

CERN-TH/96-91  
LPTENS/96/25  
SISSA/60/96/EP  
hep-th/9605011

## ON THE HETEROTIC EFFECTIVE ACTION AT ONE LOOP GAUGE COUPLINGS AND THE GRAVITATIONAL SECTOR

E. KIRITSIS<sup>1</sup>, C. KOUNNAS<sup>1,\*</sup>, P.M. PETROPOULOS<sup>1,\*</sup>

and

J. RIZOS<sup>1,2,◇</sup>

<sup>1</sup>*Theory Division, CERN  
1211 Geneva 23, Switzerland*

and

<sup>2</sup>*International School for Advanced Studies, SISSA  
Via Beirut 2-4, 34013 Trieste, Italy*

### Abstract

We present in detail the procedure for calculating the heterotic one-loop effective action. We focus on gravitational and gauge couplings. We show that the two-derivative couplings of the gravitational sector are not renormalized at one loop when the ground state is supersymmetric. Arguments are presented that this non-renormalization theorem persists to all orders in perturbation theory. We also derive the full one-loop correction to the gauge coupling. For a class of  $N = 2$  ground states, namely those that are obtained by toroidal compactification to four dimensions of generic six-dimensional  $N = 1$  models, we give an explicit formula for the gauge-group independent thresholds, and show that these are equal within the whole family.

To appear in the proceedings of the *5th Hellenic School and Workshops on Elementary Particle Physics*, Corfu, Greece, 3–24 September 1995.

CERN-TH/96-91  
LPTENS/96/25  
SISSA/60/96/EP  
April 1996

---

\* On leave from *Centre National de la Recherche Scientifique*, France.

◇ On leave from *Division of Theoretical Physics, Physics Department, University of Ioannina*, Greece.

## 1. Introduction

In the past several years, there has been significant progress in trying to compare low energy predictions of string theory with data [1]–[18]. String theory gives us the possibility of unifying gauge, Yukawa and gravitational interactions. The presence of supersymmetry is usually required in order to deal with hierarchy problems (although in the context of supergravity and strings this is not automatic, due to the presence of gravity [19]). The standard folklore demands  $N = 1$  supersymmetry in order for the theory to possess chiral fermions. There seem however to be flaws in this popular wisdom [20].

The quantities that are most easily comparable to experimental data are effective gauge couplings of the observable sector, as well as Yukawa couplings. It is well known that the low-energy world is not supersymmetric. Thus supersymmetry has to be broken spontaneously at some scale of the order of 1 TeV (for hierarchy reasons). Although there are ways to break supersymmetry in string theory [21, 22, 23], it is fair to say that none so far has yielded a phenomenologically acceptable model. Although the issue of supersymmetry breaking is an open problem, if we assume that its scale is of the order of 1 TeV and the superpartner masses are around that scale, then non-supersymmetric thresholds are not very important for dimensionless couplings (which include gauge and Yukawa couplings). Thus, it makes sense to compute them and compare them with data in the context of unbroken supersymmetry.

There are several procedures to compute the one-loop corrections to dimensionless couplings in string theory. The most powerful and unambiguous one was described in [12, 24]. It amounts to turning on gravitational background fields that provide the ground state in question with a mass gap  $\Delta m^2$ , and further background fields (magnetic fields, curvature and auxiliary  $F$  fields) in order to perform a background-field calculation of the relevant one-loop corrections.

The above procedure involves the following steps:

- (i) We first regulate the infra-red by introducing a mass gap in the relevant ground state. This is done by replacing the flat four-dimensional conformal field theory with the wormhole one,  $\mathbb{R}_Q \times SO(3)_{\frac{k}{2}}$  [25]. The mass gap is given by  $\Delta m^2 = \frac{M_s^2}{2(k+2)}$ , where  $M_s = \frac{1}{\sqrt{\alpha'}}$  and  $k$  is a (dimensionless) non-negative even integer.
- (ii) We then turn on appropriate background fields, which are exact solutions of the string equations of motion. Such backgrounds include curvature, magnetic fields and auxiliary  $F$  fields\*.
- (iii) We calculate the one-loop vacuum amplitude as a function of these background fields.
- (iv) We identify these background fields as solutions of the tree-level effective action. By substituting them into the one-loop effective action and comparing with the string calculation of the free energy, we can extract the renormalization constants at one loop.

In the following we will apply the aforementioned techniques to the calculation of string loop corrections for gauge and gravitational couplings. For heterotic ground states with at least  $N = 1$  supersymmetry, we will demonstrate that Newton's constant is not renormalized, and derive the full one-loop gauge coupling. We will in particular obtain an explicit formula for the universal part of the threshold corrections. Finally, we will show how, for the whole

---

\*These are relevant for the study of the Kähler potential renormalization. For more details see [15].

class of  $N = 2$  ground states that come from two-torus compactification of six-dimensional  $N = 1$  theories, these thresholds are equal and fully determined as a consequence of an anomaly-cancellation constraint in six dimensions. This summarizes our main results.

## 2. Infra-red regularization and background fields

As mentioned in the introduction, an infra-red regulated version of a given heterotic ground state is provided by substituting four-dimensional flat space with the  $\mathbb{R}_Q \times SO(3)_{\frac{k}{2}}$  conformal field theory<sup>†</sup>. More details can be found in [12, 24]. There is a linear dilaton in the time direction

$$\Phi = \frac{tM_s}{\sqrt{k+2}}, \quad (1)$$

necessary for making the total central charge equal to that of flat space. The mass gap can be read off from the left worldsheet Hamiltonian (in the Euclidean)

$$L_0 = -\frac{1}{2} + \frac{1}{4(k+2)} + \frac{p_t^2}{2} + \frac{j(j+1)}{k+2} + \dots, \quad (2)$$

to be  $\Delta m^2 = \frac{\mu^2}{2}$  with  $\mu = \frac{M_s}{\sqrt{k+2}}$ .

In this geometry there are several marginal deformations, which turn on background fields. For magnetic fields we use

$$V_i^{\text{magn}} \propto (J^3 + i : \psi^1 \psi^2 :) \bar{J}_i. \quad (3)$$

This turns on a magnetic field in the third space direction;  $\bar{J}_i$  is a right-moving affine current in the Cartan of the  $i$ th gauge group simple factor (picking out a single Cartan direction will be enough for our purposes), and  $J^3$  belongs to the  $SO(3)_{\frac{k}{2}}$  affine Lie algebra. There is also a gravitational perturbation generated by

$$V^{\text{grav}} \propto (J^3 + i : \psi^1 \psi^2 :) \bar{J}^3. \quad (4)$$

The currents  $J^3$ ,  $\bar{J}^3$  and  $\bar{J}_i$  are normalized so that<sup>‡</sup>

$$J^3(z) J^3(0) = \frac{k}{2z^2} + \dots, \quad \bar{J}^3(\bar{z}) \bar{J}^3(0) = \frac{k}{2\bar{z}^2} + \dots, \quad \bar{J}_i(\bar{z}) \bar{J}_i(0) = \frac{k_i}{\bar{z}^2} + \dots. \quad (5)$$

All the above perturbations are products of left times right Abelian currents and thus preserve conformal invariance. This implies that the new backgrounds satisfy the string equations of motion at tree level to all orders in the  $\alpha'$  expansion.

The vacuum amplitude at one loop, i.e. the free energy, in the presence of these backgrounds can be readily calculated:

$$\alpha'^2 F_{\text{one loop}}^{\text{string}} = \frac{1}{2(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im } \tau)^2} D_{\text{one loop}}^{\text{string}} = \frac{1}{2(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im } \tau)^2} \left\langle e^{-2\pi \text{Im } \tau \delta(L_0 + \bar{L}_0)} \right\rangle, \quad (6)$$

<sup>†</sup>The group  $SO(3)$  is required instead of  $SU(2)$  for spin-statistics consistency [24].

<sup>‡</sup>Notice that there is a factor of 2 difference between the normalization used here and that used in [12, 24]. Our present normalization is the one widely used in the literature; it corresponds to the situation where the highest root of the algebra has length squared  $\psi^2 = 2$ .

with

$$\begin{aligned} \delta L_0 = \delta \bar{L}_0 &= \frac{\sqrt{1 + \mathcal{F}^2 + \mathcal{R}^2} - 1}{2} \left( \frac{(Q + I^3)^2}{k + 2} + \frac{1}{\mathcal{R}^2 + \mathcal{F}^2} \left( \mathcal{R} \frac{\bar{I}^3}{\sqrt{k}} + \mathcal{F} \frac{\bar{P}_i}{\sqrt{2k_i}} \right)^2 \right) \\ &+ \frac{Q + I^3}{\sqrt{k + 2}} \left( \mathcal{R} \frac{\bar{I}^3}{\sqrt{k}} + \mathcal{F} \frac{\bar{P}_i}{\sqrt{2k_i}} \right). \end{aligned} \quad (7)$$

Here  $I^3, \bar{I}^3$  stand for the zero modes of the respective  $SO(3)_{\frac{k}{2}}$  currents,  $Q$  is the zero mode of the  $i : \psi^1 \psi^2 :$  current and  $\bar{P}_i$  is the zero mode of the  $\bar{J}_i$  current. We also assume that the gauge background does not correspond to an anomalous  $U(1)$ . This case can also be treated, but is more complicated. Since anomalous  $U(1)$ 's are broken at scales comparable with the string scale, their running is irrelevant for low-energy physics. Expanding to second order in the background fields, we find:

$$\begin{aligned} D_{\text{one loop}}^{\text{string}} &= \langle 1 \rangle \\ &+ \frac{8\pi^2 (\text{Im } \tau)^2 \mathcal{R}^2}{k(k+2)} \left\langle \left( (Q + I^3)^2 (\bar{I}^3)^2 - \frac{k}{8\pi \text{Im } \tau} \left( (Q + I^3)^2 + \frac{k+2}{k} (\bar{I}^3)^2 \right) \right) \right\rangle \\ &+ \frac{4\pi^2 (\text{Im } \tau)^2 \mathcal{F}^2}{k_i(k+2)} \left\langle \left( (Q + I^3)^2 \bar{P}_i^2 - \frac{k_i}{4\pi \text{Im } \tau} \left( (Q + I^3)^2 + \frac{k+2}{2k_i} \bar{P}_i^2 \right) \right) \right\rangle \\ &+ \dots, \end{aligned} \quad (8)$$

where the dots stand for higher orders in  $\mathcal{F}$  and  $\mathcal{R}$ .

Here we will assume that our ground state has at least  $N = 1$  supersymmetry<sup>§</sup>. In such ground states, terms in (8) that do not contain the helicity operator  $Q$  vanish because of the presence of the fermionic zero modes, and terms linear in  $Q$  vanish due to rotational invariance,  $\langle I^3 \rangle = 0$ . Thus for  $N = 1$  ground states, (8) becomes

$$D_{\text{one loop}}^{\text{string}} = \frac{8\pi^2 (\text{Im } \tau)^2}{k+2} \left\langle Q^2 \left( \frac{\mathcal{R}^2}{k} \left( (\bar{I}^3)^2 - \frac{k}{8\pi \text{Im } \tau} \right) + \frac{\mathcal{F}^2}{2k_i} \left( \bar{P}_i^2 - \frac{k_i}{4\pi \text{Im } \tau} \right) \right) \right\rangle + \dots \quad (9)$$

The generic  $N = 1$  four-dimensional vacuum amplitude has the form

$$\langle 1 \rangle = \frac{1}{\text{Im } \tau |\eta|^4} \sum_{a,b=0,1} \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} C \begin{bmatrix} a \\ b \end{bmatrix} \Gamma(k) = 0, \quad (10)$$

where  $C \begin{bmatrix} a \\ b \end{bmatrix}$  is the contribution of the internal conformal field theory, and

$$\Gamma(k) = 4\sqrt{x} \frac{\partial}{\partial x} \left[ \varrho(x) - \varrho(x/4) \right] \Big|_{x=k+2} \quad \text{with} \quad \varrho(x) = \sqrt{x} \sum_{m,n \in \mathbb{Z}} e^{-\frac{\pi x}{\text{Im } \tau} |m+n\tau|^2} \quad (11)$$

stands for the  $SO(3)_{\frac{k}{2}}$  partition function normalized so that  $\lim_{k \rightarrow \infty} \Gamma(k) = 1$ . This extra factor ensures the convergence of integrals such as those appearing in (6), at large values of

<sup>§</sup>The general formula in the absence of supersymmetry can be found in [12].

Im  $\tau$ . Expression (10) allows us to recast (9) as follows:

$$D_{\text{one loop}}^{\text{string}} = -\frac{4\pi i}{k+2} \frac{\text{Im } \tau}{|\eta|^4} \sum_{a,b=0,1} \left\{ \frac{\mathcal{F}^2}{k_i} \partial_\tau \left( \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right) \left( \overline{P}_i^2 - \frac{k_i}{4\pi \text{Im } \tau} \right) C \begin{bmatrix} a \\ b \end{bmatrix} \Gamma(k) \right. \\ \left. - \frac{\mathcal{R}^2}{6k} \partial_\tau \left( \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right) C \begin{bmatrix} a \\ b \end{bmatrix} \left( \widehat{E}_2 + \frac{2(k+2)}{i\pi} \partial_{\bar{\tau}} \right) \Gamma(k) \right\} + \dots, \quad (12)$$

where  $\overline{P}_i^2$  acts as  $\frac{i}{\pi} \frac{\partial}{\partial \bar{\tau}}$  on the appropriate subfactor of the 32 right-moving-fermion contribution, and

$$\widehat{E}_2 \equiv \frac{6i}{\pi} \partial_{\bar{\tau}} \log \left( \text{Im } \tau \eta^2 \right) = \overline{E}_2 - \frac{3}{\pi \text{Im } \tau}; \quad (13)$$

$E_2$  is an Eisenstein holomorphic function (see (64)) and  $\widehat{E}_2$  is modular-covariant of degree 2. Since we are interested in the large- $k$  limit, we can expand (12) in powers of  $1/k$ . In the next-to-leading order, the above expression reads:

$$D_{\text{one loop}}^{\text{string}} = -\frac{4\pi i}{k} \frac{\text{Im } \tau}{|\eta|^4} \Gamma(k) \sum_{a,b=0,1} \left\{ \frac{\mathcal{F}^2}{k_i} \partial_\tau \left( \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right) \left( \overline{P}_i^2 - \frac{k_i}{4\pi \text{Im } \tau} \right) C \begin{bmatrix} a \\ b \end{bmatrix} \left( 1 - \frac{2}{k} \right) \right. \\ \left. - \frac{\mathcal{R}^2}{6k} \partial_\tau \left( \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right) \widehat{E}_2 C \begin{bmatrix} a \\ b \end{bmatrix} + \mathcal{O} \left( \frac{1}{k^2} \right) \right\} + \dots. \quad (14)$$

This is the form of the one-loop density that we will use for our subsequent calculations.

### 3. One-loop effective action and the gravitational sector

The tree-level heterotic effective action is given by<sup>¶</sup>

$$S_{\text{tree}} = \frac{1}{\alpha'} \int d^4x \sqrt{G} e^{-2\Phi} \left( R + 4(\nabla\Phi)^2 - \frac{1}{12} \widehat{H}^2 - \alpha' \sum_{i,a} \frac{1}{4g_i^2} F_{i\mu\nu}^a F_i^{a\mu\nu} + \dots \right), \quad (15)$$

where the dots stand for two-derivative terms that include the scalars and fermions, as well as higher-derivative terms. The tree-level cosmological constant is set to zero since we consider ground states with the appropriate value of the central charge. The index  $i$  labels the various simple components of the gauge group, while  $a$  spans the corresponding adjoint representations. The tree-level couplings  $g_i$  are dimensionless and are given here by  $g_i = \frac{1}{\sqrt{k_i}}$ , where  $k_i$  are the integer levels of the appropriate affine algebras responsible for the gauge group. Note that the physical gauge couplings contain also the string coupling  $g_{\text{string}} = \exp\langle\Phi\rangle$ ; this will be restored in the next section. We have also introduced

$$\widehat{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - \alpha' \sum_i \frac{1}{2g_i^2} C S_{i\mu\nu\rho}, \quad (16)$$

---

<sup>¶</sup>Expression (15) holds in the  $\sigma$ -model frame, which is the natural frame for perturbative string calculations.

where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic permutations} \quad (17)$$

and

$$C S_{i\mu\nu\rho} = \sum_a A_{i\mu}^a F_{i\nu\rho}^a - \frac{1}{3} \sum_{a,b,c} f_i^{abc} A_{i\mu}^a A_{i\nu}^b A_{i\rho}^c + \text{cyclic permutations}. \quad (18)$$

We will now translate the gravitational and magnetic backgrounds described in the previous section in terms of conformal field theory into the language of the effective action. We will use the Euler-angle parametrization of  $SO(3)$ . In this parametrization we have three angles:  $\beta \in [0, \pi\sqrt{\alpha'k}]$  and  $\alpha, \gamma \in [0, 2\pi\sqrt{\alpha'k}]$ . The three coordinates  $\alpha, \beta$  and  $\gamma$  as well as  $t$  have dimensions of length, and  $\hat{\beta} = \beta/\sqrt{\alpha'k}$  is dimensionless. The fields  $G_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\Phi$  are dimensionless, and the gauge fields have dimensions of mass.

It is not difficult to verify [24] that the conformal field theory backgrounds of section 2 correspond to the following metric, antisymmetric tensor and dilaton:

$$\begin{aligned} G_{tt} = 1, \quad G_{\beta\beta} = \frac{1}{4}, \quad G_{\alpha\alpha} &= \frac{1}{4} \frac{(\lambda^2 + 1)^2 - (8H^2\lambda^2 + (\lambda^2 - 1)^2) \cos^2 \hat{\beta}}{(\lambda^2 + 1 + (\lambda^2 - 1) \cos \hat{\beta})^2}, \\ G_{\gamma\gamma} &= \frac{1}{4} \frac{(\lambda^2 + 1)^2 - 8H^2\lambda^2 - (\lambda^2 - 1)^2 \cos^2 \hat{\beta}}{(\lambda^2 + 1 + (\lambda^2 - 1) \cos \hat{\beta})^2}, \\ G_{\alpha\gamma} &= \frac{1}{4} \frac{4\lambda^2(1 - 2H^2) \cos \hat{\beta} - (\lambda^4 - 1) \sin^2 \hat{\beta}}{(\lambda^2 + 1 + (\lambda^2 - 1) \cos \hat{\beta})^2}; \end{aligned} \quad (19)$$

$$B_{\alpha\gamma} = \frac{1}{4} \frac{\lambda^2 - 1 + (\lambda^2 + 1) \cos \hat{\beta}}{\lambda^2 + 1 + (\lambda^2 - 1) \cos \hat{\beta}}; \quad (20)$$

$$\Phi = \frac{t}{\sqrt{\alpha'k}} - \frac{1}{2} \log \left( \lambda + \frac{1}{\lambda} + \left( \lambda - \frac{1}{\lambda} \right) \cos \hat{\beta} \right) + \log g_{\text{string}}. \quad (21)$$

The non-vanishing gauge field components are those corresponding to the Cartan direction that we have chosen in the  $i$ th simple group factor:

$$A_{i\alpha}^a = \frac{2g_i}{\sqrt{\alpha'}} \frac{H\lambda \cos \hat{\beta}}{\lambda^2 + 1 + (\lambda^2 - 1) \cos \hat{\beta}}, \quad A_{i\gamma}^a = \frac{2g_i}{\sqrt{\alpha'}} \frac{H\lambda}{\lambda^2 + 1 + (\lambda^2 - 1) \cos \hat{\beta}}. \quad (22)$$

We can check that these background fields satisfy the equations of motion stemming from the tree-level effective action (15). Note also that the exact solution (to all orders) for the dilaton is obtained by shifting  $k \rightarrow k + 2$  in eq. (21). The relation of the effective parameters  $H$  and  $\lambda$  to the conformal field theory parameters  $\mathcal{F}$  and  $\mathcal{R}$  is summarized in the following equations:

$$H^2 = \frac{1}{2} \frac{\mathcal{F}^2}{\mathcal{F}^2 + 2(1 + \sqrt{1 + \mathcal{F}^2 + \mathcal{R}^2})} = \frac{\mathcal{F}^2}{8} (1 + \mathcal{O}(\mathcal{F}^2, \mathcal{R}^2)), \quad (23)$$

$$\lambda^2 = \frac{1 + \sqrt{1 + \mathcal{F}^2 + \mathcal{R}^2} + \mathcal{R}}{1 + \sqrt{1 + \mathcal{F}^2 + \mathcal{R}^2} - \mathcal{R}} = 1 + \mathcal{R} + \frac{\mathcal{R}^2}{2} + \mathcal{O}(\mathcal{F}^3, \mathcal{R}^3). \quad (24)$$

Let us now turn to the corrections that the above effective action receives at one loop. The bosonic part of the latter in the universal and gauge sectors can be parametrized as

$$S_{\text{one loop}} = \frac{1}{\alpha'} \int d^4x \sqrt{G} \left( \frac{\Lambda_{\text{one loop}}}{\alpha'} + Z_R R + 4Z_\Phi (\nabla\Phi)^2 - \frac{Z_H}{12} \left( H - \alpha' \sum_i \frac{Z_C}{2g_i^2} C S_i \right)^2 - \alpha' \sum_{i,a} \frac{Z_F}{4g_i^2} F_{i\mu\nu}^a F_i^{a\mu\nu} \right). \quad (25)$$

All renormalization coefficients  $Z_K$  are dimensionless. They encode the one-loop corrections to the various effective couplings. There are some extra couplings associated with anomalous  $U(1)$ 's but we will not consider this case here. Since the above action is the torus contribution, there is no overall dilaton-dependent factor. Assuming unbroken supersymmetry amounts to a vanishing  $\Lambda_{\text{one loop}}$ . Moreover, we know from previous studies [10] that there is no Chern-Simons coupling at one loop, which in turn implies that  $Z_C = 0$ .

What we need to do next is to compute the free energy associated with the background fields studied above. This can be done by evaluating the corresponding one-loop action. Since the various backgrounds do not depend on the Killing coordinates, the four-dimensional measure, once normalized with respect to its flat-space limit  $V_{\text{fl.sp.}} = \frac{\sqrt{\alpha' k}}{4} \int dt d\alpha d\gamma$ , becomes

$$\int d^4x \sqrt{G} \rightarrow \sqrt{1-2H^2} \int_0^\pi d\hat{\beta} \frac{\sin \hat{\beta}}{\left( \lambda + \frac{1}{\lambda} + \left( \lambda - \frac{1}{\lambda} \right) \cos \hat{\beta} \right)}. \quad (26)$$

We can now evaluate the various terms that are relevant for the action (25). After some calculation we obtain to leading order in the  $1/k$  expansion:

$$\frac{1}{\alpha'} \int \frac{d^4x}{V_{\text{fl.sp.}}} \sqrt{G} R = \frac{1}{k} \sqrt{1-2H^2} (3+2H^2) \left( \lambda + \frac{1}{\lambda} \right) = \frac{1}{k} \left( 6 - \frac{\mathcal{F}^2}{4} + \frac{3\mathcal{R}^2}{4} + \dots \right), \quad (27)$$

$$\frac{1}{\alpha'} \int \frac{d^4x}{V_{\text{fl.sp.}}} \sqrt{G} 4(\nabla\Phi)^2 = \frac{2}{k} \sqrt{1-2H^2} \left( \lambda + \frac{1}{\lambda} \right) = \frac{1}{k} \left( 4 - \frac{\mathcal{F}^2}{2} + \frac{\mathcal{R}^2}{2} + \dots \right), \quad (28)$$

$$\frac{1}{\alpha'} \int \frac{d^4x}{V_{\text{fl.sp.}}} \sqrt{G} \frac{H^2}{12} = \frac{1}{k} \frac{1}{\sqrt{1-2H^2}} \left( \lambda + \frac{1}{\lambda} \right) = \frac{1}{k} \left( 2 + \frac{\mathcal{F}^2}{4} + \frac{\mathcal{R}^2}{4} + \dots \right), \quad (29)$$

$$\int \frac{d^4x}{V_{\text{fl.sp.}}} \sqrt{G} \frac{F_i^2}{4g_i^2} = \frac{4}{k} H^2 \sqrt{1-2H^2} \left( \lambda + \frac{1}{\lambda} \right) = \frac{1}{k} \mathcal{F}^2 + \dots. \quad (30)$$

We have used relations (23) and (24) in the right-hand side above. It should be noted that all such lowest-order contributions are of order  $1/k$ . In fact the expansion in powers of  $1/k$  organizes the various orders of derivatives in the effective action. For example,  $R^2$  terms come in at order  $1/k^2$ :

$$\begin{aligned} \int \frac{d^4x}{V_{\text{fl.sp.}}} \sqrt{G} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} &= \frac{1}{k^2} \frac{\sqrt{1-2H^2}}{3} \left( \lambda + \frac{1}{\lambda} \right) \times \\ &\times \left( (33 + 44H^2 + 132H^4) \left( \lambda^2 + \frac{1}{\lambda^2} \right) 16 (3 + 4H^2) \right) \end{aligned}$$

$$= \frac{1}{k^2} \left( 12 + \frac{\mathcal{F}^2}{2} + \frac{47\mathcal{R}^2}{2} + \dots \right). \quad (31)$$

Putting together eqs. (25) and (27)–(30), we obtain the one-loop correction to the free energy in the effective field theory:

$$\begin{aligned} \alpha'^2 F_{\text{one loop}}^{\text{effective}} &= \frac{S_{\text{one loop}}}{V_{\text{fl. sp.}}} \\ &= \frac{1}{k} \left( 2(3Z_R + 2Z_\Phi - Z_H) - (Z_R + 2Z_\Phi + Z_H + 4Z_F) \frac{\mathcal{F}^2}{4} \right. \\ &\quad \left. + (3Z_R + 2Z_\Phi - Z_H) \frac{\mathcal{R}^2}{4} + \text{higher orders in } \mathcal{F} \text{ and } \mathcal{R} \right) + \mathcal{O}\left(\frac{1}{k^2}\right). \end{aligned} \quad (32)$$

In order to determine the various renormalization constants, we have to compare the effective field theory result (32) with the string calculation of the one-loop free energy given by eqs. (6) and (14). We first note that the absence of a term independent of  $\mathcal{F}$  and  $\mathcal{R}$  in (14) leads to

$$3Z_R + 2Z_\Phi - Z_H = 0. \quad (33)$$

In turn, this relation implies through (32) that it should not be any  $\mathcal{R}^2$  term<sup>||</sup> at order  $1/k$  in (14), which is indeed the case.

Before proceeding further with the computation of the string-induced renormalizations, it is interesting to observe that, independently of any string-based consideration, relation (33) is a consequence of space-time supersymmetry. The argument is the following. The tree plus one-loop action for the universal sector is given from (15) and (25):

$$\begin{aligned} S_{\text{tree \& one loop}} &= \frac{1}{\alpha'} \int d^4x \sqrt{G} \left( (e^{-2\Phi} + Z_R) R \right. \\ &\quad \left. + 4(e^{-2\Phi} + Z_\Phi) (\nabla\Phi)^2 - \frac{1}{12} (e^{-2\Phi} + Z_H) H^2 \right). \end{aligned} \quad (34)$$

By performing the transformation

$$G_{\mu\nu} = \frac{1}{e^{-2\Phi} + Z_R} g_{\mu\nu}, \quad (35)$$

we can go to the Einstein frame, where the above action reads:

$$\begin{aligned} S_{\text{tree \& one loop}}^{\text{Einstein}} &= \frac{1}{\alpha'} \int d^4x \sqrt{g} \left( R - 2 \left( 1 - 2(Z_\Phi + 2Z_R)e^{2\Phi} \right) (\nabla\Phi)^2 \right. \\ &\quad \left. - \frac{e^{-4\Phi}}{12} \left( 1 + (Z_R + Z_H)e^{2\Phi} \right) H^2 \right). \end{aligned} \quad (36)$$

Only when relation (33) is true does the action above, upon the field redefinition

$$\Phi' = \Phi - \frac{Z_\Phi + 2Z_R}{2} e^{2\Phi} + \mathcal{O}(e^{4\Phi}), \quad (37)$$

---

<sup>||</sup>Equation (31) suggests, however, that  $\mathcal{R}^2$  terms are present at the order  $1/k^2$ , which is again in agreement with the string result (14). This makes it possible for the determination of the one-loop renormalization constant  $Z_{R^2}$ , leading in particular to the gravitational anomaly.



become the tree-level action in the Einstein frame, which is fixed by supersymmetry.

In fact, this argument generalizes to higher orders in perturbation theory, thus leading, at any order, to relations among renormalization constants similar to (33). Let

$$S = \frac{1}{\alpha'} \int d^4x \sqrt{G} e^{-2\Phi} \left( F_R(\Phi) R + 4F_\phi(\Phi) (\nabla\Phi)^2 - \frac{1}{12} F_H(\Phi) H^2 \right) \quad (38)$$

be the all-order effective action for the universal sector in the  $\sigma$ -model frame, where the functions  $F_R, F_\phi$  and  $F_H$  have the perturbative expansion

$$F_K(\Phi) = 1 + \sum_{n=1}^{\infty} Z_K^{(n)} e^{2n\Phi}. \quad (39)$$

Then  $N = 1$  supersymmetry implies that

$$\log(F_H(\Phi) F_R(\Phi)) = -4 \int_{-\infty}^{\Phi} dx \left( \sqrt{3 \left( 1 - \frac{1}{2} \frac{F'_R(x)}{F_R(x)} \right)^2 - 2 \frac{F_\phi(x)}{F_R(x)} - 1} \right), \quad (40)$$

which at the one-loop level leads precisely to (33).

Let us now come back to the string computation and show that at one loop

$$Z_R = Z_H = 0. \quad (41)$$

In order to do this, we will go beyond the calculation that we presented in section 2. Indeed, we have to study the two-point amplitude of graviton, antisymmetric tensor and dilaton, at one loop. The piece quadratic in momenta in such an amplitude determines the quadratic part of the associated one-loop action, and therefore the corresponding renormalization constant. The relevant heterotic vertex operator is

$$V(\epsilon, p) \propto \epsilon_{\mu\nu} (\partial x^\mu + i(p \cdot \psi) \psi^\mu) \bar{\partial} x^\nu e^{ip \cdot x}, \quad (42)$$

with  $p^2 = 0$ . For comparison, the vertex operator of a gauge boson is

$$V_i^a(\epsilon, p) \propto \epsilon_\mu (\partial x^\mu + i(p \cdot \psi) \psi^\mu) \bar{J}_i^a e^{ip \cdot x}. \quad (43)$$

The two-point  $S$ -matrix element on the torus is (up to an overall normalization)

$$S_{1 \rightarrow 2} \propto \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im } \tau)^2} \int d^2z \langle V(\epsilon^{(1)}, p^{(1)})(z, \bar{z}) V(\epsilon^{(2)}, p^{(2)})(0) \rangle, \quad (44)$$

with  $p^{(1)\mu} p_\mu^{(1)} = p^{(2)\mu} p_\mu^{(2)} = \mathbf{p}^{(1)} + \mathbf{p}^{(2)} = 0$ . It is known that such an on-shell amplitude is zero, which is consistent with the fact that there is no one-loop mass shift for such massless particles. However, for our purposes, we have to go off shell in order to pick out the terms quadratic in momenta. There is such a prescription [1, 10], which amounts to keeping  $p^{(1)\mu} p_\mu^{(1)} = p^{(2)\mu} p_\mu^{(2)} = 0$  but allowing  $\mathbf{p}^{(1)} + \mathbf{p}^{(2)}$  to be arbitrary, without destroying conformal or modular invariance. Furthermore, due to  $N = 1$  supersymmetry, the only term that

contributes a non-zero result in (44) is the one containing the four worldsheet fermions; since we are interested in terms with two derivatives, we can set the exponential  $e^{ip \cdot x}$  to 1. Thus

$$S_{1 \rightarrow 2} \propto \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im } \tau)^2} \int d^2 z \epsilon_{\eta\kappa}^{(1)} \epsilon_{\mu\nu}^{(2)} p_{\rho}^{(1)} p_{\sigma}^{(2)} \langle \psi^{\rho}(z) \psi^{\eta}(z) \psi^{\sigma}(0) \psi^{\mu}(0) \rangle \langle \bar{\partial} x^{\kappa}(\bar{z}) \bar{\partial} x^{\nu}(0) \rangle + \mathcal{O}(p^4) \quad (\text{off shell}). \quad (45)$$

It is obvious from the above expression, that the integrand is a total holomorphic derivative of a function that is periodic and regular on the torus. This implies that this integral vanishes, which proves relation (41). We should note here that, for the gravitational sector, the relevant integrals are finite in the infra-red so that no regularization is needed.

Another way to see the vanishing is to compare with the gauge-field case, where the associated amplitude has the form

$$S_{i,a \rightarrow j,b} \propto \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im } \tau)^2} \int d^2 z \epsilon_{\mu}^{(1)} \epsilon_{\nu}^{(2)} p_{\rho}^{(1)} p_{\sigma}^{(2)} \langle \psi^{\rho}(z) \psi^{\mu}(z) \psi^{\sigma}(0) \psi^{\nu}(0) \rangle \langle \bar{\mathcal{J}}_i^a(\bar{z}) \bar{\mathcal{J}}_j^b(0) \rangle + \mathcal{O}(p^4) \quad (\text{off shell}). \quad (46)$$

The left-moving fermionic correlation function reduces to the standard helicity trace while the integrated correlation function of the right-moving affine currents gives  $\text{Tr} \left( \bar{\mathcal{Q}}_i^2 - \frac{k_i}{4\pi \text{Im } \tau} \right)$ , where  $\bar{\mathcal{Q}}_i^2$  is the gauge-group quadratic Casimir. Upon inspection, we can see that there is a similar formula for the gravitational case with  $k_i = 1$  and  $\bar{\mathcal{Q}}_i$  replaced by the (continuous) momentum in a single direction. For continuous momentum,  $\text{Tr} \left( \bar{\mathcal{P}}^2 \right) \propto \frac{1}{\text{Im } \tau}$ , and by modular invariance it must cancel the second term.

The above argument generalizes to higher loops in the presence of  $N = 1$  supersymmetry. Again only the four-fermion term contributes and the integrand is always a total derivative. We can thus conclude that, to all orders in perturbation theory, Newton's constant is not renormalized around heterotic ground states with at least  $N = 1$  space-time supersymmetry. Similarly, there are no perturbative corrections to the dilaton and antisymmetric tensor kinetic terms.

#### 4. One-loop gauge couplings and universal thresholds

The one-loop correction to the gauge coupling can be calculated using the results of the previous section. We will describe the general structure of these corrections in the  $\overline{DR}$  scheme for a generic supersymmetric four-dimensional model, and will eventually concentrate on a specific family of  $N = 2$  ground states that are two-torus compactifications of arbitrary  $N = 1$  theories in six dimensions. In that case, it turns out that the universal, i.e. group-factor independent, part of the thresholds is *truly universal*: it does not depend on the  $(4, 0)$  internal conformal field theory that is used to reach six dimensions starting from ten.

Equations (32), (33) and (41) imply

$$\alpha'^2 F_{\text{one loop}}^{\text{effective}} = \frac{1}{k} \left( -Z_F \mathcal{F}^2 + \text{higher orders in } \mathcal{F} \text{ and } \mathcal{R} \right) + \mathcal{O} \left( \frac{1}{k^2} \right), \quad (47)$$

where  $Z_F$  can be determined by comparison with eqs. (6) and (14). Note that the normalizations for the effective field theory are chosen such that the highest roots of the group algebra have length squared equal to 1<sup>\*\*</sup>. Since our string computation was performed with  $\psi^2 = 2$ , the net result for  $Z_F$  reads:

$$Z_F = \frac{i}{16\pi^3 k_i} \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau} \frac{\Gamma(k)}{|\eta|^4} \sum_{a,b=0,1} \partial_\tau \left( \frac{\vartheta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \right) \left( \overline{P}_i^2 - \frac{k_i}{4\pi \text{Im}\tau} \right) C \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right]. \quad (48)$$

From (15) and (25) we can derive the effective one-loop string-corrected coupling  $g_{\text{eff},i}$ :

$$\begin{aligned} \frac{16\pi^2}{g_{\text{eff},i}^2} &= k_i \frac{16\pi^2}{g_{\text{string}}^2} + 16\pi^2 k_i Z_F \\ &= k_i \frac{16\pi^2}{g_{\text{string}}^2} + \frac{i}{\pi} \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau} \frac{\Gamma(k)}{|\eta|^4} \sum_{a,b=0,1} \partial_\tau \left( \frac{\vartheta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \right) \left( \overline{P}_i^2 - \frac{k_i}{4\pi \text{Im}\tau} \right) C \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right]. \end{aligned} \quad (49)$$

The latter has to be identified with the corresponding field theory one-loop gauge coupling, regulated in the infra-red in a similar fashion as the string theory. As expected, the infra-red divergence cancels between string theory and field theory results. The effective field theory has also to be supplied with an ultraviolet cut-off; expressing the field theory bare coupling in terms of the running coupling  $g_i(\mu)$ , we obtain [13]:

$$\frac{16\pi^2}{g_i^2(\mu)} = k_i \frac{16\pi^2}{g_{\text{string}}^2} + b_i \log \frac{M_s^2}{\mu^2} + \Delta_i \quad (50)$$

in the  $\overline{DR}$  scheme, where

$$b_i = \lim_{\text{Im}\tau \rightarrow \infty} \frac{i}{\pi} \frac{1}{|\eta|^4} \sum_{a,b=0,1} \partial_\tau \left( \frac{\vartheta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \right) \left( \overline{P}_i^2 - \frac{k_i}{4\pi \text{Im}\tau} \right) C \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] \quad (51)$$

are the usual beta functions, and

$$\Delta_i = \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau} \left( \frac{i}{\pi} \frac{1}{|\eta|^4} \sum_{a,b=0,1} \partial_\tau \left( \frac{\vartheta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \right) \left( \overline{P}_i^2 - \frac{k_i}{4\pi \text{Im}\tau} \right) C \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] - b_i \right) + b_i \log \frac{2e^{1-\gamma}}{\pi\sqrt{27}}. \quad (52)$$

Part of the threshold correction is universal (gauge-group independent). We can thus split (52) as

$$\Delta_i = \hat{\Delta}_i - k_i Y. \quad (53)$$

The universal piece  $Y$  contains, among other things, contributions from the universal sector (gravity in particular). Such contributions are absent in grand unified theories. Thus  $Y$  is a finite correction to the “bare” string coupling  $g_{\text{string}}$ . Moreover it is infra-red-finite, which in particular means that it remains finite when extra states become massless at some special values of the moduli. Thus we can write (50) as

$$\frac{16\pi^2}{g_i^2(\mu)} = k_i \frac{16\pi^2}{g_{\text{renorm}}^2} + b_i \log \frac{M_s^2}{\mu^2} + \hat{\Delta}_i, \quad (54)$$

---

<sup>\*\*</sup>These are the usual normalizations that lead in particular to the tree-level relation  $M_s = \frac{g_{\text{string}}}{\sqrt{32\pi G_N}}$ .

where we have defined a “renormalized” string coupling by

$$g_{\text{renorm}}^2 = \frac{g_{\text{string}}^2}{1 - \frac{Y}{16\pi^2} g_{\text{string}}^2}. \quad (55)$$

Moreover, for  $N = 2$  ground states

$$\hat{\Delta}_i = b_i \Delta, \quad (56)$$

where  $\Delta$  does not depend on the group factor. Then (54) becomes

$$\frac{16\pi^2}{g_i^2(\mu)} = k_i \frac{16\pi^2}{g_{\text{renorm}}^2} + b_i \log \frac{M_s^2 e^\Delta}{\mu^2}, \quad (57)$$

and the couplings are unified at  $\mu = M_s e^{\frac{\Delta}{2}}$ .

Let us now concentrate on a particular class of  $N = 2$  ground states, namely those that come from toroidal compactification of generic six-dimensional  $N = 1$  string theories<sup>††</sup>. Here, there is a universal two-torus, which provides the (perturbative) central charges of the  $N = 2$  algebra. Therefore (52) becomes

$$\Delta_i^{N=2} = \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \left( \frac{\Gamma_{2,2}(T, U, \bar{T}, \bar{U})}{\bar{\eta}^{24}} \left( \bar{P}_i^2 - \frac{k_i}{4\pi \text{Im } \tau} \right) \bar{\Omega} - b_i \right) + b_i \log \frac{2e^{1-\gamma}}{\pi\sqrt{27}}, \quad (58)$$

where  $T$  and  $U$  are the complex moduli of the two-torus,  $\bar{\Omega}$  is an antiholomorphic function and

$$\Gamma_{2,2}(T, U, \bar{T}, \bar{U}) = \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}} \exp \left( 2\pi i \tau (m_1 n^1 + m_2 n^2) - \frac{\pi \text{Im } \tau}{\text{Im } T \text{Im } U} |Tn^1 + TUn^2 - Um_1 + m_2|^2 \right). \quad (59)$$

From (58), we observe that the function

$$\bar{F}_i = \frac{1}{\bar{\eta}^{24}} \left( \bar{P}_i^2 - \frac{k_i}{4\pi \text{Im } \tau} \right) \bar{\Omega} \quad (60)$$

is modular invariant. Consider the associated function that appears in the  $R^2$ -term renormalization (see eq. (14) or ref. [7] for more details),

$$\bar{F}_{\text{grav}} = \frac{\hat{E}_2}{12} \frac{\bar{\Omega}}{\bar{\eta}^{24}} = \frac{1}{\bar{\eta}^{24}} \left( \frac{i}{\pi} \partial_{\bar{\tau}} \log \bar{\eta} - \frac{1}{4\pi \text{Im } \tau} \right) \bar{\Omega}, \quad (61)$$

which is also modular invariant, and eventually leads to the gravitational anomaly. The difference  $\bar{F}_i - k_i \bar{F}_{\text{grav}}$  is an antiholomorphic function, which is modular invariant. It has at most a simple pole at  $\tau \rightarrow i\infty$  (associated with the heterotic unphysical tachyon) and is finite inside the moduli space of the torus. This implies that

$$\bar{F}_i = k_i \bar{F}_{\text{grav}} + A_i \bar{j}(\bar{\tau}) + B_i, \quad (62)$$

---

<sup>††</sup>Note that this is not the most general four-dimensional  $N = 2$  theory.

where  $A_i$  and  $B_i$  are constants to be determined, and  $j(\tau) = \frac{1}{q} + 744 + \mathcal{O}(q)$ ,  $q = \exp(2\pi i\tau)$  is the standard  $j$ -function. The modular invariance of  $\overline{F}_{\text{grav}}$  implies that  $\overline{\Omega}$  is a modular form of weight 10, which is finite inside the moduli space. This property fixes

$$\overline{\Omega} = \xi \overline{E}_4 \overline{E}_6, \quad (63)$$

where  $E_{2n}$  is the  $n$ th Eisenstein series:

$$E_2 = \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}, \quad (64)$$

$$E_4 = \frac{1}{2} (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad (65)$$

$$E_6 = \frac{1}{2} (\vartheta_2^4 + \vartheta_3^4) (\vartheta_3^4 + \vartheta_4^4) (\vartheta_4^4 - \vartheta_2^4) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \quad (66)$$

Putting everything together in (58) we obtain:

$$\Delta_i^{N=2} = \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \left( \Gamma_{2,2}(T, U, \overline{T}, \overline{U}) \left( \frac{\xi k_i \widehat{E}_2 \overline{E}_4 \overline{E}_6}{12 \overline{\eta}^{24}} + A_i \overline{j} + B_i \right) - b_i \right) + b_i \log \frac{2 e^{1-\gamma}}{\pi \sqrt{27}}. \quad (67)$$

There are two constraints that allow us to fix the constants  $A_i, B_i$ . The first is that the  $1/q$  pole is absent from the group trace, which gives

$$A_i = -\frac{\xi k_i}{12}. \quad (68)$$

The second is (51), which implies

$$744 A_i + B_i - b_i + k_i b_{\text{grav}} = 0, \quad (69)$$

where

$$b_{\text{grav}} = \lim_{\text{Im } \tau \rightarrow \infty} \overline{F}_{\text{grav}} = -22 \xi \quad (70)$$

is the gravitational anomaly in units where a hypermultiplet contributes  $\frac{1}{12}$  [7]. Plugging (68)–(70) in (67), we finally obtain:

$$\begin{aligned} \Delta_i^{N=2} &= b_i \left( \log \frac{2 e^{1-\gamma}}{\pi \sqrt{27}} + \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} (\Gamma_{2,2}(T, U, \overline{T}, \overline{U}) - 1) \right) \\ &\quad + \frac{\xi k_i}{12} \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \Gamma_{2,2}(T, U, \overline{T}, \overline{U}) \left( \frac{\widehat{E}_2 \overline{E}_4 \overline{E}_6}{\overline{\eta}^{24}} - \overline{j} + 1008 \right). \end{aligned} \quad (71)$$

The first integral in (71) was computed explicitly in [3] and recently generalized in [17]:

$$\int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} (\Gamma_{2,2}(T, U, \overline{T}, \overline{U}) - 1) = -\log (|\eta(T)|^4 |\eta(U)|^4 \text{Im } T \text{Im } U) - \log \frac{8\pi e^{1-\gamma}}{\sqrt{27}}. \quad (72)$$

Therefore, as advertised above, we can write

$$\Delta_i^{N=2} = b_i \Delta - k_i Y, \quad (73)$$

with

$$\Delta = -\log \left( 4\pi^2 |\eta(T)|^4 |\eta(U)|^4 \text{Im } T \text{Im } U \right) \quad (74)$$

and

$$Y = -\frac{\xi}{12} \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \Gamma_{2,2} (T, U, \bar{T}, \bar{U}) \left( \left( \bar{E}_2 - \frac{3}{\pi \text{Im } \tau} \right) \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008 \right) \quad (75)$$

(we have used eq. (13)). This form of the universal term was determined for the case of  $Z_2 \times Z_2$  orbifolds in [13] and further generalized for a larger class of models in [14].

The coefficient  $\xi$  can be related to the number of massless vector multiplets  $N_V$  and hypermultiplets  $N_H$  via the gravitational anomaly ( $b_{\text{grav}}$ ), which can also be computed from the low-energy theory of massless states. In units where a scalar contributes 1, the graviton contributes 212, the antisymmetric tensor 91, the gravitino  $-\frac{233}{4}$ , a vector  $-13$  and a Majorana fermion  $\frac{7}{4}$ ; therefore the  $N = 2$  supergravity multiplet contributes  $212 - 2\frac{233}{4} - 13 = \frac{165}{2}$ , the tensor multiplet contributes  $-13 + 2\frac{7}{4} + 1 + 91 = \frac{165}{2}$ , a vector multiplet  $-13 + 2\frac{7}{4} + 2 = -\frac{15}{2}$  and a hypermultiplet  $2\frac{7}{4} + 4 = \frac{15}{2}$ . Thus in the units of (70),

$$b_{\text{grav}} = \frac{22 - N_V + N_H}{12}, \quad (76)$$

and hence

$$\xi = -\frac{1}{264} (22 - N_V + N_H). \quad (77)$$

Therefore, the universal contribution (75) reads:

$$Y = \frac{22 - N_V + N_H}{3168} \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im } \tau} \Gamma_{2,2} (T, U, \bar{T}, \bar{U}) \left( \left( \bar{E}_2 - \frac{3}{\pi \text{Im } \tau} \right) \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008 \right). \quad (78)$$

As an example, let us consider the case of the  $Z_2$  orbifold, where we have a gauge group  $E_8 \times E_7 \times SU(2) \times U(1)^2$  and thus  $N_V = 386$ . The number of massless hypermultiplets is  $N_H = 628$ . Using these numbers in (77) we obtain in this case  $\xi = -1$ . As expected by supersymmetry, the corresponding universal threshold is twice as big as a single-plane contribution of the symmetric  $Z_2 \times Z_2$  orbifold.

We come finally to the important observation that the result  $\xi = -1$  applies to more general situations than the above example. One can indeed show that  $N_H - N_V$  is a *universal constant* for the whole class of four-dimensional  $N = 2$  models obtained by toroidal compactification of *any*  $N = 1$  ground state in six dimensions. The argument is the following. From the six-dimensional point of view, the models at hand must obey an anomaly-cancellation constraint, which reads:  $N_H - N_V|_{\text{six dim}} = 244$ , and does not depend on the kind of compactification that has been performed from ten to six dimensions<sup>‡‡</sup> [27]. After two-torus

<sup>‡‡</sup>Actually, this constraint, which ensures that  $\text{Tr } R^4$  vanishes, holds even when there occurs a symmetry enhancement originated from non-perturbative effects, provided the number of tensor multiplets remains  $N_T = 1$ . Note that this six-dimensional anomaly-cancellation constraint is also used in [26], in relation to four-dimensional quantities.

compactification to four dimensions, two extra  $U(1)$ 's appear, leading to the relation

$$N_H - N_V = 242 \tag{79}$$

between the numbers of vector multiplets and hypermultiplets. In turn, eq. (77) implies that for this class of ground states  $\xi = -1$ , as advertised previously. As a consequence, all  $N = 2$  models under consideration have equal universal thresholds, given by (78) and (79).

## 5. Conclusions

We have applied the background field method to analyze the response of a string, supplied with an appropriate curvature-induced infra-red cut-off, to magnetic and gravitational marginal deformations. This has allowed us to obtain the exact (i.e. to all orders in  $\alpha'$ ) genus-1 free energy of the string as an expansion with respect to the space-time curvature parameter  $1/k$ . Comparison with the effective field theory  $\sigma$ -model action has led to definite results for the one-loop corrections to gauge and gravitational couplings. Indeed, we have demonstrated the absence of corrections to Newton's constant for supersymmetric ground states, and argued how this result can be extended to all orders in  $g_{\text{string}}$ . We have then derived the full one-loop gauge coupling in the  $\overline{DR}$  scheme, for  $N = 1$  supersymmetric theories.

For the class of  $N = 2$  four-dimensional theories that come from torus compactification of six-dimensional  $N = 1$  ground states, using the relation between gauge and  $R^2$ -term renormalizations, we have obtained an explicit formula for the universal part of the threshold corrections. It is quite remarkable that the latter turns out to be related to the quantity  $N_H - N_V$ , which is *fully determined* as a consequence of *the anomaly cancellation* (gauge, gravitational and mixed) in the underlying six-dimensional theory. Therefore, the whole class of models under consideration have equal universal thresholds. Note, however, that although  $N_H - N_V$  is not expected to receive any non-perturbative contribution as long as  $N_T = 1$ , the universal thresholds in general are.

We would like, finally, to note that the above results can be further generalized to the contributions of the  $N = 2$  supersymmetric sectors in  $N = 1$  string models [14]. In these models, which are phenomenologically interesting, the universal thresholds will play a role for the issue of string unification [13].

## Acknowledgements

C. Kounnas was supported in part by EEC contracts SC1\*-0394C and SC1\*-CT92-0789. P.M. Petropoulos acknowledges financial support from the EEC contracts CHRX-CT93-0340 and SC1-CT92-0792. J. Rizos would like to thank the CERN Theory Division for hospitality and acknowledges financial support from the EEC contract ERBCHBGCT940634.

## References

- [1] J. Minahan, Nucl. Phys. **B298** (1988) 36.
- [2] V.S. Kaplunovsky, Nucl. Phys. **B307** (1988) 145.

- [3] L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. **B355** (1991) 649.
- [4] V.S. Kaplunovsky and J. Louis, Nucl. Phys. **B422** (1994) 57 and **B444** (1995) 191.
- [5] J.-P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, Nucl. Phys. **B372** (1992) 145 and Phys. Lett. **271B** (1991) 307;  
G. Lopes Cardoso and B.A. Ovrut, Nucl. Phys. **B369** (1992) 351;  
S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, Nucl. Phys. **B365** (1991) 431.
- [6] I. Antoniadis, K.S. Narain and T.R. Taylor, Phys. Lett. **276B** (1991) 37;  
I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. **B407** (1993) 706, **B413** (1994) 162 and **B432** (1994) 187;  
I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. **B447** (1995) 35.
- [7] I. Antoniadis, E. Gava and K.S. Narain, Phys. Lett. **283B** (1992) 209 and Nucl. Phys. **B383** (1992) 93.
- [8] I. Antoniadis, J. Ellis, R. Lacaze and D.V. Nanopoulos, Phys. Lett. **268B** (1991) 188;  
I. Antoniadis, J. Ellis, S. Kelley and D.V. Nanopoulos, Phys. Lett. **281B** (1992) 374;  
L. Dolan and J.T. Liu, Nucl. Phys. **B387** (1992) 86;  
A. Faraggi, Phys. Lett. **302B** (1993) 202;  
M.K. Gaillard and R. Xiu, Phys. Lett. **296B** (1992) 71;  
L.E. Ibáñez, Phys. Lett. **318B** (1993) 73.
- [9] P. Mayr and S. Stieberger, Nucl. Phys. **B407** (1993) 725 and Phys. Lett. **355B** (1995) 107;  
P. Mayr, H.-P. Nilles and S. Stieberger, Phys. Lett. **317B** (1993) 53;  
H.-P. Nilles and S. Stieberger, Phys. Lett. **367B** (1996) 126;  
H.-P. Nilles, hep-ph/9601241.
- [10] P. Mayr and S. Stieberger, Nucl. Phys. **B412** (1994) 502.
- [11] G. Lopes Cardoso, D. Lüst and B. Ovrut, Nucl. Phys. **B436** (1995) 65;  
G. Lopes Cardoso, D. Lüst and T. Mohaupt, Nucl. Phys. **B450** (1995) 115 and **B455** (1995) 131;  
B. de Wit, V.S. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. **B451** (1995) 53.
- [12] E. Kiritsis and C. Kounnas, Nucl. Phys. **B442** (1995) 472, **B41** [Proc. Sup.] (1995) 331, **B45** [Proc. Sup.] (1996) 207 and hep-th/9507051.
- [13] P.M. Petropoulos and J. Rizos, hep-th/9601037, to appear in Phys. Lett. **B**.
- [14] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, CERN-TH/96-90 preprint, April 1996.
- [15] P.M. Petropoulos, in the proceedings of the *5th Hellenic School and Workshops on Elementary Particle Physics*, Corfu, Greece, 3–24 September 1995.



- [16] D. Bailin and A. Love, Phys. Lett. **292B** (1992) 315;  
D. Bailin, A. Love, W.A. Sabra and S. Thomas, Phys. Lett. **320B** (1994) 21, Mod. Phys. Lett. **A9** (1994) 67 and **A10** (1995) 337;  
C. Bachas, C. Fabre and T. Yanagida, hep-th/9510094;  
M. Chemtob, hep-th/9506178.
- [17] J.A. Harvey and G. Moore, hep-th/9510182.
- [18] K. Dienes and A. Faraggi, Phys. Rev. Lett. **75** (1995) 2646 and Nucl. Phys. **B457** (1995) 409;  
K. Dienes, A. Faraggi and J. March-Russell, hep-th/9510223;  
K. Dienes, hep-th/9602045.
- [19] S. Ferrara, C. Kounnas and F. Zwirner, Nucl. Phys. **B429** (1994) 589 and ERRATUM ibid. **B433** (1995) 255;  
C. Kounnas, I. Pavel and F. Zwirner, Phys. Lett. **335B** (1994) 403.
- [20] E. Kiritsis and C. Kounnas, to appear.
- [21] H.-P. Nilles, Phys. Lett. **115B** (1982) 193 and Nucl. Phys. **B217** (1983) 366;  
S. Ferrara, L. Girardello and H.-P. Nilles, Phys. Lett. **125B** (1983) 457;  
J.-P. Derendinger, L.E. Ibáñez and H.-P. Nilles, Phys. Lett. **155B** (1985) 65;  
M. Dine, R. Rohm, N. Seiberg and E. Witten, Phys. Lett. **156B** (1985) 55;  
C. Kounnas and M. Porrati, Phys. Lett. **191B** (1987) 91.
- [22] C. Kounnas and M. Porrati, Nucl. Phys. **B310** (1988) 355;  
S. Ferrara, C. Kounnas, M. Porrati and F. Zwirner, Nucl. Phys. **B318** (1989) 75;  
I. Antoniadis, C. Bachas, D. Lewellen and T. Tomaras, Phys. Lett. **207B** (1988) 441;  
C. Kounnas and B. Rostand, Nucl. Phys. **B341** (1990) 641;  
I. Antoniadis, Phys. Lett. **246B** (1990) 377;  
I. Antoniadis and C. Kounnas, Phys. Lett. **261B** (1991) 369.
- [23] C. Bachas, hep-th/9503030.
- [24] E. Kiritsis and C. Kounnas, Nucl. Phys. **B456** (1995) 699 and hep-th/9509043.
- [25] C. Callan, J. Harvey and A. Strominger, Nucl. Phys. **B359** (1991) 611;  
C. Kounnas, Phys. Lett. **321B** (1994) 26, Proceedings of the *International Lepton-Photon Symposium and Europhysics Conference on High Energy Physics*, Geneva, 1991, Vol. 1, pp. 302–306, and Proceedings of the *International Workshop on String Theory, Quantum Gravity and Unification of Fundamental Interactions*, Rome, 21–26 September 1992;  
E. Kiritsis, C. Kounnas and D. Lüst, Int. J. Mod. Phys. **A9** (1994) 1361;  
I. Antoniadis, S. Ferrara and C. Kounnas, Nucl. Phys. **B421** (1994) 343.
- [26] G. Aldazabal, A. Font, L.E. Ibáñez and F. Quevedo, hep-th/9602097.
- [27] J.H. Schwarz, hep-th/9512053.