) \longrightarrow Sp(2\overline{n}, \mathrm{I\!R}) \tag{3.1}$$

In particular, focusing on the isometry group of the canonical metric defined on $\frac{\mathcal{G}}{\mathcal{H}}$[2]: $\mathcal{I}\left(\frac{\mathcal{G}}{\mathcal{H}}\right) = \mathcal{G}$ we must consider the embedding:

$$\iota_\delta : \mathcal{G} \longrightarrow Sp(2\overline{n}, \mathrm{I\!R}) \tag{3.2}$$

That in eq. 3.1 is a homomorphism of finite dimensional Lie groups and as such it constitutes a problem that can be solved in explicit form. What we just need to know is

[2]Actually, in order to be true, the equation $\mathcal{I}(\frac{\mathcal{G}}{\mathcal{H}}) = \mathcal{G}$ requires that that the normaliser of $\mathcal{H}$ in $\mathcal{G}$ be the identity group, a condition that is verified in all the relevant examples

the dimension of the symplectic group, namely the number $\overline{n}$ of gauge fields appearing in the theory. Without supersymmetry the dimension m of the scalar manifold (namely the possible choices of $\frac{\mathcal{G}}{\mathcal{H}}$) and the number of vectors $\overline{n}$ are unrelated so that the possibilities covered by eq. 3.2 are infinitely many. In supersymmetric theories, instead, the two numbers m and $\overline{n}$ are related, so that there are finitely many cases to be studied corresponding to the possible embeddings of given groups $\mathcal{G}$ into a symplectic group $Sp(2\overline{n},\mathbb{R})$ of fixed dimension $\overline{n}$. Actually taking into account further conditions on the holonomy of the scalar manifold that are also imposed by supersymmetry, the solution for the symplectic embedding problem is unique for all extended supergravities with $N \geq 3$ as we have already remarked (see for instance [31]).

Apart from the details of the specific case considered once a symplectic embedding is given there is a general formula one can write down for the *period matrix* $\mathcal{N}$ that guarantees symmetry ($\mathcal{N}^T = \mathcal{N}$) and the required transformation property 2.38. This is the result we want to review. It will be useful in the sequel for comparison with the formulae of special geometry in the case the considered special manifold is homogeneous (see appendix C, in particular).

The real symplectic group $Sp(2\overline{n},\mathbb{R})$ is defined as the set of all *real* $2\overline{n} \times 2\overline{n}$ matrices $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying equation 2.25, namely

$$\Lambda^T \, \mathbb{C} \, \Lambda \, = \, \mathbb{C} \tag{3.3}$$

where $\mathbb{C} \equiv \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix}$ If we relax the condition that the matrix should be real but we still impose eq. 3.3 we obtain the definition of the complex symplectic group $Sp(2\overline{n},\mathbb{C})$. It is a well known fact that the following isomorphism is true:

$$Sp(2\overline{n},\mathbb{R}) \sim Usp(\overline{n},\overline{n}) \equiv Sp(2\overline{n},\mathbb{C}) \cap U(\overline{n},\overline{n}) \tag{3.4}$$

By definition an element $\mathcal{S} \in Usp(\overline{n},\overline{n})$ is a complex matrix that satisfies simultaneously eq. 3.3 and a pseudo–unitarity condition, that is:

$$\mathcal{S}^T \, \mathbb{C} \, \mathcal{S} \, = \, \mathbb{C} \qquad ; \qquad \mathcal{S}^\dagger \, \mathbb{H} \, \mathcal{S} \, = \, \mathbb{H} \tag{3.5}$$

where $\mathbb{H} \equiv \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & -\mathbb{1} \end{pmatrix}$. The general block form of the matrix $\mathcal{S}$ is:

$$\mathcal{S} \, = \, \begin{pmatrix} T & V^\star \\ V & T^\star \end{pmatrix} \tag{3.6}$$

and eq.s 3.5 are equivalent to:

$$T^\dagger \, T \, - \, V^\dagger \, V \, = \, \mathbb{1} \qquad ; \qquad T^\dagger \, V^\star \, - \, V^\dagger \, T^\star \, = \, \mathbf{0} \tag{3.7}$$

The isomorphism of eq. 3.4 is explicitly realized by the so called Cayley matrix:

$$\mathcal{C} \, \equiv \, \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathrm{i}\mathbb{1} \\ \mathbb{1} & -\mathrm{i}\mathbb{1} \end{pmatrix} \tag{3.8}$$

via the relation:

$$\mathcal{S} \, = \, \mathcal{C} \, \Lambda \, \mathcal{C}^{-1} \tag{3.9}$$

which yields:

$$T = \frac{1}{2}(A+D) - \frac{\mathrm{i}}{2}(B-C) \qquad ; \qquad V = \frac{1}{2}(A-D) - \frac{\mathrm{i}}{2}(B+C) \tag{3.10}$$

When we set $V = 0$ we obtain the subgroup $U(\overline{n}) \subset Usp(\overline{n},\overline{n})$, that in the real basis is given by the subset of symplectic matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. The basic idea, to obtain the general formula for the period matrix, is that the symplectic embedding of the isometry group $\mathcal{G}$ will be such that the isotropy subgroup $\mathcal{H} \subset \mathcal{G}$ gets embedded into the maximal compact subgroup $U(\overline{n})$, namely:

$$\mathcal{G} \xrightarrow{\iota_\delta} Usp(\overline{n},\overline{n}) \qquad ; \qquad \mathcal{G} \supset \mathcal{H} \xrightarrow{\iota_\delta} U(\overline{n}) \subset Usp(\overline{n},\overline{n}) \tag{3.11}$$

If this condition is realized let $L(\phi)$ be a parametrization of the coset $\mathcal{G}/\mathcal{H}$ by means of coset representatives. Relying on the symplectic embedding of eq. 3.11 we obtain a map:

$$L(\phi) \longrightarrow \mathcal{O}(\phi) = \begin{pmatrix} U_0(\phi) & U_1^\star(\phi) \\ U_1(\phi) & U_0^\star(\phi) \end{pmatrix} \in Usp(\overline{n},\overline{n}) \tag{3.12}$$

that associates to $L(\phi)$ a coset representative of $Usp(\overline{n},\overline{n})/U(\overline{n})$. By construction if $\phi' \neq \phi$ *no* unitary $\overline{n} \times \overline{n}$ matrix W *can exist* such that:

$$\mathcal{O}(\phi') = \mathcal{O}(\phi) \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & W^\star \end{pmatrix} \tag{3.13}$$

On the other hand let $\xi \in \mathcal{G}$ be an element of the isometry group of $\mathcal{G}/\mathcal{H}$. Via the symplectic embedding of eq. 3.11 we obtain a $Usp(\overline{n},\overline{n})$ matrix

$$\mathcal{S}_\xi = \begin{pmatrix} T_\xi & V_\xi^\star \\ V_\xi & T_\xi^\star \end{pmatrix} \tag{3.14}$$

such that

$$\mathcal{S}_\xi \, \mathcal{O}(\phi) = \mathcal{O}(\xi(\phi)) \begin{pmatrix} W(\xi,\phi) & \mathbf{0} \\ \mathbf{0} & W^\star(\xi,\phi) \end{pmatrix} \tag{3.15}$$

where $\xi(\phi)$ denotes the image of the point $\phi \in \mathcal{G}/\mathcal{H}$ through ξ and $W(\xi,\phi)$ is a suitable $U(\overline{n})$ compensator depending both on ξ and ϕ. Combining eq.s 3.15, 3.12, with eq.s 3.10 we immediately obtain:

$$\begin{aligned} U_0^\dagger(\xi(\phi)) + U_1^\dagger(\xi(\phi)) &= W\left[U_0^\dagger(\phi)\left(A^T + \mathrm{i}B^T\right) + U_1^\dagger(\phi)\left(A^T - \mathrm{i}B^T\right)\right] \\ U_0^\dagger(\xi(\phi)) - U_1^\dagger(\xi(\phi)) &= W\left[U_0^\dagger(\phi)\left(D^T - \mathrm{i}C^T\right) - U_1^\dagger(\phi)\left(D^T + \mathrm{i}C^T\right)\right] \end{aligned} \tag{3.16}$$

Setting:

$$\mathcal{N} \equiv \mathrm{i}\left[U_0^\dagger + U_1^\dagger\right]^{-1}\left[U_0^\dagger - U_1^\dagger\right] \tag{3.17}$$

and using the result of eq. 3.16 one checks that the transformation rule 2.38 is verified. It is also an immediate consequence of the analogue of eq.s 3.7 satisfied by U_0 and U_1 that the matrix in eq. 3.17 is symmetric

$$\mathcal{N}^T = \mathcal{N} \tag{3.18}$$

Eq. 3.17 is the master formula derived in 1981 by Gaillard and Zumino [46]. It explains the structure of the gauge field kinetic terms in all $N \geq 3$ extended supergravity theories and also in those $N = 2$ theories where the Special Kähler manifold $\mathcal{SM}$ is a homogeneous manifold $\mathcal{G}/\mathcal{H}$.

3.1 Symplectic embedding of the $\mathcal{ST}[m,n]$ homogeneous manifolds

Because of their relevance in superstring compactifications let us illustrate the general procedure with the following class of homogeneous manifolds:

$$\mathcal{ST}[m,n] \equiv \frac{SU(1,1)}{U(1)} \otimes \frac{SO(m,n)}{SO(m)\otimes SO(n)} \tag{3.19}$$

The isometry group of the $\mathcal{ST}[m,n]$ manifolds defined in eq. 3.19 contains a factor ($SU(1,1)$) whose transformations act as non–perturbative S–dualities and another factor ($SO(m,n)$) whose transformations act as T–dualities, holding true at each order in string perturbation theory. The field S is obtained by combining together the *dilaton* D and the *axion* $\mathcal{A}$:

$$\begin{aligned} S &= \mathcal{A} - \mathrm{i}\exp[D] \\ \partial^\mu \mathcal{A} &\equiv \varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu\, B_{\rho\sigma} \end{aligned} \tag{3.20}$$

while t^i is the name usually given to the moduli–fields of the compactified target space. Now in string and supergravity applications S will be identified with the complex coordinate on the manifold $\frac{SU(1,1)}{U(1)}$, while t^i will be the coordinates of the coset space $\frac{SO(m,n)}{SO(m)\otimes SO(n)}$. The case $\mathcal{ST}[6,n]$ is the scalar manifold in $N=4$ supergravity, while the case $\mathcal{ST}[2,n]$ is a very interesting instance of special Kähler manifold appearing in superstring compactifications. Although as differentiable and metric manifolds the spaces $\mathcal{ST}[m,n]$ are just direct products of two factors (corresponding to the above mentioned different physical interpretation of the coordinates S and t^i), from the point of view of the symplectic embedding and duality rotations they have to be regarded as a single entity. This is even more evident in the case $m=2, n=$ arbitrary, where the following theorem has been proven by Ferrara and Van Proeyen [48]: $\mathcal{ST}[2,n]$ are the only special Kähler manifolds with a direct product structure. The definition of special Kähler manifolds is given in the next section, yet the anticipation of this result should make clear that the special Kähler structure (encoding the duality rotations in the $N=2$ case) is not a property of the individual factors but of the product as a whole. Neither factor is by itself a special manifold although the product is.

At this point comes the question of the correct symplectic embedding. Such a question has two aspects:

1. Intrinsically inequivalent embeddings
2. Symplectically equivalent embeddings that become inequivalent after gauging

The first issue in the above list is group–theoretical in nature. When we say that the group $\mathcal{G}$ is embedded into $Sp(2\overline{n},\mathbb{R})$ we must specify how this is done from the point of view of irreducible representations. Group–theoretically the matter is settled by specifying how the fundamental representation of $Sp(2\overline{n})$ splits into irreducible representations of $\mathcal{G}$:

$$\mathbf{2\overline{n}} \xrightarrow{\mathcal{G}} \oplus_{i=1}^{\ell}\ \mathbf{D}_i \tag{3.21}$$

Once eq. 3.21 is given (in supersymmetric theories such information is provided by supersymmetry) the only arbitrariness which is left is that of conjugation by arbitrary

$Sp(2\overline{n},\mathrm{I\!R})$ matrices. Suppose we have determined an embedding ι_δ that obeys the law in eq. 3.21, then:

$$\forall \mathcal{S} \in Sp(2\overline{n},\mathrm{I\!R}) : \iota_\delta' \equiv \mathcal{S} \circ \iota_\delta \circ \mathcal{S}^{-1} \tag{3.22}$$

will obey the same law. That in eq. 3.22 is a symplectic transformation that corresponds to an allowed duality–rotation/field–redefinition in the abelian theory of type in eq. 2.5 discussed in the previous subsection. Therefore all abelian Lagrangians related by such transformations are physically equivalent.

The matter changes in presence of *gauging*. When we switch on the gauge coupling constant and the electric charges, symplectic transformations cease to yield physically equivalent theories. This is the second issue in the above list. The choice of a symplectic gauge becomes physically significant. The construction of supergravity theories proceeds in two steps. In the first step, one constructs the abelian theory: at that level the only relevant constraint is that encoded in eq. 3.21 and the choice of a symplectic gauge is immaterial. Actually one can write the entire theory in such a way that *symplectic covariance* is manifest. In the second step one *gauges* the theory. This *breaks symplectic covariance* and the choice of the correct symplectic gauge becomes a physical issue. This issue has been recently emphasized by the results in [29] where it has been shown that whether N=2 supersymmetry can be spontaneously broken to N=1 or not depends on the symplectic gauge.

These facts being cleared we proceed to discuss the symplectic embedding of the $\mathcal{ST}[m,n]$ manifolds.

Let η be the symmetric flat metric with signature (m,n) that defines the $SO(m,n)$ group, via the relation

$$L \in SO(m,n) \Longleftrightarrow L^T \eta L = \eta \tag{3.23}$$

Both in the $N=4$ and in the $N=2$ theory, the number of gauge fields in the theory is given by:

$$\#\text{vector fields} = m \oplus n \tag{3.24}$$

m being the number of *graviphotons* and n the number of *vector multiplets*. Hence we have to embed $SO(m,n)$ into $Sp(2m+2n,\mathrm{I\!R})$ and the explicit form of the decomposition in eq. 3.21 required by supersymmetry is:

$$\mathbf{2m}+\mathbf{2n} \stackrel{SO(m,n)}{\longrightarrow} \mathbf{m}+\mathbf{n} \oplus \mathbf{m}+\mathbf{n} \tag{3.25}$$

where $\mathbf{m}+\mathbf{n}$ denotes the fundamental representation of $SO(m,n)$. Eq. 3.25 is easily understood in physical terms. $SO(m,n)$ must be a T–duality group, namely a symmetry holding true order by order in perturbation theory. As such it must rotate electric field strengths into electric field strengths and magnetic field strengths into magnetic field field strengths. The two irreducible representations into which the the fundamental representation of the symplectic group decomposes when reduced to $SO(m,n)$ correspond precisely to electric and magnetic sectors, respectively. In the *simplest gauge* the symplectic embedding satisfying eq. 3.25 is block–diagonal and takes the form:

$$\forall L \in SO(m,n) \quad \stackrel{\iota_\delta}{\hookrightarrow} \quad \begin{pmatrix} L & \mathbf{0} \\ \mathbf{0} & (L^T)^{-1} \end{pmatrix} \in Sp(2m+2n,\mathrm{I\!R}) \tag{3.26}$$

Consider instead the group $SU(1,1) \sim SL(2,\mathrm{I\!R})$. This is the factor in the isometry group of $\mathcal{ST}[m,n]$ that is going to act by means of S–duality non perturbative rotations.

Typically it will rotate each electric field strength into its homologous magnetic one. Correspondingly supersymmetry implies that its embedding into the symplectic group must satisfy the following condition:

$$\mathbf{2m} + \mathbf{2n} \stackrel{SL(2,\mathbb{R})}{\longrightarrow} \oplus_{i=1}^{m+n} \ \mathbf{2} \tag{3.27}$$

where $\mathbf{2}$ denotes the fundamental representation of $SL(2,\mathbb{R})$. In addition it must commute with the embedding of $SO(m,n)$ in eq. 3.26 . Both conditions are fulfilled by setting:

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \ \stackrel{\iota_\delta}{\hookrightarrow} \ \begin{pmatrix} a\,\mathbb{1} & b\,\eta \\ c\,\eta & d\,\mathbb{1} \end{pmatrix} \in Sp(2m+2n,\mathbb{R}) \tag{3.28}$$

Utilizing eq.s 3.9 the corresponding embeddings into the group $Usp(m+n,m+n)$ are immediately derived:

$$\begin{aligned}
\forall\, L \in SO(m,n) \ &\stackrel{\iota_\delta}{\hookrightarrow} \ \begin{pmatrix} \frac{1}{2}(L+\eta L\eta) & \frac{1}{2}(L-\eta L\eta) \\ \frac{1}{2}(L-\eta L\eta) & \frac{1}{2}(L+\eta L\eta) \end{pmatrix} \in Usp(m+n,m+n) \\
\forall \begin{pmatrix} t & v^\star \\ v & t^\star \end{pmatrix} \in SU(1,1) \ &\stackrel{\iota_\delta}{\hookrightarrow} \ \begin{pmatrix} \mathrm{Re}t\mathbb{1} + \mathrm{iIm}t\eta & \mathrm{Re}v\mathbb{1} - \mathrm{iIm}v\eta \\ \mathrm{Re}v\mathbb{1} + \mathrm{iIm}v\eta & \mathrm{Re}t\mathbb{1} - \mathrm{iIm}t\eta \end{pmatrix} \in Usp(m+n,m+n)
\end{aligned} \tag{3.29}$$

where the relation between the entries of the $SU(1,1)$ matrix and those of the corresponding $SL(2,\mathbb{R})$ matrix are provided by the relation in eq. 3.10.

Equipped with these relations we can proceed to derive the explicit form of the *period matrix* $\mathcal{N}$.

The homogeneous manifold $SU(1,1)/U(1)$ can be conveniently parametrized in terms of a single complex coordinate S, whose physical interpretation will be that of *axion–dilaton*, according to eq. 3.20. The coset parametrization appropriate for comparison with other constructions (special geometry or $N=4$ supergravity) is given by the family of matrices:

$$M(S) \equiv \frac{1}{n(S)} \begin{pmatrix} \mathbb{1} & \frac{\mathrm{i}-S}{\mathrm{i}+S} \\ \frac{\mathrm{i}+\overline{S}}{\mathrm{i}-\overline{S}} & \mathbb{1} \end{pmatrix} \quad : \quad n(S) \equiv \sqrt{\frac{4\mathrm{Im}S}{1+|S|^2+2\mathrm{Im}S}} \tag{3.30}$$

To parametrize the coset $SO(m,n)/SO(m)\times SO(n)$ we can instead take the usual coset representatives (see for instance [31]):

$$L(X) \equiv \begin{pmatrix} \left(\mathbb{1}+XX^T\right)^{1/2} & X \\ X^T & \left(\mathbb{1}+X^TX\right)^{1/2} \end{pmatrix} \tag{3.31}$$

where the $m\times n$ real matrix X provides a set of independent coordinates. Inserting these matrices into the embedding formulae of eq.s 3.29 we obtain a matrix:

$$\iota_\delta\left(M(S)\right) \circ \iota_\delta\left(L(X)\right) = \begin{pmatrix} U_0(S,X) & U_1^\star(S,X) \\ U_1(S,X) & U_0^\star(S,X) \end{pmatrix} \in Usp(n+m,n+m) \tag{3.32}$$

that inserted into the master formula of eq. 3.17 yields the following result:

$$\mathcal{N} = \mathrm{iIm}S\,\eta L(X)L^T(X)\eta + \mathrm{Re}S\,\eta \tag{3.33}$$

Alternatively, remarking that if $L(X)$ is an $SO(m,n)$ matrix also $L(X)' = \eta L(X)\eta$ is such a matrix and represents the same equivalence class, we can rewrite 3.33 in the simpler form:

$$\mathcal{N} = \mathrm{iIm}S\,L(X)'L^{T\prime}(X) + \mathrm{Re}S\,\eta \tag{3.34}$$

4 Special Kähler Geometry

The first discovery that the self-interaction of Wess–Zumino multiplets is governed by Kähler geometry is due to Zumino [49] (1979). Independently, the parametrization of the coupling of Wess–Zumino multiplets to supergravity in terms of a real function, later identified with the Kähler potential, was obtained in [50, 51] (1978), shortly after that supergravity had been discovered by Freedman, Ferrara and van Nieuwenhuizen [52] (1976) and recast in first order formalism by Deser and Zumino [53] (1976).

The complete form of standard N=1 supergravity, determined by means of the superconformal calculus, was obtained in [54] (1983), while the geometric interpretation of the coupling structure is due to Bagger and Witten [55, 56] (1983).

Special Kähler geometry in special coordinates was introduced in 1984–85 by B. de Wit et al. in [57, 34] and E. Cremmer et al. in [58], where the coupling of N=2 vector multiplets to N=2 supergravity was fully determined. The more intrinsic definition of special Kähler geometry in terms of symplectic bundles is due to Strominger [59] (1990), who obtained it in connection with the moduli spaces of Calabi–Yau compactifications. The coordinate-independent description and derivation of special Kähler geometry in the context of N=2 supergravity is due to Castellani, D'Auria, Ferrara [27] and to D'Auria, Ferrara, Fre' [24] (1991). Recently Ceresole, D'Auria, Ferrara and Van Proeyen [4] have shown how one can and in important instances must dispense of the notion of holomorphic prepotential $F(X)$. Let us begin by reviewing the notions of Kähler and Hodge–Kähler manifolds that are the prerequisites to introduce the notion of Special Kähler manifolds. Once again we do this in order to fix our notations.

4.1 Hodge–Kähler manifolds

Consider a *line bundle* $\mathcal{L}\xrightarrow{\pi}\mathcal{M}$ over a Kähler manifold. By definition this is a holomorphic vector bundle of rank $r = 1$. For such bundles the only available Chern class is the first:

$$c_1(\mathcal{L}) \;=\; \frac{i}{2\pi}\,\overline{\partial}\,\left(h^{-1}\,\partial\,h\right) \;=\; \frac{i}{2\pi}\,\overline{\partial}\,\partial\log h \tag{4.1}$$

where the 1-component real function $h(z,\overline{z})$ is some hermitian fibre metric on $\mathcal{L}$. Let $f(z)$ be a holomorphic section of the line bundle $\mathcal{L}$: noting that under the action of the operator $\overline{\partial}\,\partial$ the term $\log\left(\overline{\xi}(\overline{z})\,\xi(z)\right)$ yields a vanishing contribution, we conclude that the formula in eq. 4.1 for the first Chern class can be re-expressed as follows:

$$c_1(\mathcal{L}) \;=\; \frac{i}{2\pi}\,\overline{\partial}\,\partial\log\,\parallel\,\xi(z)\,\parallel^2 \tag{4.2}$$

where $\parallel\,\xi(z)\,\parallel^2 \;=\; h(z,\overline{z})\,\overline{\xi}(\overline{z})\,\xi(z)$ denotes the norm of the holomorphic section $\xi(z)$.

Eq. 4.2 is the starting point for the definition of Hodge Kähler manifolds, an essential notion in supergravity theory.

A Kähler manifold $\mathcal{M}$ is a Hodge manifold if and only if there exists a line bundle $\mathcal{L}\;\longrightarrow\;\mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler 2-form K:

$$c_1(\mathcal{L}) \;=\; [\,K\,] \tag{4.3}$$

In local terms this means that there is a holomorphic section $W(z)$ such that we can write

$$K = \frac{i}{2\pi} g_{ij^\star} dz^i \wedge d\overline{z}^{j^\star} = \frac{i}{2\pi} \overline{\partial} \partial \log \parallel W(z) \parallel^2 \tag{4.4}$$

Recalling the local expression of the Kähler metric in terms of the Kähler potential $g_{ij^\star} = \partial_i \partial_{j^\star} \mathcal{K}(z, \overline{z})$, it follows from eq. 4.4 that if the manifold $\mathcal{M}$ is a Hodge manifold, then the exponential of the Kähler potential can be interpreted as the metric $h(z, \overline{z}) = \exp(\mathcal{K}(z, \overline{z}))$ on an appropriate line bundle $\mathcal{L}$.

This structure is precisely that advocated by the Lagrangian of $N = 1$ matter coupled supergravity: the holomorphic section $W(z)$ of the line bundle $\mathcal{L}$ is what, in N=1 supergravity theory, is named the superpotential and the logarithm of its norm $\log \parallel W(z) \parallel^2 = \mathcal{K}(z, \overline{z}) + \log | W(z) |^2 = G(z, \overline{z})$ is precisely the invariant function in terms of which one writes the potential and Yukawa coupling terms of the supergravity action (see [54] and for a review [31]).

4.2 Special Kähler Manifolds: general discussion

There are in fact two kinds of special Kähler geometry: the local and the rigid one. The former describes the scalar field sector of vector multiplets in $N = 2$ supergravity while the latter describes the same sector in rigid $N = 2$ Yang–Mills theories. Since $N = 2$ includes $N = 1$ supersymmetry, local and rigid special Kähler manifolds must be compatible with the geometric structures that are respectively enforced by local and rigid $N = 1$ supersymmetry in the scalar sector. The distinction between the two cases deals with the first Chern–class of the line–bundle $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$, whose sections are the possible superpotentials. In the local theory $c_1(\mathcal{L}) = [K]$ and this restricts $\mathcal{M}$ to be a Hodge–Kähler manifold. In the rigid theory, instead, we have $c_1(\mathcal{L}) = 0$. At the level of the Lagrangian this reflects into a different behaviour of the fermion fields. These latter are sections of $\mathcal{L}^{1/2}$ and couple to the canonical hermitian connection defined on $\mathcal{L}$:

$$\theta \equiv h^{-1} \partial h = \tfrac{1}{h} \partial_i h \, dz^i \quad ; \quad \overline{\theta} \equiv h^{-1} \overline{\partial} h = \tfrac{1}{h} \partial_{i^\star} h \, d\overline{z}^{i^\star} \tag{4.5}$$

In the local case where

$$\left[\overline{\partial} \theta \right] = c_1(\mathcal{L}) = [K] \tag{4.6}$$

the fibre metric h can be identified with the exponential of the Kähler potential and we obtain:

$$\theta = \partial \mathcal{K} = \partial_i \mathcal{K} dz^i \quad ; \quad \overline{\theta} = \overline{\partial} \mathcal{K} = \partial_{i^\star} \mathcal{K} d\overline{z}^{i^\star} \tag{4.7}$$

In the rigid case, $\mathcal{L}$ is instead a flat bundle and its metric is unrelated to the Kähler potential. Actually one can choose a vanishing connection:

$$\theta = \overline{\theta} = 0 \tag{4.8}$$

The distinction between rigid and local special manifolds is the $N = 2$ generalization of this difference occurring at the $N = 1$ level. In the $N = 2$ case, in addition to the line–bundle $\mathcal{L}$ we need a flat holomorphic vector bundle $\mathcal{SV} \longrightarrow \mathcal{M}$ whose sections can be identified with the superspace *fermi–fermi* components of electric and magnetic field–strengths (see appendix B). In this way, according to the discussion of previous sections the diffeomorphisms of the scalar manifolds will be lifted to produce an action on the gauge–field strengths as well. In a supersymmetric theory where scalars and gauge fields belong to the same multiplet this is a mandatory condition. However this symplectic bundle structure must be made compatible with the line–bundle structure already requested by $N = 1$ supersymmetry. This leads to the existence of two kinds of special geometry. Another essential distinction between the two kind of geometries arises from the different number of vector fields in the theory. In the rigid case this number equals that of the vector multiplets so that

$$\begin{array}{rcl} \#\,\text{vector fields} \equiv \overline{n} & = & n \\ \#\,\text{vector multiplets} \equiv n & = & \dim_{\mathbf{C}} \mathcal{M} \\ \text{rank}\, \mathcal{SV} \equiv 2\overline{n} & = & 2n \end{array} \tag{4.9}$$

On the other hand, in the local case, in addition to the vector fields arising from the vector multiplets we have also the graviphoton coming from the graviton multiplet. Hence we conclude:

$$\begin{array}{rcl} \#\,\text{vector fields} \equiv \overline{n} & = & n+1 \\ \#\,\text{vector multiplets} \equiv n & = & \dim_{\mathbf{C}} \mathcal{M} \\ \text{rank}\, \mathcal{SV} \equiv 2\overline{n} & = & 2n+2 \end{array} \tag{4.10}$$

In the sequel we make extensive use of covariant derivatives with respect to the canonical connection of the line–bundle $\mathcal{L}$. Let us review its normalization. As it is well known there exists a correspondence between line–bundles and $U(1)$–bundles. If $\exp[f_{\alpha\beta}(z)]$ is the transition function between two local trivializations of the line–bundle $\mathcal{L} \longrightarrow \mathcal{M}$, the transition function in the corresponding principal $U(1)$–bundle $\mathcal{U} \longrightarrow \mathcal{M}$ is just $\exp[\mathrm{i}\mathrm{Im} f_{\alpha\beta}(z)]$ and the Káhler potentials in two different charts are related by:

$$\mathcal{K}_\beta = \mathcal{K}_\alpha + f_{\alpha\beta} + \overline{f}_{\alpha\beta} \tag{4.11}$$

. At the level of connections this correspondence is formulated by setting:

$$U(1)\text{–connection} \equiv \mathcal{Q} \, = \, \mathrm{Im}\theta = -\frac{\mathrm{i}}{2}\left(\theta - \overline{\theta}\right) \tag{4.12}$$

If we apply the above formula to the case of the $U(1)$–bundle $\mathcal{U} \longrightarrow \mathcal{M}$ associated with the line–bundle $\mathcal{L}$ whose first Chern class equals the Kähler class, we get:

$$\mathcal{Q} = -\frac{\mathrm{i}}{2}\left(\partial_i \mathcal{K} dz^i - \partial_{i^\star} \mathcal{K} d\overline{z}^{i^\star}\right) \tag{4.13}$$

Let now $\Phi(z, \overline{z})$ be a section of $\mathcal{U}^p$. By definition its covariant derivative is

$$\nabla \Phi = (d + ip\mathcal{Q})\Phi \tag{4.14}$$

or, in components,

$$\nabla_i \Phi \;=\; (\partial_i + \tfrac{1}{2} p \partial_i \mathcal{K})\Phi \quad ; \quad \nabla_{i^*} \Phi \;=\; (\partial_{i^*} - \tfrac{1}{2} p \partial_{i^*} \mathcal{K})\Phi \tag{4.15}$$

A covariantly holomorphic section of $\mathcal{U}$ is defined by the equation: $\nabla_{i^*}\Phi = 0$. We can easily map each section $\Phi(z, \overline{z})$ of $\mathcal{U}^p$ into a section of the line–bundle $\mathcal{L}$ by setting:

$$\widetilde{\Phi} = e^{-p\mathcal{K}/2}\Phi \,. \tag{4.16}$$

With this position we obtain:

$$\nabla_i \widetilde{\Phi} \;=\; (\partial_i + p\partial_i \mathcal{K})\widetilde{\Phi} \quad ; \quad \nabla_{i^*}\widetilde{\Phi} \;=\; \partial_{i^*}\widetilde{\Phi} \tag{4.17}$$

Under the map of eq. 4.16 covariantly holomorphic sections of $\mathcal{U}$ flow into holomorphic sections of $\mathcal{L}$ and viceversa.

4.3 Special Kähler manifolds: the local case

We are now ready to give the definition of local special Kähler manifolds and illustrate their properties. A first definition that does not make direct reference to the symplectic bundle is the following:

Definition 4.1 *A Hodge Kähler manifold is* **Special Kähler (of the local type)** *if there exists a completely symmetric holomorphic 3-index section W_{ijk} of $(T^\star\mathcal{M})^3 \otimes \mathcal{L}^2$ (and its antiholomorphic conjugate $W_{i^*j^*k^*}$) such that the following identity is satisfied by the Riemann tensor of the Levi–Civita connection:*

$$\begin{aligned}
\partial_{m^*} W_{ijk} &= 0 \quad \partial_m W_{i^*j^*k^*} = 0 \\
\nabla_{[m} W_{i]jk} &= 0 \quad \nabla_{[m} W_{i^*]j^*k^*} = 0 \\
\mathcal{R}_{i^*j\ell^*k} &= g_{\ell^*j}g_{ki^*} + g_{\ell^*k}g_{ji^*} - e^{2\mathcal{K}} W_{i^*\ell^*s^*} W_{tkj} g^{s^*t}
\end{aligned} \tag{4.18}$$

In the above equations ∇ denotes the covariant derivative with respect to both the Levi–Civita and the $U(1)$ holomorphic connection of eq. 4.13. In the case of W_{ijk}, the $U(1)$ weight is $p = 2$.

The holomorphic sections W_{ijk} have two different physical interpretations in the case that the special manifold is utilized as scalar manifold in an N=1 or N=2 theory. In the first case they correspond to the Yukawa couplings of Fermi families [60]. In the second case they provide the coefficients for the anomalous magnetic moments of the gauginos, since they appear in the Pauli–terms of the $N = 2$ effective action. Out of the W_{ijk} we can construct covariantly holomorphic sections of weight 2 and - 2 by setting:

$$C_{ijk} \,=\, W_{ijk}\, e^{\mathcal{K}} \quad ; \quad C_{i^\star j^\star k^\star} \,=\, W_{i^\star j^\star k^\star}\, e^{\mathcal{K}} \tag{4.19}$$

Next we can give the second more intrinsic definition that relies on the notion of the flat symplectic bundle. Let $\mathcal{L} \longrightarrow \mathcal{M}$ denote the complex line bundle whose first Chern class equals the Kähler form K of an n-dimensional Hodge–Kähler manifold $\mathcal{M}$. Let $\mathcal{SV} \longrightarrow \mathcal{M}$ denote a holomorphic flat vector bundle of rank $2n+2$ with structural group $Sp(2n+2, \mathbb{R})$. Consider tensor bundles of the type $\mathcal{H} = \mathcal{SV} \otimes \mathcal{L}$. A typical holomorphic section of such a bundle will be denoted by Ω and will have the following structure:

$$\Omega \,=\, \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad \Lambda, \Sigma = 0, 1, \ldots, n \tag{4.20}$$

By definition the transition functions between two local trivializations $U_i \subset \mathcal{M}$ and $U_j \subset \mathcal{M}$ of the bundle $\mathcal{H}$ have the following form:

$$\begin{pmatrix} X \\ F \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} X \\ F \end{pmatrix}_j \tag{4.21}$$

where f_{ij} are holomorphic maps $U_i \cap U_j \rightarrow \mathbb{C}$ while M_{ij} is a constant $Sp(2n+2, \mathbb{R})$ matrix. For a consistent definition of the bundle the transition functions are obviously subject to the cocycle condition on a triple overlap:

$$\begin{aligned} e^{f_{ij}+f_{jk}+f_{ki}} &= 1 \\ M_{ij} M_{jk} M_{ki} &= 1 \end{aligned} \tag{4.22}$$

Let $i\langle \mid \rangle$ be the compatible hermitian metric on $\mathcal{H}$

$$i\langle \Omega \mid \overline{\Omega} \rangle \equiv -i\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \overline{\Omega} \tag{4.23}$$

Definition 4.2 *We say that a Hodge–Kähler manifold $\mathcal{M}$ is* **special Kähler of the local type** *if there exists a bundle $\mathcal{H}$ of the type described above such that for some section $\Omega \in \Gamma(\mathcal{H}, \mathcal{M})$ the Kähler two form is given by:*

$$K = \frac{i}{2\pi} \partial \overline{\partial} \log \left(\mathrm{i} \langle \Omega \mid \overline{\Omega} \rangle \right). \tag{4.24}$$

From the point of view of local properties, eq. 4.24 implies that we have an expression for the Kähler potential in terms of the holomorphic section Ω:

$$\mathcal{K} = -\log \left(\mathrm{i} \langle \Omega \mid \overline{\Omega} \rangle \right) = -\log \left[\mathrm{i} \left(\overline{X}^\Lambda F_\Lambda - \overline{F}_\Sigma X^\Sigma \right) \right] \tag{4.25}$$

The relation between the two definitions of special manifolds is obtained by introducing a non–holomorphic section of the bundle $\mathcal{H}$ according to:

$$V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{\mathcal{K}/2} \Omega = e^{\mathcal{K}/2} \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \tag{4.26}$$

so that eq. 4.25 becomes:

$$1 = \mathrm{i} \langle V \mid \overline{V} \rangle = \mathrm{i} \left(\overline{L}^\Lambda M_\Lambda - \overline{M}_\Sigma L^\Sigma \right) \tag{4.27}$$

Since V is related to a holomorphic section by eq. 4.26 it immediately follows that:

$$\nabla_{i^\star} V = \left(\partial_{i^\star} - \frac{1}{2} \partial_{i^\star} \mathcal{K} \right) V = 0 \tag{4.28}$$

On the other hand, from eq. 4.27, defining:

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2} \partial_i \mathcal{K} \right) V \equiv \begin{pmatrix} f_i^\Lambda \\ h_{\Sigma|i} \end{pmatrix} \tag{4.29}$$

it follows that:

$$\nabla_i U_j = \mathrm{i} C_{ijk}\, g^{k\ell^\star}\, \overline{U}_{\ell^\star} \tag{4.30}$$

where ∇_i denotes the covariant derivative containing both the Levi–Civita connection on the bundle $\mathcal{TM}$ and the canonical connection θ on the line bundle $\mathcal{L}$. In eq. 4.30 the symbol C_{ijk} denotes a covariantly holomorphic ($\nabla_{\ell^\star} C_{ijk} = 0$) section of the bundle $\mathcal{TM}^3 \otimes \mathcal{L}^2$ that is totally symmetric in its indices. This tensor can be identified with the tensor of eq. 4.19 appearing in eq. 4.18. Alternatively, the set of differential equations:

$$\nabla_i V = U_i \tag{4.31}$$
$$\nabla_i U_j = \mathrm{i} C_{ijk} g^{k\ell^\star} U_{\ell^\star} \tag{4.32}$$
$$\nabla_{i^\star} U_j = g_{i^\star j} V \tag{4.33}$$
$$\nabla_{i^\star} V = 0 \tag{4.34}$$

with V satisfying eq.s 4.26, 4.27 give yet another definition of special geometry. This is actually what one obtains from the $N = 2$ solution of Bianchi identities (see appendix A). In particular it is easy to find eq. 4.18 as integrability conditions of 4.34 The *period matrix* is now introduced via the relations:

$$\overline{M}_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} \overline{L}^\Sigma \quad ; \quad h_{\Sigma|i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma_i \tag{4.35}$$

which can be solved introducing the two $(n+1) \times (n+1)$ vectors

$$f^\Lambda_I = \begin{pmatrix} f^\Lambda_i \\ \overline{L}^\Lambda \end{pmatrix} \quad ; \quad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \overline{M}_\Lambda \end{pmatrix} \tag{4.36}$$

and setting:

$$\overline{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^I_{\ \Sigma} \tag{4.37}$$

As a consequence of its definition the matrix $\mathcal{N}$ transforms, under diffeomorphisms of the base Kähler manifold exactly as it is requested by the rule in eq. 2.38. Indeed this is the very reason why the structure of special geometry has been introduced. The existence of the symplectic bundle $\mathcal{H} \longrightarrow \mathcal{M}$ is required in order to be able to pull–back the action of the diffeomorphisms on the field strengths and to construct the kinetic matrix $\mathcal{N}$.

From the previous formulae it is easy to derive a set of useful relations among which we quote the following [20]:

$$\begin{aligned}
\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} L^\Lambda \overline{L}^\Sigma &= -\frac{1}{2} && (4.38)\\
\langle V, U_i \rangle &= \langle V, U_{i^\star} \rangle = 0 && (4.39)\\
U^{\Lambda\Sigma} \equiv f^\Lambda_i\, f^\Sigma_{j^\star}\, g^{ij^\star} &= -\frac{1}{2}\left(\mathrm{Im}\mathcal{N}\right)^{-1|\Lambda\Sigma} - \overline{L}^\Lambda L^\Sigma && (4.40)\\
g_{ij^\star} &= -\mathrm{i}\langle U_i \,|\, \overline{U}_{j^\star} \rangle = -2 f^\Lambda_i \mathrm{Im}\, \mathcal{N}_{\Lambda\Sigma} f^\Sigma_{j^\star}\ ; && (4.41)\\
C_{ijk} &= \langle \nabla_i U_j \,|\, U_k \rangle = f^\Lambda_i \partial_j \overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma_k = (\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} f^\Lambda_i \partial_j f^\Sigma_k && (4.42)
\end{aligned}$$

In particular eq.s 4.42 express the Kähler metric and the anomalous magnetic moments in terms of symplectic invariants. It is clear from our discussion that nowhere we have assumed the base Kähler manifold to be a homogeneous space. So, in general, special manifolds are not homogeneous spaces. Yet there is a subclass of homogenous special manifolds. The homogeneous symmetric ones were classified by Cremmer and Van Proeyen in [61] and are displayed in table 1. It goes without saying that for homogeneous special manifolds the two constructions of the period matrix, that provided by the master formula in eq. 3.17 and that given by eq. 4.37 must agree.In Appendix C we shall shortly verify it in the case of the manifolds $\mathcal{ST}[2,n]$ that correspond to the second infinite family of homogeneous special manifolds displayed in table 1.

Anyhow, since special geometry guarantees the existence of a kinetic period matrix with the correct covariance property it is evident that to each special manifold we can associate a duality covariant bosonic Lagrangian of the type considered in eq. 2.5. However special geometry contains more structures than just the period matrix $\mathcal{N}$ and the scalar metric $g_{ij^\star}$. All the other items of the construction do have a place and play an essential role in the supergravity Lagrangian and the supersymmetry transformation rules.

4.4 Special Kähler manifolds: the rigid case

Let $\mathcal{M}$ be a Kähler manifold with $\dim_{\mathbf{C}} \mathcal{M} = n$ and let $\mathcal{L} \longrightarrow \mathcal{M}$ be a **flat** line bundle $c_1(\mathcal{L}) = 0$[3]. Let $\mathcal{SV} \longrightarrow \mathcal{M}$ denote a holomorphic **flat** vector bundle of rank $2n$ with structural group $ISp(2n,\mathbb{R})$. Consider tensor bundles of the type $\mathcal{H} = \mathcal{SV} \otimes \mathcal{L}$. A typical holomorphic section of such a bundle will be denoted by Ω and will have the following structure:

$$\Omega = \begin{pmatrix} Y^I \\ F_J \end{pmatrix} \quad I,J = 1,\dots,n \tag{4.43}$$

By definition the transition functions between two local trivializations $U_i \subset \mathcal{M}$ and $U_j \subset \mathcal{M}$ of the bundle $\mathcal{H}$ have the following form:

$$\begin{pmatrix} Y \\ F \end{pmatrix}_i = e^{\widehat{f}_{ij}} \widehat{M}_{ij} \begin{pmatrix} Y \\ F \end{pmatrix}_j \tag{4.44}$$

where $\widehat{f}_{ij} \in \mathbb{C}$ are purely imaginary complex numbers while $\widehat{M}_{ij}$ denotes the action of an element $(\widehat{M}, c) \in ISp(2n,\mathbb{R})$ on Ω. $\widehat{M}$ is a symplectic matrix $\widehat{M} \in Sp(2n,\mathbb{R})$ and c is a n-vector:

$$(\,\widehat{M} \quad c\,) \begin{pmatrix} Y \\ F \end{pmatrix} = \widehat{M} \begin{pmatrix} V \\ F \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} \ . \tag{4.45}$$

For a consistent definition of the bundle the transition functions are obviously subject to the cocycle condition on a triple overlap:

$$\begin{aligned} e^{\widehat{f}_{ij} + \widehat{f}_{jk} + \widehat{f}_{ki}} &= 1 \\ \widehat{M}_{ij} \widehat{M}_{jk} \widehat{M}_{ki} &= \mathbb{1} \end{aligned} \tag{4.46}$$

Let $i\langle \mid \rangle$ be the compatible hermitian metric on $\mathcal{H}$

$$\mathrm{i}\langle \Omega \,|\, \overline{\Omega} \rangle \equiv -\mathrm{i}\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \overline{\Omega} \tag{4.47}$$

[3]the holomorphic sections of $\mathcal{L}$ would be the possible superpotentials if $\mathcal{M}$ were used as scalar manifold in an $N = 1$ globally supersymmetric theory.

Definition 4.3 *We say that a Hodge–Kähler manifold* $\mathcal{M}$ *is* **special Kähler of the rigid type** *if there exists a bundle* $\mathcal{H}$ *of the type described above such that for some section* $\widehat{\Omega} \in \Gamma(\mathcal{H}, \mathcal{M})$ *the Kähler two form is given by:*

$$K = -\frac{i}{2\pi}\partial\overline{\partial}\left(\mathrm{i}\langle\widehat{\Omega}\,|\,\overline{\widehat{\Omega}}\rangle\right). \tag{4.48}$$

Just as in the local case eq. 4.48 yields an expression for the Kähler potential in terms of the holomorphic section $\widehat{\Omega}$:

$$\mathcal{K} \;=\; \left(\mathrm{i}\langle\widehat{\Omega}\,|\,\overline{\widehat{\Omega}}\rangle\right) \;=\; \left[\mathrm{i}\left(\overline{Y}^I F_I - \overline{F}_J Y^J\right)\right] \tag{4.49}$$

Similarly defining

$$\widehat{U}_i = \partial_i\widehat{\Omega} \equiv \begin{pmatrix} f_i^I \\ h_{J|i} \end{pmatrix} \tag{4.50}$$

one finds:

$$D_i\widehat{U}_j = \mathrm{i}C_{ijk}\, g^{k\ell^\star}\, \overline{\widehat{U}}_{\ell^\star} \tag{4.51}$$

where D_i is the covariant derivative with respect to the Levi–Civita connection on $\mathcal{TM}$ and where C_{ijk} is a totally symmetric holomorphic section of the bundle $\mathcal{TM}^3 \otimes \mathcal{L}^2$: $\partial_{\ell^\star}C_{ijk} = 0$. Just as in the local case we may alternatively define the rigid special geometry by the following set of differential equations:

$$\partial_{i^\star}\widehat{\Omega} = 0 \tag{4.52}$$
$$\widehat{U}_i = \partial_i\widehat{\Omega} \tag{4.53}$$
$$D_i\widehat{U}_j = \mathrm{i}C_{ijk}\, g^{k\ell^\star}\, \overline{\widehat{U}}_{\ell^\star} \tag{4.54}$$

. The integrability condition of eq. 4.54 is similar but different from eq. 4.18 due to the replacement of the covariant derivative on $\mathcal{TM} \times \mathcal{L}$ by that on $\mathcal{TM}$, due to the flatness of $\mathcal{L}$. We get

$$\begin{aligned} \partial_{m^*}C_{ijk} &= 0 \quad \partial_m C_{i^*j^*k^*} = 0 \\ \nabla_{[m}C_{i]jk} &= 0 \quad \nabla_{[m}C_{i^*]j^*k^*} = 0 \\ \mathcal{R}_{i^*j\ell^*k} &= -C_{i^*\ell^*s^*}C_{tkj}g^{s^*t} \end{aligned} \tag{4.55}$$

which are the rigid counterpart of 4.18. The definition of the *period matrix* is obtained in full analogy to eq. 4.35:

$$h_{I|i} = \overline{\mathcal{N}}_{IJ} f_i^J \tag{4.56}$$

that yields:

$$\overline{\mathcal{N}}_{IJ} = h_{I|i} \circ \left(f^{-1}\right)^i_{\;J} \tag{4.57}$$

Finally we observe that, exactly as in the local case, the metric and the magnetic moments can be expressed in terms of the symplectic sections:

$$g_{ij^\star} \;=\; -\mathrm{i}\langle\, \widehat{U}_i\,|\,\overline{\widehat{U}}_{j^\star}\,\rangle \quad ; \quad C_{ijk} \;=\; \langle\, \partial_i\widehat{U}_j\,|\,\widehat{U}_k\,\rangle \tag{4.58}$$

4.5 Special Kähler manifolds: the issue of special coordinates

So far no privileged coordinate system has been chosen on the base Kähler manifold $\mathcal{M}$ and no mention has been made of the holomorphic prepotential $F(X)$ that is ubiquitous in the $N=2$ literature. The simultaneous avoidance of privileged coordinates and of the prepotential is not accidental. Indeed, when the definition of special Kähler manifolds is given in intrinsic terms, as we did in the previous subsection, the holomorphic prepotential $F(X)$ can be dispensed of. Whether a prepotential $F(X)$ exists or not depends on the choice of a symplectic gauge which is immaterial in the abelian theory but not in the gauged one. Actually, in the local case, it appears that some physically interesting cases are precisely instances where $F(X)$ does not exist. On the contrary the prepotential $F(X)$ seems to be a necessary ingredient in the tensor calculus constructions of $N=2$ theories that for this reason are not completely general. This happens because tensor calculus uses special coordinates from the very start. Let us then see how the notion of $F(X)$ emerges if we resort to special coordinate systems.

Note that under a Kähler transformation $\mathcal{K} \rightarrow \mathcal{K} + f(z) + \overline{f}(\overline{z})$ the holomorphic section transforms, in the local case, as $\Omega \rightarrow \Omega\, e^{-f}$, so that we have $X^\Lambda \rightarrow X^\Lambda\, e^{-f}$. This means that, at least locally, the upper half of Ω (associated with the electric field strengths) can be regarded as a set X^Λ of homogeneous coordinates on $\mathcal{M}$, provided that the jacobian matrix

$$e^I_i(z) = \partial_i \left(\frac{X^I}{X^0}\right) \quad ; \quad a = 1, \ldots, n \tag{4.59}$$

is invertible. In this case, for the lower part of the symplectic section Ω we obtain $F_\Lambda = F_\Lambda(X)$. Recalling eq.s 4.39, in particular:

$$0 \;=\; \langle\, V \,|\, U_i \,\rangle \;=\; X^\Lambda\, \partial_i F_\Lambda - \partial_i X^\Lambda\, F_\Lambda \tag{4.60}$$

we obtain:

$$X^\Sigma\, \partial_\Sigma F_\Lambda(x) \;=\; F_\Lambda(X) \tag{4.61}$$

so that we can conclude:

$$F_\Lambda(X) \;=\; \frac{\partial}{\partial X^\Lambda} F(X) \tag{4.62}$$

where $F(X)$ is a homogeneous function of degree 2 of the homogeneous coordinates X^Λ. Therefore,when the determinant of the Jacobian 4.59 is non vanishing, we can use the *special coordinates*:

$$t^I \;\equiv\; \frac{X^I}{X^0} \tag{4.63}$$

and the whole geometric structure can be derived by a single holomorphic prepotential:

$$\mathcal{F}(t) \;\equiv\; (X^0)^{-2} F(X) \tag{4.64}$$

In particular, eq. 4.25 for the Kähler potential becomes

$$\mathcal{K}(t,\overline{t}) = -\log \mathrm{i}\Big[2\left(\mathcal{F} - \overline{\mathcal{F}}\right) - \left(\partial_I \mathcal{F} + \partial_{I^\star} \overline{\mathcal{F}}\right)\left(t^I - \overline{t}^{I^\star}\right)\Big] \tag{4.65}$$

while eq. 4.42 for the magnetic moments simplifies into

$$W_{IJK} = \partial_I \partial_J \partial_K \mathcal{F}(t) \tag{4.66}$$

Finally we note that in the rigid case the Jacobian from a generic parametrisation to special coordinates

$$e^I_i(z) = \partial_i \left(\frac{X^I}{X^0}\right) = A + B\overline{\mathcal{N}} \tag{4.67}$$

cannot have zero eingenvalues, and therefore the function F always exist. In this case the matrix $\overline{\mathcal{N}}$ coincides with $\frac{\partial^2 F}{\partial X^I \partial X^J}$.

5 Hypergeometry

Next we turn to the hypermultiplet sector of an $N = 2$ theory. Here there are 4 real scalar fields for each hypermultiplet and, at least locally, they can be regarded as the four components of a quaternion. The locality caveat is, in this case, very substantial because global quaternionic coordinates can be constructed only occasionally even on those manifolds that are denominated quaternionic in the mathematical literature [62], [23]. Anyhow, what is important is that, in the hypermultiplet sector, the scalar manifold $\mathcal{HM}$ has dimension multiple of four:

$$\dim_{\mathbf{R}} \mathcal{HM} \;=\; 4\, m \;\equiv\; 4\, \#\,\text{of hypermultiplets} \tag{5.1}$$

and, in some appropriate sense, it has a quaternionic structure.

As *Special Kähler* is the collective name given to the vector multiplet geometry both in the rigid and in the local case, in the same way we name *Hypergeometry* that pertaining to the hypermultiplet sector, irrespectively whether we deal with global or local N=2 theories. Yet in the very same way as there are two kinds of special geometries, there are also two kinds of hypergeometries and for a very similar reason. Supersymmetry requires the existence of a principal $SU(2)$–bundle

$$\mathcal{SU} \longrightarrow \mathcal{HM} \tag{5.2}$$

that plays for hypermultiplets the same role played by the the line–bundle $\mathcal{L} \longrightarrow \mathcal{SM}$ in the case of vector multiplets. As it happens there the bundle $\mathcal{SU}$ is **flat** in the *rigid case* while its curvature is proportional to the Kähler forms in the *local case*.

The difference with the case of vector multiplets is that rigid and local hypergeometries were already known in mathematics prior to their use [26], [63], [24], [64], [65] in the context of $N = 2$ supersymmetry and had the following names:

$$\begin{aligned} \text{rigid hypergeometry} \;&\equiv\; \text{HyperKähler geom.} \\ \text{local hypergeometry} \;&\equiv\; \text{Quaternionic geom.} \end{aligned} \tag{5.3}$$

5.1 Quaternionic, versus HyperKähler manifolds

Both a quaternionic or a HyperKähler manifold $\mathcal{HM}$ is a $4m$-dimensional real manifold endowed with a metric h:

$$ds^2 = h_{uv}(q) dq^u \otimes dq^v \quad ; \quad u, v = 1, \ldots, 4m \tag{5.4}$$

and three complex structures

$$(J^x) \;:\;\; T(\mathcal{HM}) \longrightarrow T(\mathcal{HM}) \qquad\qquad (x = 1, 2, 3) \tag{5.5}$$

that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} \mathbb{1} + \epsilon^{xyz} J^z \tag{5.6}$$

and respect to which the metric is hermitian:

$$\forall \mathbf{X}, \mathbf{Y} \in T\mathcal{HM} : \quad h\left(J^x \mathbf{X}, J^x \mathbf{Y}\right) = h\left(\mathbf{X}, \mathbf{Y}\right) \qquad (x = 1, 2, 3) \tag{5.7}$$

From eq. 5.7 it follows that one can introduce a triplet of 2-forms

$$K^x \;=\; K^x_{uv} dq^u \wedge dq^v \quad ; \quad K^x_{uv} \;=\; h_{uw} (J^x)^w_v \tag{5.8}$$

that provides the generalization of the concept of Kähler form occurring in the complex case. The triplet K^x is named the *HyperKähler* form. It is an $SU(2)$ Lie–algebra valued 2–form in the same way as the Kähler form is a $U(1)$ Lie–algebra valued 2–form. In the complex case the definition of Kähler manifold involves the statement that the Kähler 2–form is closed. At the same time in Hodge–Kähler manifolds (those appropriate to local supersymmetry) the Kähler 2–form can be identified with the curvature of a line–bundle which in the case of rigid supersymmetry is flat. Similar steps can be taken also here and lead to two possibilities: either HyperKähler or Quaternionic manifolds.

Let us introduce a principal $SU(2)$–bundle $\mathcal{SU}$ as defined in eq. 5.2. Let ω^x denote a connection on such a bundle. To obtain either a HyperKähler or a quaternionic manifold we must impose the condition that the HyperKähler 2–form is covariantly closed with respect to the connection ω^x:

$$\nabla K^x \equiv dK^x + \epsilon^{xyz} \omega^y \wedge K^z \;=\; 0 \tag{5.9}$$

The only difference between the two kinds of geometries resides in the structure of the $\mathcal{SU}$–bundle.

Definition 5.1 *A HyperKähler manifold is a* $4m$*–dimensional manifold with the structure described above and such that the* $\mathcal{SU}$*–bundle is* **flat**

Defining the $\mathcal{SU}$–curvature by:

$$\Omega^x \;\equiv\; d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z \tag{5.10}$$

in the HyperKähler case we have:

$$\Omega^x \;=\; 0 \tag{5.11}$$

Viceversa

Definition 5.2 *A quaternionic manifold is a* $4m$*–dimensional manifold with the structure described above and such that the curvature of the* $\mathcal{SU}$*–bundle is proportional to the HyperKähler 2–form*

Hence, in the quaternionic case we can write:

$$\Omega^x \;=\; \lambda \, K^x \tag{5.12}$$

where λ is a non vanishing real number.

As a consequence of the above structure the manifold $\mathcal{HM}$ has a holonomy group of the following type:

$$\begin{aligned} \mathrm{Hol}(\mathcal{HM}) &= SU(2) \otimes \mathcal{H} \quad \text{(quaternionic)} \\ \mathrm{Hol}(\mathcal{HM}) &= \mathbb{1} \otimes \mathcal{H} \quad \text{(HyperKähler)} \\ \mathcal{H} &\subset Sp(2m, \mathbb{R}) \end{aligned} \tag{5.13}$$

In both cases, introducing flat indices $\{A, B, C = 1, 2\}\{\alpha, \beta, \gamma = 1, .., 2m\}$ that run, respectively, in the fundamental representations of $SU(2)$ and $Sp(2m, \mathbb{R})$, we can find a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u(q) dq^u \tag{5.14}$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} \epsilon_{AB} \tag{5.15}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are, respectively, the flat $Sp(2m)$ and $Sp(2) \sim SU(2)$ invariant metrics. The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the $SU(2)$-connection ω^z and to some $Sp(2m, \mathbb{R})$-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\begin{aligned} \nabla \mathcal{U}^{A\alpha} &\equiv d\mathcal{U}^{A\alpha} + \frac{i}{2} \omega^x (\epsilon \sigma_x \epsilon^{-1})^A{}_B \wedge \mathcal{U}^{B\alpha} \\ &+ \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0 \end{aligned} \tag{5.16}$$

where $(\sigma^x)_A{}^B$ are the standard Pauli matrices. Furthermore $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{B\beta} \tag{5.17}$$

Eq.5.17 defines the rule to lower the symplectic indices by means of the flat symplectic metrics ϵ_{AB} and $\mathbb{C}_{\alpha\beta}$. More specifically we can write a stronger version of eq. 5.15[56]:

$$\begin{aligned} (\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u) \mathbb{C}_{\alpha\beta} &= h_{uv} \epsilon^{AB} \\ (\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u) \epsilon_{AB} &= h_{uv} \frac{1}{m} \mathbb{C}^{\alpha\beta} \end{aligned} \tag{5.18}$$

We have also the inverse vielbein $\mathcal{U}^u_{A\alpha}$ defined by the equation

$$\mathcal{U}^u_{A\alpha} \mathcal{U}^{A\alpha}_v = \delta^u_v \tag{5.19}$$

Flattening a pair of indices of the Riemann tensor $\mathcal{R}^{uv}{}_{ts}$ we obtain

$$\mathcal{R}^{uv}{}_{ts} \mathcal{U}^{\alpha A}_u \mathcal{U}^{\beta B}_v = -\frac{\mathrm{i}}{2} \Omega^x_{ts} \epsilon^{AC} (\sigma_x)_C{}^B \mathbb{C}^{\alpha\beta} + \mathbb{R}^{\alpha\beta}_{ts} \epsilon^{AB} \tag{5.20}$$

where $\mathbb{R}^{\alpha\beta}_{ts}$ is the field strength of the $Sp(2m)$ connection:

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta} \mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{ts} dq^t \wedge dq^s \tag{5.21}$$

Eq. 5.20 is the explicit statement that the Levi Civita connection associated with the metric h has a holonomy group contained in $SU(2) \otimes Sp(2m)$. Consider now eq.s 5.6, 5.8 and 5.12. We easily deduce the following relation:

$$h^{st} K^x_{us} K^y_{tw} = -\delta^{xy} h_{uw} + \epsilon^{xyz} K^z_{uw} \tag{5.22}$$

that holds true both in the HyperKähler and in the Quaternionic case. In the latter case, using eq. 5.12, eq. 5.22 can be rewritten as follows:

$$h^{st}\Omega^x_{us}\Omega^y_{tw} = -\lambda^2\delta^{xy}h_{uw} + \lambda\epsilon^{xyz}\Omega^z_{uw} \tag{5.23}$$

Eq. 5.23 implies that the intrinsic components of the curvature 2-form Ω^x yield a representation of the quaternion algebra. In the HyperKähler case such a representation is provided only by the HyperKähler form. In the quaternionic case we can write:

$$\Omega^x_{A\alpha,B\beta} \equiv \Omega^x_{uv}\mathcal{U}^u_{A\alpha}\mathcal{U}^v_{B\beta} = -i\lambda\mathbb{C}_{\alpha\beta}(\sigma_x)_A{}^C\epsilon_{CB} \tag{5.24}$$

Alternatively eq.5.24 can be rewritten in an intrinsic form as

$$\Omega^x = -\mathrm{i}\,\lambda\mathbb{C}_{\alpha\beta}(\sigma_x)_A{}^C\epsilon_{CB}\mathcal{U}^{\alpha A}\wedge\mathcal{U}^{\beta B} \tag{5.25}$$

whence we also get:

$$\frac{i}{2}\Omega^x(\sigma_x)_A{}^B = \lambda\mathcal{U}_{A\alpha}\wedge\mathcal{U}^{B\alpha} \tag{5.26}$$

Homogeneous symmetric quaternionic spaces are displayed in Table 2.

6 The Gauging

With the above discussion of HyperKähler and Quaternionic manifolds we have completed the review of the geometric structures involved in the construction of an *abelian, ungauged* $N=2$ supergravity or of an abelian $N=2$ rigid gauge theory. As we are going to see in the next section, the bosonic Lagrangian of $N=2$ supergravity coupled to n abelian vector multiplets and m hypermultiplets is the following:

$$\begin{aligned}\mathcal{L}^{SUGRA|Bose}_{ungauged} &= \sqrt{-g}\Big[R[g] + g_{ij^\star}(z,\overline{z})\,\partial^\mu z^i\,\partial_\mu\overline{z}^{j^\star} - \lambda\,h_{uv}(q)\,\partial^\mu q^u\,\partial_\mu q^v \\ &\quad + \mathrm{i}\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma|\mu\nu}\right)\Big]\end{aligned} \tag{6.1}$$

where the n complex fields z^i span some *special Kähler manifold of the local type* $\mathcal{SM}$ and the $4m$ real fields q^u span a quaternionic manifold $\mathcal{HM}$. By $g_{ij^\star}$ and h_{uv} we have denoted the metrics on these two manifolds. The proportionality constant between the $SU(2)$ curvature and the HyperKähler form appearing in the Lagrangian is fixed to the value $\lambda=-1$ if we want canonical kinetic terms for the hypermultiplet scalars. The period matrix $\mathcal{N}_{\Lambda\Sigma}$ depends only on the special manifold coordinates $z^i,\overline{z}^{j^\star}$ and it is expressed in terms of the symplectic sections of the flat symplectic bundle by eq. 4.37. On the other hand the bosonic Lagrangian of a rigid $N=2$ abelian gauge theory containing n vector multiplets and coupled to m hypermultiplets is the following one:

$$\begin{aligned}\mathcal{L}^{YM|Bose}_{ungauged} &= g_{ij^\star}(z,\overline{z})\,\partial^\mu z^i\,\partial_\mu\overline{z}^{j^\star} + h_{uv}(q)\,\partial^\mu q^u\,\partial_\mu q^v \\ &\quad + \mathrm{i}\left(\overline{\mathcal{N}}_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-J|\mu\nu} - \mathcal{N}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J|\mu\nu}\right)\end{aligned} \tag{6.2}$$

where the n complex fields z^i span some *special Kähler manifold of the rigid type* $\mathcal{SM}$ and the $4m$ real fields q^u span a HyperKähler manifold $\mathcal{HM}$. By $g_{ij^\star}$ and h_{uv} we have denoted the metrics on these two manifolds. The period matrix $\mathcal{N}_{IJ}$ depends only on the special manifold coordinates $z^i, z^{j^\star}$ and it is expressed in terms of the symplectic sections of the flat symplectic bundle by eq. 4.57. In both theories there are no electric or magnetic currents and we have on shell *symplectic covariance*. By means of the first homomorphism in eq. 2.33 any diffeomorphism of the scalar manifold can be lifted to a symplectic transformation on the electric–magnetic field strengths, the *period* matrix transforming, by construction, covariantly as required by eq. 2.38. Under this lifting any isometry of the scalar manifold becomes a symmetry of the differential system made by the equations of motions plus Bianchi identities. There are in fact three type of these isometries:

1. *The classical symmetries*, namely those isometries $\xi \in \mathcal{I}\left(\mathcal{M}_{scalar}\right)$ whose image in the symplectic group is block–diagonal:

$$\iota_\delta(\xi) = \begin{pmatrix} A_\xi & 0 \\ 0 & (A_\xi^T)^{-1} \end{pmatrix} \tag{6.3}$$

These transformations are exact ordinary symmetries of the Lagrangian. They clearly form a subgroup

$$\mathcal{C}las\left(\mathcal{M}_{scalar}\right) \subset \mathcal{I}\left(\mathcal{M}_{scalar}\right) \tag{6.4}$$

2. *The perturbative symmetries*, namely those isometries $\xi \in \mathcal{I}\left(\mathcal{M}_{scalar}\right)$ whose image in the symplectic group is lower triangular:

$$\iota_\delta(\xi) = \begin{pmatrix} A_\xi & 0 \\ C_\xi & (A_\xi^T)^{-1} \end{pmatrix} \tag{6.5}$$

These transformations map the electric field strengths into linear combinations of the electric field strengths and can be reduced to linear transformations of the gauge potentials. They are almost invariances of the action. Indeed the only non–invariance comes from the transformation of the period matrix

$$\mathcal{N} \longrightarrow (A_\xi^T)^{-1}\mathcal{N}(A_\xi)^{-1} + C_\xi\,(A_\xi^T)^{-1} \tag{6.6}$$

Denoting collectively all the fields of the theory by Φ and utilizing eq.s 2.5, 2.8, 2.10, 2.11, 2.38, under a perturbative transformation the action changes as follows:

$$\begin{aligned} \int \mathcal{L}(\Phi)\,d^4x \quad &\rightarrow \quad \int \mathcal{L}(\Phi')\,d^4x + \Delta\theta_{\Lambda\Sigma}\int F^\Lambda\wedge F^\Sigma \\ \Delta\theta_{\Lambda\Sigma} \quad &= \quad \frac{1}{2}\left[C_\xi\,(A_\xi^T)^{-1}\right]_{\Lambda\Sigma} \end{aligned} \tag{6.7}$$

The added term is a total derivative and does not affect the field equations. Quantum mechanically, however, it is relevant. It corresponds to a redefinition of the *theta*–angle. It yields a symmetry of the path–integral as long as the added term is an integer multiple of $2\pi\hbar$. This consideration will restrict the possible perturbative transformations to a discrete subgroup. In any case the group of perturbative isometries defined by eq. 6.5 contains the group of classical isometries as a subgroup: $\mathcal{I}\left(\mathcal{M}_{scalar}\right) \supset \mathcal{P}ert\left(\mathcal{M}_{scalar}\right) \supset \mathcal{C}las\left(\mathcal{M}_{scalar}\right)$.

3. *The non–perturbative symmetries* namely those isometries $\xi \in \mathcal{I}(\mathcal{M}_{scalar})$ whose image in the symplectic group is of the form:

$$\iota_\delta(\xi) = \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \tag{6.8}$$

with $B_\xi \neq 0$. These transformations are neither a symmetry of the classical action nor of the perturbative path integral. Yet they are a symmetry of the quantum theory. They exchange electric field strengths with magnetic ones, electric currents with magnetic ones and hence elementary excitations with soliton states.

The above discussion of duality symmetries may be intriguing for the following reason. How can we talk about non–perturbative symmetries that exchange electric charges with magnetic charges if, so far, in the abelian theories described by eq.s 6.1 and 6.2 there are neither electric nor magnetic couplings? The answer is that the same general form of abelian theories encoded in these equations can be taken to represent two quite different things:

1. The fundamental theory prior to the gauging. It is neutral and abelian since the non–abelian interactions and the electric charges are introduced only by the gauging, but it contains all the fundamental fields.

2. The effective theory of the massless modes of the non–abelian theory. It is abelian and neutral because the only fields which remain massless are, apart from the graviton, the multiplets in the Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ of the gauge group and the neutral hypermultiplets corresponding to flat directions of the scalar potential.

What distinguishes the two cases is the type of scalar manifolds and their isometries.

In case 1) we have:

$$\begin{aligned} \dim_{\mathbf{C}} \mathcal{SM} &= n \equiv \dim \mathcal{G} \\ \frac{1}{4}\dim_{\mathbf{R}} \mathcal{HM} &= \widehat{m} \equiv \#\,\text{of all hypermul.} \end{aligned} \tag{6.9}$$

while in case 2) we have instead:

$$\begin{aligned} \dim_{\mathbf{C}} \mathcal{SM} &= r \equiv \operatorname{rank} \mathcal{G} \\ \frac{1}{4}\dim_{\mathbf{R}} \mathcal{HM} &= m \equiv \#\,\text{of moduli hypermul.} \end{aligned} \tag{6.10}$$

As far as the gauging of the $N=2$ theory is concerned, the problem consists in identifying the gauge group $\mathcal{G}$ as a subgroup, at most of dimension $n+1$ of the isometries of the product space

$$\mathcal{SM} \times \mathcal{HM} \ . \tag{6.11}$$

Here we shall mainly consider two cases even if more general situations are possible. The first is when the gauge group G is non abelian, the second is when it is the abelian group $G = U(1)^{n_V+1}$. In the first case supersymmetry requires that G be a subgroup of the isometries of $\mathcal{M}$, since the scalars (more precisely, the sections L^Λ) must belong to the adjoint representation of G. In such case the hypermultiplet space will generically split into[35]

$$n_H = \sum_i n_i R_i + \frac{1}{2} \sum_l n_l^P R_l^P \tag{6.12}$$

where R_i and R_l^P are a set of irreducible representations of G and R_l^P denote pseudoreal representations.

In the abelian case, the special manifold is not required to have any isometry and if the hypermultiplets are charged with respect to the $n_V + 1$ $U(1)$'s, then the $\mathcal{Q}$ manifold should at least have $n_V + 1$ abelian isometries.

As a consequence of gauging the Lagrangians in eq.s 6.1 and 6.2 get modified by the replacement of ordinary derivatives with covariant derivatives and by the introduction of new terms that are of two types:

1. fermion–fermion bilinears with scalar field dependent coefficients

2. A scalar potential V

It is particularly nice and rewarding that all the modifications of the Lagrangian and of the supersymmetry transformation rules can be described in terms of a very general geometric construction associated with the action of Lie–Groups on manifolds that admit a symplectic structure: *the momentum map*. In supersymmetry indeed, the geometric notion of *momentum map* has an exact correspondence with the notion of *gauge multiplet auxiliary fields* or *D–fields*. Next section is devoted to a review of the momentum map and to its applications in N=2 theories.

7 The Momentum Map

The momentum map is a construction that applies to all manifolds with a symplectic structure, in particular to Kähler, HyperKähler and Quaternionic manifolds.

Let us begin with the Kähler case, namely with the momentum map of holomorphic isometries. The HyperKähler and quaternionic case correspond, instead, to the momentum map of triholomorphic isometries.

7.1 Holomorphic momentum map on Kähler manifolds

Let $g_{ij^\star}$ be the Kähler metric of a Kähler manifold $\mathcal{M}$: it appears in the kinetic term of the scalar fields: the Wess–Zumino multiplet scalars in N=1 theories, the vector multiplet scalars in N=2 theories. If the metric $g_{ij^\star}$ has a non trivial group of continuous isometries $\mathcal{G}$ generated by Killing vectors k^i_Λ ($\Lambda = 1, \ldots, \dim \mathcal{G}$), then the kinetic Lagrangian admits $\mathcal{G}$ as a group of global space–time symmetries. Indeed under an infinitesimal variation

$$z^i \to z^i + \epsilon^\Lambda k^i_\Lambda(z) \tag{7.1}$$

$\mathcal{L}_{kin}$ remains invariant. Furthermore if all the couplings of the scalar fields are performed in a diffeomorphic invariant way, then any isometry of $g_{ij^\star}$ extends from a symmetry of $\mathcal{L}_{kin}$ to a symmetry of the whole Lagrangian. Diffeomorphic invariance means that the scalar fields can appear only through the metric, the Christoffel symbol in the covariant derivative and through the curvature. Alternatively they can appear through sections of vector bundles constructed over $\mathcal{M}$. Typical case is the dependence on the scalar fields introduced by the *period matrix* $\mathcal{N}$.

Let $k^i_\Lambda(z)$ be a basis of holomorphic Killing vectors for the metric $g_{ij^\star}$. Holomorphicity means the following differential constraint:

$$\partial_{j^*} k^i_\Lambda(z) = 0 \leftrightarrow \partial_j k^{i^*}_\Lambda(\overline{z}) = 0 \tag{7.2}$$

while the generic Killing equation (suppressing the gauge index Λ):

$$\nabla_\mu k_\nu + \nabla_\mu k_\nu = 0 \tag{7.3}$$

in holomorphic indices reads as follows:

$$\nabla_i k_j + \nabla_j k_i \quad = \quad 0 \quad ; \quad \nabla_{i^*} k_j + \nabla_j k_{i^*} \quad = \quad 0 \tag{7.4}$$

where the covariant components are defined as $k_j = g_{ji^*} k^{i^*}$ (and similarly for k_{i^*}).

The vectors k^i_Λ are generators of infinitesimal holomorphic coordinate transformations:

$$\delta z^i = \epsilon^\Lambda k^i_\Lambda(z) \tag{7.5}$$

which leave the metric invariant. In the same way as the metric is the derivative of a more fundamental object, the Killing vectors in a Kähler manifold are the derivatives of suitable prepotentials. Indeed the first of eq.s 7.4 is automatically satisfied by holomorphic vectors and the second equation reduces to the following one:

$$k^i_\Lambda = i g^{ij^*} \partial_{j^*} \mathcal{P}_\Lambda, \quad \mathcal{P}^*_\Lambda = \mathcal{P}_\Lambda \tag{7.6}$$

In other words if we can find a real function $\mathcal{P}^\Lambda$ such that the expression $i g^{ij^*} \partial_{j^*} \mathcal{P}_{(\Lambda)}$ is holomorphic, then eq. 7.6 defines a Killing vector.

The construction of the Killing prepotential can be stated in a more precise geometrical formulation which involves the notion of *momentum map*. Let us review this construction which reveals another deep connection between supersymmetry and geometry.

Consider a Kählerian manifold $\mathcal{M}$ of real dimension $2n$. Consider a compact Lie group $\mathcal{G}$ acting on $\mathcal{M}$ by means of Killing vector fields $\mathbf{X}$ which are holomorphic with respect to the complex structure J of $\mathcal{M}$; then these vector fields preserve also the Kähler 2-form

$$\left.\begin{array}{l} \mathcal{L}_{\mathbf{X}} g = 0 \quad \leftrightarrow \quad \nabla_{(\mu} X_{\nu)} = 0 \\ \mathcal{L}_{\mathbf{X}} J = 0 \end{array}\right\} \Rightarrow \quad 0 = \mathcal{L}_{\mathbf{X}} K = i_{\mathbf{X}} dK + d(i_{\mathbf{X}} K) = d(i_{\mathbf{X}} K) \tag{7.7}$$

Here $\mathcal{L}_{\mathbf{X}}$ and $i_{\mathbf{X}}$ denote respectively the Lie derivative along the vector field $\mathbf{X}$ and the contraction (of forms) with it.

If $\mathcal{M}$ is simply connected, $d(i_{\mathbf{X}} K) = 0$ implies the existence of a function $\mathcal{P}_{\mathbf{X}}$ such that

$$-\frac{1}{2\pi} d\mathcal{P}_{\mathbf{X}} = i_{\mathbf{X}} K \tag{7.8}$$

The function $\mathcal{P}_{\mathbf{X}}$ is defined up to a constant, which can be arranged so as to make it equivariant:

$$\mathbf{X} \mathcal{P}_{\mathbf{Y}} = \mathcal{P}_{[\mathbf{X},\mathbf{Y}]} \tag{7.9}$$

$\mathcal{P}_{\mathbf{X}}$ constitutes then a *momentum map*. This can be regarded as a map

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathbb{R} \otimes \mathbb{G}^* \tag{7.10}$$

where $\mathbb{G}^*$ denotes the dual of the Lie algebra $\mathbb{G}$ of the group $\mathcal{G}$. Indeed let $x \in \mathbb{G}$ be the Lie algebra element corresponding to the Killing vector $\mathbf{X}$; then, for a given $m \in \mathcal{M}$

$$\mu(m) \, : \, x \longrightarrow \mathcal{P}_{\mathbf{X}}(m) \, \in \, \mathbb{R} \tag{7.11}$$

is a linear functional on $\mathbb{G}$. If we expand $\mathbf{X} = a^\Lambda k_\Lambda$ in a basis of Killing vectors k_Λ such that

$$[k_\Lambda, k_\Gamma] = f_{\Lambda\Gamma}{}^{\Delta} k_\Delta \tag{7.12}$$

we have also

$$\mathcal{P}_{\mathbf{X}} \, = \, a^\Lambda \mathcal{P}_\Lambda \tag{7.13}$$

In the following we use the shorthand notation $\mathcal{L}_\Lambda, i_\Lambda$ for the Lie derivative and the contraction along the chosen basis of Killing vectors k_Λ.

From a geometrical point of view the prepotential, or momentum map, $\mathcal{P}_\Lambda$ is the Hamiltonian function providing the Poissonian realization of the Lie algebra on the Kähler manifold. This is just another way of stating the already mentioned *equivariance*. Indeed the very existence of the closed 2-form K guarantees that every Kähler space is a symplectic manifold and that we can define a Poisson bracket.

Consider Eqs. 7.6. To every generator of the abstract Lie algebra $\mathbb{G}$ we have associated a function $\mathcal{P}_\Lambda$ on $\mathcal{M}$; the Poisson bracket of $\mathcal{P}_\Lambda$ with $\mathcal{P}_\Sigma$ is defined as follows:

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} \equiv 4\pi K(\Lambda, \Sigma) \tag{7.14}$$

where $K(\Lambda, \Sigma) \equiv K(\vec{k}_\Lambda, \vec{k}_\Sigma)$ is the value of K along the pair of Killing vectors.

In reference [24] we proved the following lemma.

Lemma 7.1 *The following identity is true:*

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} = f_{\Lambda\Sigma}{}^{\Gamma} \mathcal{P}_\Gamma + C_{\Lambda\Sigma} \tag{7.15}$$

where $C_{\Lambda\Sigma}$ is a constant fulfilling the cocycle condition

$$f_{\Lambda\Pi}{}^{\Gamma} C_{\Gamma\Sigma} + f_{\Pi\Sigma}{}^{\Gamma} C_{\Gamma\Lambda} + f_{\Sigma\Lambda}{}^{\Gamma} C_{\Gamma\Pi} = 0 \tag{7.16}$$

If the Lie algebra $\mathbb{G}$ has a trivial second cohomology group $H^2(\mathbb{G}) = 0$, then the cocycle $C_{\Lambda\Sigma}$ is a coboundary; namely we have

$$C_{\Lambda\Sigma} = f_{\Lambda\Sigma}{}^{\Gamma} C_\Gamma \tag{7.17}$$

where C_Γ are suitable constants. Hence, assuming $H^2(\mathbb{G}) = 0$ we can reabsorb C_Γ in the definition of $\mathcal{P}_\Lambda$:

$$\mathcal{P}_\Lambda \rightarrow \mathcal{P}_\Lambda + C_\Lambda \tag{7.18}$$

and we obtain the stronger equation

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} = f_{\Lambda\Sigma}{}^{\Gamma} \mathcal{P}_\Gamma \tag{7.19}$$

Note that $H^2(\mathbb{G}) = 0$ is true for all semi-simple Lie algebras. Using eq. 7.15, eq. 7.19 can be rewritten in components as follows:

$$\frac{i}{2} g_{ij^*}(k^i_\Lambda k^{j^*}_\Sigma - k^i_\Sigma k^{j^*}_\Lambda) = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}_\Gamma \tag{7.20}$$

Equation 7.20 is identical with the equivariance condition in eq. 7.9.

Comparing the definition of the Kähler potential in eq. C.54 with the definition of the momentum function in eq. 7.6, we obtain an expression for the momentum map function in terms of derivatives of the Kähler potential:

$$\mathrm{i}\,\mathcal{P}_\Lambda \;=\; \frac{1}{2}\left(k^i_\Lambda\,\partial_i\mathcal{K} - k^{i^\star}_\Lambda\,\partial_{i^\star}\mathcal{K}\right) \;=\; k^i_\Lambda\,\partial_i\mathcal{K} \;=\; -k^{i^\star}_\Lambda\,\partial_{i^\star}\mathcal{K} \tag{7.21}$$

Eq. 7.21 is true if the Kähler potential is exactly invariant under the transformations of the isometry group $\mathcal{G}$ and not only up to a Kähler transformation as defined in eq. C.55. In other words eq. 7.21 is true if

$$0 \;=\; \mathcal{L}^\Lambda\,\mathcal{K} \;=\; k^i_\Lambda\,\partial_i\mathcal{K} + k^{i^\star}_\Lambda\,\partial_{i^\star}\mathcal{K} \tag{7.22}$$

Not all the isometries of a general Kähler manifold have such a property, but those that in a suitable coordinate frame display a linear action on the coordinates certainly do. However, in Hodge Kähler manifolds, eq. 7.22 can be replaced by the following one which is certainly true:

$$\begin{aligned} 0 \;&=\; \mathcal{L}^\Lambda\,G \;=\; k^i_\Lambda\,\partial_i G + k^{i^\star}_\Lambda\,\partial_{i^\star} G \\ G(z,\overline{z}) \;&\equiv\; \log\;\parallel W(z)\parallel^2 = \;\mathcal{K}(z,\overline{z}) \;+\; \mathrm{Re}\,W(z) \end{aligned} \tag{7.23}$$

where the *superpotential* $W(z)$ is any holomorphic section of the Hodge line–bundle. Indeed the transformation under the isometry of the Kähler potential is compensated by the transformation of the superpotential. Consequently, in Hodge–Kähler manifolds eq. 7.21 can be rewritten as

$$\mathrm{i}\,\mathcal{P}_\Lambda \;=\; \frac{1}{2}\left(k^i_\Lambda\,\partial_i G - k^{i^\star}_\Lambda\,\partial_{i^\star} G\right) \;=\; k^i_\Lambda\,\partial_i G \;=\; -k^{i^\star}_\Lambda\,\partial_{i^\star} G \tag{7.24}$$

and holds true for any isometry.

In $N = 1$ supersymmetry the Kählerian momentum maps $\mathcal{P}_\Gamma$ appear as auxiliary fields of the vector multiplets. For $N = 1$ supergravity the scalar manifold is of the Hodge type and eq. 7.24 can always be employed.

On the other hand, in $N = 2$ supersymmetry the auxiliary fields of the vector multiplets, that form an $SU(2)$ triplet, are given by the momentum map of triholomorphic isometries on the hypermultiplet manifold (HyperKählerian or quaternionic depending on the local or rigid nature of supersymmetry). The triholomorphic momentum map is discussed in the subsection after the next. Yet, although not identified with the auxiliary fields, the holomorphic momentum map plays a role also in $N = 2$ theories in the *gauging* of the $U(1)$ connection 4.13, as we show shortly from now.

7.2 Holomorphic momentum map on Special Kähler manifolds

Here the Kähler manifold is not only Hodge but it is special. Correspondingly we can write a formula for $\mathcal{P}_\Lambda$ in terms of symplectic invariants. In this context, to distinguish the holomorphic momentum map from the triholomorphic one $\mathcal{P}^x_\Lambda$ that carries an $SU(2)$ index $x = 1, 2, 3$, we adopt the notation $\mathcal{P}^0_\Lambda$. The request that the isometry group should be embedded into the symplectic group is formulated by writing:

$$\mathcal{L}_\Lambda V \equiv k^i_\Lambda \partial_i V + k^{i^\star}_\Lambda \partial_{i^\star} V = T_\Lambda V + V f_\Lambda(z) \tag{7.25}$$

where V is the covariantly holomorphic section of the vector bundle $\mathcal{H} \longrightarrow \mathcal{M}$ defined in eq. 4.27,

$$T_\Lambda = \begin{pmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{pmatrix} \in \mathbf{Sp(2n+2, \mathbb{R})} \tag{7.26}$$

is some element of the real symplectic Lie algebra and $f_\Lambda(z)$ corresponds to an infinitesimal Kähler transformation.

The classical or perturbative isometries ($b_\Lambda = 0$) that are relevant to the gauging procedure are normally characterized by

$$f_\Lambda(z) = 0 \tag{7.27}$$

Under condition 7.27, recalling eq.s 4.25 and 4.26, from eq. 7.25 we obtain:

$$\mathcal{L}^\Lambda \mathcal{K} = k^i_\Lambda \partial_i \mathcal{K} + k^{i^\star}_\Lambda \partial_{i^\star} \mathcal{K} = 0 \tag{7.28}$$

that is identical with eq. 7.22. Hence we can use eq. 7.21, that we rewrite as:

$$\mathrm{i}\, \mathcal{P}^0_\Lambda = k^i_\Lambda \partial_i \mathcal{K} = -k^{i^\star}_\Lambda \partial_{i^\star} \mathcal{K} \tag{7.29}$$

Utilizing the definition in eq. 4.29 we easily obtain:

$$k^i_\Lambda U^i = T_\Lambda V \exp[f_\Lambda(z)] + \mathrm{i}\, \mathcal{P}^0_\Lambda V \tag{7.30}$$

Taking the symplectic scalar product of eq. 7.30 with $\overline{V}$ and recalling eq. 4.27 we finally [4] get:

$$\mathcal{P}^0_\Lambda = \langle \overline{V} \,|\, T_\Lambda V \rangle = \langle V \,|\, T_\Lambda \overline{V} \rangle = \exp[\mathcal{K}] \, \langle \overline{\Omega} \,|\, T_\Lambda \Omega \rangle \tag{7.31}$$

In the gauging procedure we are interested in groups the symplectic image of whose generators is block–diagonal and coincides with the adjoint representation in each block. Namely

$$T_\Lambda = \begin{pmatrix} f^\Sigma{}_{\Lambda\Delta} & \mathbf{0} \\ \mathbf{0} & -f^\Sigma{}_{\Lambda\Delta} \end{pmatrix} \tag{7.32}$$

Then eq. 7.31 becomes

$$\mathcal{P}^0_\Lambda = e^{\mathcal{K}} \left(F_\Delta f^\Delta{}_{\Lambda\Sigma} \overline{X}^\Sigma + \overline{F}_\Delta f^\Delta{}_{\Lambda\Sigma} X^\Sigma \right) \tag{7.33}$$

[4] The following and the next two formulae have been obtained in private discussions of one of us (P.Fré) with A. Van Proeyen and B. de Wit

7.3 The triholomorphic momentum map on HyperKähler and Quaternionic manifolds

Next we turn to a discussion of isometries of the manifold $\mathcal{HM}$ associated with hypermultiplets. As we know, it can be either HyperKählerian or quaternionic. For applications to $N = 2$ theories we must assume that on $\mathcal{HM}$ we have an action by triholomorphic isometries of the same Lie group $\mathcal{G}$ that acts on the Special Kähler manifold $\mathcal{SM}$. This means that on $\mathcal{HM}$ we have Killing vectors

$$\vec{k}_\Lambda = k^u_\Lambda \frac{\vec{\partial}}{\partial q^u} \tag{7.34}$$

satisfying the same Lie algebra as the corresponding Killing vectors on $\mathcal{SM}$. In other words

$$\widehat{\vec{k}}_\Lambda = k^i_\Lambda \vec{\partial}_i + k^{i^*}_\Lambda \vec{\partial}_{i^*} + k^u_\Lambda \vec{\partial}_u \tag{7.35}$$

is a Killing vector of the block diagonal metric:

$$\widehat{g} = \begin{pmatrix} g_{ij^*} & 0 \\ 0 & h_{uv} \end{pmatrix} \tag{7.36}$$

defined on the product manifold $\mathcal{SM} \otimes \mathcal{HM}$. Triholomorphicity means that the Killing vector fields leave the HyperKähler structure invariant up to $SU(2)$ rotations in the $SU(2)$–bundle defined by eq. 5.2. Namely:

$$\mathcal{L}_\Lambda K^x \;=\; \epsilon^{xyz} K^y W^z_\Lambda \quad ; \quad \mathcal{L}_\Lambda \omega^x \;=\; \nabla W^x_\Lambda \tag{7.37}$$

where W^x_Λ is an $SU(2)$ compensator associated with the Killing vector k^u_Λ. The compensator W^x_Λ necessarily fulfils the cocycle condition:

$$\mathcal{L}_\Lambda W^x_\Sigma - \mathcal{L}_\Sigma W^x_\Lambda + \epsilon^{xyz} W^y_\Lambda W^z_\Sigma = f_{\Lambda\Sigma}^{\cdot\cdot\Gamma} W^x_\Gamma \tag{7.38}$$

In the HyperKähler case the $SU(2)$–bundle is flat and the compensator can be reabsorbed into the definition of the HyperKähler forms. In other words we can always find a map

$$\mathcal{HM} \;\longrightarrow\; L^x{}_y(q) \;\in\; SO(3) \tag{7.39}$$

that trivializes the $\mathcal{SU}$–bundle globally. Redefining:

$$K^{x\prime} \;=\; L^x{}_y(q)\, K^y \tag{7.40}$$

the new HyperKähler form obeys the stronger equation:

$$\mathcal{L}_\Lambda K^{x\prime} \;=\; 0 \tag{7.41}$$

On the other hand, in the quaternionic case, the non–triviality of the $\mathcal{SU}$–bundle forbids to eliminate the W–compensator completely. Due to the identification between HyperKähler forms and $SU(2)$ curvatures eq. 7.37 is rewritten as:

$$\mathcal{L}_\Lambda \Omega^x \;=\; \epsilon^{xyz} \Omega^y W^z_\Lambda \quad ; \quad \mathcal{L}_\Lambda \omega^x \;=\; \nabla W^x_\Lambda \tag{7.42}$$

In both cases, anyhow, and in full analogy with the case of Kähler manifolds, to each Killing vector we can associate a triplet $\mathcal{P}^x_\Lambda(q)$ of 0-form prepotentials. Indeed we can set:

$$\mathbf{i}_\Lambda K^x = -\nabla \mathcal{P}^x_\Lambda \equiv -(d\mathcal{P}^x_\Lambda + \epsilon^{xyz}\omega^y \mathcal{P}^z_\Lambda) \tag{7.43}$$

where ∇ denotes the $SU(2)$ covariant exterior derivative.

As in the Kähler case eq. 7.43 defines a momentum map:

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathbb{R}^3 \otimes \mathbb{G}^* \tag{7.44}$$

where $\mathbb{G}^*$ denotes the dual of the Lie algebra $\mathbb{G}$ of the group $\mathcal{G}$. Indeed let $x \in \mathbb{G}$ be the Lie algebra element corresponding to the Killing vector $\mathbf{X}$; then, for a given $m \in \mathcal{M}$

$$\mu(m) \,:\, x \longrightarrow \mathcal{P}_{\mathbf{X}}(m) \in \mathbb{R}^3 \tag{7.45}$$

is a linear functional on $\mathbb{G}$. If we expand $\mathbf{X} = a^\Lambda k_\Lambda$ on a basis of Killing vectors k_Λ such that

$$[k_\Lambda, k_\Gamma] = f_{\Lambda\Gamma}{}^{\Delta} k_\Delta \tag{7.46}$$

and we also choose a basis $\mathbf{i}_x$ $(x = 1, 2, 3)$ for $\mathbb{R}^3$ we get:

$$\mathcal{P}_{\mathbf{X}} \,=\, a^\Lambda \mathcal{P}^x_\Lambda \, \mathbf{i}_x \tag{7.47}$$

Furthermore we need a generalization of the equivariance defined by eq. 7.9

$$\mathbf{X} \circ \mathcal{P}_{\mathbf{Y}} \,=\, \mathcal{P}_{[\mathbf{X},\mathbf{Y}]} \tag{7.48}$$

In the HyperKähler case, the left–hand side of eq. 7.48 is defined as the usual action of a vector field on a 0–form:

$$\mathbf{X} \circ \mathcal{P}_{\mathbf{Y}} \,=\, \mathbf{i}_{\mathbf{X}}\, d\mathcal{P}_{\mathbf{Y}} \,=\, X^u \frac{\partial}{\partial q^u} \mathcal{P}_{\mathbf{Y}} \tag{7.49}$$

The equivariance condition implies that we can introduce a triholomorphic Poisson bracket defined as follows:

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \equiv 2K^x(\Lambda, \Sigma) \tag{7.50}$$

leading to the triholomorphic Poissonian realization of the Lie algebra:

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \,=\, f^{\Delta}{}_{\Lambda\Sigma}\, \mathcal{P}^x_\Delta \tag{7.51}$$

which in components reads:

$$K^x_{uv}\, k^u_\Lambda\, k^v_\Sigma \,=\, \frac{1}{2}\, f^{\Delta}{}_{\Lambda\Sigma}\, \mathcal{P}^x_\Delta \tag{7.52}$$

In the quaternionic case, instead, the left–hand side of eq. 7.48 is interpreted as follows:

$$\mathbf{X} \circ \mathcal{P}_{\mathbf{Y}} \,=\, \mathbf{i}_{\mathbf{X}}\, \nabla\mathcal{P}_{\mathbf{Y}} \,=\, X^u\, \nabla_u\, \mathcal{P}_{\mathbf{Y}} \tag{7.53}$$

where ∇ is the $SU(2)$–covariant differential. Correspondingly, the triholomorphic Poisson bracket is defined as follows:

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \equiv 2K^x(\Lambda, \Sigma) - \lambda\, \varepsilon^{xyz}\, \mathcal{P}^y_\Lambda\, \mathcal{P}^z_\Sigma \tag{7.54}$$

and leads to the Poissonian realization of the Lie algebra

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \,=\, f^{\Delta}{}_{\Lambda\Sigma}\, \mathcal{P}^x_\Delta \tag{7.55}$$

which in components reads:

$$K^x_{uv}\, k^u_\Lambda\, k^v_\Sigma \,-\, \frac{\lambda}{2}\, \varepsilon^{xyz}\, \mathcal{P}^y_\Lambda\, \mathcal{P}^z_\Sigma \,=\, \frac{1}{2}\, f^{\Delta}{}_{\Lambda\Sigma}\, \mathcal{P}^x_\Delta \tag{7.56}$$

Eq. 7.56, which is the most convenient way of expressing equivariance in a coordinate basis, plays a fundamental role in the construction of the supersymmetric action, supersymmetry transformation rules and of the superpotential for $N=2$ supergravity on a general quaternionic manifold. It is also very convenient to retrieve the rigid supersymmetry limit. Indeed, using physical units, we may set $\lambda = \frac{\widehat{\lambda}}{\mu^2}$ where μ is the Planck mass (see section 9); letting $\mu \to \infty$ eq. 7.56 reduces to eq. 7.52. Eq. 7.56 was introduced in the physical literature in [24] where the general form of $N=2$ supergravity beyond the limitations of tensor calculus was given.

7.4 Gauging of the composite connections

Using the concepts and the geometric structures introduced in the previous sections the form of the Lagrangian and of the transformation rules for $N=2$ supergravity can now be given. The essential thing is that the fermions of the theory, behave as sections of the bundles we have introduced so far. In particular he gravitino field ψ^A_μ apart from being a spinor–valued 1–form on space–time, behaves as a section of the bundle $\mathcal{L}\otimes\mathcal{SU}$. The gaugino field $\lambda^{i|A}$ apart from being a section of the spinor bundle, behaves as a section of $\mathcal{L}\otimes\mathcal{TSM}\otimes\mathcal{SU}$. Finally the hyperino field ζ^α is a section of the rank $2m$ vector bundle with structural group $Sp(2m,\mathbb{R})$ that one obtains by deleting the $SU(2)$ part of the holonomy group on $\mathcal{HM}$. In other words it is a section of the bundle $\mathcal{THM}\otimes\mathcal{SU}^{-1}$. Correspondingly the covariant derivatives of the fermions appearing in the action and in the transformation rules involves the composite connections $\mathcal{Q}$, $\Gamma^i{}_j$, ω^x and $\Delta^{\alpha\beta}$ defined on these bundles. Gauging just modifies these composite connections by means of Killing vectors and momentum map functions. Explicitly we have:

$$\begin{array}{ccccc} \mathcal{TSM} & : & \text{tangent bundle} & \Gamma^i{}_j & \rightarrow & \widehat{\Gamma}^i{}_j = \Gamma^i{}_j + g\, A^\Lambda\, \partial_j k^i_\Lambda \\ \mathcal{L} & : & \text{line bundle} & \mathcal{Q} & \rightarrow & \widehat{\mathcal{Q}} = \mathcal{Q} + g\, A^\Lambda\, \mathcal{P}^0_\Lambda \\ \mathcal{SU} & : & SU(2)\ \text{bundle} & \omega^x & \rightarrow & \widehat{\omega}^x = \omega^x + g\, A^\Lambda\, \mathcal{P}^x_\Lambda \\ \mathcal{SU}^{-1}\otimes\mathcal{THM} & : & Sp(2m)\ \text{bundle} & \Delta^{\alpha\beta} & \rightarrow & \widehat{\Delta}^{\alpha\beta} = \Delta^{\alpha\beta} + g\, A^\Lambda\, \partial_u k^v_\Lambda\, \mathcal{U}^{u|\alpha A}\, \mathcal{U}^\beta_{v|A} \end{array} \tag{7.57}$$

Correspondingly the gauged curvatures are:

$$\begin{aligned} \widehat{R}^i{}_j &= R^i{}_{j\ell^\star k}\, \nabla\overline{z}^{\ell^\star}\wedge\nabla z^k \,+\, g\, F^\Lambda\, \partial_j k^i_\Lambda \\ \widehat{K} &= K_{ij^\star}\, \nabla\overline{z}^i\wedge\nabla z^{j^\star} \,+\, g\, F^\Lambda\, \mathcal{P}^0_\Lambda \\ \widehat{\Omega}^x &= \Omega^x_{uv}\, \nabla q^u\wedge\nabla q^v \,+\, g\, F^\Lambda\, \mathcal{P}^x_\Lambda \\ \widehat{\mathbb{R}}^{\alpha\beta} &= \mathbb{R}^{\alpha\beta}_{uv}\, \nabla q^u\wedge\nabla q^v \,+\, g\, A^\Lambda\, \partial_u k^v_\Lambda\, \mathcal{U}^{u|\alpha A}\, \mathcal{U}^\beta_{v|A} \end{aligned} \tag{7.58}$$

8 The Complete N=2 Supergravity Theory

In this section we write the supersymmetric invariant action and supersymmetry transformation rules for a completely general $N = 2$ supergravity.

Such a theory includes

1. *the gravitational multiplet*, described by the vielbein 1–form V^a, $(a = 0, 1, 2, 3)$, the spin–connection 1–form ω^{ab}, the $SU(2)$ doublet of gravitino 1-forms ψ^A, ψ_A ($A = 1, 2$ and the upper or lower position of the index denotes left, respectively right chirality), the graviphoton 1-form A^0

2. *n vector multiplets*. Each vector multiplet contains a gauge boson 1–form A^I ($I = 1, \dots, n$), a doublet of gauginos (0–form spinors) λ^{iA}, $\lambda^{i^\star}_A$, and a complex scalar field (0–form) z^i ($i =, 1, \dots, n$). The scalar fields z^i can be regarded as coordinates on a special manifold $\mathcal{SM}$ which can be chosen *arbitrarily*.
$$\dim_{\mathbf{C}} \mathcal{SM} = n \tag{8.1}$$

3. *m hypermultiplets*. Each hypermultiplet contains a doublet of 0–form spinors, that is the hyperinos ζ^α ($\alpha = 1, \dots, 2m$ and here the lower or upper position of the index denotes left, respectively right chirality) and four real scalar fields q^u ($u = 1, \dots, 4m$), that can be regarded as coordinates of a quaternionic manifold $\mathcal{HM}$ which can be chosen arbitrarily.
$$\dim_{\mathbb{Q}} \mathcal{HM}_m = m \qquad \dim_{\mathbb{R}} \mathcal{HM}_m = 4m \tag{8.2}$$
As explained in the previous sections any quaternionic manifold has a holonomy group:
$$\mathcal{H}ol\,(\mathcal{HM}_m) \subset SU(2) \otimes Sp(2m, \mathbb{R}) \tag{8.3}$$
and the index α of the hyperinos transforms in the *fundamental* representation of $Sp(2m, \mathbb{R})$

Using the information collected in the previous sections we can immediately write down the definition of the curvatures and covariant derivatives for all the fields. The definition of curvatures in the gravitational sector is given by:

$$\begin{aligned}
T^a &\equiv \mathcal{D}V^a - \mathrm{i}\,\overline{\psi}_A \wedge \gamma^a \psi^A & (8.4)\\
\rho_A &\equiv d\psi_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi_A + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}} \wedge \psi_A + \widehat{\omega}_A{}^B \wedge \psi_B \equiv \nabla\psi_A & (8.5)\\
\rho^A &\equiv d\psi^A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi^A - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}} \wedge \psi^A + \widehat{\omega}^A{}_B \wedge \psi^B \equiv \nabla\psi^A & (8.6)\\
R^{ab} &\equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} & (8.7)
\end{aligned}$$

where $\omega_A{}^B = \frac{\mathrm{i}}{2}\omega^x(\sigma_x)_A^B$ and $\omega^A{}_B = \epsilon^{AC}\epsilon_{DB}\omega_C{}^D$, and where the gauged connections for the $\mathcal{SU}$ and $\mathcal{L}$ bundles were introduced in eq.s 7.57. In all the above formulae the pull–back on space–time through the maps

$$z^i \;:\; M_4 \longrightarrow \mathcal{SM} \quad ; \quad q^u \;:\; M_4 \longrightarrow \mathcal{HM} \tag{8.8}$$

is obviously understood. In this way the composite connections become 1–forms on space–time.

In the vector multiplet sector the curvatures and covariant derivatives are:

$$\begin{aligned}
\nabla z^i &= dz^i + g\, A^\Lambda\, k^i_\Lambda(z) && (8.9)\\
\nabla \overline{z}^{i^\star} &= d\overline{z}^{i^\star} + g\, A^\Lambda\, k^{i^\star}_\Lambda(\overline{z}) && (8.10)\\
\nabla\lambda^{iA} &\equiv d\lambda^{iA} - \frac{1}{4}\gamma_{ab}\,\omega^{ab}\lambda^{iA} - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\lambda^{iA} + \widehat{\Gamma}^i{}_j\lambda^{jA} + \widehat{\omega}^A{}_B \wedge \lambda^{iB}\\
\nabla\lambda^{i^\star}_A &\equiv d\lambda^{i^\star}_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab}\lambda^{i^\star}_A + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\lambda^{i^\star}_A + \widehat{\Gamma}^{i^\star}{}_{j^\star}\lambda^{j^\star}_A + \widehat{\omega}_A{}^B \wedge \lambda^{i^\star}_B\\
F^\Lambda &\equiv dA^\Lambda + \frac{1}{2}\, g\, f^\Lambda{}_{\Sigma\Gamma}\, A^\Sigma \wedge A^\Gamma + \overline{L}^\Lambda \overline{\psi}_A \wedge \psi_B \epsilon^{AB} + L^\Lambda \overline{\psi}^A \wedge \psi^B \epsilon_{AB} && (8.11)
\end{aligned}$$

where the gauged Levi–Civita connection $\widehat{\Gamma}^i{}_j$ on $\mathcal{SM}$ is also given by eq. 7.57 and where $L^\Lambda = e^{\frac{\mathcal{K}}{2}} X^\Lambda$ is the upper half (electric) of the symplectic section of $\mathcal{H}$ introduced in equation 4.26. The lower part M_Λ of such a symplectic section would appear in the magnetic field strengths if we did introduce them.

Finally in the hypermultiplet sector the covariant derivatives are:

$$\begin{aligned}
\mathcal{U}^{A\alpha} &\equiv \mathcal{U}^{A\alpha}_v \nabla q^v \equiv \mathcal{U}^{A\alpha}_v \left(d\, q^v + g\, A^\Lambda\, k^v_\Lambda(q) \right) && (8.12)\\
\nabla\zeta_\alpha &\equiv d\zeta_\alpha - \frac{1}{4}\omega^{ab}\,\gamma_{ab}\,\zeta_\alpha - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\zeta_\alpha + \widehat{\Delta}_\alpha{}^\beta\zeta_\beta && (8.13)\\
\nabla\zeta^\alpha &\equiv d\zeta^\alpha - \frac{1}{4}\omega^{ab}\,\gamma_{ab}\,\zeta^\alpha + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\zeta^\alpha + \widehat{\Delta}^\alpha{}_\beta\zeta^\beta && (8.14)
\end{aligned}$$

where $\widehat{\Delta}_\alpha{}^\beta$ is the gauged Levi–Civita connection on $\mathcal{HM}$ defined in eq. 7.57, satisfying the condition to be $Sp(2m, \mathbb{R})$ Lie–algebra valued and

$$\widehat{\Delta}_\alpha{}^\beta \equiv \widehat{\Delta}^{\gamma\beta}\,\mathbb{C}_{\gamma\alpha}\,;\; \widehat{\Delta}^\alpha{}_\beta \equiv \mathbb{C}_{\beta\gamma}\,\widehat{\Delta}^{\alpha\gamma} \tag{8.15}$$

Let us note that the definition of the generalized curvatures as given in eq.s 8.4-8.7 and 8.11 has been chosen in such a way that when all the p-forms are extended to superforms in superspace they give the correct supercurvatures of the $N = 2$ superalgebra; that means that if we set all supercurvatures to zero the corresponding equations represent the $N = 2$ superalgebra in dual form. Given these definitions our next task is to write down the space-time Lagrangian and the supersymmetry transformation laws of the fields. The method employed for this construction is based on the geometrical approach: for a review see [31]. The rheonomic derivation of the N=2 theory is explained in Appendix A. Actually one solves the Bianchi identities in $N = 2$ superspace and then constructs the rheonomic superspace Lagrangian in such a way that the superspace "curvatures" given by the solution of the Bianchi identities are reproduced by the variational equations of motion derived from the Lagrangian. After this procedure is completed the space-time Lagrangian is immediately retrieved by restricting the superspace p-forms to space-time.

Using the results of Appendix B one finds the space-time $N = 2$ supergravity action that can be split in the following way:

$$S = \int \sqrt{-g}\, d^4 x \left[\mathcal{L}_k + \mathcal{L}_{4f} + \mathcal{L}'_g \right] ,$$

$$
\begin{aligned}
\mathcal{L}_k &= \mathcal{L}^{inv}_{kin} + \mathcal{L}_{Pauli} \ , \\
\mathcal{L}_{4f} &= \mathcal{L}^{inv}_{4f} + \mathcal{L}^{non\,inv}_{4f} \ , \\
\mathcal{L}'_g &= \mathcal{L}_{mass} - V(z,\overline{z},q) \ ,
\end{aligned}
\tag{8.16}
$$

where $\mathcal{L}^{inv}_{kin}$ consists of the true kinetic terms as well as Pauli-like terms containing the derivatives of the scalar fields. The modifications due to the gauging are contained not only in $\mathcal{L}'_g$ but also in the gauged covariant derivatives in the rest of the lagrangian. We collect the various terms of (8.16) in the table below.

N=2 Supergravity lagrangian

$$
\begin{aligned}
\mathcal{L}^{inv}_{kin} &= -\frac{1}{2}R + g_{ij^\star}\nabla^\mu z^i \nabla_\mu \overline{z}^{j^\star} + h_{uv}\nabla_\mu q^u \nabla^\mu q^v + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\overline{\psi}^A_\mu \gamma_\sigma \rho_{A\nu\lambda} - \overline{\psi}_{A\mu}\gamma_\sigma \rho^A_{\nu\lambda}\right) \\
&- \frac{\mathrm{i}}{2} g_{ij^\star}\left(\overline{\lambda}^{iA}\gamma^\mu \nabla_\mu \lambda^{j^\star}_A + \overline{\lambda}^{j^\star}_A \gamma^\mu \nabla_\mu \lambda^{iA}\right) - \mathrm{i}\left(\overline{\zeta}^\alpha \gamma^\mu \nabla_\mu \zeta_\alpha + \overline{\zeta}_\alpha \gamma^\mu \nabla_\mu \zeta^\alpha\right) \\
&+ \mathrm{i}\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) + \Big\{ - g_{ij^\star}\nabla_\mu \overline{z}^{j^\star}\overline{\psi}^\mu_A \lambda^{iA} \\
&- 2\mathcal{U}^{A\alpha}_u \nabla_\mu q^u \overline{\psi}^\mu_A \zeta_\alpha + g_{ij^\star}\nabla_\mu \overline{z}^{j^\star}\overline{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\nu} + 2\mathcal{U}^{\alpha A}_u \nabla_\mu q^u \overline{\zeta}_\alpha \gamma^{\mu\nu}\psi_{A\nu} + \text{h.c.}\Big\}
\end{aligned}
\tag{8.17}
$$

$$
\begin{aligned}
\mathcal{L}_{Pauli} &= \Big\{\mathcal{F}^{-\Lambda}_{\mu\nu}\left(\mathrm{Im}\,\mathcal{N}\right)_{\Lambda\Sigma}[4L^\Sigma \overline{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB} - 4\mathrm{i}\overline{f}^\Sigma_{i^\star}\overline{\lambda}^{i^\star}_A \gamma^\nu \psi^\mu_B \epsilon^{AB} + \\
&+ \frac{1}{2}\nabla_i f^\Sigma_j \overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\epsilon_{AB} - L^\Sigma \overline{\zeta}_\alpha \gamma^{\mu\nu}\zeta_\beta \mathbb{C}^{\alpha\beta}] + \text{h.c.}\Big\}
\end{aligned}
\tag{8.18}
$$

$$
\begin{aligned}
\mathcal{L}^{inv}_{4f} &= \frac{\mathrm{i}}{2}\left(g_{ij^\star}\overline{\lambda}^{iA}\gamma_\sigma \lambda^{j^\star}_B - 2\delta^A_B \overline{\zeta}^\alpha \gamma_\sigma \zeta_\alpha\right)\overline{\psi}_{A\mu}\gamma_\lambda \psi^B_\nu \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}} \\
&- \frac{1}{6}\left(C_{ijk}\overline{\lambda}^{iA}\gamma^\mu \psi^B_\mu \overline{\lambda}^{jC}\lambda^{kD}\epsilon_{AC}\epsilon_{BD} + h.c.\right) \\
&- 2\overline{\psi}^A_\mu \psi^B_\nu \overline{\psi}^\mu_A \psi^\nu_B + 2g_{ij^\star}\overline{\lambda}^{iA}\gamma_\mu \psi^B_\nu \overline{\lambda}^{i^\star}_A \gamma^\mu \psi^\nu_B \\
&+ \frac{1}{4}\left(R_{ij^\star lk^\star} + g_{ik^\star}g_{lj^\star} - \frac{3}{2}g_{ij^\star}g_{lk^\star}\right)\overline{\lambda}^{iA}\lambda^{lB}\overline{\lambda}^{j^\star}_A \lambda^{k^\star}_B \\
&+ \frac{1}{4}g_{ij^\star}\overline{\zeta}^\alpha \gamma_\mu \zeta_\alpha \overline{\lambda}^{iA}\gamma^\mu \lambda^{j^\star}_A + \frac{1}{2}\mathcal{R}^\alpha_{\beta ts}\mathcal{U}^t_{A\gamma}\mathcal{U}^s_{B\delta}\epsilon^{AB}C^{\delta\eta}\overline{\zeta}_\alpha \zeta_\eta \overline{\zeta}^\beta \zeta^\gamma \\
&- \left[\frac{\mathrm{i}}{12}\nabla_m C_{jkl}\overline{\lambda}^{jA}\lambda^{mB}\overline{\lambda}^{kC}\lambda^{lD}\epsilon_{AC}\epsilon_{BD} + h.\,c.\right] \\
&+ g_{ij^\star}\overline{\psi}^A_\mu \lambda^{j^\star}_A \overline{\psi}^\mu_B \lambda^{iB} + 2\overline{\psi}^A_\mu \zeta^\alpha \overline{\psi}^\mu_A \zeta_\alpha + \left(\epsilon_{AB}\mathbb{C}_{\alpha\beta}\overline{\psi}^A_\mu \zeta^\alpha \overline{\psi}^{B|\mu}\zeta^\beta + h.c.\right)
\end{aligned}
\tag{8.19}
$$

$$
\begin{aligned}
\mathcal{L}^{non\,inv}_{4f} &= \Big\{\left(\mathrm{Im}\,\mathcal{N}\right)_{\Lambda\Sigma}\Big[2L^\Lambda L^\Sigma \left(\overline{\psi}^A_\mu \psi^B_\nu\right)^- \left(\overline{\psi}^C_\mu \psi^D_\nu\right)^- \epsilon_{AB}\epsilon_{CD} \\
&- 8\mathrm{i}\,L^\Lambda \overline{f}^\Sigma_{i^\star}\left(\overline{\psi}^A_\mu \psi^B_\nu\right)^- \left(\overline{\lambda}^{i^\star}_A \gamma^\nu \psi^\mu_B\right)^- \\
&- 2\overline{f}^\Lambda_{i^\star}\overline{f}^\Sigma_{j^\star}\left(\overline{\lambda}^{i^\star}_A \gamma^\nu \psi^\mu_B\right)^- \left(\overline{\lambda}^{j^\star}_C \gamma_\nu \psi_{D|\mu}\right)^- \epsilon^{AB}\epsilon^{CD} \\
&+ \frac{\mathrm{i}}{2}L^\Lambda \overline{f}^\Sigma_{\ell^\star}g^{k\ell^\star}C_{ijk}\left(\overline{\psi}^A_\mu \psi^B_\nu\right)^- \overline{\lambda}^{iC}\gamma^{\mu\nu}\lambda^{jD}\epsilon_{AB}\epsilon_{CD}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \overline{f}^{\Lambda}_{m^\star}\overline{f}^{\Sigma}_{\ell^\star}\, g^{k\ell^\star}\, C_{ijk}\left(\overline{\lambda}^{m^\star}_{A}\gamma_\nu\psi_{B\mu}\right)^{-}\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB} \\
&- \quad L^{\Lambda}L^{\Sigma}\left(\overline{\psi}^{A}_{\mu}\psi^{B}_{\nu}\right)^{-}\overline{\zeta}_{\alpha}\gamma^{\mu\nu}\zeta_{\beta}\,\epsilon_{AB}\,\mathbb{C}^{\alpha\beta} \\
&+ \quad \mathrm{i}L^{\Lambda}\overline{f}^{\Sigma}_{i^\star}\left(\overline{\lambda}^{i^\star}_{A}\gamma^{\nu}\psi^{\mu}_{B}\right)^{-}\overline{\zeta}_{\alpha}\gamma_{\mu\nu}\zeta_{\beta}\,\epsilon^{AB}\mathbb{C}^{\alpha\beta} \\
&- \quad \frac{1}{32}\,C_{ijk}\,C_{lmn}g^{kr^\star}\,g^{ns^\star}\,\overline{f}^{\Lambda}_{r^\star}\,\overline{f}^{\Sigma}_{s^\star}\,\overline{\lambda}^{iA}\,\gamma_{\mu\nu}\,\lambda^{jB}\,\overline{\lambda}^{kC}\,\gamma^{\mu\nu}\,\lambda^{lD}\,\epsilon_{AB}\epsilon_{CD} \\
&- \quad \frac{1}{8}\,L^{\Lambda}\nabla_{i}f^{\Sigma}_{j}\overline{\zeta}_{\alpha}\gamma_{\mu\nu}\zeta_{\beta}\,\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\;\epsilon_{AB}\,\mathbb{C}^{\alpha\beta} \\
&+ \quad \frac{1}{8}\,L^{\Lambda}\,L^{\Sigma}\overline{\zeta}_{\alpha}\gamma_{\mu\nu}\zeta_{\beta}\,\overline{\zeta}_{\gamma}\gamma^{\mu\nu}\zeta_{\delta}\,\mathbb{C}^{\alpha\beta}\,\mathbb{C}^{\gamma\delta}\Big] + \text{h.c.}\Big\} \qquad (8.20)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{mass} &= g\Big[2S_{AB}\overline{\psi}^{A}_{\mu}\gamma^{\mu\nu}\psi^{B}_{\nu} + \mathrm{i}g_{ij^\star}W^{iAB}\overline{\lambda}^{j^\star}_{A}\gamma_{\mu}\psi^{\mu}_{B} + 2\mathrm{i}N^{A}_{\alpha}\overline{\zeta}^{\alpha}\gamma_{\mu}\psi^{\mu}_{A} \\
&+ \mathcal{M}^{\alpha\beta}\overline{\zeta}_{\alpha}\zeta_{\beta} + \mathcal{M}^{\alpha}{}_{iB}\overline{\zeta}_{\alpha}\lambda^{iB} + \mathcal{M}_{iA\ell B}\overline{\lambda}^{iA}\lambda^{\ell B}\Big] + \text{h.c.} \qquad (8.21) \\
\mathrm{V}(z,\overline{z},q) &= g^{2}\Big[\left(g_{ij^\star}k^{i}_{\Lambda}k^{j^\star}_{\Sigma} + 4h_{uv}k^{u}_{\Lambda}k^{v}_{\Sigma}\right)\overline{L}^{\Lambda}L^{\Sigma} + g^{ij^\star}f^{\Lambda}_{i}f^{\Sigma}_{j^\star}\mathcal{P}^{x}_{\Lambda}\mathcal{P}^{x}_{\Sigma} - 3\overline{L}^{\Lambda}L^{\Sigma}\mathcal{P}^{x}_{\Lambda}\mathcal{P}^{x}_{\Sigma}\Big] \qquad (8.22)
\end{aligned}
$$

where $\mathcal{F}^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}(\mathcal{F}^{\Lambda}_{\mu\nu} \pm \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\rho\sigma})$ and $(...)^{-}$ denotes the self dual part of the fermion bilinears. The mass–matrices are given by:

$$
\begin{aligned}
S_{AB} &= \frac{\mathrm{i}}{2}(\sigma_x)_{A}{}^{C}\epsilon_{BC}\mathcal{P}^{x}_{\Lambda}L^{\Lambda} \\
W^{iAB} &= \epsilon^{AB}\,k^{i}_{\Lambda}\overline{L}^{\Lambda} \,+\, \mathrm{i}(\sigma_x)_{C}{}^{B}\epsilon^{CA}\mathcal{P}^{x}_{\Lambda}g^{ij^\star}\overline{f}^{\Lambda}_{j^\star} \\
N^{A}_{\alpha} &= 2\,\mathcal{U}^{A}_{\alpha u}\,k^{u}_{\Lambda}\,\overline{L}^{\Lambda} \\
\mathcal{M}^{\alpha\beta} &= -\,\mathcal{U}^{\alpha A}_{u}\,\mathcal{U}^{\beta B}_{v}\,\varepsilon_{AB}\,\nabla^{[u}k^{v]}_{\Lambda}\,L^{\Lambda} \\
\mathcal{M}^{\alpha}{}_{iB} &= -4\,\mathcal{U}^{\alpha}_{Bu}\,k^{u}_{\Lambda}\,f^{\Lambda}_{i} \\
\mathcal{M}_{iA|\ell B} &= \frac{1}{3}\left(\varepsilon_{AB}\,g_{ij^\star}k^{j^\star}_{\Lambda}\,f^{\Lambda}_{\ell} + \mathrm{i}(\sigma_x\epsilon^{-1})_{AB}\,\mathcal{P}^{x}_{\Lambda}\,\nabla_{\ell}f^{\Lambda}_{i}\right) \qquad (8.23)
\end{aligned}
$$

The coupling constant g in $\mathcal{L}'_{g}$ is just a symbolic notation to remind that these terms are entirely due to the gauging and vanish in the ungauged theory, where also all gauged covariant derivatives reduce to ordinary ones. Note that in general there is not a single coupling constant, but rather there are as many independent coupling constants as the number of factors in the gauge group. The normalization of the kinetic term for the quaternions depends on the scale λ of the quaternionic manifold, appearing in eq. (5.10), for which we have chosen the value $\lambda = -1$.

Furthermore, using the geometric approach, the form of the supersymmetry transformation laws is also easily deduced from the solution of the Bianchi identities in superspace (see Appendix A). One gets

Supergravity transformation rules of the Fermi fields

$$
\delta\,\psi_{A\mu} \;=\; \mathcal{D}_{\mu}\,\epsilon_{A} - \frac{1}{4}\left(\partial_{i}\,K\overline{\lambda}^{iB}\epsilon_{B} \,-\, \partial_{i^\star}\,K\overline{\lambda}^{i^\star}_{B}\epsilon^{B}\right)\psi_{A\mu}
$$

$$
\begin{aligned}
&\quad -\omega_{Av}{}^{B}\,\mathcal{U}^{v}_{C\alpha}\,\left(\epsilon^{CD}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_{\beta}\,\epsilon_{D}\,+\,\overline{\zeta}^{\alpha}\,\epsilon^{C}\right)\psi_{B\mu} \\
&\quad +\left(A_{A}{}^{\nu B}\eta_{\mu\nu}+A'_{A}{}^{\nu B}\gamma_{\mu\nu}\right)\epsilon_{B} \\
&\quad +\left[\mathrm{i}\,g\,S_{AB}\eta_{\mu\nu}+\epsilon_{AB}(T^{-}_{\mu\nu}\,+\,U^{+}_{\mu\nu})\right]\gamma^{\nu}\epsilon^{B} \qquad (8.24)\\
\delta\,\lambda^{iA} &= \frac{1}{4}\left(\partial_{j}\,K\overline{\lambda}^{jB}\epsilon_{B}\,-\,\partial_{j^\star}\,K\overline{\lambda}^{j^\star}_{B}\epsilon^{B}\right)\lambda^{iA} \\
&\quad -\omega^{A}{}_{Bv}\,\mathcal{U}^{v}_{C\alpha}\,\left(\epsilon^{CD}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_{\beta}\,\epsilon_{D}\,+\,\overline{\zeta}^{\alpha}\,\epsilon^{C}\right)\lambda^{iB} \\
&\quad -\Gamma^{i}{}_{jk}\overline{\lambda}^{kB}\epsilon_{B}\,\lambda^{jA}+\mathrm{i}\,\left(\nabla_{\mu}\,z^{i}-\overline{\lambda}^{iA}\psi_{A\mu}\right)\gamma^{\mu}\epsilon^{A} \\
&\quad +G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon_{B}\epsilon^{AB}\,+\,D^{iAB}\epsilon_{B} \qquad (8.25)\\
\delta\,\zeta_{\alpha} &= -\Delta_{\alpha v}{}^{\beta}\,\mathcal{U}^{v}_{\gamma A}\,\left(\epsilon^{AB}\,\mathbb{C}^{\gamma\delta}\,\overline{\zeta}_{\delta}\,\epsilon_{B}\,+\,\overline{\zeta}^{\gamma}\,\epsilon^{A}\right)\zeta_{\beta} \\
&\quad +\frac{1}{4}\left(\partial_{i}\,K\overline{\lambda}^{iB}\epsilon_{B}\,-\,\partial_{i^\star}\,K\overline{\lambda}^{i^\star}_{B}\epsilon^{B}\right)\zeta_{\alpha} \\
&\quad +\mathrm{i}\,\left(\mathcal{U}^{B\beta}_{u}\,\nabla_{\mu}\,q^{u}\,-\epsilon^{BC}\,\mathbb{C}^{\beta\gamma}\,\overline{\zeta}_{\gamma}\,\psi_{C}\,-\,\overline{\zeta}^{\beta}\,\psi^{B}\right)\gamma^{\mu}\epsilon^{A}\epsilon_{AB}\,\mathbb{C}_{\alpha\beta}\,+\,g\,N^{A}_{\alpha}\,\epsilon_{A} \qquad (8.26)
\end{aligned}
$$

Supergravity transformation rules of the Bose fields

$$
\begin{aligned}
\delta\,V^{a}_{\mu} &= -\mathrm{i}\,\overline{\psi}_{A\mu}\,\gamma^{a}\,\epsilon^{A}-\mathrm{i}\,\overline{\psi}^{A}_{\mu}\,\gamma^{a}\,\epsilon_{A} \qquad (8.27)\\
\delta\,A^{\Lambda}_{\mu} &= 2\overline{L}^{\Lambda}\overline{\psi}_{A\mu}\epsilon_{B}\epsilon^{AB}\,+\,2L^{\Lambda}\overline{\psi}^{A}_{\mu}\epsilon^{B}\epsilon_{AB} \\
&\quad +\,\left(\mathrm{i}\,f^{\Lambda}_{i}\,\overline{\lambda}^{iA}\gamma_{\mu}\epsilon^{B}\,\epsilon_{AB}+\mathrm{i}\,\overline{f}^{\Lambda}_{i^\star}\,\overline{\lambda}^{i^\star}_{A}\gamma_{\mu}\epsilon_{B}\,\epsilon^{AB}\right) \qquad (8.28)\\
\delta\,z^{i} &= \overline{\lambda}^{iA}\epsilon_{A} \qquad (8.29)\\
\delta\,z^{i^\star} &= \overline{\lambda}^{i^\star}_{A}\epsilon^{A} \qquad (8.30)\\
\delta\,q^{u} &= \mathcal{U}^{u}_{\alpha A}\left(\overline{\zeta}^{\alpha}\epsilon^{A}+\mathbb{C}^{\alpha\beta}\epsilon^{AB}\overline{\zeta}_{\beta}\epsilon_{B}\right) \qquad (8.31)
\end{aligned}
$$

where we have:

Supergravity values of the auxiliary fields

$$
\begin{aligned}
A_{A}{}^{\mu B} &= -\frac{\mathrm{i}}{4}\,g_{k^\star\ell}\,\left(\overline{\lambda}^{k^\star}_{A}\,\gamma^{\mu}\lambda^{\ell B}\,-\,\delta^{B}_{A}\,\overline{\lambda}^{k^\star}_{C}\,\gamma^{\mu}\lambda^{\ell C}\right) \qquad (8.32)\\
A'_{A}{}^{\mu B} &= \frac{\mathrm{i}}{4}\,g_{k^\star\ell}\,\left(\overline{\lambda}^{k^\star}_{A}\,\gamma^{\mu}\lambda^{\ell B}-\frac{1}{2}\,\delta^{B}_{A}\,\overline{\lambda}^{k^\star}_{C}\,\gamma^{\mu}\lambda^{C\ell}\right)\,-\,\frac{\mathrm{i}}{4}\,\delta^{B}_{A}\,\overline{\zeta}_{\alpha}\gamma^{\mu}\zeta^{\alpha} \qquad (8.33)
\end{aligned}
$$

$$
\begin{aligned}
T^{-}_{\mu\nu} &= 2\mathrm{i}\,(\mathrm{Im}\,\mathcal{N})_{\Lambda\Sigma}\,L^{\Sigma}\left(\widetilde{F}^{\Lambda-}_{\mu\nu}+\frac{1}{8}\nabla_{i}\,f^{\Lambda}_{j}\,\overline{\lambda}^{iA}\gamma_{\mu\nu}\,\lambda^{jB}\,\epsilon_{AB}-\frac{1}{4}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_{\alpha}\gamma_{\mu\nu}\,\zeta_{\beta}\,L^{\Lambda}\right) \qquad (8.34)\\
T^{+}_{\mu\nu} &= 2\mathrm{i}\,(\mathrm{Im}\,\mathcal{N})_{\Lambda\Sigma}\,\overline{L}^{\Sigma}\left(\widetilde{F}^{\Lambda+}_{\mu\nu}+\frac{1}{8}\nabla_{i^\star}\,\overline{f}^{\Lambda}_{j^\star}\,\overline{\lambda}^{i^\star}_{A}\gamma_{\mu\nu}\,\lambda^{j^\star}_{B}\epsilon^{AB}-\frac{1}{4}\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^{\alpha}\gamma_{\mu\nu}\,\zeta^{\beta}\,\overline{L}^{\Lambda}\right) \qquad (8.35)
\end{aligned}
$$

$$U^{-}_{\mu\nu} \;=\; -\frac{\mathrm{i}}{4}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_{\alpha}\gamma_{\mu\nu}\,\zeta_{\beta} \tag{8.36}$$

$$U^{+}_{\mu\nu} \;=\; -\frac{\mathrm{i}}{4}\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^{\alpha}\gamma_{\mu\nu}\,\zeta^{\beta} \tag{8.37}$$

$$\begin{aligned}
G^{i-}_{\mu\nu} \;&=\; -g^{ij^\star}\overline{f}^{\Gamma}_{j^\star}\,(\mathrm{Im}\,\mathcal{N})_{\Gamma\Lambda}\left(\widetilde{F}^{\Lambda-}_{\mu\nu} + \frac{1}{8}\nabla_k f^{\Lambda}_{\ell}\overline{\lambda}^{kA}\gamma_{\mu\nu}\,\lambda^{\ell B}\epsilon_{AB}\right.\\
&\qquad\left. -\frac{1}{4}\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_{\alpha}\gamma_{\mu\nu}\,\zeta_{\beta}\,L^{\Lambda}\right)
\end{aligned} \tag{8.38}$$

$$\begin{aligned}
G^{i^\star+}_{\mu\nu} \;&=\; -g^{i^\star j}f^{\Gamma}_{j}\,(\mathrm{Im}\,\mathcal{N})_{\Gamma\Lambda}\left(\widetilde{F}^{\Lambda+}_{\mu\nu} + \frac{1}{8}\nabla_{k^\star}\overline{f}^{\Lambda}_{\ell^\star}\overline{\lambda}^{k^\star}_{A}\gamma_{\mu\nu}\,\lambda^{\ell^\star}_{B}\epsilon^{AB}\right.\\
&\qquad\left. -\frac{1}{4}\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^{\alpha}\gamma_{\mu\nu}\,\zeta^{\beta}\,\overline{L}^{\Lambda}\right)
\end{aligned} \tag{8.39}$$

$$D^{iAB} \;=\; \frac{\mathrm{i}}{2}g^{ij^\star}C_{j^\star k^\star \ell^\star}\overline{\lambda}^{k^\star}_{C}\,\lambda^{\ell^\star}_{D}\epsilon^{AC}\epsilon^{BD} \;+\; W^{iAB} \tag{8.40}$$

In eqs. (8.34), (8.35), (8.38), (8.39) we have denoted by $\widetilde{F}_{\mu\nu}$ the supercovariant field strength defined by:

$$\widetilde{F}^{\Lambda}_{\mu\nu} = \mathcal{F}^{\Lambda}_{\mu\nu} \;+\; L^{\Lambda}\overline{\psi}^{A}_{\mu}\psi^{B}_{\nu}\,\epsilon_{AB} \;+\overline{L}^{\Lambda}\overline{\psi}_{A\mu}\psi_{B\nu}\epsilon^{AB} \;-\mathrm{i}\,f^{\Lambda}_{i}\,\overline{\lambda}^{iA}\gamma_{[\nu}\psi^{B}_{\mu]}\,\epsilon_{AB} \;-\mathrm{i}\,\overline{f}^{\Lambda}_{i^\star}\,\overline{\lambda}^{i^\star}_{A}\gamma_{[\nu}\psi_{B\mu]}\,\epsilon^{AB}\;. \tag{8.41}$$

Let us make some observation about the structure of the Lagrangian and of the transformation laws.

i) We note that all the terms of the Lagrangian are given in terms of purely geometric objects pertaining to the Special and quaternionic geometries. Furthermore the Lagrangian does not rely on the existence of a prepotential function $F = F\,(X)$ and it is valid for any choice of the quaternionic manifold.

ii) The Lagrangian is not invariant under symplectic duality transformations. However, in absence of gauging ($g = 0$), if we restrict the Lagrangian to configurations where the vectors are on shell, it becomes symplectic invariant (ref). This allows us to fix the terms appearing in $\mathcal{L}^{non\,inv}_{4\,ferm}$ in a way independent from supersymmetry arguments.

Here we report only the results of the application of the method of [4],[67]in our case. For a complete treatment see [4],[67]. The non-invariant part of the Lagrangian is:

$$\mathcal{L} \;=\; \mathcal{L}^{vectors}_{Kin} \;+\; \mathcal{L}^{non\,inv}_{Pauli} \;+\; \mathcal{L}^{non\,inv}_{4f} \tag{8.42}$$

where: $\mathcal{L}^{vectors}_{Kin} \;=\; \mathrm{i}\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}\mathcal{F}^{-\Sigma} \;-\; h.c.\right)$. The part $\mathcal{L}^{non\,inv}_{4f}$ of the 4–fermi Lagrangian is fixed by the requirement of on-shell vector invariance.Indeed, imposing the equation of m motion for the gauge fields,with straightforward calculations one finds that$\mathcal{L}^{non\,inv}$ can be written as follows:

$$\mathcal{L}^{non\,inv}_{on\,shell} \;=\; \frac{1}{2}\left(\mathcal{F}^{-\Lambda}\mathcal{H}^{-}_{\Lambda} \;+\; h.c.\right) \;+\; \mathcal{L}^{non\,inv}_{4\,ferm} \tag{8.43}$$

where:

$$\mathcal{H}^{-}_{\Lambda|\mu\nu} \;=\; \left(\mathcal{N} - \overline{\mathcal{N}}\right)_{\Lambda\Sigma}\tau^{-\Sigma}_{\mu\nu} \tag{8.44}$$

with:

$$
\begin{aligned}
\tau^{-\Sigma}_{\mu\nu} &= \Big[-2\mathrm{i}\, L^{\Sigma}\left(\overline{\psi}^{A|\mu}\psi^{B|\nu}\right)^{-}\epsilon_{AB} - 2\overline{f}^{\Sigma}_{i^\star}\left(\overline{\lambda}^{i^\star}_{A}\gamma^{\nu}\psi^{\mu}_{B}\right)^{-}\epsilon^{AB} \\
&\quad - \frac{\mathrm{i}}{4}\nabla_i f^{\Sigma}_{j}\,\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\,\epsilon_{AB} + \frac{\mathrm{i}}{2}L^{\Sigma}\overline{\zeta}_{\alpha}\gamma^{\mu\nu}\zeta_{\beta}\,C^{\alpha\beta}\Big]
\end{aligned}
\tag{8.45}
$$

From duality arguments it then follows ([4],[67]) that the non invariant 4 fermion terms can be written as the following perfect square:

$$
\mathcal{L}^{noninv}_{4ferm} = +\frac{\mathrm{i}}{4}\mathcal{H}^{-}_{\Lambda|\mu\nu}\,\tau^{-\Lambda|\mu\nu} + h.c. = +\frac{\mathrm{i}}{4}\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}\tau^{-\Lambda}_{\mu\nu}\tau^{-\Sigma|\mu\nu} + h.c. \tag{8.46}
$$

This result was in fact employed as a useful consistency check in the calculations to construct the Lagrangian.

iii) We note that the field strengths $\mathcal{F}^{\Lambda\,-}_{\mu\nu}$ originally introduced in the Lagrangian are the free gauge field strengths.The interacting field strengths which are supersymmetry eigenstates are defined as the objects appearing in the transformation laws of the gravitinos and gauginos fields,respectively,namely the bosonic part of $T^{-}_{\mu\nu}$ and $G^{-i}_{\mu\nu}$ defined in eq.s 8.34, 8.38

9 Comments on the scalar potential

A general Ward identity[39] of N-extended supergravity establishes the following formulae for the scalar potential $V(\phi)$ of the theory (in appropriate normalizations for the generic fermionic shifts $\delta\chi^a$)

$$
Z_{ab}\delta_A\chi^a\delta^B\overline{\chi}^b - 3\mathcal{M}_{AC}\overline{\mathcal{M}}^{CB} = \delta^A_{\ B}V(\phi) \quad A,B = 1,\ldots,N \tag{9.47}
$$

where $\delta_A\chi^a$ is the extra contribution, due to the gauging, to the spin $\frac{1}{2}$ supersymmetry variations of the scalar vev's, Z_{ab} is the (scalar dependent) kinetic term normalization and $\mathcal{M}_{AC}$ is the (scalar dependent) gravitino mass matrix. Since in the case at hand ($N=2$) all terms in question are expressed in terms of Killing vectors and prepotentials, contracted with the symplectic sections, we will be able to derive a completely geometrical formula for $V(z,\overline{z},q)$. The relevant terms in the fermionic transformation rules are

$$
\begin{aligned}
\delta\psi_{A\mu} &= igS_{AB}\gamma_{\mu}\epsilon^{B} \ , \\
\delta\lambda^{iA} &= gW^{iAB}\epsilon_{B} \ , \\
\delta\zeta_{\alpha} &= gN^{A}_{\alpha}\epsilon_{A} \ .
\end{aligned}
\tag{9.48}
$$

In our normalization the previous Ward identity gives

$$
V = (g_{ij^\star}k^i_{\Lambda}k^{j^\star}_{\Sigma} + 4h_{uv}k^u_{\Lambda}k^v_{\Sigma})\overline{L}^{\Lambda}L^{\Sigma} + (U^{\Lambda\Sigma} - 3\overline{L}^{\Lambda}L^{\Sigma})\mathcal{P}^x_{\Lambda}\mathcal{P}^x_{\Sigma} \ . \tag{9.49}
$$

with $U^{\Lambda\Sigma}$ is defined in (4.40). Above, the first two terms are related to the gauging of isometries of $\mathcal{SK}\otimes\mathcal{Q}$. For an abelian group, the first term is absent. The negative term is the gravitino mass contribution, while the one in $U^{\Lambda\Sigma}$ is the gaugino shift contribution due to the quaternionic prepotential.

Eq. (9.49) can be rewritten in a suggestive form as

$$V = (k_\Lambda, k_\Sigma)\overline{L}^\Lambda L^\Sigma + (U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma)(\mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma - \mathcal{P}_\Lambda \mathcal{P}_\Sigma) \ , \tag{9.50}$$

where

$$(k_\Lambda, k_\Sigma) = \begin{pmatrix} k^i_\Lambda, & k^{i^\star}_\Lambda, & k^u_\Lambda \end{pmatrix} \begin{pmatrix} 0 & g_{ij^\star} & 0 \\ g_{i^\star j} & 0 & 0 \\ 0 & 0 & 2h_{uv} \end{pmatrix} \begin{pmatrix} k^j_\Sigma \\ k^{j^\star}_\Sigma \\ k^v_\Sigma \end{pmatrix} \tag{9.51}$$

is the scalar product of the Killing vector and we have used eq. (7.6) and the relation

$$k^i_\Lambda L^\Lambda = k^{i^\star}_\Lambda \overline{L}^\Lambda = \mathcal{P}_\Lambda L^\Lambda = \mathcal{P}_\Lambda \overline{L}^\Lambda = 0 \ . \tag{9.52}$$

$\mathcal{P}^x_\Lambda$ are the quaternionic (triplet) prepotentials and $U^{\Lambda\Sigma}$, L^Λ are special geometry data.

In a theory with only abelian vectors, the potential may still be non-zero due to Fayet-Iliopoulos terms:

$$\mathcal{P}^x_\Lambda = \xi^x_\Lambda \ (\text{constant}); \quad \epsilon^{xyz}\xi^y_\Lambda \xi^z_\Sigma = 0 \ . \tag{9.53}$$

In this case

$$V(z, \overline{z}) = (U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma)\xi^x_\Lambda \xi^x_\Sigma \ . \tag{9.54}$$

Examples with $V(z, \overline{z}) = 0$ but non-vanishing gravitino mass (with $N = 2$ supersymmetry broken to $N = 0$) were given in [36], then generalizing to $N = 2$ the no scale models of $N = 1$ supergravity [42]. These models were obtained by taking a $\xi^x_\Lambda = (\xi_0, 0, 0)$. In this case the expression

$$V = U^{00} - 3\overline{L}^0 L^0 \tag{9.55}$$

reduces to

$$V = (\partial_i K g^{ij^\star} \partial_{j^\star} K - 3)e^K \tag{9.56}$$

which is the $N = 1$ supergravity potential, with solution ($V = 0$) the cubic holomorphic prepotential

$$F(X) = d_{ABC}\frac{X^A X^B X^C}{X^0} \qquad A = 1, \ldots, n \ . \tag{9.57}$$

Another solution is obtained by taking the $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(n)}$ coset in the $SO(2, n)$ symmetric parametrization of the symplectic sections ($X^\Lambda, F_\Lambda = \eta_{\Lambda\Sigma} S X^\Sigma$; $X^\Lambda X^\Sigma \eta_{\Lambda\Sigma} = 0$, $\eta_{\Lambda\Sigma} = (1, 1, -1, \ldots, -1)$) where a prepotential F does not exist. In this case

$$U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma = -\frac{1}{i(S - \overline{S})}\eta_{\Lambda\Sigma} \tag{9.58}$$

where we have used the fact that

$$\mathcal{N}_{\Lambda\Sigma} = (S - \overline{S})(\Phi_\Lambda \overline{\Phi}_\Sigma + \overline{\Phi}_\Lambda \Phi_\Sigma) + \overline{S}\eta_{\Lambda\Sigma} \quad , \ \Phi^\Lambda = \frac{X^\Lambda}{(X^\Lambda \overline{X}_\Lambda)^{1/2}} \ . \tag{9.59}$$

The identity (9.58) allows one to prove that the tree level potential of an arbitrary heterotic string compactification (including orbifolds with twisted hypermultiplets) is semi-positive definite provided we don't gauge the graviphoton and the gravidilaton vectors (i.e. $\mathcal{P}^x_\Lambda = 0$ for $\Lambda = 0, 1$, $\mathcal{P}^x_\Lambda \neq 0$ for $\Lambda = 2, \ldots, n_V$). On the other hand, it also proves that tree level supergravity breaking may only occurr if $\mathcal{P}^x_\Lambda \neq 0$ for $\Lambda = 0, 1$. This instance is related to models with Scherk-Schwarz mechanism studied in the literature [40, 41].

A vanishing potential can be obtained if $\xi^x_\Lambda = (\xi_\Lambda, 0, 0)$ with

$$\xi_\Lambda \xi_\Sigma \eta^{\Lambda\Sigma} = 0 \ . \tag{9.60}$$

In this case we may also consider the gauge group to be $U(1)^{p+2} \otimes G(n_V - p)$ and introduce $\xi_\Lambda = (\xi_0, \ldots, \xi_{p+1}, 0, \ldots, 0)$ such that $\xi_\Lambda \xi_\Sigma \eta^{\Lambda\Sigma} = 0$ where $\eta^{\Lambda\Sigma}$ is the $SO(2,p)$ Lorentzian metric. The potential is now:

$$V = k^i_\Lambda g_{ij^\star} k^{j^\star}_\Sigma \overline{L}^\Lambda L^\Sigma \quad ; \quad (U^{\Lambda\Sigma} - 3\overline{L}^\Lambda L^\Sigma) \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma = 0 \tag{9.61}$$

where $k^i_\Lambda L^\Lambda = 0$ for $\Lambda \leq p+1$. The gravitino have equal mass

$$\mid m_{3/2} \mid \simeq e^{K/2} \mid \xi_\Lambda X^\Lambda \mid \tag{9.62}$$

with $\xi_\Lambda \xi_\Sigma \eta^{\Lambda\Sigma} = 0$, $\Lambda = 0, \ldots, p+1$.

It is amusing to note that the gravitino mass, as a function of the $O(2,p)/O(2) \otimes O(p)$ moduli and of the F-I terms, just coincides with the central charge formula for the level $N_L = 1$ in heterotic string (H-monopoles), if the F-I terms are identified with the $O(2,p)$ lattice electric charges.

Note that, because of the special form of the gauged $\widehat{Q}$, $\widehat{\omega}^x$, we see that whenever $\mathcal{P}_\Lambda \neq 0$ the gravitino is charged with respect to the $U(1)$ factor and whenever $\mathcal{P}^x_\Lambda \neq 0$ the gravitino is charged with respect to the $SU(2)$ factor of the $U(1) \otimes SU(2)$ automorphism group of the supersymmetry algebra. In the case of $U(1)^p$ gauge fields with non-vanishing F-I terms $\xi^x_p = (0, 0, \xi_p)$ the gauge field $A^\Lambda_\mu \xi_\Lambda = A_\mu$ gauge a $U(1)$ subgroup of $SU(2)_L$ susy algebra.

Models with breaking of $N = 2$ to $N = 1$ [29] necessarily require k^u_Λ not to be zero. The minimal model where this happens with $V = 0$ is the one based on

$$\mathcal{SK} \otimes \mathcal{Q} = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(4,1)}{SO(4)} \ , \tag{9.63}$$

where a $U(1) \otimes U(1)$ isometry of $\mathcal{Q}$ is gauged. In this case the vanishing of V requires a compensation of the $\delta\lambda, \delta\zeta$ variations with the gravitino contribution

$$4 k^u_\Lambda k^v_\Sigma h_{uv} + U^{\Lambda\Sigma} \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma = 3 \overline{L}^\Lambda L^\Sigma \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \ . \tag{9.64}$$

The moduli space of vacua satisfying (9.64) is a four dimensional subspace of (9.63).

One may wonder where are the explicit mass terms for hypermultiplets. In $N = 2$ supergravity, since the hypermultiplet mass is a central charge, which is gauged, such term corresponds to the gauging of a $U(1)$ charge. This is best seen if we consider the case where no vector multiplets (and then gauginos) are present. In this case $L^\Lambda = L^0 = 1$ and the potential becomes

$$V = 4 h_{uv} k^u k^v - 3 \mathcal{P}^x \mathcal{P}^x \tag{9.65}$$

where k^u is the Killing vector of a $U(1)$ symmetry of $\mathcal{Q}$, gauged by the graviphoton and $\mathcal{P}^x$ is the associated prepotential. For $\frac{SO(4,1)}{SO(4)}$ this reproduces the Zachos model [69]. The gauged $U(1)$ in this model is contained in $SU_R(2)$ which commutes with the symmetry $SU_L(2)$ in the decomposition of $SO(4) = SU_L(2) \otimes SU_R(2)$. This model has a local minimum at vanishing hypermultiplet vev at which $U(1)$ is unbroken, and the extrema (at $u = 1$) (maxima) which break $U(1)$. The extremal model is when both $n_H = n_V = 0$. Still we may have a pure F-I term

$$V = -3\xi^2 \quad \xi = (\xi, 0, 0) \tag{9.66}$$

This corresponds to the gauging of a $U(1) \subset SU(2)_L$ and gravitinos have charged coupling. This model corresponds to anti-De Sitter $N = 2$ supergravity [44].

10 The rigid limit: N=2 matter coupled Yang–Mills theory

In this section we consider the rigid limit of matter coupled N=2 supergravity. The aim is that of obtaining the most general form of matter coupled N=2 super Yang–Mills theory. By this we mean the rigid supersymmetric N=2 theory of n vector multiplets coupled to m hypermultiplets interacting through a generic *rigid special manifold* and a generic *hyperKähler manifold*. Such a theory, in general, is not renormalizable: renormalizability obtains only in the case of a *flat* special manifold and a *flat* hyperKähler manifold. Yet it is very interesting as an effective low energy lagrangian. Seiberg Witten lagrangian [1], is just an instance in this general class. One could derive this type of theory by direct methods solving Bianchi identities in flat superspace and then constructing the corresponding rheonomic action. It is however much simpler to derive it through a suitable *scaling limit* from the N=2 supergravity theory. The contraction parameter is obviously the Planck mass μ and the limit must be performed in such a way that local special geometry flows to rigid special geometry and quaternionic geometry flows to hyperKähler geometry. We already know how this can happen: the curvature of the line and $SU(2)$ bundles must flow to zero in the limit. In the next subsection we describe the appropriate rescalings. Then in a further subsection we report the final result written in space–time component formalism for the benefit of the reader who does not want to be involved with the rheonomy formalism.

10.1 Planck mass rescalings

We begin with the special geometry sector. Here we consider the covariantly holomorphic symplectic section 4.26 and we write:

$$V \equiv \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv \exp\left[\mathcal{K}/2\right] \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} = \exp\left[\widehat{\mathcal{K}}/(2\mu^2)\right] \left(\Omega_0 + \frac{1}{\mu}\widehat{\Omega} + \frac{1}{\mu^3}\Omega_3\right) \tag{10.1}$$

where:

$$\Omega_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \mathbf{0} \\ -\frac{\mathrm{i}}{\sqrt{2}} \\ \mathbf{0} \end{pmatrix} \qquad \widehat{\Omega} = \begin{pmatrix} 0 \\ Y^I \\ 0 \\ F_J \end{pmatrix} \qquad \Omega_3 = \begin{pmatrix} \widehat{Y}_0 \\ \mathbf{0} \\ \widehat{F}_0 \\ \mathbf{0} \end{pmatrix} \tag{10.2}$$

The hatted objects are those that survive in the infinite Planck mass limit $\mu \to \infty$. Recalling eq. 4.25 we obtain:

$$\begin{aligned}
\widehat{\mathcal{K}} &= -\lim_{\mu\to\infty} \mu^2 \log \left[\mathrm{i}\,\langle \Omega | \overline{\Omega}\rangle\right] \\
&= -\lim_{\mu\to\infty} \mu^2 \log \left[1 + \frac{\mathrm{i}}{\mu^2}\left(\overline{Y}^I F_I - \overline{F}_J Y^J\right) + \frac{\sqrt{2}}{\mu^3}\left(\mathrm{Re}Y^0 - \mathrm{Im}F_0\right)\right. \\
&\quad \left. + \frac{\mathrm{i}}{\mu^6}\left(\overline{Y}^0 F_0 - \overline{F}_0 Y^0\right)\right] \\
&= -\mathrm{i}\left(\overline{Y}^I F_I - \overline{F}_J Y^J\right) \\
&= -\mathrm{i}\,\langle \widehat{\Omega} | \widehat{\overline{\Omega}}\rangle \equiv \mathrm{i}\widehat{\Omega}^T \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix} \widehat{\overline{\Omega}}
\end{aligned} \tag{10.3}$$

which reproduces eq. 4.49 for the Kähler potential of rigid special geometry. An observation here is in order. The last line in eq. 10.3 still differs from eq. 4.49 in one respect: the symplectic metric and the symplectic sections in 10.3 are $(2n+2)$–dimensional while those in eq. 4.48 are $2n$–dimensional. Yet the entries of the symplectic sections in the two additional dimensions are always zero so that we can safely reduce the bundle and its structural group from $Sp(2n+2,\mathbb{R})$ to $Sp(2n,\mathbb{R})$.

Let us next consider the symplectic vector U_i defined in eq. 4.29. Using the above rescalings we obtain:

$$U_i = \frac{1}{\mu}\widehat{U}_i + \frac{1}{\mu^2}\begin{pmatrix} \frac{1}{2\sqrt{2}}\partial_i\widehat{\mathcal{K}} \\ \mathbf{0} \\ \frac{-\mathrm{i}}{2\sqrt{2}}\partial_i\widehat{\mathcal{K}} \\ \mathbf{0} \end{pmatrix} + \frac{1}{\mu^3}\begin{pmatrix} \dots \\ \dots \\ \dots \\ \dots \end{pmatrix} \tag{10.4}$$

where

$$\widehat{U}_i = \begin{pmatrix} 0 \\ \partial_i Y^I \\ 0 \\ \partial_i F_J \end{pmatrix} = \partial_i \widehat{\Omega} \tag{10.5}$$

So we have retrieved eq. 4.50, apart from the identically zero extra entries. Hence we can set:

$$g_{ij^\star} = \frac{1}{\mu^2}\widehat{g}_{ij^\star} \tag{10.6}$$

which is consistent with

$$\widehat{g}_{ij^\star} = -\mathrm{i}\,\langle \widehat{U}_i | \widehat{U}_{j^\star}\rangle \tag{10.7}$$

that reproduces the first of eq.s 4.58: the second of such equations is retrieved by setting:

$$C_{ijk} = \frac{1}{\mu^2}\widehat{C}_{ijk} + \mathcal{O}\left(\frac{1}{\mu^3}\right) \quad \Longrightarrow \quad \widehat{C}_{ijk} = \langle \partial_i \widehat{U}_j | \widehat{U}_k\rangle \tag{10.8}$$

Finally we observe that the Levi–Civita connection Γ^i_{jk} is not rescaled by any power of the Planck mass since it contains a metric and an inverse metric (see eq. C.56). This implies the following rescaling for the Riemann tensor of the special manifold:

$$R_{ij^\star k\ell^\star} = g_{ip^\star} R^{p^\star}{}_{j^\star k\ell^\star} = \frac{1}{\mu^2}\widehat{R}_{ij^\star k\ell^\star} \tag{10.9}$$

and the fundamental identity of local special geometry 4.18 becomes

$$\widehat{R}_{ij^\star k\ell^\star} = \frac{1}{\mu^2} \left(\widehat{g}_{ij^\star} \widehat{g}_{k\ell^\star} + \widehat{g}_{kj^\star} \widehat{g}_{i\ell^\star} \right) + \widehat{C}_{iks} \, \widehat{C}_{t^\star j^\star \ell^\star} \, \widehat{g}^{st^\star} \tag{10.10}$$

that in the limit $\mu \to \infty$ reproduces the fundamental identity of rigid special geometry (eq. 4.55).

Summarizing we have:

Rescalings in the Special geometry sector

$$\begin{array}{ll}
L^0 \to \frac{1}{2} + \mathcal{O}\left(\frac{1}{\mu^2}\right) & L^I \to \frac{1}{\mu} Y^I + \mathcal{O}\left(\frac{1}{\mu^2}\right) \\
g_{ij^\star} \to \frac{1}{\mu^2} \widehat{g}_{ij^\star} + \mathcal{O}\left(\frac{1}{\mu^3}\right) & C_{ijk} \to \frac{1}{\mu^2} \widehat{C}_{ijk} + \mathcal{O}\left(\frac{1}{\mu^3}\right) \\
R_{ij^\star k\ell^\star} \to \frac{1}{\mu^2} \widehat{R}_{ij^\star k\ell^\star} + \mathcal{O}\left(\frac{1}{\mu^3}\right) & z^i \to \widehat{z}^i \\
f_i^0 \to \frac{1}{\mu^2} \widehat{f}_i^0 + \mathcal{O}\left(\frac{1}{\mu^3}\right) & f_i^I \to \frac{1}{\mu} \widehat{f}_i^I + \mathcal{O}\left(\frac{1}{\mu^3}\right) \\
\Gamma^i_{jk} \to \widehat{\Gamma}^i_{jk} & \mathcal{Q} \to \frac{1}{\mu^2} \widehat{\mathcal{Q}}
\end{array} \tag{10.11}$$

Next we consider the rescalings in the quaternionic manifold sector. Here we set

Rescalings in the quaternionic manifold sector

$$\begin{array}{lll}
\mathcal{U}^{\alpha A} \to \frac{1}{\mu} \widehat{\mathcal{U}}^{\alpha A} & h_{uv} \to \frac{1}{\mu^2} \widehat{h}_{uv} & q^u \to \widehat{q}^u \\
K^x \to \frac{1}{\mu^2} \widehat{K}^x & \Omega^x \to \widehat{\Omega}^x & \mathcal{P}^x_\Lambda \to \frac{1}{\mu^2} \widehat{\mathcal{P}}^x_\Lambda
\end{array} \tag{10.12}$$

Using these rescalings the quaternionic algebra 5.22 is satisfied by the rescaled hyperKähler structures $\widehat{K}^x_{uv}$ as much as by the unrescaled ones K^x_{uv}: however the relation 5.12 between the $SU(2)$ curvatures and the hyperKähler structures $\widehat{K}^x_{uv}$ becomes:

$$\widehat{\Omega}^x = \frac{\lambda}{\mu^2} \widehat{K}^x \tag{10.13}$$

and in the limit $\mu \to \infty$ we obtain $\widehat{\Omega}^x = 0$, as indeed we expect in the case of a hyperKähler manifold. Indeed we can rephrase this result by saying that, upon restoration of physical units, the $SU(2)$–curvature scale is

$$\lambda = \frac{\widehat{\lambda}}{\mu^2} \tag{10.14}$$

and in the infinite Planck mass limit goes to zero. Indeed when we fixed $\lambda = -1$ to obtain canonical kinetic terms this value had to be interpreted in squared Planck mass units (namely $\widehat{\lambda} = -1$). Eq.s 10.12 are consistent with the definition

$$\mathbf{i}_\Lambda \widehat{K}^x \mathcal{P}^x_\Lambda = \nabla \widehat{\mathcal{P}}^x_\Lambda = d\widehat{\mathcal{P}}^x_\Lambda \tag{10.15}$$

of the triholomorphic momentum map on hyperKähler manifolds. The last equality in eq. 10.15 is justified by the vanishing of the $SU(2)$ curvature that is obtained in the limit $\mu \to \infty$. Finally the rescaled form of the quaternionic equivariance eq. 7.54 is

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \equiv 2K^x(\Lambda, \Sigma) - \frac{\lambda}{\mu^2}\,\varepsilon^{xyz}\,\mathcal{P}^y_\Lambda\,\mathcal{P}^z_\Sigma \tag{10.16}$$

and in the infinite Planck mass limit it flows into the equivariance condition of momentum maps for hyperKähler manifolds, that is eq. 7.50.

To complete our rigid limit programme we have to prescribe the appropriate Planck mass rescalings for the space–time fields and the fermions. These are as follows:

Rescalings of space–time fields and fermions

$V^a \to \frac{1}{\mu}\widehat{V}^a$	$g_{\mu\nu} \to \frac{1}{\mu^2}\,\widehat{g}_{\mu\nu}$	$x^\mu \to \widehat{x}^\mu$
$\omega^{ab} \to \widehat{\omega}^{ab}$	$A^0 \to \frac{1}{\mu}\widehat{A}^0$	$A^I \to \frac{1}{\mu^2}\widehat{A}^I$
$\psi_A \to \frac{1}{\sqrt{\mu}}\,\widehat{\psi}_A$	$\lambda^{iA} \to \sqrt{\mu}\,\widehat{\lambda}^{iA}$	$\zeta^\alpha \to \frac{1}{\sqrt{\mu}}\,\widehat{\zeta}^\alpha$

(10.17)

Utilizing the rescalings of eq.s 10.11, 10.12 and 10.17 in the curvature definitions 8.4, 8.5, 8.6, 8.7, 8.9, 8.10, 8.11, 8.11, 8.11, 8.12, 8.13, 8.14 and in the curvature rheonomic parametrization given in Appendix B, by performing the limit $\mu \to \infty$ we obtain the rheonomic parametrization and curvature definition of the rigid theory. Indeed the first four equations A.23, A.24, A.25 become:

$$\begin{aligned}
T^a &\equiv dV^a - \omega^{ab} \wedge V^c\,\eta_{bc} = 0\\
\rho_A &\equiv d\psi_A - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\,\psi_A = 0\\
\rho^A &\equiv d\psi^A - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\,\psi^A = 0\\
R^{ab} &\equiv d\omega^{ab} - \omega^{ac} \wedge \omega^{cd}\,\eta_{cd} = 0
\end{aligned} \tag{10.18}$$

that are the structural equations of N=2 rigid superspace if they are completed with

$$F^0 \equiv dA^0 + \frac{1}{\sqrt{2}}\left[\overline{\psi}_A \wedge \psi_B\,\varepsilon^{AB} + \overline{\psi}^A \wedge \psi^B\,\varepsilon_{AB}\right] = 0 \tag{10.19}$$

Eq. 10.19 is precisely what we obtain in the $\mu \to \infty$ limit from the case $\Lambda = 0$ of eq.s A.27 and 8.11. Algebraically eq. 10.19 tells us that the graviphoton one–form is the dual of the central charge generator. The case $\Lambda = I$ of the same equations provides the definition and rheonomic parametrization of the Yang–Mills curvatures in rigid superspace:

$$
\begin{aligned}
F^I &\equiv dA^I + \frac{1}{2}\, g\, f^I{}_{JK}\, A^J \wedge A^K, + \overline{Y}^I \overline{\psi}_A \wedge \psi_B \epsilon^{AB} + Y^I \overline{\psi}^A \wedge \psi^B \epsilon_{AB} \\
&= F^I_{ab} V^a \wedge V^b + \left(\mathrm{i}\, f^I_i\, \overline{\lambda}^{iA} \gamma_a \psi^B\, \epsilon_{AB} + \mathrm{i}\, \overline{f}^I_{i^\star}\, \overline{\lambda}^{i^\star}_A \gamma_a \psi_B\, \epsilon^{AB} \right) \wedge V^a
\end{aligned}
\tag{10.20}
$$

From the $\mu \to \infty$ limit of eq. A.28 and A.29 we obtain the gaugino curvature parametrizations:

$$
\begin{aligned}
\nabla \lambda^{iA} &= \nabla_a \lambda^{iA} V^a + \mathrm{i}\, Z^i_a \gamma^a \psi^A + G^{-i}_{ab} \gamma^{ab} \psi_B \epsilon^{AB} + D^{i|AB} \psi_B \\
\nabla \lambda^{i^\star}_A &= \nabla_a \lambda^{i^\star}_A V^a + \mathrm{i}\, \overline{Z}^{i^\star}_a \gamma^a \psi_A + G^{+i^\star}_{ab} \gamma^{ab} \psi^B \epsilon_{AB} + D^{i^\star}{}_{AB}\, \psi^B
\end{aligned}
\tag{10.21}
$$

where Z^i_a and $Z^{i^\star}_a$ are defined by eq. A.30 and its complex conjugate that survive unmodified in the limit while $G^{\pm i^\star}_{ab}$ and the auxiliary fields $D^{i|AB}$, $D^{i^\star}{}_{AB}$ are given in eq.s 10.35. As usual the rheonomic parametrizations correspond to the supersymmetry transformation rules that we have collected in the next subsection together with the space–time action for the benefit of those readers who doe not want to get involved with the rheonomy formalism. Also the rheonomic parametrizations A.32, A.33, A.34 mantain the same form in the rigid limit, but the hyperino shifts N^α_A, N^A_α are now given by eq.s 10.35. Using the same scaling limit one obtains the rigid rheonomic action (which we do not report) from which one retrieves the space–time action reported in the next subsection.

10.2 Summary of the rigid N=2 Yang–Mills theory

Let us then summarize our results by writing the final most general form of N=2 matter coupled Yang–Mills theory. Such a theory arises from a generic choice of the rigid special manifold $\mathcal{SM}_{rig}$, a generic choice of the Hyperkähler manifold $\mathcal{HM}_{rig}$ and a generic choice of the *gauging*.

Let:

$$
\mathcal{F}^I \equiv dA^I + \frac{1}{2} f^I{}_{JK} A^J \wedge A^K = \mathcal{F}^I_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{10.22}
$$

be the field–strengths of the gauge group $\mathcal{G}$. Let z^i be the coordinates of the rigid special manifold $\mathcal{SM}_{rig}$, whose complex dimension n equals the real dimension of the gauge group and let q^u be the $4\,m$ coordinates of the Hyperkähler manifold $\mathcal{HM}_{rig}$. In addition let $\lambda^{iA}, \lambda^{i^\star}_A$ be the two chiral projections of the gaugino field and $\zeta^\alpha, \zeta_\alpha$ the two chiral projections of the hyperino field. Let us moreover define :

the covariant derivatives of the Bose fields

$$
\begin{aligned}
\nabla_\mu z^i &= \partial_\mu z^i + g\, A^I_\mu\, k^i_I \\
\nabla_\mu z^i &= \partial_\mu \overline{z}^i + g\, A^I_\mu\, k^{i^\star}_I \\
\nabla_\mu q^u &= \partial_\mu q^u + g\, A^I_\mu\, k^u_I
\end{aligned}
$$

and

the covariant derivatives of the Fermi fields

$$
\begin{aligned}
\nabla_\mu \lambda^{iA} &= \partial_\mu \lambda^{iA} + \left(\Gamma^i_{jk}\, \nabla_\mu z^j + g\, A^I_\mu\, \partial_j k^i_I \right) \lambda^{jA} \\
\nabla_\mu \lambda^{i^\star}_A &= \partial_\mu \lambda^{i^\star}_A + \left(\overline{\Gamma}^{i^\star}_{j^\star k^\star}\, \nabla_\mu \overline{z}^{j^\star} + g\, A^I_\mu\, \partial_{j^\star} k^{i^\star}_I \right) \lambda^{j^\star}_A
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\mu \zeta^\alpha &= \partial_\mu \zeta^\alpha + \left(\Delta_u^{\alpha\beta}\, \nabla_\mu q^u + g\, A_\mu^I\, \partial_u k_I^v\, \mathcal{U}^{u|\alpha A}\, \mathcal{U}_v^{\beta B}\, \varepsilon_{AB}\right) \mathbb{C}_{\beta\gamma}\, \zeta^\gamma \\
\nabla_\mu \zeta_\gamma &= \partial_\mu \zeta_\gamma + \mathbb{C}_{\gamma\alpha} \left(\Delta_u^{\alpha\beta}\, \nabla_\mu q^u + g\, A_\mu^I\, \partial_u k_I^v\, \mathcal{U}^{u|\alpha A}\, \mathcal{U}_v^{\beta B}\, \varepsilon_{AB}\right) \zeta_\beta
\end{aligned}
\tag{10.23}
$$

In terms of these field strengths and derivatives and of all the geometric structures pertaining to rigid special manifolds and to hyperKähler manifolds discussed in previous sections the most general N=2 supersymmetric invariant lagrangian has the following form:

Matter coupled N=2 Yang Mills action

$$
\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{Pauli} + \mathcal{L}_{massmatrix} + \mathcal{L}_{potential} + \mathcal{L}_{4fermi} \tag{10.24}
$$

where

$$
\begin{aligned}
\mathcal{L}_{kin} &= \mathrm{i}\left(\overline{\mathcal{N}}_{IJ}\, \mathcal{F}^{I-}_{\mu\nu}\, \mathcal{F}^{J-|\mu\nu} - \mathcal{N}_{IJ}\, \mathcal{F}^{I+}_{\mu\nu}\, \mathcal{F}^{J+|\mu\nu}\right) \\
&\quad + g_{ij^\star}\, \nabla^\mu z^i\, \nabla_\mu \overline{z}^{j^\star} + h_{uv}\, \nabla^\mu q^u\, \nabla_\mu q^v \\
&\quad - \frac{\mathrm{i}}{2}\, g_{ij^\star} \left(\overline{\lambda}^{iA}\, \gamma^\mu\, \nabla_\mu \lambda^{j^\star}_A + \overline{\lambda}^{j^\star}_A\, \gamma^\mu\, \nabla_\mu \lambda^{iA}\right) \\
&\quad - \mathrm{i}\left(\overline{\zeta}^\alpha\, \gamma^\mu\, \nabla_\mu \zeta_\alpha + \overline{\zeta}_\alpha\, \gamma^\mu\, \nabla_\mu \zeta^\alpha\right) && (10.25) \\
\mathcal{L}_{Pauli} &= \mathrm{i}\frac{1}{2}\, C_{ijk} \left(g^{k\ell^\star}\, \overline{f}^J_{\ell^\star}\, \mathrm{Im}\mathcal{N}_{IJ}\, \mathcal{F}^{-I}_{\mu\nu}\right) \overline{\lambda}^{iA}\, \gamma^{\mu\nu}\, \lambda^{jB}\, \varepsilon_{AB} \\
&\quad - \mathrm{i}\frac{1}{2}\, \overline{C}_{i^\star j^\star k^\star} \left(g^{k^\star \ell}\, f^J_\ell\, \mathrm{Im}\mathcal{N}_{IJ}\, \mathcal{F}^{+I}_{\mu\nu}\right) \overline{\lambda}^{i^\star}_A\, \gamma^{\mu\nu}\, \lambda^{j^\star}_B\, \varepsilon^{AB} && (10.26) \\
\mathcal{L}_{massmatrix} &= \mathcal{M}^{\alpha|\beta}\, \overline{\zeta}_\alpha\, \zeta_\beta + \mathcal{M}_{\alpha|\beta}\, \overline{\zeta}^\alpha\, \zeta^\beta \\
&\quad + \mathcal{M}^{\alpha|}{}_{|iB}\, \overline{\zeta}_\alpha\, \lambda^{iB} + \mathcal{M}_{\alpha|i^\star}{}^{|}{}^{B}\, \overline{\zeta}^\alpha\, \lambda^{i^\star}_B \\
&\quad + \mathcal{M}_{iA|\ell B}\, \overline{\lambda}^{iA} \lambda^{\ell B} + \mathcal{M}_{i^\star}{}^{A|}{}_{|\ell^\star}{}^{B}\, \overline{\lambda}^{i^\star}_A \lambda^{\ell^\star}_B && (10.27) \\
\mathcal{L}_{potential} &= -\mathrm{V}(z, \overline{z}, q) && (10.28) \\
\mathcal{L}_{4fermi} &= \frac{1}{4}\, R_{ij^\star \ell k^\star}\, \overline{\lambda}^{iA}\, \lambda^{\ell B}\, \overline{\lambda}^{j^\star}_A\, \lambda^{k^\star}_B \\
&\quad + \frac{1}{2}\, \mathbb{R}^\alpha{}_{\beta|ts}\, \mathcal{U}^t_{A\gamma}\, \mathcal{U}^s_{B\delta}\, \varepsilon^{AB}\, \mathbb{C}^{\delta\eta}\, \overline{\zeta}_\alpha\, \zeta_\eta\, \overline{\zeta}^\beta\, \zeta^\gamma \\
&\quad - \frac{1}{32}\, \mathrm{Im}\mathcal{N}_{IJ}\, C_{ijk}\, C_{\ell mn}\, g^{kr^\star} g^{ns^\star}\, \overline{f}^I_{r^\star}\, \overline{f}^J_{s^\star} \overline{\lambda}^{iA} \gamma_{\mu\nu} \lambda^{jB}\, \overline{\lambda}^{\ell C}\, \gamma^{\mu\nu} \lambda^{mD}\, \varepsilon_{AB}\varepsilon_{CD} \\
&\quad - \frac{1}{32}\, \mathrm{Im}\mathcal{N}_{IJ}\, \overline{C}_{i^\star j^\star k^\star} \overline{C}_{\ell^\star m^\star n^\star}\, g^{k^\star r} g^{n^\star s}\, f^I_r\, f^J_s\, \overline{\lambda}^{i^\star}_A \gamma_{\mu\nu} \lambda^{j^\star}_B\, \overline{\lambda}^{k^\star}_C\, \gamma^{\mu\nu} \lambda^{\ell^\star}_D\, \varepsilon^{AB}\varepsilon^{CD} && (10.29) \\
& && (10.30)
\end{aligned}
$$

where the mass–matrices and the scalar potential are given by:

N=2 Yang Mills mass matrices and scalar potential

$$
\begin{aligned}
\mathcal{M}^{\alpha|\beta} &= -\, g\, \mathcal{U}^{\alpha A}_u\, \mathcal{U}^{\beta B}_v\, \varepsilon_{AB}\, \nabla^{[u}\, k^{v]}_I\, Y^I \\
\mathcal{M}_{\alpha|\beta} &= -\, g\, \mathcal{U}_{\alpha A|u}\, \mathcal{U}_{\beta B|v}\, \varepsilon^{AB}\, \nabla^{[u}\, k^{v]}_I\, \overline{Y}^I \\
\mathcal{M}^{\alpha|}{}_{|iB} &= 4\, g\, \mathcal{U}^{\alpha A}_u\, k^u_I\, f^I_i\, \varepsilon_{AB} \\
\mathcal{M}_{\alpha|i^\star}{}^{|\ \ B} &= -\, 4\, g\, \mathcal{U}_{\alpha A|u}\, k^u_I\, \overline{f}^I_{i^\star}\, \varepsilon^{AB} \\
\mathcal{M}_{iA|\ell B} &= \frac{1}{3}\, g\, \left(\varepsilon_{AB}\, g_{ij^\star} k^{j^\star}_I\, f^I_\ell + \mathrm{i}(\sigma_x)_A{}^C \epsilon_{BC}\, \mathcal{P}^x_I\, \nabla_\ell f^I_i\right) \\
\mathcal{M}_{i^\star}{}^{A|}{}_{|\ell^\star}{}^{B} &= \frac{1}{3}\, g\, \left(\varepsilon^{AB}\, g_{i^\star j} k^j_I \overline{f}^I_{\ell^\star} - \mathrm{i}\epsilon^{AC} (\sigma_x)_C{}^B\, \mathcal{P}^x_I\, \nabla_{\ell^\star} \overline{f}^I_{i^\star}\right)
\end{aligned}
\tag{10.31}
$$

$$
\begin{aligned}
\mathrm{V}(z, \overline{z}, q) &= g^2\, \left(g_{ij^\star} k^i_I\, k^{j^\star}_J\ +\ 4\, h_{uv}\, k^u_I k^v_J\right) \overline{Y}^I\, Y^J \\
&\quad + g^{ij^\star}\, f^I_i\, \overline{f}^J_{j^\star}\, \sum_{x=1}^{3}\, \mathcal{P}^x_I\, \mathcal{P}^x_J
\end{aligned}
\tag{10.32}
$$

The coupling constant in front of the mass–matrices and of the potential is just a symbolic notation to remind the reader that these terms are entirely due to the gauging and vanish in the ungauged theory. In general there is not a single coupling constant rather there are as many independent coupling constants as mutually commuting subgroups in the gauge group. For instance if $\mathcal{G}$ is a product or r $U(1)$–factors, there are r independent coupling constants that can be reabsorbed into the definition of the killing vectors k^i_I, k^u_I.

The supersymmetry transformation rules with respect to which the lagrangian 10.24 is invariant are the following ones:

N=2 rigid transformation rules of Bose fields

$$
\begin{aligned}
\delta\, A^I_\mu &= +\mathrm{i}\,\left(f^I_i\, \overline{\lambda}^{iA}\, \gamma_\mu\, \epsilon^B\, \varepsilon_{AB} \,+\, \overline{f}^I_{i^\star} \overline{\lambda} i^\star \quad {}_A\, \gamma_\mu\, \epsilon_B\, \varepsilon^{AB} \right) \\
\delta z^i &= +\overline{\lambda}^{iA}\, \epsilon_A \\
\delta \overline{z}^{i^\star} &= +\overline{\lambda}^{i^\star}_A\, \epsilon^A \\
\mathcal{U}^{\alpha A}_u(q)\, \delta q^u &= \varepsilon^{AB}\, \mathbb{C}^{\alpha\beta}\, \overline{\epsilon}_B \zeta_\beta + \overline{\epsilon}^A \zeta^\alpha
\end{aligned}
\tag{10.33}
$$

N=2 rigid transformation rules of Fermi fields

$$
\begin{aligned}
\delta\lambda^{iA} &= \mathrm{i}\,\nabla_\mu z^i\, \gamma^\mu\, \epsilon^A \,+\, G^{-i}_{\mu\nu}\, \gamma^{\mu\nu}\, \epsilon_B\, \varepsilon^{AB} \,+\, D^{i|AB}\, \epsilon_B \\
\delta\lambda^{i^\star}_A &= \mathrm{i}\,\nabla_\mu \overline{z}^{i^\star}\, \gamma^\mu\, \epsilon_A \,+\, G^{+i^\star}_{\mu\nu}\, \gamma^{\mu\nu}\, \epsilon^B\, \varepsilon_{AB} \,+\, D^{i^\star}{}_{|AB}\, \epsilon^B \\
\delta\zeta_\alpha &= \mathrm{i}\mathcal{U}^{\beta B}_u\, \nabla_\mu q^u\, \gamma^\mu \epsilon^A\, \varepsilon_{AB}\, \mathbb{C}_{\alpha\beta} \,+\, N^A_\alpha\, \epsilon_A \\
\delta\zeta^\alpha &= \mathrm{i}\mathcal{U}_{\beta B|u}\, \nabla_\mu q^u\, \gamma^\mu \epsilon_A\, \varepsilon^{AB}\, \mathbb{C}^{\alpha\beta} \,+\, N^\alpha_A\, \epsilon^A
\end{aligned}
\tag{10.34}
$$

where:

N=2 rigid values of the auxiliary fields

$$
\begin{aligned}
G^{-i}_{\mu\nu} &= \mathrm{i}\, g^{ij^\star} \overline{f}^I_{j^\star}\, \mathrm{Im}\mathcal{N}_{IJ}\, \left(\mathcal{F}^{-J}_{\mu\nu} + \frac{1}{8} \nabla_k f^J_\ell\, \overline{\lambda}^{kA}\, \gamma_{\mu\nu}\, \lambda^{\ell B}\, \varepsilon_{AB} \right) \\
G^{+i^\star}_{\mu\nu} &= \mathrm{i}\, g^{i^\star j} f^I_j\, \mathrm{Im}\mathcal{N}_{IJ}\, \left(\mathcal{F}^{+J}_{\mu\nu} + \frac{1}{8} \nabla_{k^\star} \overline{f}^J_{\ell^\star}\, \overline{\lambda}^{k^\star}_A\, \gamma_{\mu\nu}\, \lambda^{\ell^\star}_B\, \varepsilon^{AB} \right) \\
D^{i|AB} &= Y^{i|AB} + W^{i|[AB]} + W^{i|(AB)} \\
D^{i^\star}{}_{|AB} &= Y^{i^\star}{}_{|AB} + W^{i^\star}{}_{|[AB]} + W^{i^\star}{}_{|(AB)} \\
Y^{i|AB} &= \mathrm{i}\frac{1}{2}\, g^{ij^\star}\, \overline{C}_{j^\star k^\star \ell^\star}\, \overline{\lambda}^{k^\star}_C\, \lambda^{\ell^\star}_D \varepsilon^{AC}\, \varepsilon^{BD} \\
Y^{i^\star}{}_{|AB} &= -\mathrm{i}\frac{1}{2}\, g^{i^\star j}\, C_{jk\ell}\, \overline{\lambda}^{kC}\, \lambda^{\ell D} \varepsilon_{AC}\, \varepsilon_{BD} \\
W^{i|[AB]} &= \varepsilon^{AB}\, k^i_I \overline{Y}^I \\
W^{i^\star}{}_{|[AB]} &= \varepsilon_{AB}\, k^{i^\star}_I\, Y^I \\
W^{i|(AB)} &= -\mathrm{i}\, \epsilon^{AC} (\sigma_x)_C{}^B\, \mathcal{P}^x_I g^{ij^\star} \overline{f}^I_{j^\star} \\
W^{i^\star}{}_{|(AB)} &= \mathrm{i}\, (\sigma_x)_A{}^C \epsilon_{BC}\, \mathcal{P}^x_I g^{i^\star j} f^I_j \\
N^A_\alpha &= 2\,\mathcal{U}^A_{\alpha|u}\, k^u_I\, \overline{Y}^I \\
N^\alpha_A &= -2\,\mathcal{U}^\alpha_{A|u}\, k^u_I\, Y^I
\end{aligned}
\tag{10.35}
$$

10.3 The renormalizable microscopic theory

As an exemplification of the general formalism and for the sake of its intrinsic interest, in this subsection we consider the case of the renormalizable microscopic N=2 (matter coupled) Yang–Mills theory. The theory is specified by the choice of the following geometrical data:

1. A flat rigid special manifold $\mathcal{SM}_{flat}$ describing the vector multiplet couplings
2. A flat Hyperkähler manifold $\mathcal{HM}_{flat}$ describing the hypermultiplet couplings

Let us briefly discuss these geometries and the corresponding form of the Lagrangian.

Flat rigid special geometry

In the vector multiplet sector the appropriate geometry is described as follows. Let θ be the theta–angle, $1/g^2$ the inverse of the squared gauge coupling constant, and $\mathbf{g}_{IJ}$ the constant Killing metric on the gauge Lie algebra. Define the complex parameter:

$$\tau = \theta + \mathrm{i}\,\frac{1}{g^2} \tag{10.36}$$

and choose as holomorphic section of the flat symplectic bundle the following one:

$$\widehat{\Omega} = \begin{pmatrix} Y^I \\ \tau\,\mathbf{g}_{IJ}\,Y^J \end{pmatrix} \qquad\qquad I, J = 1, \ldots, n = \mathrm{dim}\mathcal{G} \tag{10.37}$$

In this case the upper half of the holomorphic section 10.37 can be taken as coordinates on the manifold (the special coordinates):

$$z^i \equiv Y^I \,. \tag{10.38}$$

The action of the gauge group on these coordinates is obviously the adjoint action:

$$\delta_I\, Y^J = f^J{}_{IK}\, Y^K \tag{10.39}$$

where $f^J{}_{IK}$ are the structure constants of the gauge Lie algebra:

$$[t_I\,,\, t_J] = f^K{}_{IJ}\, t_K \tag{10.40}$$

t_I being a basis of generators. Hence using eq.s 4.58 and 4.57 we obtain

$$\begin{array}{rclcrcl} \mathcal{N}_{IJ} & = & \overline{\tau}\,\mathbf{g}_{IJ} & & g_{ij^\star} & = & 2\,\mathrm{Im}\tau\,\mathbf{g}_{IJ} \\ \mathrm{Im}\mathcal{N}_{IJ} & = & -\,\mathrm{Im}\tau\,\mathbf{g}_{IJ} & & f^I_i & = & \delta^I_i \\ C_{ijk} & = & 0 & & k^j_I & = & f^J{}_{IK}\,Y^K \end{array} \tag{10.41}$$

Flat HyperKähler geometry

In the hypermultiplet sector we arrange the $4m$ coordinates q^u of $\mathcal{HM}_{flat} = \mathbb{R}^{4m}$ into a $4m$ column vector:

$$\mathbf{q} \equiv q^{a|t} \qquad\qquad \begin{cases} a = 0, 1, 2, 3 \\ t = 1, 2, \ldots\, m \end{cases} \tag{10.42}$$

that is regarded as an element of the tensor product $\mathbb{R}^4 \otimes \mathbb{R}^m \sim \mathbb{R}^{4m}$. Let

$$
\begin{array}{rclcrcl}
J^{+|1} & = & \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} & & J^{-|1} & = & \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
J^{+|2} & = & \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} & & J^{-|2} & = & \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
J^{+|3} & = & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} & & J^{-|3} & = & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\end{array}
\tag{10.43}
$$

be the two triplets of self–dual and antiself dual 't Hooft matrices satisfying the quaternionic algebra:

$$
\begin{aligned}
J^{\pm|x}\, J^{\pm|y} &= -\,\delta^{xy}\, \mathbb{1}_{4\times 4} + \varepsilon^{xyz}\, J^{\pm|z} \\
J^{\pm|x}_{ab} &= \pm \frac{1}{2}\, \varepsilon_{abcd}\, J^{\pm|x}_{cd} \\
0 &= \left[J^{+|x} \,,\, J^{-|y} \right] \qquad \forall x, y
\end{aligned}
\tag{10.44}
$$

Let, furthermore

$$
e_a = \begin{cases} e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ e_x = \begin{cases} e_1 = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \\ e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ e_3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 1 & \mathrm{i} \end{pmatrix} \end{cases} \end{cases}
\tag{10.45}
$$

be a complete basis of two matrices for the expansion of a generic quaternion:

$$
Q \equiv q^a\, e_a
\tag{10.46}
$$

e_x, being the three imaginary units. The flat HyperKähler metric and the corresponding triplet of HyperKähler 2–forms are given by:

$$
\begin{aligned}
ds^2 &\equiv h_{uv}\, dq^u\, dq^v = d\mathbf{q}^T \left(\mathbb{1}_{4\times 4} \otimes \mathbb{1}_{m\times m} \right) d\mathbf{q} \\
K^x &= d\mathbf{q}^T \wedge \left(J^{+|x} \otimes \mathbb{1}_{m\times m} \right) d\mathbf{q}
\end{aligned}
\tag{10.47}
$$

Alternatively in the above formula one can use the triplet of antiself dual t'Hooft matrices to define the HyperKähler structure. Using the identities:

$$
\begin{cases} J^{+|x}_{ab} = \frac{1}{2}\, \mathrm{tr}(e_a \overline{e}_b e_x^T) \\ J^{-|x}_{ab} = -\,\frac{1}{2}\, \mathrm{tr}(e_a e_x^T \overline{e}_b) \end{cases}
\tag{10.48}
$$

and rearranging the $4m$ coordinates $q^{a|t}$ into an m-vector of quaternions:

$$\mathbf{Q} = \begin{pmatrix} Q^1 & = & q^{a|1}\, e_a \\ Q^2 & = & q^{a|2}\, e_a \\ & \dots & \\ Q^t & = & q^{a|t}\, e_a \\ & \dots & \end{pmatrix} \tag{10.49}$$

eq.s 10.47 can be rewritten as follows:

$$\begin{aligned} ds^2 &= \frac{1}{2}\, \mathrm{tr}\, \left(d\mathbf{Q}^\dagger\, \mathbb{1}_{m\times m} d\mathbf{Q} \right) \\ K &= \frac{1}{2}\, d\mathbf{Q}^T \wedge \mathbb{1}_{m\times m} d\overline{\mathbf{Q}} = \frac{1}{2}\, K^x\, e_x^T \end{aligned} \tag{10.50}$$

The action of the gauge group $\mathcal{G}$ on the hypermultiplets is assumed to be linear and be generated by a set of $4m \times 4m$ matrices T_I. Namely we set:

$$\delta_I \mathbf{q} = T_I\, \mathbf{q} \quad \longrightarrow \quad k_I^u = \left(T_I\right)^u{}_v\, q^v \tag{10.51}$$

In order for this action to be an isometry of the Euclidean diagonal metric 10.47 it is necessary and sufficient that the matrices T_I belong to the orthogonal Lie algebra $SO(4m)$, namely:

$$T_I^T = -T_I \tag{10.52}$$

The action of $\mathcal{G}$ however is not only required to be isometrical but also to be triholomorphic. This means:

$$\ell_I\, K^x \equiv \mathbf{i}_I\, dK^x + d\, \mathbf{i}_I\, K^x = d\, \mathbf{i}_I\, K^x = 0 \tag{10.53}$$

A straightforward calculation yields:

$$d\, \mathbf{i}_I\, K^x = -\, d\mathbf{q}^T \wedge \left[T_I\, ,\, J^{+|x} \otimes \mathbb{1}_{m\times m} \right] d\mathbf{q} \tag{10.54}$$

so that the triholomorphicity condition is that the generators T_I should commute with the tensor product of the 't Hooft matrices with the unit matrix in m–dimensions. When this last condition is verified we can write the momentum maps as:

$$\mathcal{P}_I^x = \mathbf{q}^T\, J^{+|x} \otimes \mathbb{1}_{m\times m}\, T_I\, \mathbf{q} \tag{10.55}$$

Alternatively using the quaternionic notation we have:

$$\mathbf{P}_I = \frac{1}{2}\, \mathcal{P}_I^x\, e_x^T = \frac{1}{2}\, \mathbf{Q}^T \mathbb{1}_{m\times m}\, T_I\, \overline{\mathbf{Q}} \tag{10.56}$$

The lagrangian

Using these ingredients the lagrangian of the microscopic renormalizable theory is immediately retrieved from the general formulae of the previous subsection. It is convenient to set:

$$\begin{array}{rclcrcl} \mathbf{Y} & \equiv & Y^I\, t_I & \quad & \overline{\mathbf{Y}} & \equiv & \overline{Y}^I\, t_I \\ \mathbf{F}_{\mu\nu} & \equiv & F^I_{\mu\nu}\, t_I & \quad & \mathrm{tr}\, (t_I t_J) & \equiv & \mathbf{g}_{IJ} \end{array} \tag{10.57}$$

t_I denoting a basis of generators of the gauge group and in this condensed notation we obtain:

$$\mathcal{L}^{microscopic}_{N=2YM} = \mathcal{L}^{microscopic}_{bosonic} + \mathcal{L}^{microscopic}_{fermionic} \tag{10.58}$$

where the bosonic lagrangian is:

$$\begin{aligned} \mathcal{L}^{microscopic}_{bosonic} &= -\,\mathrm{Im}\tau \,\mathrm{tr}\,(\mathbf{F}_{\mu\nu}\,\mathbf{F}_{\mu\nu}) + \frac{1}{2}\,\mathrm{Re}\tau\,\mathrm{tr}\,(\mathbf{F}_{\mu\nu}\,\mathbf{F}_{\rho\sigma})\,\varepsilon^{\mu\nu\rho\sigma} \\ &\quad +2\,\mathrm{Im}\tau\,\mathrm{tr}\,\left(\nabla_\mu \mathbf{Y}\,\nabla_\mu \overline{\mathbf{Y}}\right) + \nabla_\mu \mathbf{q}^T\,\nabla_\mu \mathbf{q} - V(\mathbf{Y},\mathbf{q}) \end{aligned} \tag{10.59}$$

$$\begin{aligned} V(\mathbf{Y},\mathbf{q}) &= 2\,\mathrm{Im}\tau\,\mathrm{tr}\,\left(\left[\mathbf{Y}\,,\,\overline{\mathbf{Y}}\right]\right)^2 - 2\,\mathbf{q}^T\left\{\mathbf{Y}\,,\,\overline{\mathbf{Y}}\right\}\mathbf{q} \\ &\quad + \frac{1}{2\mathrm{Im}\tau}\sum_{x=1}^{3}\mathcal{P}^x_I\,\mathcal{P}^x_J\,\mathbf{g}^{IJ} \end{aligned} \tag{10.60}$$

The formula for the scalar potential exhibits in a clear fashion the flat directions associated with the moduli fields $\mathbf{Y}$ in the Cartan subalgebra $\mathcal{H}$ of the gauge algebra. Actually the potential is just homogeneous of degree four in all the scalar fields as expected from renormalizability.

The fermionic part of the lagrangian also simplifies very much since it just contains the kinetic part and the mass terms induced by the gauging. The Pauli terms and the 4–fermi terms are all zero, since the tensor C_{ijk} vanishes and the Riemann tensors of the special and HyperKähler manifolds also vanish. The evaluation of the mass matrices is straightforward by inserting the explicit form of the Killing vectors and of the momentum maps into eq.s 10.31. The only item that is still missing in such a calculation is the explicit form of the quaternionic vielbein. This is very easily given. We set:

$$\mathcal{U}^{A\alpha} \equiv \mathcal{U}^{A\alpha}_{b|s}\,dq^{b|s} = d\mathbf{Q} = dq^{a|t}\,(e_a)^A{}_B \tag{10.61}$$

and we identify the symplectic index α running on $2m$ values with the pair of indices B, t ($B = 1,2$; $t = 1,\dots,m$). In this way we obtain:

$$\mathcal{U}^{A\ \ |t}_{\ B|b|s} = \delta^t_s\,(e_b)^A{}_B \tag{10.62}$$

Appendix A: The solution of the Bianchi identities and the supersymmetry transformation laws

In this Appendix we describe the geometric approach for the derivation of the $N = 2$ supersymmetry transformation laws of the physical fields. As it will appear in the following this requires the preliminary solution of Bianchi identities in superspace.

The first step to perform is to extend the physical fields to superfields in $N = 2$ superspace: that means that the space–time 1-forms $\omega^{a\,b}$, V^a,ψ^A, ψ_A, A^Λ and the space–time 0–forms λ^{iA}, $\lambda^{i^\star}_A$, z^i, $z^{i^\star}$, ζ^α, ζ_α, q^u defined in section 8 are promoted to 1–superforms and 0–superforms in $N = 2$ superspace, respectively.

The definition of the superspace curvatures actually coincides with that given in eq.s 8.4–8.14 provided all the p–forms ($p = 0, 1, 2$) are thought as p–superforms (here and in the following by "curvatures" we mean not only 2–forms, but also the 1–forms

defined as covariant differentials of the 0–form superfields).
We note that the definition of superspace curvatures in the gravitational sector, namely:

$$
\begin{aligned}
T^a &\equiv \mathcal{D}V^a - \mathrm{i}\,\overline{\psi}_A \wedge \gamma^a \psi^A && \text{(A.1)}\\
\rho_A &\equiv d\psi_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi_A + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}} \wedge \psi_A + \widehat{\omega}_A{}^B \wedge \psi_B \equiv \nabla\psi_A && \text{(A.2)}\\
\rho^A &\equiv d\psi^A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi^A - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}} \wedge \psi^A + \widehat{\omega}^A{}_B \wedge \psi^B \equiv \nabla\psi^A && \text{(A.3)}\\
R^{ab} &\equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} && \text{(A.4)}\\
F^0 &\equiv dA^0 + \overline{L}^0\overline{\psi}_A \wedge \psi_B\epsilon^{AB} + L^0\overline{\psi}^A \wedge \psi^B\epsilon_{AB} && \text{(A.5)}
\end{aligned}
$$

where F^0 denotes the graviphoton, has been chosen in such a way that by setting $R^{ab} = T^a = \rho^A = \rho_A = F^0 = 0$, deleting the composite connections $\widehat{\mathcal{Q}}$, $\widehat{\omega}^A_B$ and normalising $L^0\,(0,0) = 1$ we obtain the Maurer–Cartan equations of the $N = 2$ Poincaré superalgebra where the one forms $\omega^{ab}, V^a, \psi^A, \psi_A, A^0$ are dual to the corresponding generators of the group.

The next step is to write down the Bianchi identities for all the curvatures and to solve them in superspace. Applying the d operator to eq.s A.1– A.4 and 8.9– 8.14 one finds:

$$
\begin{aligned}
\mathcal{D}T^a &+ R^{ab} \wedge V^b - \mathrm{i}\overline{\psi}^A \wedge \gamma^a \rho_A + \mathrm{i}\overline{\rho}^A \wedge \gamma^a \psi_A = 0 && \text{(A.6)}\\
\nabla\rho_A &+ \frac{1}{4}\gamma_{ab}\,R^{ab} \wedge \psi_A - \frac{\mathrm{i}}{2}\widehat{K} \wedge \psi_A - \frac{\mathrm{i}}{2}\widehat{R}_A{}^B \wedge \psi_B = 0 && \text{(A.7)}\\
\nabla\rho^A &+ \frac{1}{4}\gamma_{ab}\,R^{ab} \wedge \psi^A + \frac{\mathrm{i}}{2}\widehat{K} \wedge \psi^A - \widehat{R}^A{}_B \wedge \psi^B = 0 && \text{(A.8)}\\
\mathcal{D}R^{ab} &= 0 && \text{(A.9)}\\
\nabla^2 z^i &- g\left(F^\Lambda - \overline{L}^\Lambda\,\overline{\psi}_A \wedge \psi_B\epsilon^{AB} - \overline{L}^\Lambda\,\overline{\psi}^A \wedge \psi^B\epsilon_{AB}\right)\,k^i_\Lambda(z) = 0 && \text{(A.10)}\\
\nabla^2 z^{i^\star} &- g\left(F^\Lambda - \overline{L}^\Lambda\,\overline{\psi}_A \wedge \psi_B\epsilon^{AB} - \overline{L}^\Lambda\,\overline{\psi}^A \wedge \psi^B\epsilon_{AB}\right)\,k^{i^\star}_\Lambda = 0 && \text{(A.11)}\\
\nabla^2\lambda^{iA} &+ \frac{1}{4}\gamma_{ab}\,R^{ab}\lambda^{iA} + \frac{\mathrm{i}}{2}\widehat{K}\lambda^{iA} + \widehat{R}^i{}_j\lambda^{jA} - \frac{\mathrm{i}}{2}\widehat{R}^A{}_B \wedge \lambda^{iB} = 0 && \text{(A.12)}\\
\nabla^2\lambda^{i^\star}_A &+ \frac{1}{4}\gamma_{ab}\,R^{ab}\lambda^{i^\star}_A - \frac{\mathrm{i}}{2}\widehat{K}\lambda^{i^\star}_A + \widehat{R}^{i^\star}{}_{j^\star}\lambda^{j^\star}_A - \frac{\mathrm{i}}{2}\widehat{R}^B{}_A \wedge \lambda^{i^\star}_B = 0 && \text{(A.13)}\\
\nabla F^\Lambda &- \nabla\overline{L}^\Lambda \wedge \overline{\psi}_A \wedge \psi_B\epsilon^{AB} - \nabla L^\Lambda \wedge \overline{\psi}^A \wedge \psi^B\epsilon_{AB}\\
&+ 2\overline{L}^\Lambda\overline{\psi}_A \wedge \rho_B\epsilon^{AB} + 2L^\Lambda\overline{\psi}^A \wedge \psi^B\epsilon_{AB} = 0 && \text{(A.14)}\\
\nabla\mathcal{U}^{A\alpha} &- g\left(F^\Lambda - \overline{L}^\Lambda\,\overline{\psi}_A \wedge \psi_B\epsilon^{AB} - \overline{L}^\Lambda\,\overline{\psi}^A \wedge \psi^B\epsilon_{AB}\right)\,k^u_\Lambda(z)\mathcal{U}^{A\alpha}_u = 0 && \text{(A.15)}\\
\nabla^2\zeta_\alpha &+ \frac{1}{4}R^{ab}\,\gamma_{ab}\,\zeta_\alpha + \frac{\mathrm{i}}{2}\widehat{K}\,\zeta_\alpha + \widehat{R}_\alpha{}^\beta\zeta_\beta = 0 && \text{(A.16)}\\
\nabla^2\zeta^\alpha &+ \frac{1}{4}R^{ab}\,\gamma_{ab}\,\zeta^\alpha - \frac{\mathrm{i}}{2}\widehat{K}\,\zeta^\alpha + \widehat{R}^\alpha{}_\beta\zeta^\beta = 0 && \text{(A.17)}\\
& && \text{(A.18)}
\end{aligned}
$$

The covariant derivatives ∇ and $\mathcal{D}$ have been defined in eq.s A.1– A.5 and include the gauged connections defined in eq. 7.57. Furthermore the hat on the scalar manifolds curvatures $\widehat{K}, \widehat{R}^i{}_j\,, \widehat{R}^\alpha{}_\beta\,, \widehat{R}^A{}_B$ denotes the gauged curvatures defined in 7.58.

The solution can be obtained as follows: first of all one requires that the expansion of the curvatures along the intrinsic p–forms basis in superspace namely: $V^a, V^a \wedge V^b, \psi, \psi \wedge V^b, \psi \wedge \psi$, is given in terms only of the physical fields (rheonomy). This insures that no new degree of freedom is introduced in the theory.
Secondly one writes down such expansion in a form which is compatible with all the symmetries of the theory, that is: covariance under $U(1)$ Kähler and $SU(2) \otimes Sp(2,m)$, Lorentz transformations and reparametrization of the scalar manifolds. Besides it is very useful to take into account the invariance under the following rigid rescalings of the fields (and their corresponding curvatures):

$$(\omega^{ab}, A^\Lambda, q^u, z^i, z^{i^\star}) \rightarrow (\omega^{ab}, A^\Lambda, q^u, z^i, z^{i^\star}) \tag{A.19}$$

$$V^a \rightarrow \ell V^a \tag{A.20}$$

$$(\psi^A, \psi_A) \rightarrow \ell^{\frac{1}{2}} (\psi^A, \psi_A) \tag{A.21}$$

$$(\lambda^{iA}, \lambda^{i^\star}_A, \zeta^\alpha, \zeta_\alpha) \rightarrow \ell^{-\frac{1}{2}} (\lambda^{iA}, \lambda^{i^\star}_A, \zeta^\alpha, \zeta_\alpha) \tag{A.22}$$

Indeed these rescalings and the corresponding ones for the curvatures leave invariant the definitions of the curvatures and the Bianchi identities.
Finally we note that we are looking for a solution of the coupled system of Bianchi identities of the gravitational sector with those of the matter sectors. The coupling is obtained by setting the auxiliary fields of the $N = 2$ multiplets to definite expressions in the physical fields compatible with all the previously mentioned requirements. This fixes completely the ansatz for the curvatures at least if we exclude higher derivative interactions.

Performing all the steps requires a lot of work. For a more detailed explanation the interested reader is referred to the standard reference of the geometrical approach [31]. The final parametrizations of the superspace curvatures, are given by:

$$T^a = 0 \tag{A.23}$$

$$\begin{aligned} \rho_A &= \widetilde{\rho}_{A|ab} V^a \wedge V^b + \left(A_A{}^{B|b} \eta_{ab} + A'_A{}^{B|b} \gamma_{ab} \right) \psi_B \wedge V^a \\ &\quad + \left[\mathrm{i}\, g\, S_{AB} \eta_{ab} + \epsilon_{AB} (T^-_{ab} + U^+_{ab}) \right] \gamma^b \psi^B \wedge V^a \end{aligned} \tag{A.24}$$

$$\begin{aligned} \rho^A &= \widetilde{\rho}^A_{|ab} V^a \wedge V^b + \left(\overline{A}^{A|}{}_B{}^b\, \eta_{ab} + \overline{A}'^{A|}{}_B{}^b\, \gamma_{ab} \right) \psi^B \wedge V^a \\ &\quad + \left[\mathrm{i}\, g\, \overline{S}^{AB} \eta_{ab} + \epsilon^{AB} \left(T^+_{ab} + U^-_{ab} \right) \right] \gamma^b \psi_B \wedge V^a \end{aligned} \tag{A.25}$$

$$\begin{aligned} R^{ab} &= \widetilde{R}^{ab}{}_{cd} V^c \wedge V^d - \mathrm{i}\, (\overline{\psi}_A \theta^{A|ab}_c + \overline{\psi}^A \theta^{ab}_{A|c}) \wedge V^c \\ &\quad + \epsilon^{abcf}\, \overline{\psi}^A \wedge \gamma_f \psi_B (A'^B{}_{A|c} - \overline{A}'{}_{A|c}{}^B) \\ &\quad + \mathrm{i}\, \epsilon^{AB}\, \overline{\psi}_A \wedge \psi_B (T^{+ab} + U^{-ab}) - \mathrm{i}\, \epsilon_{AB} \overline{\psi}^A \wedge \psi^B (T^{-ab} + U^{+ab}) \\ &\quad - g S_{AB}\, \overline{\psi}^A \wedge \gamma^{ab} \psi^B - g \overline{S}^{AB}\, \overline{\psi}_A \wedge \gamma^{ab} \psi_B \end{aligned} \tag{A.26}$$

$$F^\Lambda = \widetilde{F}^\Lambda_{ab} V^a \wedge V^b + \left(\mathrm{i}\, f^\Lambda_i\, \overline{\lambda}^{iA} \gamma_a \psi^B\, \epsilon_{AB} + \mathrm{i}\, \overline{f}^\Lambda_{i^\star}\, \overline{\lambda}^{i^\star}_A \gamma_a \psi_B\, \epsilon^{AB} \right) \wedge V^a \tag{A.27}$$

$$\nabla \lambda^{iA} = \widetilde{\nabla_a \lambda}^{iA} V^a + \mathrm{i}\, \widetilde{Z}^i_a \gamma^a \psi^A + G^{-i}_{ab} \gamma^{ab} \psi_B \epsilon^{AB} + \left(Y^{iAB} + g\, W^{iAB} \right) \psi_B \tag{A.28}$$

$$\nabla \lambda^{i^\star}_A = \widetilde{\nabla_a \lambda}^{i^\star}_A V^a + \mathrm{i}\, \overline{\widetilde{Z}}{}^{i^\star}_a \gamma^a \psi_A + G^{+i^\star}_{ab} \gamma^{ab} \psi^B \epsilon_{AB} + \left(Y^{i^\star}{}_{AB} + g\, W^{i^\star}{}_{AB} \right) \psi^B \tag{A.29}$$

$$
\begin{aligned}
\nabla z^i &= \widetilde{Z}^i_a V^a + \overline{\lambda}^{iA}\psi_A && \text{(A.30)}\\
\nabla \overline{z}^{i^\star} &= \widetilde{Z}^{i^\star}_a V^a + \overline{\lambda}^{i^\star}_A \psi^A && \text{(A.31)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{U}^{A\alpha} &= \widetilde{\mathcal{U}}^{A\alpha}_a \, V^a \, + \, \epsilon^{AB}\,\mathbb{C}^{\alpha\beta}\,\overline{\psi}_B\zeta_\beta \, + \, \overline{\psi}^A\zeta^\alpha && \text{(A.32)}\\
\nabla\zeta_\alpha &= \widetilde{\nabla_a\zeta_\alpha} V^a \, + \, \mathrm{i}\,\widetilde{\mathcal{U}}^{B\beta}_a\,\gamma^a\psi^A\epsilon_{AB}\,\mathbb{C}_{\alpha\beta} \, + \, g\,N^A_\alpha\,\psi_A && \text{(A.33)}\\
\nabla\zeta^\alpha &= \widetilde{\nabla_a\zeta^\alpha} V^a \, + \, \mathrm{i}\,\widetilde{\mathcal{U}}^{A\alpha}_a\,\gamma^a\psi_A \, + \, g\,N^\alpha_A\,\psi^A && \text{(A.34)}
\end{aligned}
$$

where:

$$
\begin{aligned}
A_A{}^{|aB} &= -\frac{\mathrm{i}}{4}\,g_{k^\star\ell}\left(\overline{\lambda}^{k^\star}_A\gamma^a\lambda^{\ell B} \, - \, \delta^B_A\,\overline{\lambda}^{k^\star}_C\,\gamma^a\lambda^{\ell C}\right) && \text{(A.35)}\\
A'_A{}^{|aB} &= \frac{\mathrm{i}}{4}\,g_{k^\star\ell}\left(\overline{\lambda}^{k^\star}_A\gamma^a\lambda^{\ell B} - \frac{1}{2}\,\delta^B_A\,\overline{\lambda}^{k^\star}_C\,\gamma^a\lambda^{C\ell}\right) \, + \, \frac{\mathrm{i}}{4}\,\lambda\,\delta^B_A\,\overline{\zeta}_\alpha\gamma^a\zeta^\alpha && \text{(A.36)}
\end{aligned}
$$

$$
\theta^{ab|c}_A = 2\gamma^{[a}\rho^{b]c}_A + \gamma^c\rho^{ab}_A; \quad \theta^{ab\ A}_c = 2\gamma^{[a}\rho^{b]c|A} + \gamma^c\rho^{ab|A} \qquad \text{(A.37)}
$$

$$
\begin{aligned}
T^-_{ab} &= \left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma} L^\Sigma\left(\widetilde{F}^{\Lambda-}_{ab} + \tfrac{1}{8}\nabla_i\,f^\Lambda_j\,\overline{\lambda}^{iA}\gamma_{ab}\,\lambda^{jB}\,\epsilon_{AB} + \tfrac{1}{4}\,\lambda\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\alpha\gamma_{ab}\,\zeta_\beta\,L^\Lambda\right) && \text{(A.38)}\\
T^+_{ab} &= \left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}\overline{L}^\Sigma\left(\widetilde{F}^{\Lambda+}_{ab} + \tfrac{1}{8}\nabla_{i^\star}\,\overline{f}^\Lambda_{j^\star}\,\overline{\lambda}^{i^\star}_A\gamma_{ab}\,\lambda^{j^\star}_B\epsilon^{AB} + \tfrac{1}{4}\,\lambda\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^\alpha\gamma_{ab}\,\zeta^\beta\,\overline{L}^\Lambda\right) && \text{(A.39)}
\end{aligned}
$$

$$
\begin{aligned}
U^-_{ab} &= \frac{\mathrm{i}}{4}\,\lambda\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\alpha\gamma_{ab}\,\zeta_\beta && \text{(A.40)}\\
U^+_{ab} &= \frac{\mathrm{i}}{4}\,\lambda\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^\alpha\gamma_{ab}\,\zeta^\beta && \text{(A.41)}
\end{aligned}
$$

$$
\begin{aligned}
G^{i-}_{ab} &= \frac{\mathrm{i}}{2}\,g^{ij^\star}\overline{f}^\Gamma_{j^\star}\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Gamma\Lambda}\left(\widetilde{F}^{\Lambda-}_{ab} + \frac{1}{8}\nabla_k f^\Lambda_\ell\overline{\lambda}^{kA}\gamma_{ab}\,\lambda^{\ell B}\epsilon_{AB}\right.\\
&\qquad\qquad \left. + \frac{1}{4}\,\lambda\,\mathbb{C}^{\alpha\beta}\,\overline{\zeta}_\alpha\gamma_{ab}\,\zeta_\beta\,L^\Lambda\right) && \text{(A.42)}\\
G^{i^\star+}_{ab} &= \frac{\mathrm{i}}{2}\,g^{i^\star j}f^\Gamma_j\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Gamma\Lambda}\left(\widetilde{F}^{\Lambda+}_{ab} + \frac{1}{8}\nabla_{k^\star}\overline{f}^\Lambda_{\ell^\star}\overline{\lambda}^{k^\star}_A\gamma_{ab}\,\lambda^{\ell^\star}_B\epsilon^{AB}\right.\\
&\qquad\qquad \left. + \frac{1}{4}\,\lambda\,\mathbb{C}_{\alpha\beta}\,\overline{\zeta}^\alpha\gamma_{ab}\,\zeta^\beta\,\overline{L}^\Lambda\right) && \text{(A.43)}\\
& && \text{(A.44)}
\end{aligned}
$$

$$
\begin{aligned}
Y^{iAB} &= \frac{\mathrm{i}}{2}g^{ij^\star}C_{j^\star k^\star\ell^\star}\overline{\lambda}^{k^\star}_C\,\lambda^{\ell^\star}_D\epsilon^{AC}\epsilon^{BD} && \text{(A.45)}\\
Y^{i^\star}_{AB} &= -\frac{\mathrm{i}}{2}g^{i^\star j}C_{jk\ell}\overline{\lambda}^{kC}\,\lambda^{\ell D}\epsilon_{AC}\epsilon_{BD} && \text{(A.46)}
\end{aligned}
$$

$$
\begin{aligned}
S_{AB} &= \frac{\mathrm{i}}{2}(\sigma_x)_A{}^C\epsilon_{BC}\mathcal{P}^x_\Lambda L^\Lambda\\
\overline{S}^{AB} &= \frac{\mathrm{i}}{2}(\sigma_x)_C{}^B\epsilon^{CA}\mathcal{P}^x_\Lambda\overline{L}^\Lambda && \text{(A.47)}
\end{aligned}
$$

$$
\begin{aligned}
N^A_\alpha &= 2\,\mathcal{U}^A_{\alpha|u}\,k^u_\Lambda\,\overline{L}^\Lambda \\
N^\alpha_A &= -\,2\,\mathcal{U}^\alpha_{A|u}\,k^u_\Lambda\,L^\Lambda
\end{aligned} \tag{A.48}
$$

$$
\begin{aligned}
W^{iAB} &= W^{i[AB]} + W^{i(AB)} \\
W^{i^\star}_{AB} &= W^{i^\star}_{[AB]} + W^{i^\star}_{(AB)} \\
\text{where :} &
\end{aligned} \tag{A.49}
$$

$$
\begin{aligned}
W^{i[AB]} &= \epsilon^{AB}\,k^i_\Lambda \overline{L}^\Lambda \\
W^{i^\star}_{[AB]} &= \epsilon_{AB}\,k^{i^\star}_\Lambda\,L^\Lambda \\
W^{i(AB)} &= \mathrm{i}(\sigma_x)_C{}^B\epsilon^{CA}\mathcal{P}^x_\Lambda g^{ij^\star}\overline{f}^\Lambda_{j^\star} \\
W^{i^\star}_{(AB)} &= \mathrm{i}(\sigma_x)_A{}^C\epsilon_{BC}\mathcal{P}^x_\Lambda g^{i^\star j}\,f^\Lambda_j
\end{aligned} \tag{A.50}
$$

As promised the solution for the curvatures is given as an expansion along the 2–form basis $(V\wedge V\,,\,V\wedge\psi\,,\,\psi\wedge\psi)$ or the 1–form basis $(V\,,\,\psi)$ with coefficients given in terms of the physical fields.
The "on–shell" auxiliary fields are given in our case by the composite connections $\widehat{\mathcal{Q}}\,,\,\widehat{\omega^A_B}$ and by $T^\mp_{ab}\,,\,W^{iAB}$ and S_{AB}.

It is important to stress that the field strengths $\widetilde{R}^{ab}{}_{cd}$, $\widetilde{\rho}_{A|ab}$, $\widetilde{F}^\Lambda_{ab}\,,\,\widetilde{\mathcal{U}}^{A\alpha}_a \equiv \mathcal{U}^{A\alpha}_u\widetilde{\nabla_a q}^u$, $\widetilde{\nabla_a\lambda}^{iA}$, $\widetilde{\nabla_a\zeta_\alpha}$ and their hermitian conjugates are not space–time field strengths since they are components along the bosonic vielbeins $V^a = V^a_\mu dx^\mu + V^a_\alpha d\theta^\alpha$ where $(V^a_\mu\,,\,V^a_\alpha)$ is a submatrix of the super–vielbein matrix $E^I\equiv(V^a\,,\,\psi)$. The physical field strengths are given by the expansion of the forms along the dx^μ-differentials and by restricting the superfields to space–time ($\theta=0$ component). For example, from the parametrization (27), expanding along the dx^μ–basis one finds:

$$
F^\Lambda_{\mu\nu} = \widetilde{F}^\Lambda_{ab}V^a_{[\mu}V^b_{\nu]} + \mathrm{i}\,f^\Lambda_i\,\overline{\lambda}^{iA}\gamma_a\psi^B_{[\mu}V^a_{\nu]}\,\epsilon_{AB} + \mathrm{i}\,\overline{f}^\Lambda_{i^\star}\,\overline{\lambda}^{i^\star}_A\gamma_a\psi_{B[\mu}V^a_{\nu]}\,\epsilon^{AB} \tag{A.51}
$$

where:

$$
F^\Lambda = \mathcal{F}^\Lambda \,+\, L^\Lambda\overline{\psi}^A\wedge\psi^B\,\epsilon_{AB} \,+\, \overline{L}^\Lambda\overline{\psi}_A\wedge\psi_B\,\epsilon^{AB} \tag{A.52}
$$

according to equations 8.11, A.27. When all the superfields are restricted to space–time we may treat the V^a_μ vielbein as the usual 4–dimensional invertible matrix converting intrinsic indices in coordinate indices and we obtain:

$$
\begin{aligned}
\widetilde{F}^\Lambda_{\mu\nu} \;&=\; \mathcal{F}^\Lambda_{\mu\nu} \,+\, L^\Lambda\overline{\psi}^A_\mu\psi^B_\nu\,\epsilon_{AB} \,+\overline{L}^\Lambda\overline{\psi}_{A\mu}\psi_{B\nu}\epsilon^{AB} \,-\mathrm{i}\,f^\Lambda_i\,\overline{\lambda}^{iA}\gamma_{[\nu}\psi^B_{\mu]}\,\epsilon_{AB} \\
&\quad -\mathrm{i}\,\overline{f}^\Lambda_{i^\star}\,\overline{\lambda}^{i^\star}_A\gamma_{[\nu}\psi_{B\mu]}\,\epsilon^{AB}
\end{aligned} \tag{A.53}
$$

By the same token we also get:

$$
\begin{aligned}
\widetilde{\nabla_\mu\lambda}^{iA} \;&=\; \nabla_\mu\lambda^{iA} \,-\mathrm{i}\left(\nabla_\mu z^i \,-\overline{\lambda}^{iB}\psi_{B|\nu}\right)\gamma^\nu\psi^A_\mu \,-\, G^{-i}_{\nu\rho}\gamma^{\nu\rho}\psi_{B|\mu}\epsilon^{AB} \\
&\quad -\left(Y^{iAB} \,+\, g\,W^{iAB}\right)\psi_{B|\mu} \\
\widetilde{\nabla_\mu\zeta_\alpha} \;&=\; \nabla_\mu\zeta_\alpha \,-\mathrm{i}\left(\mathcal{U}^{B\beta}_u\nabla_\nu q^u \,-\,\epsilon^{BC}\,\mathbb{C}^{\beta\gamma}\,\overline{\psi}_{C|\nu}\zeta_\gamma \,-\,\overline{\psi}^B_\nu\,\zeta^\beta\right)\gamma^\nu\psi^A_\mu\epsilon_{AB}\,\mathbb{C}_{\alpha\beta} \,-\, g\,N^A_\alpha\,\psi_{A|\mu} \\
\widetilde{Z}^i_\mu \;&=\; \nabla_\mu z^i \,-\overline{\lambda}^{iA}\psi_{A|\mu} \\
\widetilde{\mathcal{U}}^{A\alpha}_\mu \;&=\; \mathcal{U}^{A\alpha}_u\nabla_\mu q^u \,-\,\epsilon^{AB}\,\mathbb{C}^{\alpha\beta}\,\overline{\psi}_{B|\mu}\zeta_\beta \,-\,\overline{\psi}^A_\mu\zeta^\alpha
\end{aligned} \tag{A.54}
$$

We note that in the component approach the "tilded" field strengths defined in the previous equations are usually referred to as the *supercovariant* field strengths.

The physical fields appearing in the parametrizations are actually further required to satisfy extra–constraints which are essentially of two types:

1. The supercovariant field strengths satisfy a set of differential constraints which are to be identified, when the fields are restricted to space–time only, with the equations of motion of the theory. Indeed the analysis of the Bianchi identities for the fermion fields give such equations (in the sector containing the 2–form basis $\overline{\psi}_A\gamma^a\psi^A$). Further the superspace derivative along the $\psi_A\left(\psi^A\right)$ directions, which amounts to a supersymmetry transformation, yields the equations of motion of the bosonic fields. This is not a surprise since the closure of the Bianchi identities is in fact equivalent to the closure of the $N=2$ supersymmetry algebra on the physical fields and we know that in general such closure implies the equations of motion for the fermion fields. Indeed in our case the usual auxiliary fields of $N=2$ theory have been determined as suitable expressions in the physical fields.

   Finally we also note that since the expressions for the curvatures imply the equations of motion it follows that in the ungauged case ($g=0$) the formulae 8.24– 8.26 are symplectic covariant since the ungauged theory is on-shell symplectic covariant.

2. The second type of constraints following from the closure of Bianchi identities is a set of differential constraints on the upper part L^Λ, $\overline{L}^\Lambda$, f_i^Λ, $\overline{f}_{i^\star}^\Lambda$ of the symplectic sections V and U_i and of the $\mathcal{TM}^3\otimes\mathcal{L}^2$ sections C_{ijk} (together with its complex conjugate $C_{i^\star j^\star k^\star}$).
   One finds:
$$\nabla_{i^\star}L^\Lambda=\nabla_i\overline{L}^\Lambda=0 \tag{A.55}$$
$$f_i^\Lambda=\nabla_iL^\Lambda;\quad \overline{f}_{i^\star}^\Lambda=\nabla_{i^\star}\overline{L}^\Lambda \tag{A.56}$$
$$\begin{aligned}\nabla_{\ell^\star}C_{ijk} &= \nabla_\ell C_{i^\star j^\star k^\star}=0 && \text{(A.57)}\\ \nabla_{[\ell}C_{i]jk} &= \nabla_{[\overline{\ell}}C_{i^\star]j^\star k^\star}=0 && \text{(A.58)}\\ \nabla_j f_k^\Lambda &= \mathrm{i}g^{i\overline{\ell}}\overline{f}_{\ell^\star}^\Lambda C_{ijk} && \text{(A.59)}\end{aligned}$$
   Using the identities of Special Geometry (4.18, 4.26, 4.30, 4.37), C_{ijk} can be written as:
$$C_{ijk}=\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}f_i^\Lambda\nabla_jf_k^\Sigma \tag{A.60}$$
   In particular equation A.59 implies the constraint given in 4.18 for the Riemann tensor of the Kähler-Hodge manifold while equations A.57–A.58 are actually equivalent to the other equations 4.18, using 4.19. Therefore the constraints A.55–A.59 imply that the Kähler-Hodge manifold we started from is actually a special Kähler manifold.
   We may also verify that the same equations A.55–A.59 hold provided we replace $L^\Lambda\to M_\Lambda$ and $f_i^\Lambda\to h_{\Lambda i}$ (together with their c.c.). Hence we have a set of symplectic covariant constraints, namely:
$$\nabla_iV = U_i$$

$$
\begin{aligned}
\nabla_j U_j &= \mathrm{i} C_{ijk} g^{k\ell^\star} U_{\ell^\star} \\
\nabla_i U_{j^\star} &= g_{ij^\star} V \\
\nabla_{i^\star} V &= 0
\end{aligned}
\tag{A.61}
$$

which give an alternative definition of Special Geometry in terms of differential constraints on a symplectic bundle of the Kähler-Hodge manifold. This definition of Special Geometry was in fact first deduced in [27] from $N=2$ Bianchi identities (i.e. for the closure of $N=2$ susy algebra). Furthermore there is a close connection, exploited in ref. [70], between the differential constraints A.61 and the Picard–Fuchs equations for the periods of a 3–dimensional Calabi–Yau manifold [71, 70] .

The determination of the superspace curvatures enables us to write down the $N=2$ SUSY transformation laws. Indeed we recall that from the superspace point of view a supersymmetry transformation is a Lie derivative along the tangent vector:

$$
\epsilon = \overline{\epsilon}^A \vec{D}_A + \overline{\epsilon}_A \vec{D}^A \tag{A.62}
$$

where the basis tangent vectors $\vec{D}_A$, $\vec{D}^A$ are dual to the gravitino 1–forms:

$$
\vec{D}_A \left(\psi^B\right) = \vec{D}^A \left(\psi_B\right) = \mathbf{1} \tag{A.63}
$$

where $\mathbf{1}$ is the unit in spinor space.

Denoting by μ^I and R^I the set of one–forms $\left(V^a,\, \psi_A,\, \psi^A,\, A^\Lambda\right)$ and of two–forms $\left(R^a,\, \rho_a,\, \rho^A,\, F^\lambda\right)$ respectively, one has:

$$
\ell\mu^I = \left(i_\epsilon d + d i_\epsilon\right)\mu^I \equiv \left(D\epsilon\right)^I + i_\epsilon R^I \tag{A.64}
$$

where D is the derivative covariant with respect to the $N=2$ Poincaré superalgebra and i_ϵ is the contraction operator along the tangent vector ϵ.

In our case:

$$
\begin{aligned}
(D\epsilon)^a &= \mathrm{i}\left(\overline{\psi}_A \gamma^a \epsilon^A + \overline{\psi}^A \gamma^a \epsilon_A\right) && \text{(A.65)} \\
(D\epsilon)^\alpha &= \nabla \epsilon^\alpha && \text{(A.66)} \\
(D\epsilon)^\Lambda &= 0 && \text{(A.67)}
\end{aligned}
$$

(here α is a spinor index)
For the 0–forms which we denote shortly as $\nu^I \equiv \left(q^u,\, z^i,\, z^{i^\star},\, \lambda^{iA},\, \lambda^{i^\star}_A,\, \zeta_\alpha,\, \zeta^\alpha\right)$ we have the simpler result:

$$
\ell_\epsilon = i_\epsilon d\nu^I = i_\epsilon \left(\nabla\nu^I - \mathit{connection\,terms}\right) \tag{A.68}
$$

Using the parametrizations given for R^I and $\nabla\nu^I$ and identifying δ_ϵ with the restriction of ℓ_ϵ to space–time it is immediate to find the $N=2$ susy laws for all the fields. The explicit formulae are given in section 8.

Appendix B: Derivation of the space time Lagrangian from the geometric approach

In Appendix A we have seen how to reconstruct the $N=2$ susy transformation laws of the physical fields from the solution of the Bianchi identities in superspace.

In principle, since the Bianchi identities imply the equations of motion, the Lagrangian could also be completely determined. However this would be a cumbersome procedure.

In this Appendix we give a short account of the construction of the Lagrangian on space–time from a geometrical Lagrangian in superspace.
In the geometric (rheonomic) approach the superspace action is a 4–form in superspace integrated on a 4–dimensional (bosonic) hypersurface $\mathcal{M}^4$ locally embedded in $\mathcal{M}^{4|8}$:

$$\mathcal{A} = \int_{\mathcal{M}^4 \subset \mathcal{M}^{4|8}} \mathcal{L} \tag{B.1}$$

Provided we do not introduce the Hodge duality operator in the construction of $\mathcal{L}$ the equations of motions derived from the generalized variational principle $\delta\mathcal{A} = 0$ are 3–form or 4–form equations independent from the particular hypersurface $\mathcal{M}^4$ on which we integrate.
These superspace equations of motion can be analyzed along the p–form basis. The components of the equations obtained along bosonic vielbeins give the differential equations for the fields which, identifying $\mathcal{M}^4$ with space–time, are the ordinary equations of motion of the theory. The components of the same equations along p–forms containing at least one gravitino (”outer components”) give instead algebraic relations which identify the components of the various ”supercurvatures” in superspace.

The Lagrangian must be constructed according to the principles of rheonomy: the ”outer components” computed from the variational equations must be all expressed in terms of the supercovariant components (components along the vielbeins basis). Actually if we have already solved the Bianchi identities this requirement is equivalent to identify the outer components of the curvatures obtained from the variational principle with those obtained from the Bianchi identities.

There are simple rules which can be used in order to write down the most general Lagrangian compatible with this requirement.
The implementation of these rules is described in detail in the literature to which we refer the interested reader. Actually one writes down the most general 4–form as a sum of terms with indeterminate coefficients in such a way that $\mathcal{L}$ be a scalar with respect to all the symmetry transformations of the theory (Lorentz invariance, $SU(2) \otimes Sp(2m)$ and $U(1)$ Kähler invariance, invariance under the rescaling A.22). Varying the action and comparing the outer equations of motion with the actual solution of the Bianchi identities one then fixes all the undetermined coefficients.

Let us perform the steps previously indicated. The most general Lagrangian has the following form:

$$\mathcal{L} = \mathcal{L}_{grav} + \mathcal{L}_{kin} + \mathcal{L}_{Pauli} + \mathcal{L}_{torsion} + \mathcal{L}_{4ferm} + \mathcal{L}_{gauging} \tag{B.2}$$

$$\begin{aligned}
\mathcal{L}_{grav} &= \epsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4 \left(\overline{\psi}^A \gamma_a \rho_A - \overline{\psi}_A \gamma_a \rho^A \right) V^a \\
\mathcal{L}_{kin} &= \beta_1 g_{ij^\star} \left[Z^i_a \left(\nabla \overline{z}^{j^\star} - \overline{\psi}^A \lambda^{j^\star}_A \right) + \overline{Z}^{j^\star}_a \left(\nabla z^i - \overline{\psi}_A \lambda^{iA} \right) \right] \wedge V^b \wedge V^c \wedge V^d \epsilon^a{}_{bcd} \\
&\quad + b_1 \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{A\alpha}_a \left(\mathcal{U}^{B\beta} - \overline{\psi}^B \zeta^\beta - \epsilon^{BC} \mathbb{C}^{\beta\gamma} \overline{\psi}_C \zeta_\gamma \right) \wedge V^b \wedge V^c \wedge V^d \epsilon^a{}_{bcd} \\
&\quad - \frac{1}{4} \left(\beta_1 g_{ij^\star} Z^i_l \overline{Z}^{j^\star}_m + \frac{1}{2} b_1 \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{A\alpha}_l \mathcal{U}^{B\beta}_m \right) \eta^{lm} \epsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d \\
&\quad + \mathrm{i}\, \beta_2 g_{ij^\star} \left(\overline{\lambda}^{iA} \gamma^a \nabla \lambda^{j^\star}_A + \overline{\lambda}^{j^\star}_A \gamma^a \nabla \lambda^{iA} \right) \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd}
\end{aligned}$$

$$
\begin{aligned}
& +\mathrm{i}\, b_2 \left(\overline{\zeta}^{\alpha}\gamma^a\nabla\zeta_\alpha \,+\, \overline{\zeta}_\alpha\gamma^a\nabla\zeta^\alpha\right)\wedge V^b\wedge V^c\wedge V^d\epsilon_{abcd} \\
& +\mathrm{i}\,\beta_3\left(\mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}_{ab} \,+\, \overline{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}_{ab}\right)\Big[F^{\Sigma} \,-\, \mathrm{i}\left(f^{\Sigma}_i\,\overline{\lambda}^{iA}\gamma_c\psi^B\epsilon_{AB}\right. \\
& \qquad\qquad\qquad\qquad\qquad\left. +\, f^{\Sigma}_{i^\star}\,\overline{\lambda}^{i^\star}_A\gamma_c\psi_B\epsilon^{AB}\right)\wedge V^c\Big]\wedge V^a\wedge V^b \\
& -\frac{1}{24}\,\beta_3\left(\overline{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}_{lm}F^{-\Sigma|lm} \,-\, \mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}_{lm}F^{+\Sigma|lm}\right)\epsilon_{abcd}V^a\wedge V^b\wedge V^c\wedge V^d
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{Pauli} \;=\; & \beta_5 F^{\Lambda}\left(\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}\overline{\psi}^A\psi^B\epsilon_{AB} \,+\, \overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^{\Sigma}\overline{\psi}_A\psi_B\epsilon^{AB}\right) \\
& +\mathrm{i}\,\beta_6 F^{\Lambda}\left(\overline{\mathcal{N}}_{\Lambda\Sigma}f^{\Sigma}_i\,\overline{\lambda}^{iA}\gamma_a\psi^B\epsilon_{AB} \,+\, \mathcal{N}_{\Lambda\Sigma}\overline{f}^{\Sigma}_{i^\star}\overline{\lambda}^{i^\star}_A\gamma_a\psi_B\epsilon^{AB}\right)\wedge V^a \\
& +\beta_7 F^{\Lambda}\left(\mathcal{N} \,-\, \overline{\mathcal{N}}\right)_{\Lambda\Sigma}\left(\nabla_i f^{\Sigma}_j\overline{\lambda}^{iA}\gamma_{ab}\lambda^{jB}\epsilon_{AB}\right. \\
& \qquad\qquad\left. -\,\nabla_{i^\star}f^{\Sigma}_{j^\star}\overline{\lambda}^{i^\star}_A\gamma_{ab}\lambda^{j^\star}_B\epsilon^{AB}\right)\wedge V^a\wedge V^b \\
& +b_5 F^{\Lambda}\left(\mathcal{N} \,-\, \overline{\mathcal{N}}\right)_{\Lambda\Sigma}\left(L^{\Sigma}\overline{\zeta}_\alpha\gamma_{ab}\zeta_\beta C^{\alpha\beta} \,-\, \overline{L}^{\Sigma}\overline{\zeta}^{\alpha}\gamma_{ab}\zeta^\beta C_{\alpha\beta}\right)\wedge V^a\wedge V^b \\
& +\beta_8 g_{ij^\star}\left(\overline{\lambda}^{iA}\gamma^{ab}\psi_A\,\nabla\overline{z}^{j^\star} \,+\, \overline{\lambda}^{j^\star}_A\gamma_{ab}\psi^A\nabla z^i\right)\wedge V^c\wedge V^d\epsilon_{abcd} \\
& +b_3\left(\overline{\zeta}_\alpha\gamma^{ab}\psi_A\,\mathcal{U}^{\alpha A} \,+\, \overline{\zeta}^{\alpha}\gamma^{ab}\psi^A\,\mathcal{U}_{\alpha A}\right)\wedge V^c\wedge V^d\epsilon_{abcd} \\
\mathcal{L}_{torsion} \;=\; & \left(\beta_4 g_{ij^\star}\,\overline{\lambda}^{iA}\gamma_b\lambda^{j^\star}_A \,+\, b_4\,\overline{\zeta}^{\alpha}\gamma_b\zeta_\alpha\right)T_a\wedge V^a\wedge V^b \\
\mathcal{L}_{4ferm} \;=\; & \alpha_1\left(L^{\Lambda}\overline{\psi}^A\psi^B\epsilon_{AB} \,+\, \overline{L}^{\Lambda}\overline{\psi}_A\psi_B\epsilon^{AB}\right)\wedge\left(\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}\overline{\psi}^C\psi^D\epsilon_{CD}\right. \\
& \qquad\qquad\left. +\overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^{\Sigma}\overline{\psi}_C\psi_D\epsilon^{CD}\right) \\
& +\alpha_2\left(f^{\Lambda}_i\,\overline{\lambda}^{iA}\gamma_a\psi^B\epsilon_{AB} \,+\, f^{\Lambda}_{i^\star}\overline{\lambda}^{i^\star}_A\gamma_a\psi_B\epsilon^{AB}\right)\wedge\left(\overline{\mathcal{N}}_{\Lambda\Sigma}f^{\Sigma}_j\,\overline{\lambda}^{jC}\gamma_b\psi^D\epsilon_{CD}\right. \\
& \qquad\qquad\qquad\left. +\mathcal{N}_{\Lambda\Sigma}f^{\Sigma}_{j^\star}\overline{\lambda}^{j^\star}_C\gamma_b\psi_D\epsilon^{CD}\right)\wedge V^a\wedge V^b \\
& +\alpha_3\left(\mathcal{N} \,-\, \overline{\mathcal{N}}\right)_{\Lambda\Sigma}\left(f^{\Lambda}_i\nabla_{k^\star}f^{\Sigma}_{j^\star}\,\overline{\lambda}^{iA}\gamma_c\psi^B\,\overline{\lambda}^{k^\star}_C\gamma_{ab}\lambda^{j^\star}_D\,\epsilon_{AB}\epsilon^{CD}\right. \\
& \qquad\left. -\,\overline{f}^{\Lambda}_{i^\star}\nabla_k f^{\Sigma}_j\,\overline{\lambda}^{i^\star}_A\gamma_c\psi_B\,\overline{\lambda}^{kC}\gamma_{ab}\lambda^{jD}\,\epsilon^{AB}\epsilon_{CD}\right)\wedge V^a\wedge V^b\wedge V^c \\
& +a_1\left(\mathcal{N} \,-\, \overline{\mathcal{N}}\right)_{\Lambda\Sigma}\left(\overline{f}^{\Lambda}_{i^\star}L^{\Sigma}\overline{\lambda}^{i^\star}_A\gamma_c\psi_B\,\overline{\zeta}_\alpha\gamma_{ab}\zeta_\beta\,\epsilon^{AB}\mathbb{C}^{\alpha\beta}\right. \\
& \qquad\left. -\,f^{\Lambda}_i\overline{L}^{\Sigma}\overline{\lambda}^{iA}\gamma_c\psi^B\,\overline{\zeta}^{\alpha}\gamma_{ab}\zeta^\beta\,\epsilon_{AB}\mathbb{C}_{\alpha\beta}\right)\wedge V^a\wedge V^b\wedge V^c \\
& +a_2\left(\overline{\psi}^A\psi^B\,\overline{\zeta}^{\alpha}\gamma_{ab}\zeta^\beta\epsilon_{AB}\mathbb{C}_{\alpha\beta} \,+\, \overline{\psi}_A\psi_B\,\overline{\zeta}_\alpha\gamma_{ab}\zeta_\beta\,\epsilon^{AB}\mathbb{C}^{\alpha\beta}\right)\wedge V^a\wedge V^b \\
& +\left(\alpha_4 g_{ij^\star}\,\overline{\lambda}^{iA}\gamma_b\lambda^{j^\star}_B \,+\, a_3\,\delta^A_B\overline{\zeta}^{\alpha}\gamma_b\zeta_\alpha\right)\overline{\psi}_A\gamma_a\psi^B\wedge V^a\wedge V^b \\
& +\alpha_5\left(C_{ijk}\overline{\lambda}^{iA}\gamma^a\psi^B\,\overline{\lambda}^{jC}\lambda^{kD}\,\epsilon_{AC}\epsilon_{BD}\right. \\
& \qquad\left. -\,C_{i^\star j^\star k^\star}\overline{\lambda}^{i^\star}_A\gamma^a\psi_B\,\overline{\lambda}^{i^\star}_C\lambda^{k^\star}_D\epsilon^{AC}\epsilon^{BD}\right)\wedge V^b\wedge V^c\wedge V^d\,\epsilon_{abcd} \\
& +\frac{1}{72}\Big[\gamma_1\left(R_{ij^\star lk^\star} \,+p\,g_{ik^\star}\,g_{lj^\star} \,+q\,g_{ij^\star}\,g_{lk^\star}\right)\overline{\lambda}^{iA}\lambda^{lB}\overline{\lambda}^{j^\star}_A\lambda^{k^\star}_B \\
& +\gamma_2\left(\nabla_m C_{jkl}\overline{\lambda}^{jA}\lambda^{mB}\overline{\lambda}^{kC}\lambda^{lD}\epsilon_{AC}\epsilon_{BD} - h.c.\right) \\
& +\gamma_3\left(\mathcal{N} \,-\, \overline{\mathcal{N}}\right)_{\Lambda\Sigma}\left(C_{ijk}\,C_{lmn}g^{k\overline{r}}\,g^{n\overline{s}}f^{\Lambda}_{\overline{r}}\,f^{\Sigma}_{\overline{s}}\,\overline{\lambda}^{iA}\gamma_{lm}\lambda^{jB}\overline{\lambda}^{kC}\gamma^{lm}\lambda^{lD}\,\epsilon_{AB}\epsilon_{CD}\right. \\
& \qquad\qquad\qquad\qquad\left. +h.c.\right) \\
& +\gamma_4\;g_{ij^\star}\,\overline{\zeta}^{\alpha}\gamma_a\zeta_\alpha\,\overline{\lambda}^{iA}\gamma^a\lambda^{j^\star}_A
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma_5\,\mathcal{R}^{\alpha}{}_{\beta ts}\,\mathcal{U}^{t}_{A\gamma}\,\mathcal{U}^{s}_{B\delta}\epsilon^{AB}\,C^{\delta\eta}\overline{\zeta}_{\alpha}\,\zeta_{\eta}\,\overline{\zeta}^{\beta}\,\zeta^{\gamma}\\
&+\gamma_6\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}\left(L^{\Lambda}\nabla_i f^{\Sigma}_{j}\overline{\zeta}_{\alpha}\gamma_{ab}\zeta_{\beta}\,\overline{\lambda}^{iA}\gamma^{ab}\lambda^{jB}\;\epsilon_{AB}\,\mathbb{C}^{\alpha\beta}+h.c.\right)\\
&+\gamma_7\left(\mathcal{N}-\overline{\mathcal{N}}\right)_{\Lambda\Sigma}\left(L^{\Lambda}\,L^{\Sigma}\overline{\zeta}_{\alpha}\gamma_{ab}\zeta_{\beta}\,\overline{\zeta}_{\gamma}\gamma^{ab}\zeta_{\delta}\,\mathbb{C}^{\alpha\beta}\,\mathbb{C}^{\gamma\delta}\right.\\
&\qquad\qquad\qquad\qquad\left.\left.+h.c.\right)\right]V^a\wedge V^b\wedge V^c\wedge V^d\,\epsilon_{abcd}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{gauging} \;=\; &-\mathrm{i}\,g\,\delta_1\left(S_{AB}\overline{\psi}^{A}\gamma_{ab}\psi^{B}+h.c.\right)V^a\wedge V^b\\
&+\mathrm{i}\,g\,\delta_2\,g_{ij^\star}\left(W^{iAB}\overline{\lambda}^{j^\star}_{A}\gamma^{a}\psi_{B}+h.c.\right)\wedge V^b\wedge V^c\wedge V^d\,\epsilon_{abcd}\\
&+\mathrm{i}\,g\,\delta_3\left(N^{A}_{\alpha}\,\overline{\zeta}^{\alpha}\gamma^{a}\psi_{A}+h.c.\right)\wedge V^b\wedge V^c\wedge V^d\,\epsilon_{abcd}\\
&+g\Big[\delta_4\nabla_u N^{\alpha}_{A}\,\mathcal{U}^{u}_{B\beta}\epsilon^{AB}\mathbb{C}^{\beta\gamma}\overline{\zeta}_{\alpha}\zeta_{\gamma}+\delta_5\nabla_i N^{\alpha}_{A}\overline{\zeta}_{\alpha}\lambda^{iA}\\
&+\delta_6\,g_{i\overline{\jmath}}\nabla_k W^{\overline{\jmath}}_{AB}\overline{\lambda}^{iA}\lambda^{kB}+\;h.c.\Big]V^a\wedge V^b\wedge V^c\wedge V^d\epsilon_{abcd}\\
&+\delta_7\,g^2\,\mathrm{V}_{potential}\,V^a\wedge V^b\wedge V^c\wedge V^d\,\epsilon_{abcd}
\end{aligned}
\tag{B.3}
$$

where:

$$
\begin{aligned}
\mathrm{V}_{potential} \;=\; &\left(g_{ij^\star}\,k^{i}_{\Lambda}\,k^{j^\star}_{\Sigma}\;+\;4\,h_{uv}k^{u}_{\Lambda}\,k^{v}_{\Sigma}\right)\overline{L}^{\Lambda}\,L^{\Sigma}\\
&+g^{ij^\star}\,f^{\Lambda}_{i}\,f^{\Sigma}_{j^\star}\,\mathcal{P}^{x}_{\Lambda}\,\mathcal{P}^{x}_{\Sigma}\;-\;3\,\overline{L}^{\Lambda}\,L^{\Sigma}\,\mathcal{P}^{x}_{\Lambda}\,\mathcal{P}^{x}_{\Sigma}
\end{aligned}
\tag{B.4}
$$

We note that the kinetic terms of the Lagrangian have been written in first–order form to avoid the Hodge–operator which would destroy the independence of the variational equations from the particular hypersurface of integration. Specifically one introduces auxiliary 0–forms namely $F^{\pm\Lambda}_{ab}$, Z^{i}_{a}, $\overline{Z}^{i^\star}_{a}$, $\mathcal{U}^{A\alpha}_{a}$ whose variational equations identify them with $\widetilde{F}^{\pm\Lambda}_{ab}$, $\widetilde{Z}^{i}_{a}$, $\widetilde{\overline{Z}}^{i^\star}_{a}$, $\widetilde{\mathcal{U}}^{A\alpha}_{a}$ defined in Appendix A. Of course also the spin connection ω^{ab} has to be treated as an independent field: indeed the term $\mathcal{L}_{\mathrm{torsion}}$ appearing in the Lagrangian has been chosen in such a way that the equation of motion of ω^{ab} gives $T^a = 0$.

The analysis of the variational equations for the other p–forms containing at least a fermionic vielbein $\psi^A\,(\psi_A)$ then fixes completely all the coefficients, except the coefficients of terms that are proportional to $V^aV^bV^cV^d\epsilon_{abcd}$, which, after variation, do not contain any $\psi^A\,(\psi_A)$ and therefore appear in the space–time equations of motion.
These undetermined coefficients, however, can be retrieved by comparing the space–time equations of motion for the 0–form fermion fields λ^{iA} , $\lambda^{i^\star A}$, ζ^{α} , ζ_{α} as obtained from the Bianchi identities with those obtained from the Lagrangian. In this way all the coefficients have been fixed. The result is:

$$
\begin{aligned}
&\beta_1\,=\,\frac{2}{3}\,;\,\beta_2\,=\,-\frac{1}{3}\,;\,\beta_3\,=\,4\,\mathrm{i}\,;\,\beta_4\,=\,-1\,;\,\beta_5\,=\,4\,;\,\beta_6\,=\,-4\,;\,\beta_7\,=\,\frac{1}{2}\,;\,\beta_8\,=\,-1\,;\\
&b_1\,=\,-\frac{4}{3}\lambda\,;\,b_2\,=\,\frac{2}{3}\lambda\,;\,b_3\,=\,2\lambda\,;\,b_4\,=\,-2\lambda\,;\,b_5\,=\,\lambda\,;\\
&\alpha_1\,=\,-2\,;\,\alpha_2\,=\,2\,;\,\alpha_3\,=\,\frac{\mathrm{i}}{2}\,;\,\alpha_4\,=\,-2\,\mathrm{i}\,;\,\alpha_5\,=\,-\frac{1}{9}\,;\\
&a_1\,=\,-\mathrm{i}\lambda\,;\,a_2\,=\,-\mathrm{i}\lambda\,;\,a_3\,=\,-4\,\mathrm{i}\lambda\,;\\
&\gamma_1\,=\,3\,;\,\gamma_2\,=\,-\mathrm{i}\,;\,\gamma_3\,=\,\frac{3\mathrm{i}}{16}\,;\,\gamma_4\,=\,-3\,\lambda\,;\,\gamma_5\,=\,-6\lambda\,p\,=\,1\,;\,q\,=\,-\frac{2}{3}\,;
\end{aligned}
$$

$$\gamma_6 \;=\; -\frac{3}{4}\mathrm{i}\,\lambda\,;\;\gamma_7 \;=\; -\frac{3}{4}\mathrm{i}\,\lambda^2\,;$$

$$\delta_1 \;=\; 4\,;\;\delta_2 \;=\; \frac{2}{3}\,;\;\delta_3 \;=\; -\frac{4}{3}\,\lambda\,;\;\delta_4 \;=\; -\frac{1}{12}\,\lambda\,;\;\delta_5 \;=\; -\frac{1}{3}\,\lambda\,;\;\delta_6 \;=\; \frac{1}{18}\,;\;\delta_7 \;=\; -\frac{1}{6} \tag{B.5}$$

In order to obtain the space–time Lagrangian the last step to perform is the restriction of the 4–form Lagrangian from superspace to space–time. Namely we restrict all the terms to the $\theta = 0\,,\, d\theta = 0$ hypersurface $\mathcal{M}^4$. In practice one first goes to the second order formalism by identifying the auxiliary 0–form fields as explained before. Then one expands all the forms along the dx^μ differentials and restricts the superfields to their lowest ($\theta = 0$) component. Finally the coefficients of:

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \;=\; \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\left(\sqrt{g}d^4x\right) \tag{B.6}$$

give the Lagrangian density written in chapter 8. The overall normalisation of the space–time action has been chosen such as to be the standard one for the Einstein term.

Appendix C: Supergravity theory on $ST[2,n] \otimes HQ[m]$

In this appendix, as an illustration of the general method and also for its interest in applications to tree level effective lagrangians of heterotic string theory, we consider the specialization of our formulae to the case where the scalar manifold of N=2 supergravity is chosen as in eq. 1.1. This choice is by no means new in the literature, but the interesting point is to utilize the symplectic gauge where the holomorphic prepotential $F(X)$ does not exist. This is the gauge chosen by string theory and also that where partial supersymmetry breaking can be obtained.

The $\mathcal{ST}[2,n]$ special manifolds and the Calabi Visentini coordinates

When we studied the symplectic embeddings of the $\mathcal{ST}[m,n]$ manifolds, defined by eq. 3.19, a study that lead us to the general formula in eq. 3.34, we remarked that the subclass $\mathcal{ST}[2,n]$ constitutes a family of special Kähler manifolds: actually a quite relevant one. Here we survey the special geometry of this class.

Besides their applications in the large radius limit of superstring compactifications, the $\mathcal{ST}[2,n]$ manifolds are interesting under another respect. They provide an example where the holomorphic prepotential can be non–existing. Furthermore it is precisely in the symplectic gauge where $F(x)$ does not exist that the model $n = 1$, $m = 1$ of eq. 1.1 exhibits partial supersymmetry breaking $N = 2 \longrightarrow N = 1$ [29].

Consider a standard parametrization of the $SO(2,n)/SO(2) \times SO(n)$ manifold, like for instance that in eq. 3.31. In the $m = 2$ case we can introduce a canonical complex structure on the manifold by setting:

$$\Phi^\Lambda(X) \;\equiv\; \frac{1}{\sqrt{2}}\left(L^\Lambda{}_0 + \mathrm{i}\,L^\Lambda{}_1\right) \qquad ; \qquad (\Lambda = 0, 1, a \quad a = 2, \ldots, n+1) \tag{C.1}$$

The relations satisfied by the upper two rows of the coset representative (consequence of $L(X)$ being pseudo–orthogonal with respect to metric $\eta_{\Lambda\Sigma} = \mathrm{diag}(+,+,-,\dots,-)$):

$$L^\Lambda{}_0 \, L^\Sigma{}_0 \, \eta_{\Lambda\Sigma} \; = \; 1 \quad ; \quad L^\Lambda{}_0 \, L^\Sigma{}_1 \, \eta_{\Lambda\Sigma} \; = \; 0 \quad ; \quad L^\Lambda{}_1 \, L^\Sigma{}_1 \, \eta_{\Lambda\Sigma} \; = \; 1 \tag{C.2}$$

can be summarized into the complex equations:

$$\overline{\Phi}^\Lambda \, \Phi^\Sigma \eta_{\Lambda\Sigma} \; = \; 1 \quad ; \quad \Phi^\Sigma \eta_{\Lambda\Sigma} \; = \; 0 \tag{C.3}$$

Eq.s C.3 are solved by posing:

$$\Phi^\Lambda \; = \; \frac{X^\Lambda}{\sqrt{\overline{X}^\Lambda \, X^\Sigma \, \eta_{\Lambda\Sigma}}} \tag{C.4}$$

where X^Λ denotes any set of complex parameters, determined up to an overall multiplicative constant and satisfying the constraint:

$$X^\Lambda \, X^\Sigma \eta_{\Lambda\Sigma} \; = \; 0 \tag{C.5}$$

In this way we have proved the identification, as differentiable manifolds, of the coset space $SO(2,n)/SO(2) \times SO(n)$ with the vanishing locus of the quadric in eq. C.5. Taking any holomorphic solution of eq. C.5, for instance:

$$X^\Lambda(y) \; \equiv \; \left(\begin{array}{c} 1/2\,(1+y^2) \\ \mathrm{i}/2\,(1-y^2) \\ y^a \end{array} \right) \tag{C.6}$$

where y^a is a set of n independent complex coordinates, inserting it into eq. C.4 and comparing with eq. C.1 we obtain the relation between whatever coordinates we had previously used to write the coset representative $L(X)$ and the complex coordinates y^a. In other words we can regard the matrix L as a function of the y^a that are named the Calabi Visentini coordinates [68].

Consider in addition the *axion–dilaton* field S that parametrizes the $SU(1,1)/U(1)$ coset according with eq. 3.30. The special geometry of the manifold $\mathcal{ST}[2,n]$ is completely specified by writing the holomorphic symplectic section Ω as follows ([4]):

$$\Omega(y,S) \; = \; \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \; = \; \begin{pmatrix} X^\Lambda(y) \\ \mathcal{S} \, \eta_{\Lambda\Sigma} X^\Sigma(y) \end{pmatrix} \tag{C.7}$$

Notice that with the above choice, it is not possible to describe F_Λ as derivatives of any prepotential. Yet everything else can be calculated utilizing the formulae we presented in the text. The Kähler potential is:

$$\mathcal{K} \; = \; \mathcal{K}_1(S) + \mathcal{K}_2(y) \; = \; -\log \mathrm{i}(\overline{S} - S) - \log X^T \eta X \tag{C.8}$$

The Kähler metric is block diagonal:

$$g_{ij^\star} \; = \; \left(\begin{array}{cc} g_{S\overline{S}} & \mathbf{0} \\ \mathbf{0} & g_{a\overline{b}} \end{array} \right) \qquad \qquad \left\{ \begin{array}{l} g_{S\overline{S}} = \partial_S \partial_{\overline{S}} \mathcal{K}_1 = \frac{-1}{(\overline{S}-S)^2} \\ g_{a\overline{b}}(y) = \partial_a \partial_{\overline{b}} \mathcal{K}_2 \end{array} \right. \tag{C.9}$$

as expected. The anomalous magnetic moments-Yukawa couplings C_{ijk} $(i = S, a)$ have a very simple expression in the chosen coordinates:

$$C_{Sab} = -\exp[\mathcal{K}]\,\delta_{ab}, \tag{C.10}$$

all the other components being zero.

Using the definition of the *period matrix* given in eq. 4.37 we obtain

$$\mathcal{N}_{\Lambda\Sigma} = (S - \overline{S})\frac{X_\Lambda \overline{X}_\Sigma + \overline{X}_\Lambda X_\Sigma}{\overline{X}^T \eta X} + \overline{S}\eta_{\Lambda\Sigma}. \tag{C.11}$$

In order to see that eq. C.11 just coincides with eq. 3.34 it suffices to note that as a consequence of its definition C.1 and of the pseudo–orthogonality of the coset representative $L(X)$, the vector Φ^Λ satisfies the following identity:

$$\Phi^\Lambda\,\overline{\Phi}^\Sigma + \Phi^\Sigma\,\overline{\Phi}^\Lambda \;=\; \frac{1}{2}\,L^\Lambda{}_\Gamma\,L^\Sigma{}_\Delta\left(\delta^{\Gamma\Delta} + \eta^{\Gamma\Delta}\right) \tag{C.12}$$

Inserting eq. C.12 into eq. C.11, formula 3.34 is retrieved.

This completes the proof that the choice C.7 of the special geometry holomorphic section corresponds to the symplectic embedding 3.26 and 3.28 of the coset manifold $\mathcal{ST}[2,n]$. In this symplectic gauge the symplectic transformations of the isometry group are the simplest possible ones and the entire group $SO(2,n)$ is represented by means of *classical* transformations that do not mix electric fields with magnetic fields. The disadvantage of this basis, if any, is that there is no holomorphic prepotential. To find an $F(X)$ it suffices to make a symplectic rotation to a different basis.

If we set:

$$\begin{aligned}
X^1 = \frac{1}{2}(1+y^2) &= -\frac{1}{2}(1-\eta_{ij}t^i t^j) \\
X^2 = i\frac{1}{2}(1-y^2) &= t^2 \\
X^a = y^a &= t^{2+a} \quad a = 1,\ldots,n-1 \\
X^{a=n} = y^n &= \frac{1}{2}(1+\eta_{ij}t^i t^j)
\end{aligned} \tag{C.13}$$

where

$$\eta_{ij} = \mathrm{diag}\,(+,-,\ldots,-)\;\; i,j = 2,\ldots,n+1 \tag{C.14}$$

Then we can show that $\exists\,\mathcal{C} \in Sp(2n+2,\mathbb{R})$ such that:

$$\mathcal{C}\begin{pmatrix} X^\Lambda \\ S\eta_{\Lambda\Sigma}X^\Lambda \end{pmatrix} = \exp[\varphi(t)]\begin{pmatrix} 1 \\ S \\ t^i \\ 2\,\mathcal{F} - t^i\frac{\partial}{\partial t^i}\mathcal{F} - S\frac{\partial}{S}\mathcal{F} \\ S\frac{\partial}{S}\mathcal{F} \\ \frac{\partial}{\partial t^i}\mathcal{F} \end{pmatrix} \tag{C.15}$$

with

$$\begin{aligned}
\mathcal{F}(S,t) &= \frac{1}{2}\,S\,\eta_{ij}t^i t^j = \frac{1}{2}\,d_{IJK}t^I t^J t^K \\
t^1 &= S \\
d_{IJK} &= \begin{cases} d_{1jk} = \eta_{ij} \\ 0 \;\; \text{otherwise} \end{cases}
\end{aligned} \tag{C.16}$$

and

$$W_{IJK} = d_{IJK} = \frac{\partial^3 \mathcal{F}(S, t^i)}{\partial t^I \partial t^J \partial t^K} \tag{C.17}$$

This means that in the new basis the symplectic holomorphic section $\mathcal{C}\Omega$ can be derived from the following cubic prepotential:

$$F(X) \;=\; \frac{1}{3!}\,\frac{d_{IJK}\,X^I\,X^J\,X^K}{X_0} \tag{C.18}$$

For instance in the case $n = 1$ the matrix which does such a job is:

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{C.19}$$

Comments on the $\mathcal{ST}[2,2]$ case: S duality and R symmetry

To conclude let us focus on the case $\mathcal{ST}[2,2]$. This manifold has two coordinates that we can either call S and t, in the parametrization of eq. C.16 or S and y in the Calabi Visentini basis. The relation between t and y simplifies enormously in this case:

$$t \;=\; i\frac{y+1}{y-1} \tag{C.20}$$

It is then a matter of choice to regard the holomorphic section in whatever basis as a function of y or of t, in addition to S. Independently from this choice the manifold $\mathcal{ST}[2,2]$ emerges as *moduli space (at tree–level)* in a locally N=2 supersymmetric gauge theory of a rank one gauge group, namely $SU(2)$. The two fields spanning the manifold have very different interpretations. The field y is the scalar partner of the gauge field that remains massless after Higgs mechanism. Its vacuum expectation value is the modulus of the gauge theory. It is the same field that occurs also in a globally supersymmetric theory. On the other hand the field S is the dilaton–axion. It plays the role of generalized coupling constant and generalized *theta*–angle. There are two $SL(2,\mathbb{R})$ groups embedded in $SP(6,\mathbb{R})$, they act as standard fractional linear transformations on the *dilaton–axion* S and on the special coordinate t for the gauge modulus. Using the Calabi–Visentini section of eq. C.7 and the embedding eq.s 3.26 and 3.28, we have that

S–duality $S \longrightarrow -1/S$ is generated by the symplectic matrix:

$$S_{duality} \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{C.21}$$

while **T–duality** $t \longrightarrow -1/t$ is generated by the symplectic matrix:

$$R_{symmetry} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.22}$$

If we think of the t–field as the *modulus* of some compact internal manifold then T–duality is just the transformation from small to large compactification radius. Looking at the same transformation in terms of the y variable its meaning becomes more clear. It is R–symmetry $y \longrightarrow -y$, an exact global symmetry of the *microscopic* lagrangian. The fact that the matrix generating T–duality or R–symmetry is block–diagonal agrees with the fact that this is a perturbative symmetry, holding at each order in perturbation theory and never exchanging electric with magnetic states. Very different is the nature of S–duality. Since it inverts the coupling constant it is by definition non–perturbative. It exchanges strong and weak coupling regimes and because of that it is supposed to exchange elementary states with soliton states. For this reason it must mix electric with magnetic field strengths and it is off–diagonal. These symmetries exist in the microscopic theory which is derived by *gauging* the abelian theories possessing continuous duality symmetries (in this case the two $SL(2,\mathbb{R})$ groups). After gauging the continuous duality symmetries will be broken. The question is will the integer valued symplectic generators of S–duality and R–symmetry survive given that they respect the Dirac quantization condition? The answer is yes, but in the effective quantum theory they will be represented by new integer valued elements of $Sp(6,\mathbb{Z})$ not derivable from the classical embedding. Since the special geometry in the effective theory is corrected by the instanton contributions and has a new complicated transcendental structure, the duality generators must change basis to adapt themselves to the new situation and be integer valued in the new non–perturbative geometry. Alternatively one can turn matters around. If we know the new quantum symplectic embedding of the discrete duality group we have essentially determined the non perturbative geometry. It is this point of view that has proven very fruitful in the very recent literature.

C.1 Momentum maps of $HQ[m]$ and mass matrices

As we are just going to see the quaternionic manifold $HQ[m]$ is the closest quaternionic analogue of a flat HyperKähler manifold and the relevant formulae for the metric and the momentum maps are almost identical, *mutatis mutandis*, with the equations surveyed in subsection 9.3, when we discussed the renormalizable microscopic N=2 super Yang–Mills lagrangian.

To describe the coset manifold $SO(4,m)/SO(4) \times SO(m)$ we use a family of coset representatives $L(q) \in SO(4,m)$. A typical choice is the $(4+m) \times (4+m)$ matrix:

$$L(q) \;=\; \begin{pmatrix} \sqrt{\mathbb{1} + q\,q^T} & q \\ q^T & \sqrt{\mathbb{1} + q^T q} \end{pmatrix} \tag{C.23}$$

function of an independent $4 \times m$ matrix q. By definition of the group $SO(4,m)$ we have:

$$L^T \, \eta \, L \; = \; \eta \qquad ; \qquad \eta \; = \; \text{diag}\,(+,+,+,+,-,\dots,-) \tag{C.24}$$

We can regard the index range in the fundamental representation of $SO(4,m)$ as split in the following way:

$$L \; = \; L^I{}_J \qquad\qquad I,J = \begin{cases} a,b = 0,1,2,3 \\ t,s = 1,2,\dots\, m \end{cases} \tag{C.25}$$

and introducing the left invariant one–form:

$$L^{-1}\, dL \; \equiv \; \Theta \tag{C.26}$$

we can split it into the vielbein and the connections on the coset manifold:

$$\Theta = \begin{pmatrix} \theta^{ab} & E^{at} \\ (E^T)^{ta} & \Delta^{st} \end{pmatrix} \qquad \begin{cases} \theta^{ab} & \text{SO(4) connection} \\ E^{at} & \text{Vielbein on the coset} \\ \Delta^{st} & \text{SO(m) connection.} \end{cases} \tag{C.27}$$

From the very definition of Θ one immediately obtains the Maurer-Cartan equations:

$$\begin{cases} \delta E^{at} + \theta^{ab} \wedge E^{bt} - \Delta^{ts} \wedge E^{as} = 0 & \text{Torsion equation} \\ \delta\theta^{ab} + \theta^{ac} \wedge \theta^{cb} = -E^{as} \wedge E^{bs} & \text{SO(4) curvature} \\ \delta\Delta^{ts} - \Delta^{tr} \wedge \Delta^{rs} = E^{at} \wedge E^{as} = 0 & \text{SO(m) curvature} \end{cases} \tag{C.28}$$

Notice that the vielbein $E^{at} = E^{at}_u dq^u$ carries a vector index $a = 0,1,2,3$ of SO(4) and an index t in the vector representation of SO(m) just as it does the coordinate $\mathbf{q}$ of the flat HyperKähler manifold discussed in eq. 10.42. Accordingly the quaternionic generalization of eq. 10.42 is obtained by setting:

$$\begin{aligned} \mathbf{l} \; &\equiv \; L^{a|t} \\ \mathbf{E} \; &\equiv \; E^{a|t} \end{aligned} \tag{C.29}$$

The quaternionic metric and the corresponding triplet of HyperKähler 2–forms are given by:

$$\begin{aligned} ds^2 \; &\equiv \; h_{uv}\, dq^u\, dq^v \; = \; \mathbf{E}^T \, \left(\mathbb{1}_{4\times 4} \otimes \mathbb{1}_{m\times m}\right) \mathbf{E} \\ K^x \; &= \; \mathbf{E}^T \wedge \left(J^{+|x} \otimes \mathbb{1}_{m\times m}\right) \mathbf{E} \; , \end{aligned} \tag{C.30}$$

which is the quaternionic counterpart of eq. 10.47 Alternatively in the above formula one can use the triplet of antiself dual t'Hooft matrices to define the HyperKähler structure. Using the identities 10.48 and rearranging the $4m$ vielbein $E^{a|t}$ into an m-vector of quaternions:

$$\mathbf{QE} \; = \; \begin{pmatrix} QE^1 \; = \; E^{a|1}\, e_a \\ QE^2 \; = \; E^{a|2}\, e_a \\ \dots \\ QE^t \; = \; E^{a|t}\, e_a \\ \dots \end{pmatrix} \tag{C.31}$$

which is the quaternionic counterpart of eq. 10.49, eq.s C.30 can be rewritten in a form completely analogous to eq.s 10.50:

$$\begin{aligned} ds^2 &= \frac{1}{2}\,\mathrm{tr}\,\left(\mathbf{QE}^\dagger\,\mathbb{1}_{m\times m}\mathbf{QE}\right) \\ K &= \frac{1}{2}\,\mathbf{QE}^T \wedge \mathbb{1}_{m\times m}\overline{\mathbf{QE}} = \frac{1}{2}\,K^x\,e_x^T \end{aligned} \tag{C.32}$$

Just as in the flat Hyperkähler case the action of the gauge group $\mathcal{G}$ on the hypermultiplets is assumed to be linear and be generated by a set of $4m \times 4m$ matrices T_I:

$$\delta_I \mathbf{l} = T_I\,\mathbf{l} \quad \longrightarrow \quad k_I^u = (T_I)^u{}_v\,q^v \tag{C.33}$$

In order for this action to be an isometry of the Euclidean diagonal metric 10.47 it is necessary and sufficient that the matrices T_I belong to the linearly realized part of the isometry algebra $SO(4,m)$, namely $SO(4) \times SO(m)$. namely:

$$T_I \in SO(4) \times SO(m) \subset SO(4,m) \tag{C.34}$$

The action of $\mathcal{G}$ however is not only required to be isometrical but also to be triholomorphic. This means:

$$\ell_I\,K^x \equiv \mathbf{i}_I\,dK^x + d\,\mathbf{i}_I\,K^x = \nabla W_I^x \tag{C.35}$$

where W_x is the infinitesimal parameter of some $SU(2)$ transformation A straightforward calculation shows that the triholomorphicity condition is that the generators T_I should commute with the tensor product of the 't Hooft matrices with the unit matrix in m–dimensions. When this last condition is verified we can write the momentum maps as:

$$\mathcal{P}_I^x = \mathbf{l}^T\,J^{+|x} \otimes \mathbb{1}_{m\times m}\,T_I\,\mathbf{l} \tag{C.36}$$

Using these ingredients the mass matrices and the scalar potential can be written down without any further difficulty. The quaternionic vielbein is given in full analogy to eq.s 10.61, 10.62, by

$$\mathcal{U}^{A\alpha} \equiv \mathcal{U}^{A\alpha}_{b|s}\,dq^{b|s} == E^{a|t}\,(e_a)^A{}_B \tag{C.37}$$

and, as before, we identify the symplectic index α running on $2m$ values with the pair of indices B, t ($B = 1,2$; $t = 1,\ldots,m$).

Appendix D: Normalizations and conventions

Minkowski metric:

$$\eta_{ab} \equiv (1,-1,-1,-1) \tag{C.38}$$

Definition of the Riemann tensor:

$$R^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu \equiv -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}dx^\rho \wedge dx^\sigma \tag{C.39}$$

Decomposition of tensors in self–dual and antiself–dual parts ($\epsilon_{0123} \equiv 1$):

$$T^\mp_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} \mp \frac{\mathrm{i}}{2}\epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}\right) \tag{C.40}$$

Clifford Algebra:

$$\begin{aligned}
\{\gamma_a, \gamma_b\} &= 2\,\eta_{ab} \\
[\gamma_a, \gamma_b] &= 2\,\gamma_{ab} \\
\gamma_5 &\equiv -\mathrm{i}\,\gamma_0\gamma_1\gamma_2\gamma_3 \\
\gamma_0^\dagger &= \gamma_0; \qquad\qquad \gamma_0\gamma_i^\dagger\gamma_0 \,=\, \gamma_i \qquad (i=1,2,3); \qquad\qquad \gamma_5^\dagger \,=\, \gamma_5 \\
\epsilon_{abcd}\gamma^{cd} &= 2\,\mathrm{i}\,\gamma_{ab}\gamma_5
\end{aligned} \tag{C.41}$$

Decomposition of fermions in chiral and antichiral parts:
the indices of the spinors also fix their chirality according to the following conventions:

$$\gamma_5 \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_A \end{pmatrix} = \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_A \end{pmatrix} , \tag{C.42}$$

$$\gamma_5 \begin{pmatrix} \lambda^{i^\star}_A \\ \zeta^\alpha \\ \psi^A \end{pmatrix} = - \begin{pmatrix} \lambda^{i^\star}_A \\ \zeta^\alpha \\ \psi^A \end{pmatrix} \tag{C.43}$$

Majorana conventions:
For any fermion ϕ :

$$\overline{\phi} \equiv \phi^\dagger\gamma_0 \,=\, \phi^T C \tag{C.44}$$

Fierz rearrangements
Let us denote by a lower or upper dot right and left chirality respectively. Then: *for 0–form spinors* χ, ξ:

$$\begin{aligned}
\chi_\bullet\overline{\xi}_\bullet &= -\frac{1}{2}\,\overline{\xi}_\bullet\chi_\bullet + \frac{1}{8}\,\gamma_{ab}\overline{\xi}_\bullet\gamma^{ab}\chi_\bullet \\
\chi_\bullet\overline{\xi}^\bullet &= -\frac{1}{2}\,\gamma_a\overline{\xi}^\bullet\gamma^a\chi_\bullet
\end{aligned} \tag{C.45}$$

for 1–form spinors ψ_A, ψ^B:

$$\begin{aligned}
\psi_A\overline{\psi}_B &= \frac{1}{2}\,\overline{\psi}_B\psi_A - \frac{1}{8}\,\gamma_{ab}\overline{\psi}_B\gamma^{ab}\psi_A \\
\psi_A\overline{\psi}^B &= \frac{1}{2}\,\gamma_a\overline{\psi}^B\gamma^a\psi_A
\end{aligned} \tag{C.46}$$

Charge conjugation matrix properties:

$$C^2 \,=\, -1; \qquad C^T \,=\, -\,C; \qquad (C\gamma^a)^T \,=\, C\gamma^a; \qquad \left(C\gamma^{ab}\right)^T \,=\, C\gamma^{ab} \tag{C.47}$$

Hermiticity of currents
for 0–form spinors:

$$(\overline{\chi}_\bullet\xi_\bullet)^\dagger \,=\, \overline{\xi}^\bullet\chi^\bullet \,=\, \overline{\chi}^\bullet\xi^\bullet \tag{C.48}$$

$$(\overline{\chi}_\bullet\gamma^a\xi^\bullet)^\dagger \,=\, \overline{\xi}_\bullet\gamma^a\chi^\bullet \,=\, -\overline{\chi}^\bullet\gamma^a\xi_\bullet \tag{C.49}$$

$$\left(\overline{\chi}_\bullet\gamma^{ab}\xi_\bullet\right)^\dagger \,=\, -\overline{\xi}^\bullet\gamma^{ab}\chi^\bullet \,=\, \overline{\chi}^\bullet\gamma^{ab}\xi^\bullet \tag{C.50}$$

for 1–form spinors:

$$\left(\overline{\psi}_A \psi_B\right)^\dagger = -\overline{\psi}^B \psi^A = \overline{\psi}^A \psi^B \tag{C.51}$$

$$\left(\overline{\psi}^A \gamma^a \psi_B\right)^\dagger = -\overline{\psi}^B \gamma^a \psi_A = -\overline{\psi}_A \gamma^a \psi^B \tag{C.52}$$

$$\left(\overline{\psi}^A \gamma^{ab} \psi^B\right)^\dagger = \overline{\psi}_B \gamma^{ab} \psi_A = \overline{\psi}_A \gamma^{ab} \psi_B \tag{C.53}$$

Conventions on Kähler geometry: The hermitean metric is locally given by:

$$g_{ij^\star} = \partial_i \partial_{j^\star} \mathcal{K} \tag{C.54}$$

where the real function $\mathcal{K} = \mathcal{K}^\star = \mathcal{K}(z, z^*)$ is named the *Kähler potential*. It is defined up to the real part of a holomorphic function $f(z)$. Indeed one sees that

$$\mathcal{K}'(z, z^{i^\star}) = \mathcal{K}(z, z^{i^\star}) + \mathrm{Re} f(z) \tag{C.55}$$

gives rise to the same metric $g_{ij^\star}$ as $\mathcal{K}$. The transformation in eq. C.55 is named a *Kähler transformation*.

To fix our notations we write the formulae for the Levi–Civita connection 1–form and Riemann curvature 2–form on a Kähler manifold:

$$\begin{array}{ccccccc}
\Gamma^i_j & = & \Gamma^i_{kj} dz^k & ; & \Gamma^i_{kj} & = & g^{i\ell^*}(\partial_j g_{k\ell^*}) \\
\Gamma^{i^*}_{j^*} & = & \Gamma^{i^*}_{k^* j^*} d\overline{z}^{k^*} & ; & \Gamma^{i^*}_{k^* j^*} & = & g^{i^*\ell}(\partial_{j^*} g_{k^*\ell}) \\
\mathcal{R}^i_j & = & \mathcal{R}^i_{jk^*\ell} d\overline{z}^{k^*} \wedge dz^\ell & ; & \mathcal{R}^i_{jk^*\ell} & = & \partial_{k^*} \Gamma^i_{j\ell} \\
\mathcal{R}^{i^*}_{j^*} & = & \mathcal{R}^{i^*}_{j^* k\ell^*} dz^k \wedge d\overline{z}^{\ell^*} & ; & \mathcal{R}^{i^*}_{j^* k\ell^*} & = & \partial_k \Gamma^{i^*}_{j^*\ell^*}
\end{array} \tag{C.56}$$

$SU(2)$ and $Sp(2n)$ metrics:

$$\epsilon^{AB} \epsilon_{BC} = -\delta^A_C; \qquad \epsilon^{AB} = -\epsilon^{BA} \tag{C.57}$$

$$\mathbb{C}^{\alpha\beta} \mathbb{C}_{\beta\gamma} = -\delta^\alpha_\gamma; \qquad \mathbb{C}^{\alpha\beta} = -\mathbb{C}^{\beta\alpha}; \tag{C.58}$$

For any $SU(2)$ vector P_A we have:

$$\epsilon_{AB} P^B = P_A; \qquad \epsilon^{AB} P_B = -P^A \tag{C.59}$$

and equivalently for $Sp(2n)$ vectors P_α:

$$\mathbb{C}_{\alpha\beta} P^\beta = P_\alpha; \qquad \mathbb{C}^{\alpha\beta} P_\beta = -P^\alpha \tag{C.60}$$

Reality condition for $SU(2)$ valued matrices H^{AB}:

$$\overline{(H^{AB})} = \epsilon^{AC} \epsilon^{BD} H^\star_{CD} \tag{C.61}$$

Table 1: *Homogeneous Symmetric Special Manifolds*

n	G/H	$Sp(2n+2)$	symp rep of G
1	$\frac{SU(1,1)}{U(1)}$	$Sp(4)$	$\underline{\mathbf{4}}$
n	$\frac{SU(1,n)}{SU(n)\times U(1)}$	$Sp(2n+2)$	$\underline{\mathbf{n+1}} \oplus \underline{\mathbf{n+1}}$
$n+1$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2)\times SO(n)}$	$Sp(2n+4)$	$\underline{\mathbf{2}} \otimes (\underline{\mathbf{n+2}} \oplus \underline{\mathbf{n+2}})$
6	$\frac{Sp(6,\mathbb{R})}{SU(3)\times U(1)}$	$Sp(14)$	$\underline{\mathbf{14}}$
9	$\frac{SU(3,3)}{S(U(3)\times U(3))}$	$Sp(20)$	$\underline{\mathbf{20}}$
15	$\frac{SO^\star(12)}{SU(6)\times U(1)}$	$Sp(32)$	$\underline{\mathbf{32}}$
27	$\frac{E_{7(-6)}}{E_6\times SO(2)}$	$Sp(56)$	$\underline{\mathbf{56}}$

Table 2: *Homogeneous symmetric quaternionic manifolds*

m	G/H
m	$\frac{Sp(2m+2)}{Sp(2)\times SP(2m)}$
m	$\frac{SU(m,2)}{SU(m)\times SU(2)\times U(1)}$
m	$\frac{SO(4,m)}{SO(4)\times SO(m)}$
2	$\frac{G_2}{S0(4)}$
7	$\frac{F_4}{Sp(6)\times Sp(2)}$
10	$\frac{E_6}{SU(6)\times U(1)}$
16	$\frac{E_7}{S0(12)\times SU(2)}$
28	$\frac{E_8}{E_7\times SU(2)}$

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