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ONE-LOOP CORRECTIONS TO COUPLING CONSTANTS IN STRING EFFECTIVE FIELD THEORY ◇

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Abstract

In the framework of a recently proposed method for computing exactly string amplitudes regularized in the infra-red, I determine the one-loop correlators for auxiliary fields in the symmetric $Z_2 \times Z_2$ orbifold model. The D-field correlation function turns out to give the one-loop corrections for the gauge couplings, which amounts to a string-theory supersymmetry Ward identity. The two-point function for uncharged F fields leads to the one-loop renormalization of the moduli Kähler metric, and eventually to the corrections for the Yukawa couplings.

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1. Introduction

String theories have been advocated as the best candidates for unifying all known and yet to be discovered interactions. This unification takes place at the Planck scale, which is the natural scale for a theory that contains quantum gravity. However, in order to make contact with low-energy physics and eventually to reach phenomenological predictions, one needs a consistent description of the massless degrees of freedom of the string and it is remarkable that this part of the spectrum can be described in terms of an effective field theory which, for superstring compactifications, turns out to be a supersymmetric Yang–Mills theory coupled to gravity.

The most general $N = 1$ supergravity action including up to two derivatives is fully specified by its Kähler potential $K(z, \bar{z})$, its superpotential $W(z)$ and its gauge function $f(z)$ [1]. These are respectively real and holomorphic functions of the chiral-superfield scalar components. The Kähler potential determines the matter kinetic terms while the superpotential and the gauge function are related to the Yukawa and gauge couplings respectively. One of the main tasks for the determination of the low-energy string effective theory is the computation of these functions in the fundamental theory. This is achieved by computing adequate string amplitudes as functions of the vev's of gauge singlet fields corresponding to the moduli of the internal manifold. In most cases (including symmetric orbifolds) such a computation can be performed at the level of the sphere and leads in particular to the so-called string unification: both gauge and Yukawa couplings are homogeneously related to the dilaton vev [2], which plays the role of the string coupling constant. Of major importance are, however, the string loop corrections, since they contain all the information that one needs at low energies about the decoupled infinite tower of massive string modes [3]. The effects of these massive states are summarized in the threshold corrections, which are non-trivial functions of the moduli and spoil the tree-level relations among couplings [4–7]. The threshold corrections are therefore important for the issue of string unification below the Planck mass [8, 9]. They may also have implications in the problem of supersymmetry breaking.

There is no general method for computing string loop amplitudes. When the space-time fields involved in these amplitudes are associated with truly marginal world-sheet vertex operators, the approach becomes more systematic. Classical background sources are introduced for these fields that generate exactly marginal deformations. Hence, if one can solve the vacuum amplitude for the deformed model, any correlation function of these operators will be reached exactly, including the back-reaction of all gravitationally coupled fields, by taking derivatives with respect to the above sources. This is possible, in general, in lattice compactifications whenever the deformation corresponds to a Lorentz boost. However, even in those cases, ambiguities related to the infra-red divergences may arise, due to the on-shell formulation of string theory in the presence of massless particles. In a recent article [7] an interesting method for computing unambiguously the string loop corrections has been proposed. The procedure that is used consists of replacing flat four-dimensional space-time with a suitably chosen curved one in a way that preserves gauge symmetry, supersymmetry and modular invariance, and with curvature that induces an infra-red cut-off.

In these notes I would like to illustrate the aforementioned methods and compute the one-loop correction to the Kähler metric for some moduli fields in a particular heterotic-string compactification: the $Z_2 \times Z_2$ orbifold model. The one-loop correction to the Kähler metric, i.e. the scalar wave-function renormalization, and its moduli dependance are of prime importance in the determination of the Yukawa couplings. In fact, as long as supersymmetry is unbroken, the superpotential does not receive any loop correction [10], and consequently the only corrections to the Yukawa couplings are induced by the wave-function renormalizations. Of course, there is experimental evidence in favor of non-supersymmetric low-energy physics. Nevertheless, assuming that the breaking of supersymmetry occurs at a scale of the order of 1 TeV, non-supersymmetric corrections to dimensionless parameters are not important, and there is some relevance in computing corrections in the context of unbroken supersymmetry.

In order to apply the background-field method successfully, I will compute string vacuum amplitudes with insertions of F auxiliary fields associated with the untwisted moduli. The corresponding vertex operators turn out to be truly marginal world-sheet operators. Their classical background fields generate Lorentz boosts, and any of their correlation functions is therefore exactly calculable. Thanks to the presence of supersymmetry, one expects the correlation functions determined in this way to be identical to those that would have been obtained by using insertions of the moduli-field kinetic terms, and therefore to lead to the moduli wave-function renormalization. Of course, this statement is based on the existence of Ward identities, which relate the renormalization of various members of a supermultiplet and that have not been strictly proved in string theory. However, in order to show the consistency of my calculation, I will proceed to the determination of amplitudes with insertions of D auxiliary fields. The result for the two-point function turns out to be in this case identical to the one obtained in [7] for the double insertion of the magnetic field operator, and thus demonstrates a supersymmetry Ward identity for the gauge multiplet.

These notes are organized as follows. Section 2 will be a reminder of the $Z_2 \times Z_2$ orbifold compactification of the heterotic string. I will essentially settle the notations and explain how to compute exactly one-loop string amplitudes of vertex operators associated with a marginal background deformation corresponding to Lorentz boosts. In section 3, I will proceed to the explicit determination of the two-point function for the D fields associated with the vector supermultiplet. This will lead to the one-loop gauge coupling correction. Section 4 will be devoted to an analogous computation for the F components of the untwisted moduli chiral superfields. I will conclude on the possible generalizations of this work in the last section.

2. Marginal background deformations and the $Z_2 \times Z_2$ orbifold

The background field method has been extensively used in field theory for the computation of various quantum corrections. It is easily adapted to string theory where the aim is to determine, order by order in the topological expansion, the low-energy effective theory parameters. These parameters are associated with four-dimensional fields that correspond to massless string excitations. Their computation therefore amounts to the evaluation of string amplitudes

$$\int_{\mathcal{F}_g} d\Omega_g \left\langle \frac{1}{16\pi^3} \int 2i\, dz\, d\bar{z}\, V_1(z,\bar{z})\, \frac{1}{16\pi^3} \int 2i\, dz\, d\bar{z}\, V_2(z,\bar{z}) \ldots \right\rangle_g \tag{1}$$

at zero external momenta. Hence, $V_j(z,\bar{z})$ are the internal conformal field theory part of some massless vertex operators. In expression (1) $d\Omega_g$ is the invariant measure for genus-g moduli, $\mathcal{F}_g$ the corresponding fundamental domain and $\langle\ \rangle_g$ denotes a conformal field theory correlation function on a genus-g surface. Vertices $V_j(z,\bar{z})$ are weight-$(1,1)$ operators, and correlators such as those appearing in (1) are in principle calculable by switching on the corresponding constant backgrounds f_j, namely by adding to the two-dimensional action the deformation

$$\Delta S = -\frac{1}{16\pi^3} \int 2i\, dz\, d\bar{z} \sum_j f_j\, V_j(z,\bar{z})\ , \tag{2}$$

and computing the deformed vacuum amplitude $Z\left(\mathbf{f}\right) = \left\langle e^{-\Delta S}\right\rangle$ (I dropped the subscript g since, from now on, I will be interested in the torus only).

In general, one wishes to compute exactly the string amplitudes (1), i.e. to all orders in α'. This is possible provided ΔS is a conformal deformation, which requires the vertices appearing in (2) to form a set of truly marginal operators. Moreover, $Z\left(\mathbf{f}\right)$ should be computed for finite backgrounds since the back-reaction of gravity, which one would like to take properly into account, appears generally in the next-to-leading order in f_j. Situations where all these requirements are satisfied appear when, in string lattice compactifications, $V_j(z,\bar{z})$ generate lattice Lorentz boosts. To clarify this issue I will concentrate on the case where the deformation is generated by $V(z,\bar{z}) = f_1\, J(z)\, \overline{J}_1(\bar{z}) + f_2\, J(z)\, \overline{J}_2(\bar{z})$. Here $J(z)$ and $\overline{J}_{1,2}(\bar{z})$ are $(1,0)$ and $(0,1)$ currents*; moreover, I assume that $\overline{J}_1(\bar{z})$ and $\overline{J}_2(\bar{z})$ commute, and thus $V(z,\bar{z})$ is an exactly marginal operator. Their lattice momenta are Q and $\overline{Q}_{1,2}$ respectively. In terms of these charges, the undeformed one-loop partition function can be written in the Hamiltonian approach as follows:

$$Z = \operatorname{Tr}\, \exp\Big(-2\pi\, \operatorname{Im}\tau \left(L_0 + \overline{L}_0\right) + 2\pi i\, \operatorname{Re}\tau \left(L_0 - \overline{L}_0\right) \Big)\ , \tag{3}$$

where τ is the genus-1 modular parameter and

$$L_0 = \frac{1}{2} Q^2 + \cdots\ ,\ \overline{L}_0 = \frac{1}{2} \overline{Q}_1^{\,2} + \frac{1}{2} \overline{Q}_2^{\,2} + \cdots\ ; \tag{4}$$

* The normalization for these currents is such that $J(z)\, J(w) = \frac{1}{(z-w)^2} + \cdots$ and similarly for the right-moving ones. If they are elements of an affine Lie algebra at level 1, this normalization implies that the higher root has $\psi^2 = 2$.

the dots stand for charges other than Q and $\overline{Q}_{1,2}$. The perturbation generated by $\Delta S = -\frac{1}{16\pi^3}\int 2i\, dz\, d\bar{z}\, V(z,\bar{z})$ turns out to be a Lorentz boost that can be easily implemented in the Hamiltonian description (3) and (4). The action of this boost on the charges is

$$\begin{pmatrix} Q' \\ \overline{Q}_1' \\ \overline{Q}_2' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 \\ \sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} Q \\ \overline{Q}_1 \\ \overline{Q}_2 \end{pmatrix} , \tag{5}$$

leading to

$$\begin{aligned} \Delta\left(L_0 + \overline{L}_0\right) = & \frac{\sqrt{1+\frac{f_1^2+f_2^2}{64\pi^4}}-1}{2}\left(Q^2 + \frac{\left(f_1\,\overline{Q}_1 + f_2\,\overline{Q}_2\right)^2}{f_1^2+f_2^2}\right) \\ & + \frac{f_1}{8\pi^2}\,Q\,\overline{Q}_1 + \frac{f_2}{8\pi^2}\,Q\,\overline{Q}_2\,, \end{aligned} \tag{6}$$

while $L_0 - \overline{L}_0$ remains invariant. To obtain eq. (6) I identified the constant backgrounds f_1 and f_2 with $8\pi^2\sinh 2\phi\cos\theta$ and $8\pi^2\sinh 2\phi\sin\theta$, respectively. The deformed partition function now reads

$$Z\left(\mathbf{f}\right) = \operatorname{Tr}\,\exp\Big(-2\pi\operatorname{Im}\tau\left(L_0 + \overline{L}_0 + \Delta\left(L_0+\overline{L}_0\right)\right) + 2\pi i\operatorname{Re}\tau\left(L_0 - \overline{L}_0\right)\Big)\,, \tag{7}$$

which is exact for any finite value of f_j, and contains the gravity back-reaction generated by these non-vanishing background fields. Derivatives with respect to f_j allow one to compute any correlation function, provided the insertion of the charges Q and $\overline{Q}_{1,2}$ into the trace is tractable, which turns out to be generally the case as will be seen in the sequel.

The method described above is general and can be applied successfully to many situations. In the following, I will stick to a particular string vacuum, namely the symmetric $Z_2 \times Z_2$ orbifold in the heterotic construction. This model has $N = 1$ space-time supersymmetry and an $E_8 \times E_6 \times U(1)^2$ gauge group, which is promoted to $E_8 \times E_6 \times SU(2) \times U(1)$ or even $E_8 \times E_6 \times SU(3)$ at some special values of the moduli describing the internal manifold. The latter is a six-dimensional torus parametrized by three pairs of complex moduli: $T_i\,,\ U_i\,,\ i = 1,2,3$. I choose this model because it has the advantage of reproducing generic features, although it is quite simple. An interesting property is, for instance, the absence of ($N = 1$)-sector contributions to the beta-function coefficients, which avoids the related holomorphic anomalies.

The partition function for the $Z_2 \times Z_2$ orbifold heterotic compactification reads

$$\begin{aligned} Z(\tau,\bar{\tau}) = & \frac{1}{\operatorname{Im}\tau(\eta\bar{\eta})^2}\,\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\,\frac{\vartheta\left[{a \atop b}\right]}{\eta} \\ & \frac{1}{4}\sum_{h_j,g_j}\frac{\vartheta\left[{a+h_1 \atop b+g_1}\right]}{\eta}\,\frac{\vartheta\left[{a+h_2 \atop b+g_2}\right]}{\eta}\,\frac{\vartheta\left[{a+h_3 \atop b+g_3}\right]}{\eta}\,Z_{2,2}\begin{bmatrix} h_1 \\ g_1 \end{bmatrix}\,Z_{2,2}\begin{bmatrix} h_2 \\ g_2 \end{bmatrix}\,Z_{2,2}\begin{bmatrix} h_3 \\ g_3 \end{bmatrix} \\ & \frac{1}{2}\sum_{\bar{a},\bar{b}}\frac{\bar{\vartheta}\left[{\bar{a}+h_1 \atop \bar{b}+g_1}\right]}{\bar{\eta}}\,\frac{\bar{\vartheta}\left[{\bar{a}+h_2 \atop \bar{b}+g_2}\right]}{\bar{\eta}}\,\frac{\bar{\vartheta}\left[{\bar{a}+h_3 \atop \bar{b}+g_3}\right]}{\bar{\eta}}\left(\frac{\bar{\vartheta}\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}}\right)^5\frac{1}{2}\sum_{\bar{c},\bar{d}}\left(\frac{\bar{\vartheta}\left[{\bar{c} \atop \bar{d}}\right]}{\bar{\eta}}\right)^8\,, \end{aligned} \tag{8}$$

where the summation over the orbifold sectors is subject to the condition $h_1+h_2+h_3=0$ and similarly for the g_j's. The first line of eq. (8) corresponds to the transverse space-time coordinates together with their left-moving supersymmetric partners; the internal bosonic contribution together with the corresponding left-moving fermionic one are given in the second line; the last line is associated with the 32 right-moving fermions that generate the gauge group as well as, together with the left-moving fermions, the internal $(2,2)$ superconformal symmetry. The internal bosons $X^4,\dots,X^9$ are compactified on three two-tori. In the following sections, I will use their complex combinations

$$\begin{aligned} Y^1 &= \frac{X^4+iX^5}{\sqrt{2}}\,, \quad Y^2 = \frac{X^6+iX^7}{\sqrt{2}}\,, \quad Y^3 = \frac{X^8+iX^9}{\sqrt{2}}\,, \\ Y^{-1} &= \frac{X^4-iX^5}{\sqrt{2}}\,, \; Y^{-2} = \frac{X^6-iX^7}{\sqrt{2}}\,, \; Y^{-3} = \frac{X^8-iX^9}{\sqrt{2}}\,, \end{aligned} \tag{9}$$

together with their fermionic partners

$$\begin{aligned} \Psi^1 &= \frac{\psi^4+i\psi^5}{\sqrt{2}}\,, \quad \Psi^2 = \frac{\psi^6+i\psi^7}{\sqrt{2}}\,, \quad \Psi^3 = \frac{\psi^8+i\psi^9}{\sqrt{2}}\,, \\ \Psi^{-1} &= \frac{\psi^4-i\psi^5}{\sqrt{2}}\,, \; \Psi^{-2} = \frac{\psi^6-i\psi^7}{\sqrt{2}}\,, \; \Psi^{-3} = \frac{\psi^8-i\psi^9}{\sqrt{2}}\,. \end{aligned} \tag{10}$$

In our conventions[†] the two-torus partition function with periodic boundary conditions for both compactified bosons reads

$$\begin{aligned} Z_{2,2}\begin{bmatrix}0\\0\end{bmatrix}\left(T_j,U_j,\overline{T}_j,\overline{U}_j\right) &= \frac{1}{(\eta\bar{\eta})^2}\Gamma_{2,2}\left(T_j,U_j,\overline{T}_j,\overline{U}_j\right) \\ &= \frac{1}{(\eta\bar{\eta})^2}\sum_{\mathbf{m},\mathbf{n}} q^{\left|P^L_j\right|^2}\bar{q}^{\left|P^R_j\right|^2}\,, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \begin{pmatrix} P^L_j \\ P^R_j \end{pmatrix} =& \frac{1}{2i\sqrt{\operatorname{Im}T_j \operatorname{Im}U_j}}\times \\ & \times\left\{ m_1\begin{pmatrix} U_j \\ \overline{U}_j\end{pmatrix} - m_2\begin{pmatrix}1\\1\end{pmatrix} + n_1\begin{pmatrix}T_j\\ \overline{T}_j\end{pmatrix} + n_2\begin{pmatrix}T_j\,U_j\\ \overline{T}_j\,U_j\end{pmatrix}\right\} \end{aligned} \tag{12}$$

are the momenta associated with the left-moving $i\partial Y^j$ and right-moving $i\bar{\partial}Y^j$ currents. Similarly $(\,P^{L*}_j \quad P^{R*}_j\,)$ correspond to $i\partial Y^{-j}$ and $i\bar{\partial}Y^{-j}$. On the other hand, when the bosons are twisted,

$$Z_{2,2}\begin{bmatrix}h_j\\g_j\end{bmatrix} = \frac{4\eta\bar{\eta}}{\sqrt{\vartheta\left[{1+h_j \atop 1+g_j}\right]\vartheta\left[{1-h_j \atop 1-g_j}\right]\bar{\vartheta}\left[{1+h_j \atop 1+g_j}\right]\bar{\vartheta}\left[{1-h_j \atop 1-g_j}\right]}}\,, \tag{13}$$

[†] Here, when the real parts of the moduli vanish, the imaginary parts read $\operatorname{Im}T_j = R^1_j\,R^2_j$ and $\operatorname{Im}U_j = R^1_j/R^2_j$, where R^1_j and R^2_j are the compactification radii for the jth plane. Notice also that $\alpha'=1$.

which is moduli-independent.

The perturbative formulation of string theory does not allow one to go easily off-shell in the string amplitudes. This does not affect so much the sphere computations, but once one goes to the loops, the presence of massless degrees of freedom leads to infra-red divergences which, on the torus, appear at large values of $\mathrm{Im}\,\tau$. In a recent article [7] a method for regulating consistently these divergences has been developed. It consists of replacing the four-dimensional flat space-time with a more general σ-model preserving gauge symmetries, supersymmetry and modular invariance and with a curvature that induces a mass gap acting as an infra-red regulator. Among other requirements, this σ-model must be $N=4$ superconformal in order to be able to accommodate up to two space-time supersymmetries *. This leads to several candidates [11] among which the simplest for the above purposes is a $(1,0)$ supersymmetric version of $W_k^{(4)} = U(1)_Q \times SU(2)_k$, with $Q = \sqrt{\frac{1}{k+2}}$ a background charge for the time coordinate chosen such that $\hat{c}_{\text{space}-\text{time}} = 4$. In practice (see [7, 12] for the details) this modification to the flat space accounts for an extra factor in the partition function (eq. (8)) corresponding to the (suitably normalized) $SU(2)_k$ partition function †

$$\Gamma(\mu) = -2\mu^2\sqrt{\mathrm{Im}\,\tau}\frac{\partial}{\partial\mu}\left[\Gamma_{1,1}(\mu) - \Gamma_{1,1}(2\mu)\right]\bigg|_{\mu=\frac{1}{\sqrt{k+2}}}, \tag{14}$$

with

$$\Gamma_{1,1}(\mu) = \sum_{m,n} e^{\frac{i\pi\tau}{2}\left(m\mu+\frac{n}{\mu}\right)^2} e^{-\frac{i\pi\bar\tau}{2}\left(m\mu-\frac{n}{\mu}\right)^2}. \tag{15}$$

This extra factor $\Gamma(\mu)$ ensures the convergence of the integrals at large values of $\mathrm{Im}\,\tau$ by introducing a universal mass gap $\Delta m^2 = \frac{\mu^2}{2}$ with $\mu = \frac{1}{\sqrt{k+2}}$ (in $M_s \equiv \frac{1}{\sqrt{\alpha'}}$ units) to all (bosonic and fermionic) string excitations. This infra-red regularization of the string (on-shell) loop amplitudes vanishes when the flat-space limit is reached since $\lim_{\mu\to 0}\Gamma(\mu) = 1$.

Before going further in the application of the methods described so far, let me comment on another consequence of the curvature in the space-time sector of the model at hand. The introduction of a curved background not only regulates the infra-red but makes it possible for vertices such as the chromo-magnetic field, which are not well-defined conformal operators on the flat space, to become truly marginal on the σ-model version. This is precisely the feature that allows the resolution of the $Z_2 \times Z_2$ model in the presence of a finite constant chromo-magnetic background, leading therefore to the

* In the above heterotic string theory, when the space-time σ-model is coupled with the internal theory, the superconformal invariance reduces to N=2.

† Actually a Z_2-orbifold version of the $SU(2)_k$ WZW model (or, put differently, a $SO(3)_{k/2}$ model) is necessary to guarantee the exact matching of all degrees of freedom at the limit $k\to\infty$, when flat space-time is reached.

exact one-loop corrections for the gauge couplings [7]. More generally, an interesting question that one can address is whether a suitable choice of space-time background is possible in order to promote a given vertex to the rank of an exact conformal operator.

3. D auxiliary fields and the renormalization of the gauge coupling

I now turn to the computation of string amplitudes involving auxiliary $D_{(\alpha)a}$ fields that belong to the vector multiplet; here α labels the gauge-group factor G_α and a is the adjoint representation index. The (zero-momentum) vertex operators for these fields are obtained by acting on the gauge-field vertex in the minus-one-ghost picture with an appropriate form [13]. The result is the operator

$$V\left(D_{(\alpha)a}\right) = \frac{1}{\sqrt{3}}\left(\Psi^{-1}\Psi^1 + \Psi^{-2}\Psi^2 + \Psi^{-3}\Psi^3\right)\frac{\overline{J}_{(\alpha)a}}{\sqrt{k_\alpha}}\,, \tag{16}$$

formally in the zero-ghost picture though it is not physical since it does not survive the GSO projection. The bilinears $\Psi^{-j}\Psi^j$ are the internal helicity operators, one for each of the three planes, and Q^j the corresponding fermionic charges. The left-moving factor of (16) is the internal $N=2$ superconformal current and $\overline{J}_{(\alpha)a}$ are elements of the affine Lie algebra that realizes the gauge group; $\overline{Q}_{(\alpha)a}$ are their lattice momenta. It deserves stressing here that these results are generic for any symmetric orbifold $(2,2)$ heterotic compactification; this is the reason why I introduced explicitly the level k_α of the affine Lie algebra*, which is equal to 1 for all group factors in the $Z_2\times Z_2$ model.

Being the product of a left times a right current, the set of operators $V\left(D_{(\alpha)a}\right)$ with index a corresponding to the Cartan subalgebra generate an exactly integrable r-dimensional deformation, where $r=\sum_\alpha r_\alpha$ is the rank of the gauge group. The deformation is given by the $SO(1,r)/SO(r)$ Lorentz boosts. To get a flavour of the effect of an auxiliary-field background on the string, it is not necessary to look at the most general deformation and I will actually restrict the following analysis to the case of a single Cartan direction in the group factor G_α. The perturbation under consideration, at the level of the action, is now

$$\Delta S_\alpha = -\frac{d_\alpha}{16\pi^3}\int 2i\,dz\,d\bar{z}\,V\left(D_{(\alpha)}\right) \tag{17}$$

(I dropped the index a by choosing a direction and there is no summation over α), and the deformed toroidal partition function $Z\left(d_\alpha\right)$ is given by eq. (7) with

$$\Delta\left(L_0+\overline{L}_0\right) = \frac{\sqrt{1+\frac{d_\alpha^2}{64\pi^4}}-1}{2}\left(\frac{\left(Q^1+Q^2+Q^3\right)^2}{3}+\frac{\overline{Q}_{(\alpha)}^2}{k_\alpha}\right)$$

* I assume again that the highest root of the algebra has length squared equal to 2. Therefore, the residue of the short-distance leading singularities of bilinears in $\frac{\overline{J}_{(\alpha)a}}{\sqrt{k_\alpha}}$ is 1. This agrees with the normalizations of section 2. Note, however, that in the effective field theory the usual normalizations for the group algebra are such that $\psi^2=1$, and one has to be careful when identifying the effective renormalization constants with the corresponding string amplitudes. This has been properly taken into account throughout these notes.

$$+ \frac{d_\alpha}{8\pi^2} \frac{Q^1+Q^2+Q^3}{\sqrt{3}} \frac{\overline{Q}_{(\alpha)}}{\sqrt{k_\alpha}} . \tag{18}$$

This expression does not vanish as long as $d_\alpha \neq 0$ since a constant D-field deformation ((16), (17)) breaks the $(2,2)$ superconformal invariance (it preserves however the $(1,0)$ world-sheet supersymmetry).

Recalling that in the low-energy effective action the vector-multiplet kinetic terms are of the form

$$\sum_{\alpha,a} \frac{1}{g_\alpha^2} \left(-\frac{1}{4} F_{(\alpha)a\mu\nu} F_{(\alpha)a}{}^{\mu\nu} + \frac{1}{2} D_{(\alpha)a} D_{(\alpha)a} \right) , \tag{19}$$

the one-loop string correction to $\frac{16\pi^2}{g_\alpha^2}$ is given by

$$Z_{2,d_\alpha} = 16\pi^2 k_\alpha \int_{\mathcal{F}} \frac{d^2\tau}{(\,\mathrm{Im}\,\tau)^2} \frac{\partial^2 Z\,(d_\alpha)}{\partial d_\alpha^2} \bigg|_{d_\alpha=0} . \tag{20}$$

Equations (7) and (18) allow one to compute the above derivative with the result:

$$\begin{aligned} 16\pi^2 k_\alpha \frac{\partial^2 Z}{\partial d_\alpha^2}\bigg|_{d_\alpha=0} = \ & \mathrm{Tr} \Bigg\{ \exp\Big(-2\pi\,\mathrm{Im}\,\tau \left(L_0+\overline{L}_0\right) + 2\pi i\,\mathrm{Re}\,\tau \left(L_0-\overline{L}_0\right)\Big) \\ & \qquad (\,\mathrm{Im}\,\tau)^2 \Bigg(\frac{\left(Q^1+Q^2+Q^3\right)^2}{3} \overline{Q}^2_{(\alpha)} \\ & \qquad - \frac{k_\alpha}{4\pi\,\mathrm{Im}\,\tau} \left[\frac{\left(Q^1+Q^2+Q^3\right)^2}{3} + \frac{\overline{Q}^2_{(\alpha)}}{k_\alpha} \right] \Bigg) \Bigg\} . \end{aligned} \tag{21}$$

The last term of the post-exponential factor vanishes since the insertion of $\overline{Q}^2_{(\alpha)}$ does not affect the supersymmetry properties of the left-moving sector. All I have to do is therefore to compute the vacuum trace with the insertion of

$$(\,\mathrm{Im}\,\tau)^2 \frac{\left(Q^1+Q^2+Q^3\right)^2}{3} \left(\overline{Q}^2_{(\alpha)} - \frac{k_\alpha}{4\pi\,\mathrm{Im}\,\tau} \right) . \tag{22}$$

This insertion can be performed by noting that it is equivalent to the action of a differential operator on the undeformed vacuum amplitude. Indeed, Q^j acts as $\frac{1}{2\pi i} \frac{\partial}{\partial v_j}\Big|_{v_j=0}$ on the ith-plane holomorphic ϑ function, while $\overline{Q}^2_{(\alpha)}$ acts as $\frac{i}{\pi}\frac{\partial}{\partial\bar{\tau}}$ on the appropriate subfactor of the 32 right-moving-fermion contribution.

I will now focus on the explicit computation of the above trace (eq. (21)) in the specific case of the $Z_2 \times Z_2$ orbifold model. The partition function for that model is given by (8), multiplied by the extra factor (14) for the infra-red regularization. One can use the generalized Jacobi identity (valid under the condition $\sum_j h_j = \sum_j g_j = 0$)

$$\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\,\vartheta\begin{bmatrix}a\\b\end{bmatrix}(v_0\mid\tau)\prod_{j=1}^{3}\vartheta\begin{bmatrix}a+h_j\\b+g_j\end{bmatrix}(v_j\mid\tau)=$$
$$=\vartheta\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{v_0-\sum_j v_j}{2}\middle|\tau\right)\prod_{j=1}^{3}\vartheta\begin{bmatrix}1-h_j\\1-g_j\end{bmatrix}\left(\frac{v_0+\cdots-v_j+\cdots}{2}\middle|\tau\right) \tag{23}$$

to recast (8) in a form where the insertion of $\frac{\left(Q^1+Q^2+Q^3\right)^2}{3}\equiv-\frac{1}{12\pi^2}\left(\sum_j\frac{\partial}{\partial v_j}\right)^2\Big|_{\mathbf{v}=0}$ is more transparent. Indeed, it becomes clear that the only non-zero contributions appear when each of the two derivatives acts on a ϑ_1. Hence, only the $N=2$ sectors contribute (notice that for other orbifold models such as the Z_3, there might exist non-vanishing ($N=1$)-sector contributions). Using the identities $\frac{\vartheta_1'(0|\tau)}{2\pi}=\eta^3(\tau)=\frac{\vartheta_2(0|\tau)\,\vartheta_3(0|\tau)\,\vartheta_4(0|\tau)}{2}$, one finally obtains

$$Z_{2,d_\alpha}(\mu)=\int_{\mathcal{F}}\frac{d^2\tau}{\operatorname{Im}\tau}\,\Gamma(\mu)\sum_j\frac{\Gamma_{2,2}(j)}{\bar\eta^{24}}\left(\overline{Q}^2_{(\alpha)}-\frac{k_\alpha}{4\pi\operatorname{Im}\tau}\right)\overline{\Omega}\,, \tag{24}$$

where $k_\alpha=1$, $\Gamma_{2,2}(j)\equiv\Gamma_{2,2}\left(T_j,U_j,\overline{T}_j,\overline{U}_j\right)$ is given by (11), $\Gamma(\mu)$ is the infra-red regulating factor (see eqs. (14) and (15)), and $\overline{\Omega}=\overline{\Omega}_8\,\overline{\Omega}_6$, with

$$\Omega_8=\frac{1}{2}\sum_a\vartheta_a^8\ \text{and}\ \Omega_6=\frac{1}{4}\left(\vartheta_2^4+\vartheta_3^4\right)\left(\vartheta_3^4+\vartheta_4^4\right)\left(\vartheta_2^4-\vartheta_4^4\right) \tag{25}$$

(these are proportional to the Eisenstein modular-covariant functions E_2 (or G_2) and E_3 (or G_3), respectively: $\Omega_8=E_2=\frac{G_2}{2\zeta(4)}$ and $-2\Omega_6=E_3=\frac{G_3}{2\zeta(6)}$ [14] †). Notice that the contribution to the partition function of the twisted bosons cancels that of the twisted fermions. Although the result (24) has been achieved in the framework of the $Z_2\times Z_2$ model, it actually applies to any $(2,2)$ symmetric orbifold once the appropriate modifications are performed at the level of the moduli-dependent function $\sum_j\Gamma_{2,2}(j)$ as well as of the modular function $\overline{\Omega}$ [15]. It is interesting to observe that the radiative corrections (24) include exactly the back-reaction of the gravitationally coupled fields; this accounts for the term $\frac{-k_\alpha}{4\pi\operatorname{Im}\tau}$, which is universal and guarantees modular invariance.

An important conclusion that the above analysis allows one to draw is the following: the one-loop correction to $\frac{16\pi^2}{g_\alpha^2}$ calculated here (expression (24)) coincides with the one that was first obtained in [7] by considering string amplitudes with two magnetic

† Some authors (as in ref. [19]) use the notation E_{2k} instead of E_k, $k\geq 2$.

field insertions. As appears from the low-energy field theory (see eq. (19)), the one-loop correction to $\frac{16\pi^2}{g_\alpha^2}$ is indeed expected to be given by

$$Z_{2,b_\alpha} = -16\pi^2 k_\alpha \int_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \frac{\partial^2 Z\,(b_\alpha)}{\partial b_\alpha^2}\bigg|_{b_\alpha=0} \tag{26}$$

provided the supersymmetry Ward identity that relates the $F_{(\alpha)a}{}^{\mu\nu}$- and $D_{(\alpha)a}$-field renormalization holds at the level of the fundamental theory. In eq. (26) $Z\,(b_\alpha)$ is the partition function in the presence of a constant background chromo-magnetic field b_α pointing for instance in the third space direction, and in some Cartan direction of the group algebra. In flat space, two-dimensional conformal invariance is broken by a constant magnetic background because of gravity back-reaction. As already mentioned in the previous section, however, in the $W_k^{(4)}$ σ-model, a constant magnetic field induces a conformal deformation generated by

$$\Delta S_\alpha = -\frac{b_\alpha}{16\pi^3}\int 2i\, dz\, d\bar{z}\, \frac{J^3 + i : \psi^1\psi^2 :}{\sqrt{\frac{k}{2}+1}}\, \frac{\overline{J}_{(\alpha)}^{\text{ any Cartan}}}{\sqrt{k_\alpha}} \tag{27}$$

and leads to the result *:

$$\begin{aligned} 16\pi^2 k_\alpha \frac{\partial^2 Z}{\partial b_\alpha^2}\bigg|_{b_\alpha=0} = \operatorname{Tr}\Bigg\{ &\exp\Big(-2\pi \operatorname{Im}\tau\,\big(L_0+\overline{L}_0\big) + 2\pi i \operatorname{Re}\tau\,\big(L_0-\overline{L}_0\big)\Big) \\ &(\operatorname{Im}\tau)^2 \bigg((I+Q)^2\, \overline{Q}^2_{(\alpha)} \\ &- \frac{k_\alpha}{4\pi \operatorname{Im}\tau}\left[(I+Q)^2 + \overline{Q}^2_{(\alpha)}\, \frac{k+2}{2k_\alpha}\right]\bigg)\Bigg\}. \end{aligned} \tag{28}$$

Here J^3 is a $SO(3)_{\frac{k}{2}}$ current, I is the corresponding zero mode and Q are the fermionic charges. As a consequence of supersymmetry properties, the insertion of I^2 and $\overline{Q}^2_{(\alpha)}$ vanish; that of IQ could contribute in the presence of $N=1$ sectors but is actually zero because of space isotropy. Finally the insertion of Q^2 amounts to the action of $-\frac{1}{4\pi^2}\,\frac{\partial^2}{\partial v_0^2}\Big|_{v_0=0}$ on the holomorphic functions $\vartheta\left[{a \atop b}\right](v_0 \mid \tau)$, which, thanks to the generalized Jacobi identity, is equivalent to the action of $\frac{1}{12\pi^2}\left(\sum_j \frac{\partial}{\partial v_j}\right)^2\Big|_{\mathbf{v}=0}$ on $\prod_{j=1}^3 \vartheta\left[{a+h_j \atop b+g_j}\right](v_j \mid \tau)$. This demonstrates that

$$Z_{2,b_\alpha} = Z_{2,d_\alpha}\,, \tag{29}$$

* In order to take into account the finite volume of the $SU(2)_k$ manifold and recover the correct smooth flat-space limit at $k\to\infty$, the relation between the magnetic field and the boost parameter is here $b_\alpha = \frac{8\pi^2 \sinh 2\phi}{\sqrt{k/2+1}}$.

which is the anticipated Ward identity, at the one-loop level.

Finally, I would like to emphasize that the one-loop corrections to the gauge couplings (24), as was extensively discussed in [7], are exact, i.e. that they contain rigorously both universal and group-factor-dependent thresholds. The former were missing in the pioneering works on string thresholds [3]. They actually contain, besides the gravity back-reaction term[†] $\frac{-k_\alpha}{4\pi\,\mathrm{Im}\,\tau}$, other contributions originated from the insertion of $\overline{Q}^2_{(\alpha)}$. These contributions, which guarantee modular invariance and infra-red finiteness, have been worked out for the symmetric $Z_2 \times Z_2$ orbifold in [9], where the effects of the (moduli-dependent) universal thresholds on string unification are also analyzed; generalization to a larger class of models can be found in [15]. As far as the infra-red regularization is concerned, I should also mention here that the regularized correlator $Z_{2,d_\alpha}(\mu)$ (eq. (24)) is related to the one-loop effective theory running gauge coupling $g_\alpha(\mu)$ in a very simple way [9]:

$$\frac{16\pi^2}{g_\alpha^2(\mu)} = k_\alpha \frac{16\pi^2}{g_s^2} + Z_{2,d_\alpha}(\mu) - \mathbf{b}_\alpha(2\gamma+2)\,, \tag{30}$$

where g_s is the string coupling and $\mathbf{b}_\alpha$ the beta-function coefficients. This expression holds in the $\overline{DR}$ scheme and despite the presence of the curvature-induced infra-red regulator $\Gamma(\mu)$ in $Z_{2,d_\alpha}(\mu)$, the thresholds are infra-red cut-off independent [9]. This shows how the above regularization procedure allows one to avoid ambiguities.

4. One-loop corrections to the Kähler metric for moduli fields

The kinetic terms for the scalar fields and the corresponding auxiliary F fields appear in the low-energy effective action as

$$K^{\bar{j}}{}_{i} \left(\partial_\mu \bar{z}_j \, \partial^\mu z^i + \overline{F}_j \, F^i \right) \,, \tag{31}$$

where $K^{\bar{j}}{}_{i} = \frac{\partial^2 K}{\partial \bar{z}_j \partial z^i}$ are the Kähler metric elements and $K(z,\bar{z})$ is the Kähler potential. An important issue is the dependance of this potential on the moduli fields, first because the Kähler geometry of the moduli space is related to the Yukawa couplings of the matter fields, and eventually because K enters the scalar potential and could influence supersymmetry-breaking mechanisms. I will focus here on the particular case of the untwisted moduli $T, \overline{T}, U, \overline{U}$ that appear in symmetric orbifold $(2,2)$ compactifications.

The moduli Kähler metric can be extracted from string scattering amplitudes involving four complex moduli fields, by solving a differential equation [16]. Although such a procedure could be helpful on the sphere, it becomes very complicated beyond tree level. Of course, one might proceed to a direct determination of the wave-function renormalization for moduli fields, by looking at amplitudes that involve vertices associated

[†] This universal term was also found in [5] for the three-point function of two gauge bosons and the modulus T.

with $\partial_\mu T, \ldots$ Unfortunately it is difficult to perform exactly this kind of computation in string theory, essentially because of its first-quantized formulation. On the other hand, advocating space-time supersymmetry, it can be argued that this is actually equivalent to computing string amplitudes for two auxiliary fields, and this is precisely what I will be analyzing in the following. This straightforward method is an alternative to the previous indirect determination of the one-loop Kähler metric performed by looking at three-point functions of two moduli and one antisymmetric tensor [6]. The actual calculation of the appropriate string amplitudes will be realized here by switching on constant F-auxiliary-field backgrounds, as I did in the previous section for the case of D fields.

In order to proceed further, let me describe the vertices associated with the moduli multiplets of the $Z_2 \times Z_2$ orbifold model. I will concentrate on the third plane where, according to the conventions (11) and (12), the correspondence is the following:

$$
\begin{aligned}
T_3 \ &: \ \frac{-i\partial Y^{-3}\,\bar{\partial} Y^3}{\operatorname{Im} T_3}\,, \ \overline{T}_3 \ : \ \frac{i\partial Y^{3} \quad \bar{\partial} Y^{-3}}{\operatorname{Im} T_3}\,, \\
U_3 \ &: \ \frac{i\partial Y^{-3}\,\bar{\partial} Y^{-3}}{\operatorname{Im} U_3}\,, \ \overline{U}_3 \ : \ \frac{-i\partial Y^{3} \quad \bar{\partial} Y^{3}}{\operatorname{Im} U_3}\,,
\end{aligned}
\tag{32}
$$

in the zero-ghost picture. The auxiliary fields are obtained by going to the minus-one-ghost picture and acting with the top form (or its complex conjugate) [13]. At this point it is convenient to introduce the left-moving $SO(4)$ level-one current algebra generated by bilinears of the first- and second-plane fermions ψ^4, ψ^5, ψ^6, ψ^7:

$$
\begin{aligned}
J^x &= \frac{:\psi^6\psi^5: + :\psi^7\psi^4:}{i\sqrt{2}}\,, \ K^x = \frac{:\psi^6\psi^5: - :\psi^7\psi^4:}{i\sqrt{2}}\,, \\
J^y &= \frac{:\psi^6\psi^4: + :\psi^5\psi^7:}{i\sqrt{2}}\,, \ K^y = \frac{:\psi^6\psi^4: - :\psi^5\psi^7:}{i\sqrt{2}}\,, \\
J^z &= \frac{:\psi^4\psi^5: + :\psi^6\psi^7:}{i\sqrt{2}}\,, \ K^z = \frac{:\psi^4\psi^5: - :\psi^6\psi^7:}{i\sqrt{2}}\,.
\end{aligned}
\tag{33}
$$

In this form the $SU(2) \times SU(2)$ structure is manifest and the normalizations are such that for each $SU(2)$ algebra the roots have length squared equal to 2. With the above definitions the vertices for the auxiliary fields read:

$$
\begin{aligned}
F_{T_3} \ &: \ \frac{-\big(\,(K^y - J^y) + i\,(K^y + J^y)\,\big)\,\bar{\partial} Y^3}{2i \operatorname{Im} T_3}\,, \\
\overline{F}_{T_3} \ &: \ \frac{-\big(\,(K^y - J^y) - i\,(K^y + J^y)\,\big)\,\bar{\partial} Y^{-3}}{2i \operatorname{Im} T_3}\,, \\
F_{U_3} \ &: \ \frac{\big(\,(K^y - J^y) + i\,(K^y + J^y)\,\big)\,\bar{\partial} Y^{-3}}{2i \operatorname{Im} U_3}\,, \\
\overline{F}_{U_3} \ &: \ \frac{\big(\,(K^y - J^y) - i\,(K^y + J^y)\,\big)\,\bar{\partial} Y^{3}}{2i \operatorname{Im} U_3}\,.
\end{aligned}
\tag{34}
$$

Obviously, both left and right factors are Abelian currents; therefore, switching on the corresponding backgrounds simultaneously does not affect conformal invariance. Moreover if f_{T_3}, $\bar{f}_{T_3}$, f_{U_3}, $\bar{f}_{U_3}$ are these constant backgrounds, the one-loop corrections to the Kähler metric read *

$$K^{\overline{T}_3\,(1)}_{\ T_3} = \int_{\mathcal{F}} \frac{d^2\tau}{(\,\mathrm{Im}\,\tau)^2} \frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3} \partial f_{T_3}}\bigg|_{\mathbf{f}=0} \,, \ K^{\overline{T}_3\,(1)}_{\ U_3} = \int_{\mathcal{F}} \frac{d^2\tau}{(\,\mathrm{Im}\,\tau)^2} \frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3} \partial f_{U_3}}\bigg|_{\mathbf{f}=0} \,, \tag{35}$$

and similarly for the other components.

There are a few observations that one can make in order to simplify the computation of quantities such as $\frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3} \partial f_{T_3}}\Big|_{\mathbf{f}=0}$, $\frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3} \partial f_{U_3}}\Big|_{\mathbf{f}=0}$, ... First, the presence of the $SU(2)\times SU(2)$ symmetry allows a rotation of J^y and K^y onto J^z and K^z. The zero modes of the latter are the combinations $-\frac{Q^1+Q^2}{\sqrt{2}}$ and $-\frac{Q^1-Q^2}{\sqrt{2}}$ of the internal helicity operator charges introduced in the previous section. When inserted in the vacuum trace, these charges act as $\frac{i}{2\pi\sqrt{2}}\left(\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_2}\right)$ and $\frac{i}{2\pi\sqrt{2}}\left(\frac{\partial}{\partial v_1}-\frac{\partial}{\partial v_2}\right)$ on the holomorphic ϑ functions of the first and second plane. Secondly, instead of (34) it is more convenient to choose the following basis of real (commuting) $(1,1)$ operators:

$$K^y\, i\bar{\partial}X^8 \,,\ J^y\, i\bar{\partial}X^8 \,,\ K^y\, i\bar{\partial}X^9 \,,\ J^y\, i\bar{\partial}X^9 \,, \tag{36}$$

and denote α, β, γ, δ the corresponding (real) backgrounds. A straightforward calculation shows that

$$\frac{\partial^2}{\partial \bar{f}_{T_3} \partial f_{T_3}} = \frac{1}{4\,(\,\mathrm{Im}\,T_3)^2}\left(-2\frac{\partial^2}{\partial\alpha\partial\delta} + 2\frac{\partial^2}{\partial\beta\partial\gamma} + \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\gamma^2} + \frac{\partial^2}{\partial\delta^2}\right), \tag{37}$$

$$\frac{\partial^2}{\partial \bar{f}_{T_3} \partial f_{U_3}} = \frac{-1}{4\,\mathrm{Im}\,T_3\,\mathrm{Im}\,U_3}\left(-2i\frac{\partial^2}{\partial\alpha\partial\gamma} - 2i\frac{\partial^2}{\partial\beta\partial\delta} + \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\gamma^2} - \frac{\partial^2}{\partial\delta^2}\right), \tag{38}$$

and similarly for the others. As far as the left-moving sector is concerned †, derivatives such as $\frac{\partial^2}{\partial\alpha^2}$, $\frac{\partial^2}{\partial\gamma^2}$ or $\frac{\partial^2}{\partial\alpha\partial\gamma}$ will involve, according to the above argument, insertions of

* Remember that $K^{\bar{j}}_{\ i} = K^{\bar{j}\,(0)}_{\ i} + g_s^2\, K^{\bar{j}\,(1)}_{\ i} + \cdots$ with $K^{\overline{T}_3\,(0)}_{\ T_3} = \frac{1}{4(\,\mathrm{Im}\,T_3)^2}$,... Notice also that operators such as those appearing in (34) do not commute with the superconformal current. Therefore, the above method becomes questionable when applied to higher-order correlators or at higher genus.

† Insertions of operators acting exclusively on the right-moving sectors vanish identically because of supersymmetry.

$\left(Q^1 - Q^2\right)^2$ into the vacuum trace. Thanks to the Jacobi identity (23), this insertion vanishes. On the other hand, the derivatives $\frac{\partial^2}{\partial\alpha\partial\delta}$ and $\frac{\partial^2}{\partial\beta\partial\gamma}$ will lead to insertions of $\left(Q^1 - Q^2\right)\left(Q^1 + Q^2\right)$. In general, these do receive contributions from the $N = 1$ sectors and it is not difficult to see that those contributions are equal for both $\frac{\partial^2}{\partial\alpha\partial\delta}$ and $\frac{\partial^2}{\partial\beta\partial\gamma}$, and thus cancel in (37) (this is of course irrelevant in the $Z_2 \times Z_2$ orbifold where there are no $N = 1$ sectors). Hence, one concludes that

$$\left.\frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3}\partial f_{T_3}}\right|_{\mathbf{f}=0} = \frac{1}{4\left(\operatorname{Im} T_3\right)^2}\left(\frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\delta^2}\right) \left. Z(\beta,\delta)\right|_{\beta=\delta=0}, \tag{39}$$

and

$$\left.\frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3}\partial f_{U_3}}\right|_{\mathbf{f}=0} = \frac{-1}{4\operatorname{Im} T_3 \operatorname{Im} U_3}\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\delta^2} - 2i\frac{\partial^2}{\partial\beta\partial\delta}\right) \left. Z(\beta,\delta)\right|_{\beta=\delta=0}. \tag{40}$$

Equations (39) and (40) show that, for the issue of the one-loop corrections to the Kähler metric, it is not necessary to analyze the most general deformation: only $Z(\beta,\delta)$ is relevant. Such a deformation has been considered in section 2 for a generic case (eqs. (6) and (7)) and those results can immediately be applied to (39) and (40):

$$\begin{aligned}\left.\frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3}\partial f_{T_3}}\right|_{\mathbf{f}=0} = \frac{(\operatorname{Im}\tau)^2}{64\pi^2\left(\operatorname{Im} T_3\right)^2} \operatorname{Tr}\Bigg\{ &\left(Q^1 + Q^2\right)^2 \left[\left|p_3^R\right|^2 - \frac{1}{4\pi\operatorname{Im}\tau}\right] \\ &\exp\Big(-2\pi\operatorname{Im}\tau\left(L_0 + \overline{L}_0\right) + 2\pi i \operatorname{Re}\tau\left(L_0 - \overline{L}_0\right)\Big)\Bigg\},\end{aligned} \tag{41}$$

$$\begin{aligned}\left.\frac{\partial^2 Z(\mathbf{f})}{\partial \bar{f}_{T_3}\partial f_{U_3}}\right|_{\mathbf{f}=0} = \frac{-(\operatorname{Im}\tau)^2}{64\pi^2\operatorname{Im} T_3\operatorname{Im} U_3} \operatorname{Tr}\Bigg\{ &\left(Q^1 + Q^2\right)^2 \left(p_3^{R*}\right)^2 \\ &\exp\Big(-2\pi\operatorname{Im}\tau\left(L_0 + \overline{L}_0\right) + 2\pi i \operatorname{Re}\tau\left(L_0 - \overline{L}_0\right)\Big)\Bigg\}.\end{aligned} \tag{42}$$

The evaluation of the traces is performed along the same lines of thought as in the D-field calculation. Now, the only non-vanishing contribution comes from the $N = 2$ sectors with twist $(h_3, g_3) = (0,0)$. After some algebra, using in particular the fact that the insertion of $\left|p_3^R\right|^2$ amounts to the operator $\frac{i}{2\pi}\frac{\partial}{\partial\bar{\tau}}$ acting on the solitonic contribution of the third two-torus, one obtains

$$K^{\overline{T}_3\,(1)}_{\ \,T_3} = \frac{-1}{16\pi^2\left(\operatorname{Im} T_3\right)^2}\int_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2}\,\Gamma(\mu)\,\frac{\overline{\Omega}}{\bar{\eta}^{24}}\,\frac{i}{2\pi}\,\frac{\partial}{\partial\bar{\tau}}\Big(\operatorname{Im}\tau\,\Gamma_{2,2}(3)\Big), \tag{43}$$

$$K^{\overline{T}_3\,(1)}_{\ \,U_3} = \frac{1}{16\pi^2\operatorname{Im} T_3\operatorname{Im} U_3}\int_{\mathcal{F}} \frac{d^2\tau}{\operatorname{Im}\tau}\,\Gamma(\mu)\,\frac{\overline{\Omega}}{\bar{\eta}^{24}}\sum_{\mathbf{m},\mathbf{n}}\left(p_3^{R*}\right)^2 q^{\left|P_3^L\right|^2}\bar{q}^{\left|P_3^R\right|^2}, \tag{44}$$

and similarly for the other elements.

The above results deserve a few comments. The computation I presented here is exact, that is it holds to all orders in α' and takes properly into account the back-reaction of gravity (terms proportional to $\frac{1}{4\pi \operatorname{Im}\tau}$ in (41) or (43)). The integrands of (43) and (44) are τ-modular-invariant functions, which are well behaved at large values of $\operatorname{Im}\tau$. Therefore, one can remove the infra-red regulator by taking the limit $\mu \to 0$ in the corresponding integrals. By using identities such as

$$\frac{1}{(\operatorname{Im} T)^2}\frac{\partial^2}{\partial\bar\tau\partial\tau}\Big(\operatorname{Im}\tau\,\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right)\Big) = \frac{1}{\operatorname{Im}\tau}\frac{\partial^2}{\partial\overline{T}\partial T}\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right) \tag{45}$$

and integrating by parts when appropriate, it is then easy to show that

$$K^{\overline{T}_3\,(1)}_{\phantom{\overline{T}_3}T_3} = \frac{\partial^2 K^{(1)}}{\partial\overline{T}_3\partial T_3}\,,\ K^{\overline{T}_3\,(1)}_{\phantom{\overline{T}_3}U_3} = \frac{\partial^2 K^{(1)}}{\partial\overline{T}_3\partial U_3}\,,\dots, \tag{46}$$

where the real function

$$K^{(1)}\left(\mathbf{T},\mathbf{U},\overline{\mathbf{T}},\overline{\mathbf{U}}\right) = \frac{1}{4\pi^2}\,Y\left(\mathbf{T},\mathbf{U},\overline{\mathbf{T}},\overline{\mathbf{U}}\right) + \kappa\big(\mathbf{T},\mathbf{U}\big) + \bar\kappa\left(\overline{\mathbf{T}},\overline{\mathbf{U}}\right) \tag{47}$$

is the one-loop correction to the Kähler potential. The first term in (47) is, as expected from supersymmetry, proportional to the universal part of the one-loop thresholds of the gauge couplings [9]:

$$\begin{aligned} Y\left(\mathbf{T},\mathbf{U},\overline{\mathbf{T}},\overline{\mathbf{U}}\right) = {} & \frac{1}{6}\sum_i \log\big|j\left(T_i\right) - j\left(U_i\right)\big| \\ & + \int_{\mathcal{F}}\frac{d^2\tau}{\operatorname{Im}\tau}\sum_i \Gamma_{2,2}(i)\left(-2\left[\frac{\partial\log\bar\eta}{\partial\log\bar q} - \frac{1}{8\pi\operatorname{Im}\tau}\right]\frac{\overline{\Omega}}{\bar\eta^{24}} + 11\right)\,. \end{aligned} \tag{48}$$

This expression is invariant under duality transformations on T_i, U_i and finite everywhere in the moduli space despite the logarithmic divergences that both terms suffer around enhanced symmetry lines. On the other hand, $\kappa(\mathbf{T},\mathbf{U})$ is an analytic function of the moduli, which is irrelevant in (46) but plays an important role for the duality covariance of $K^{(1)}$. Finally, it is interesting to observe that due to the identity

$$\frac{\Omega}{\eta^{24}} = \frac{j - j(i)}{2}\left(\frac{\partial\log j}{\partial\log q}\right)^{-1}\,, \tag{49}$$

the corrections to the Kähler metric I determined for the specific $N=1$ model turn out to be identical to those obtained in [17] for a class of $N=2$ string vacua. Expression (48), and consequently expression (47), can be further generalized to a larger class of $N=2$ four-dimensional theories, namely those that come from toroidal compactification of generic six-dimensional $N=1$ ground states. These theories always possess a universal two-torus. Therefore, advocating τ-modular invariance and infra-red finiteness all over the moduli space, one can draw the conclusion that for these models the universal thresholds are proportional to the contribution of a single plane in the $Z_2 \times Z_2$ orbifold. The coefficient can be related via the gravitational anomaly to the quantity $N_v - N_h$, where N_v and N_h are the number of massless vector multiplets and the number of massless hypermultiplets, respectively. Taking into account that for this class of ground states the cancellation of anomalies originated from six dimensions implies that $N_v - N_h$ is a universal constant [20], the net result reads [15]:

$$
\begin{aligned}
Y\left(T,U,\overline{T},\overline{U}\right) = &\frac{1}{3}\log\left|j(T)-j(U)\right| \\
&+ 2\int_{\mathcal{F}} \frac{d^2\tau}{\operatorname{Im}\tau}\,\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right)\left(-2\left[\frac{\partial\log\bar{\eta}}{\partial\log\bar{q}} - \frac{1}{8\pi\operatorname{Im}\tau}\right]\frac{\overline{\Omega}}{\bar{\eta}^{24}} + 11\right). \quad (50)
\end{aligned}
$$

In the presence of local $N = 2$ space-time supersymmetry, the Kähler potential is described in terms of an analytic function, the prepotential. The moduli dependance of the latter, as well as its properties under modular transformations, is non-trivial and has attracted much attention in the rapidly growing subject of dualities [17–19]. This is another motivation for a careful study of the one-loop Kähler potential.

5. Conclusions

By applying the background field method to situations where the corresponding deformations are truly conformal, it has been possible to compute exact one-loop string amplitudes for both vector D and chiral F fields. The former lead to a supersymmetry Ward identity relating amplitudes of various members of the vector supermultiplet *; in the framework of strings, this is a simple consequence of the Jacobi identity between ϑ functions. The F-field insertions, on the other hand, give the corrections to the Kähler metric for the moduli fields, which turn out to contain information about the Kähler potential.

The one-loop corrections to the Kähler potential are determined by the universal thresholds of the gauge couplings as well as by the real part of an analytic function $\kappa(\mathbf{T},\mathbf{U})$. An interesting open problem is the exact determination, i.e. up to a degree-two polynomial, of this analytic function. In fact, the knowledge of the Kähler potential as a function of the moduli is useful (i) for phenomenological purposes since it is related to the Yukawa couplings and enters the scalar potential where it could play a role when, in case of supersymmetry breaking, the moduli are no longer flat directions; (ii) because, for $N = 2$ supersymmetric vacua, it is related to the prepotential. Moreover, as I explained at the end of the previous section, although the calculation I presented here strictly holds for the $Z_2 \times Z_2$ orbifold, the results seem to have a wider application

* A similar Ward identity would have been reached by calculating H-field string amplitudes and comparing them with R-tensor insertions: they both lead to the gravitational coupling corrections (for a discussion see [15]).

either in the case of some $N = 2$ superstring compactifications, or for the $(N = 2)$-sector contributions of other $N = 1$ string vacua.

Finally, it can be observed that amplitudes involving F-auxiliary fields associated with the moduli do not suffer from infra-red divergences. However, it is clear from the D-field computation that this is not a generic feature, and that for scalar fields other than the moduli, one generally needs to keep μ finite and treat these infra-red divergences as is done for the gauge couplings [7, 9]. This would happen in particular in the case of charged (twisted or untwisted) chiral multiplets for which it would be interesting to compute the Kähler metric corrections and their effects on the Yukawa couplings [21].

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