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Phases of Antisymmetric Tensor Field Theories

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We study the different phases of field theories of compact antisymmetric tensors of rank $h - 1$ in arbitrary space-time dimensions $D = d + 1$. Starting in a ‘Coulomb’ phase, topological defects of dimension $d - h - 1$ ($(d - h - 1)$ -branes) may condense leading to a generalized ‘confinement’ phase. If the dual theory is also compact the model may also have a third, generalized ‘Higgs’ phase, driven by the condensation of the dual $(h - 2)$ -branes. Developing on the work of Julia and Toulouse for ordered solid-state media, we obtain the low energy effective action for these phases. Each phase has two dual descriptions in terms of antisymmetric tensors of different ranks, which are massless for the Coulomb phase but massive for the Higgs and confinement phases. We illustrate our prescription in detail for compact QED in 4D. Compact QED and $O(2)$ models in 3D, as well as a periodic scalar field in 2D (strings on a circle), are also discussed. In this last case we show how T -duality is maintained if one considers both worldsheet instantons and their duals. We also unify various approaches to the problem of the axion mass in 4D string models. Finally we discuss possible implications of our results for non-perturbative issues in string theory.

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1. Introduction

Antisymmetric tensor theories have been thoroughly studied during the past years [1,2,3,4,5,6]. They are the natural extension of free scalar field theories and Abelian gauge theories, and share some properties which makes them easy to study. In particular the powerful property of strong–weak coupling duality [7], known for electromagnetism and free scalar field theories, can be easily generalized to antisymmetric tensor theories in any dimension.

Antisymmetric tensors also appear very naturally in supersymmetric field theories and in string theories [8]. They play an important role in the realization of the various strong–weak coupling dualities among string theories [9,10]. An antisymmetric tensor of rank $h-1$ couples naturally to an elementary extended object of dimension $h-2$, a $(h-2)$ -brane [11]. These objects, however, may also appear as solitonic excitations [12] of an underlying theory if the antisymmetric tensors are compact variables.

It is well known that the condensation of topological defects may drive phase transitions, the prototype of this phenomenon being the vortex-driven Kosterlitz–Thouless transition in two space dimensions [13,14].

When analyzing phase transitions induced by topological defects, two questions have to be answered. The first is to establish if a certain kind of topological defect does indeed condense, and for which values of the temperature or the coupling constants this happens. The second is to establish the nature of the new phase with a finite condensate of topological defects. In this paper we shall concentrate on this second aspect for generic antisymmetric tensor field theories in $D = d + 1$ space-time dimensions.

Nearly twenty years ago, Julia and Toulouse [15] tackled this problem in the framework of ordered solid-state media. They considered models supporting stable topological defects, with homotopy group Z [16], and characterized by a length scale $1/M$, where the mass M is considered as a cut-off for the low-energy effective field theory.

The idea of Julia and Toulouse is that the condensation of these topological defects generates new hydrodynamical modes for the low-energy effective theory: these new modes are essentially the long wavelength fluctuations of the continuous distribution of topological defects. Moreover, Julia and Toulouse proposed also a generic prescription to identify these new modes. However, in the framework of ordered solid-state media, it is difficult to write down an action for the phase with a condensate of topological defects due to the non-linearity of the topological currents, the lack of relativistic invariance, and the need to

introduce dissipation terms. Here, we will apply and develop the idea of Julia and Toulouse for generic compact antisymmetric field theories, for which none of the above problems is present.

In this framework we will show that the Julia–Toulouse prescription can be made much more precise and that it leads also to an explicit form for the action in the finite condensate phase. We will also show that this phase is the natural generalization of the confinement phase for a vector gauge field and that the Julia–Toulouse mechanism is the exact *dual* of the Higgs mechanism in its pristine Stückelberg formulation [17]. Thus the generalized confinement phase for the original tensor is equivalent to a generalized Higgs–Stückelberg phase for its dual tensor. Our results generalize the well-known ’t Hooft–Mandelstam duality [18] to any p -form theory in D space-time dimensions.

We also present several concrete examples, among which a detailed discussion of 4D compact quantum electrodynamics (QED) [19], of T -duality in compactified strings [21] and of the axion mass in 4D string models [22,23,24]. Finally we discuss possible implications of our results for non-perturbative issues in string theories.

This paper is organized as follows. In section 2 we present the original Julia–Toulouse prescription, while in section 3 we adapt it to antisymmetric tensor field theories. In section 4 we discuss the example of compact QED and we demonstrate the confinement mechanism. Section 5 is devoted to the derivation of the duality between the Julia–Toulouse mechanism and the Higgs–Stückelberg mechanism. In section 6 we discuss various examples. Finally we draw our conclusions in section 7.

2. The Julia–Toulouse prescription

In this section we shall present the Julia–Toulouse prescription as originally developed for ordered solid-state media [15].

The low-energy excitations of such systems are generically described by a field theory for an order parameter. In addition to these propagating modes, there are also massive classical solutions describing the *topological defects* of the medium.

We thus consider a generic $(d + 1)$ -dimensional field theory with symmetry group G spontaneously broken down to H . The homotopy groups $\Pi_h(G/H)$ classify the topological defects that can arise in this theory [16]. A non-trivial $\Pi_h(G/H)$ for $h < d$ describes *solitons* of dimension $d - h - 1$; for $h = d$, instead, it describes *instantons* of the Euclidean version of the model. These finite-energy (action) classical solutions are characterized by

radii $r_i = 1/M_i$, where M_i are the various masses associated with spontaneous symmetry breakdown. From the point of view of the low-energy effective theory with symmetry group H , valid on energy scales much smaller than $\min\{M_i\}$, they can be viewed essentially as singularities of dimension $d - h - 1$ in R^d (for solitons) or point singularities in R^{d+1} (for instantons) [12]. The low-energy effective action is then well-defined only outside these singularities, which contain lumps of energy (action) involving higher-lying fields.

From now on we shall specialize to *stable topological defects*, for which the relevant homotopy groups $\Pi_h(G/H) = \mathbb{Z}$. In this case there is an analytic topological invariant that can be written as

$$\int_{S_h} \omega_h, \quad (2.1)$$

where S_h is an h -dimensional sphere surrounding the singularity on an $(h+1)$ -dimensional hyperplane perpendicular to it and ω_h is an h -form, which is exact: $\omega_h = d\phi_{h-1}$, ‘outside’ S_h . Both ϕ_{h-1} and ω_h are constructed in terms of the fundamental fields of the low-energy effective theory with symmetry group H .

Consider now this low-energy effective theory in the presence of a few topological defects. Essentially we have a model defined on $(d+1)$ -dimensional Minkowski space-time with a few $(d-h-1)$ -dimensional holes in space where the topological defects live. In the case of instantons we would have correspondingly a model on $(d+1)$ -dimensional Euclidean space with a few holes at the location of the instantons. When the number of topological defects grows, the manifold on which the low-energy effective action is well defined soon becomes very complicated. The question we would like to address is what happens if, for a certain range of parameters, the dynamics favours the formation of a macroscopic number of topological excitations with a finite density. The Julia–Toulouse theory provides a prescription to identify the additional hydrodynamical (long-lived) modes of a solid state medium due to the continuous distribution of topological defects. In the framework of relativistic field theories these modes would be additional fields in the new phase of the low-energy theory at finite density of topological defects.

We start with the following observation. Associated with the integral invariant (2.1) there is a $(d-h)$ -form $J_{d-h} = \Omega_{h+1}^* = (d\omega_h)^*$. When considered as forms defined only on the manifold with holes where the low-energy effective theory lives, both Ω_{h+1} and J_{d-h} vanish identically since $\omega_h = d\phi_{h-1}$ there. If they are extended to the whole manifold R^{d+1} , however, they pick up delta-like singularities at the locations of the topological

defects. In this case the totally antisymmetric components $J^{\mu\nu\dots\alpha}$ of J_{d-h} describe the topologically conserved “currents” of topological defects,

$$\partial_\mu J^{\mu\nu\dots\alpha} = 0. \tag{2.2}$$

For point-like solitons $h = d - 1$ and J is a 1-form whose components J^μ form a proper current. For string-like vortices $h = d - 2$ and J_{d-h} is a 2-form with components $J^{\mu\nu}$: the three pure space components correspond to the density of vortices, while the three mixed components correspond to the current density of vortices. For instantons $h = d$ and J_{d-h} is a 0-form describing the density of instantons in Euclidean space: in this case there is clearly no differential conservation law.

The idea of Julia and Toulouse to identify the additional low-energy modes in the phase with a finite density of topological defects consists in promoting the closed form ω_h defined on the very complicated manifold ‘outside’ the many defects to a fundamental form defined everywhere on R^{d+1} . In this way, the components of J_{d-h} become smooth fields on R^{d+1} describing the conserved fluctuations of the continuous distribution of topological defects. These fluctuations constitute the new hydrodynamical modes. Actually, since $J_{d-h} = (d\omega_h)^*$, these new degrees of freedom are associated only with the *gauge-invariant* part of the new fundamental h -form ω_h . For example, for vortices in (3+1) dimensions, we generically obtain 2 new degrees of freedom.

3. Application to antisymmetric tensor field theories

While the Julia–Toulouse prescription is simple and appealing for the identification of the additional low-energy modes due to a continuous distribution of topological defects, it is not so simple to extract the dynamics of these new modes and their coupling to the original fields of the low-energy effective model. Two are the difficulties in the framework of ordered solid-state media. First, in the generic case, the form ω_h is a complicated expression in terms of the fundamental fields of the original theory; secondly, one has to introduce appropriate dissipation terms for the new modes.

In this section we wish to point out that there is a class of relativistic field theories for which the Julia–Toulouse prescription can be made much more precise: in these cases we will also obtain a simple form for the effective action in the phase with finite density of topological defects. These field theories are relevant to confinement physics, the low-energy limit of fundamental string theories and possibly cosmology.

The class of field theories we have in mind contain in their low-energy limit a *compact* [20] *fundamental* $(h-1)$ -form ϕ_{h-1} with (generalized) gauge invariance under transformations $\phi_{h-1} \rightarrow \phi_{h-1} + d\lambda_{h-2}$. In particular we will consider the following generic low-energy effective action:

$$S = \int \frac{(-1)^{h-1}}{e^2} d\phi_{h-1} \wedge (d\phi_{h-1})^* + \kappa \phi_{h-1} \wedge j_{h-1}^* + S_M, \quad (3.1)$$

where j_{h-1} describes a conserved (tensor) current of fields whose dynamics is governed by the action S_M . For convenience we will take the canonical mass dimension of ϕ_{h-1} as $(d-1)/2$, so that e^2 is a dimensionless coupling constant. Gauge fixing is always understood. For $j_{h-1} = 0$ this action describes

$$N_{\phi_{h-1}} = \binom{d}{h-1} - \binom{d}{h-2} + \binom{d}{h-3} + \dots + (-1)^{h-1} \binom{d}{0} = \binom{d-1}{h-1} \quad (3.2)$$

massless physical degrees of freedom.

The compactness [20] of the form ϕ_{h-1} ensures the presence of $(d-h-1)$ -dimensional singularities describing the topological defects. The origin of these can be thought of as spontaneous symmetry breaking at a very high energy scale or as a different mechanism. For example, the field theories describing the low-energy limit of fundamental string theories typically contain higher-rank tensor fields [8]: in this case the topological defects of the low-energy field theory may be thought of as lumps of energy (action) involving higher-lying string modes and describe essentially $(d-h-1)$ -branes [11]. For our purposes, however, the origin of the topological defects is inessential; the important point is that in this case the topological invariant (2.1) can be formulated directly in terms of the form $\omega_h = d\phi_{h-1}$, which is *linear* in the fundamental field ϕ_{h-1} of the low-energy effective field theory. Moreover, in the framework of relativistic field theories one clearly does not have the problem of introducing dissipation terms.

In the following we shall follow closely the idea of Julia and Toulouse, i.e. we shall consider that a condensation of topological defects generates a new low-energy mode described by the gauge-invariant part of an h -form ω_h : however, we will take at first ω_h to be a new, additional field, which is not related to $d\phi_{h-1}$. Therefore the action in the phase with finite density of topological defects will depend on this additional field ω_h , as well as on the original low-energy fields ϕ_{h-1} .

In order to write down this action, we start by noting that the condensation of topological defects generates a new scale Λ in the theory. Suppose that solitons of dimension

$d - h - 1$ condense. Take any $(h + 1)$ -dimensional hyperplane in R^d : generically, the intersections of this hyperplane with the solitons will be point-like. The new parameter can be taken to represent the average density ρ of these points on the chosen hyperplane. Since ρ has dimension $(\text{mass})^{h+1}$, the new mass scale Λ is essentially given by $\Lambda \propto \rho^{1/h+1}$. This formula is valid also in the case of instantons, for which $h = d$ and ρ describes the density of instantons in Euclidean space.

With this point of view there are three requirements on the effective action in the phase at finite density of topological defects. The first is *gauge invariance*. Actually two gauge symmetries must be present: the first is the original gauge symmetry under transformations $\phi_{h-1} \rightarrow \phi_{h-1} + d\lambda_{h-2}$; the second is a new gauge symmetry under transformations $\omega_h \rightarrow \omega_h + d\psi_{h-1}$, which ensures that only the gauge invariant part of ω_h describes a new physical degree of freedom. The second requirement is *relativistic invariance*. The third is that in the limit $\Lambda \rightarrow 0$ one has to recover the original low-energy theory. Up to two derivatives in the new fundamental form ω_h we thus obtain the effective action

$$S_{d-h-1} = \int \frac{(-1)^h}{\Lambda^2} \Omega_{h+1} \wedge \Omega_{h+1}^* + \frac{(-1)^{h-1}}{e^2} (\omega_h - d\phi_{h-1}) \wedge (\omega_h - d\phi_{h-1})^* + \kappa (\omega_h - d\phi_{h-1}) \wedge T_h^* + S_M , \quad (3.3)$$

where we index the new action by the dimension of the condensing topological defects and $\Omega_{h+1} \equiv d\omega_h$. Relativistic invariance and the two gauge symmetries are manifest. Actually, transformations $\omega_h \rightarrow \omega_h + d\psi_{h-1}$ must be accompanied by corresponding transformations $\phi_{h-1} \rightarrow \phi_{h-1} + \psi_{h-1}$, so that the full new gauge symmetry is given by

$$\begin{aligned} \omega_h &\rightarrow \omega_h + d\psi_{h-1} , \\ \phi_{h-1} &\rightarrow \phi_{h-1} + \psi_{h-1} . \end{aligned} \quad (3.4)$$

Correspondingly, the original conserved $(h - 1)$ -form current must be promoted to an h -form T_h such that

$$\partial_\mu T^{\mu\nu\dots\alpha} = \frac{1}{h} j^{\nu\dots\alpha} . \quad (3.5)$$

In the limit $\Lambda \rightarrow 0$ the only contributions to the partition function come from configurations for which $\Omega_{h+1} = 0$. The solution to this constraint is $\omega_h = d\psi_{h-1}$: the field ψ_{h-1} can then be absorbed into ϕ_{h-1} and one recovers (upon an integration by parts) the original low-energy effective action (3.1).

Let us now have a closer look at the action (3.3). Clearly the new gauge symmetry (3.4) has to be gauge fixed, i.e. one has to divide the partition function corresponding to

(3.3) by the gauge volume. As usual for Abelian systems, this means that the functional integration over the original field ϕ_{h-1} can be dropped after having absorbed $d\phi_{h-1}$ into a redefinition of the new field ω_h . The gauge-fixed action can thus be formulated exclusively in terms of the new field ω_h :

$$S_{d-h-1} = \int \frac{(-1)^h}{\Lambda^2} \Omega_{h+1} \wedge \Omega_{h+1}^* + \frac{(-1)^{h-1}}{e^2} \omega_h \wedge \omega_h^* + \kappa \omega_h \wedge T_h^* + S_M . \quad (3.6)$$

For $T_h = j_{h-1} = 0$ the equations of motion following from this action are given by

$$\begin{aligned} \partial_\mu \partial^{[\mu} \omega^{\alpha_1 \dots \alpha_h]} + \frac{\Lambda^2}{e^2} \omega^{\alpha_1 \dots \alpha_h} &= 0 , \\ \partial_{[\mu} \omega_{\alpha_1 \dots \alpha_h]} &\equiv \partial_\mu \omega_{\alpha_1 \dots \alpha_h} + (-1)^h \partial_{\alpha_1} \omega_{\alpha_2 \dots \alpha_h \mu} \\ &+ \partial_{\alpha_2} \omega_{\alpha_3 \dots \mu \alpha_1} + \dots + (-1)^h \partial_{\alpha_h} \omega_{\mu \alpha_1 \dots \alpha_{h-1}} . \end{aligned} \quad (3.7)$$

Taking derivatives with respect to all variables x^{α_i} we then obtain

$$\partial_{\alpha_i} \omega^{\alpha_1 \dots \alpha_h} = 0 , \quad \forall i . \quad (3.8)$$

Inserting this back into (3.7) we finally get

$$\begin{aligned} (\partial^2 + m^2) \omega^{\alpha_1 \dots \alpha_h} &= 0 , \\ m &= \frac{\Lambda}{e} , \end{aligned} \quad (3.9)$$

showing explicitly that the physical content of (3.6) consists of

$$N_{\omega_h} = \binom{d}{h} \quad (3.10)$$

massive degrees of freedom ¹.

What has happened here is the exact contrary of the familiar Higgs mechanism, where the original gauge field ‘eats’ the Goldstone mode due to the condensation of the Higgs field thereby acquiring a longitudinal part and a mass. Here it is the new gauge field, due to the condensation of topological defects, which ‘eats’ the original gauge field, thereby acquiring $\binom{d-1}{h}$ ‘longitudinal’ degrees of freedom and a mass. This renders much more precise the original Julia–Toulouse prescription of ‘promoting the $(h-1)$ -form ϕ_{h-1} to a new fundamental h -form ω_h ’. Actually, the relation between this mechanism, which

¹ Massive antisymmetric tensors have been previously studied in [4,5,22].

we shall call the *Julia–Toulouse mechanism* and the Higgs mechanism can be made more precise. Indeed, in the next section we will show that the Julia–Toulouse mechanism is the exact *dual* to the Higgs mechanism.

The question of additional couplings remains to be discussed. As we have shown above, the original, conserved $(h - 1)$ -form current j_{h-1} must be promoted to an h -form T_h satisfying (3.5). By integrating out the field ω_h we obtain an induced action for T_h . The non-local kernel in this action is short-range since ω_h is a massive field. By taking the local limit we obtain a local, low-energy, induced action $S_{\text{ind}}(T_h)$ for T_h . This example will clarify that the Julia–Toulouse mechanism describes a confinement phase for the original $(h - 1)$ -form ϕ_{h-1} .

We still have to discuss the possible *topological terms* that we have neglected until now. Depending on the space-time dimensionality, there are three possible types of gauge invariant topological terms:

$$\begin{aligned}
d = 2h - 2 , & & \phi_{h-1} \wedge d\phi_{h-1} , \\
d = 2h - 1 , & & (\omega_h - d\phi_{h-1}) \wedge (\omega_h - d\phi_{h-1}) , \\
d = 2h , & & \omega_h \wedge d\omega_h .
\end{aligned} \tag{3.11}$$

The first is a (generalized) Chern–Simons term for the original gauge field ϕ_{h-1} . We do not include such a term in our generic action (3.1) since it produces a confinement mechanism for the topological defects which prevents a condensation phase. This is well known [25] for the familiar vector Chern–Simons term in $d = 2$, which suppresses the instantons of the model. The same type of argument, however, leads to similar results in higher-dimensional theories. The second gauge-invariant topological term comprises actually three terms: a topological mass term $\omega_h \wedge \omega_h$ for ω_h , a so-called BF-coupling $\omega_h \wedge d\phi_{h-1}$, and a (generalized) θ -term $d\phi_{h-1} \wedge d\phi_{h-1}$ for ϕ_{h-1} . According to our construction this gauge-invariant combination appears in the action at finite density of topological defects if the θ -term $d\phi_{h-1} \wedge d\phi_{h-1}$ is present in the original low-energy effective action. We will show below, how this affects the mass (3.9) in a specific example. The third possible topological term is a (generalized) Chern-Simons term for ω_h . This term cannot be excluded on general grounds: it could appear in the actions (3.3) and (3.6) if the condensation of topological defects violates some discrete symmetries of the original theory or if these were anyhow violated by the coupling with j .

We would like to conclude this section by stressing that the realization of a phase with a condensate of topological defects remains a dynamical issue, which cannot be solved

within the present framework. Here we have only discussed the form of the field theories at finite density of topological defects, assuming that a condensation indeed takes place. The best way to address the condensation mechanism is via a lattice formulation of the antisymmetric tensor theories [13,26,19,20,27,6,28] although a full renormalization group analysis is available only in 2D [13].

4. Compact QED in (3+1) dimensions

Before pursuing our general analysis we shall pause to describe a first concrete example, namely *compact* quantum electrodynamics (QED)[19,20].

Compact QED (in 3+1 dimensions) is QED with *magnetic monopoles* [29]. It can be thought of as a cut-off theory describing the low-energy sector of an $SO(3)$ Georgi–Glashow model with spontaneously broken symmetry $SO(3) \rightarrow U(1)$. From the point of view of the low-energy $U(1)$ theory the 't Hooft–Polyakov monopoles of the model appear essentially as singular Dirac monopoles labelled by $\Pi_2(SO(3)/U(1)) = \Pi_1(U(1)) = \mathbb{Z}$.

It is by now generally accepted that the condensation of monopoles in compact QED drives a transition from a weak coupling Coulomb phase to a strong coupling *confinement phase*, characterized by a ‘massive photon’ and an area law for the Wilson loop [20]. An analytical proof of this mechanism has been recently given for $N = 2$ supersymmetric Yang-Mills theories [30], for which the role of the Higgs field is played by the scalar in the $N = 2$ vector multiplet.

In the Coulomb phase, magnetic monopoles are dilute. Away from the singularities the action is given simply by

$$S = \int d^4x \frac{-1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu} F^{\mu\nu*} , \quad (4.1)$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu , \quad F^{\mu\nu*} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} ,$$

where we have included also a θ -term to illustrate how this affects the Julia–Toulouse mechanism. The θ -term can be rewritten as a total derivative: in the absence of magnetic monopoles it can be dropped altogether; in their presence, however, it produces non-trivial effects, notably it assigns an electric charge $q = e\theta/2\pi$ to elementary magnetic monopoles [31]; θ is an angular variable: elementary monopoles with $\theta = 2\pi n + \theta'$ are equivalent to dyons with electric charge en and parameter θ' . Since we are interested here in the condensation of magnetic monopoles, we shall restrict θ to the range $0 \leq \theta < 2\pi$.

Following the general construction outlined in the preceding sections we can now immediately write down the action for compact QED in the phase with a monopole condensate. The gauge-fixed action is formulated in terms of an antisymmetric tensor $\omega_{\mu\nu}$ and reads

$$S_1 = \int d^4x \frac{1}{12\Lambda^2} \Omega_{\mu\nu\alpha} \Omega^{\mu\nu\alpha} - \frac{1}{4e^2} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{\theta}{64\pi^2} \omega_{\mu\nu} \epsilon^{\mu\nu\alpha\beta} \omega_{\alpha\beta} , \quad (4.2)$$

$$\Omega_{\mu\nu\alpha} \equiv \partial_\mu \omega_{\nu\alpha} + \partial_\nu \omega_{\alpha\mu} + \partial_\alpha \omega_{\mu\nu} .$$

The equations of motion following from this action are given by

$$\partial_\mu \Omega^{\mu\alpha\beta} + \frac{\Lambda^2}{e^2} \omega^{\alpha\beta} - \frac{\Lambda^2 \theta}{16\pi^2} \epsilon^{\alpha\beta\gamma\delta} \omega_{\gamma\delta} = 0 . \quad (4.3)$$

Actually, only the three equations for the space-space components ω^{ij} are true equations of motion. The remaining three equations are constraints enforced by the three Lagrange multipliers ω^{0i} .

Contracting (4.3) with ∂_α we obtain the conditions

$$\partial_\mu \omega^{\mu\nu} + \frac{e^2 \theta}{8\pi^2} \Omega^\nu = 0 , \quad (4.4)$$

$$\Omega^\mu \equiv \frac{1}{6} \epsilon^{\mu\nu\alpha\beta} \Omega_{\nu\alpha\beta} .$$

Contracting then the equations of motion (4.3) with $\epsilon_{\nu\gamma\alpha\beta} \partial^\gamma$ and using the above conditions we finally obtain

$$(\partial^2 + m_\theta^2) \Omega^\mu = 0 , \quad (4.5)$$

$$m_\theta = \frac{e\Lambda}{4\pi} \sqrt{\left(\frac{4\pi}{e^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2} .$$

As expected S_1 describes a *massive vector particle* ('massive photon'). Note that the mass of this particle is determined by the same modular parameter $\tau = (\theta/2\pi) + i(4\pi/e^2)$, which enters the mass formula for the monopoles in the BPS limit [7].

In the following we shall consider matter couplings. To this end we add a coupling term

$$S_{\text{coup}}^E = i \int d^4x \omega_{\mu\nu} T_{\mu\nu} , \quad (4.6)$$

$$\partial_\nu T_{\mu\nu} = \frac{1}{2} j_\mu ,$$

to the Euclidean version of the action (4.2) and we integrate out $\omega_{\mu\nu}$ to obtain an induced action for $T_{\mu\nu}$. Note that for $\Lambda \rightarrow 0$ we have $\omega_{\mu\nu} \rightarrow \partial_\mu A_\nu - \partial_\nu A_\mu$ and therefore (4.6) reduces to the standard photon coupling to the conserved matter current j_μ .

The result of the Gaussian integration is the Euclidean induced action

$$S_{\text{ind}}^E = \int d^4x \Lambda^2 T_{\mu\nu} \frac{1}{m_\theta^2 - \nabla^2} T_{\mu\nu} + 2e^2 \partial_\nu T_{\mu\nu} \frac{1}{m_\theta^2 - \nabla^2} \partial_\alpha T_{\mu\alpha} + i \frac{e^2 \Lambda^2 \theta}{16\pi^2} T_{\mu\nu} \frac{\epsilon_{\mu\nu\alpha\beta}}{m_\theta^2 - \nabla^2} T_{\alpha\beta} . \quad (4.7)$$

On distance scales much smaller than $1/\Lambda$ we can neglect terms Λ^2/∇^2 and the induced action reduces to

$$S_{\text{ind}}^E = \int d^4x \frac{e^2}{2} j_\mu \frac{1}{-\nabla^2} j_\mu , \quad (4.8)$$

which shows that at short distances we have the usual Coulomb interactions between charges. In the following, however, we shall be mainly interested in the opposite limit, the long-distance (low-energy) limit.

In order to establish the induced action in this limit we first note that monopoles are expected to condense at strong coupling. In the following we shall assume $e \gg 1$, so that we have well separated scales e/m_θ , $1/m_\theta$, and $1/em_\theta$. Moreover, we are assuming that $n \equiv \theta/2\pi \gg 4\pi/e^2$. We are seeking then a low-energy induced action which is valid for scales $R \gg 1/em_\theta$ but including scales $R = O(e/m_\theta)$, $O(1/m_\theta)$. Technically this means that in (4.7) we can neglect terms $\nabla^2/(em_\theta)^2$ but we have to keep terms $e^2\nabla^2/m_\theta^2$ and ∇^2/m_θ^2 . In this *local limit* the induced action reduces to

$$S_{\text{ind}}^E = \int d^4x \frac{\Lambda^2}{m_\theta^2} T_{\mu\nu} T_{\mu\nu} + \frac{2e^2}{m_\theta^2} \partial_\nu T_{\mu\nu} \partial_\alpha T_{\mu\alpha} + i \frac{e^2 \Lambda^2 \theta}{16\pi^2 m_\theta^2} T_{\mu\nu} \epsilon_{\mu\nu\alpha\beta} (1 + \nabla^2/m_\theta^2) T_{\alpha\beta} . \quad (4.9)$$

4.1. Point particles

In order to expose the nature of the monopole condensate phase we first consider external point particles and we compute the induced static potential. In this case we have

$$j_\mu = \int d\tau \frac{dx_\mu}{d\tau} \delta^4(\mathbf{x} - \mathbf{x}(\tau)) , \quad (4.10)$$

where $\mathbf{x}(\tau)$ parametrizes a closed curve in the four-dimensional Euclidean space. Correspondingly we have

$$\begin{aligned} T_{\mu\nu} &= \frac{-1}{2} \int d^2\sigma X_{\mu\nu}(\sigma) \delta^4(\mathbf{x} - \mathbf{x}(\sigma)) , \\ X_{\mu\nu} &= \epsilon^{ab} \frac{\partial x_\mu}{\partial \sigma^a} \frac{\partial x_\nu}{\partial \sigma^b} , \end{aligned} \quad (4.11)$$

where $\mathbf{x}(\sigma)$ parametrizes a surface bounded by the closed curve $\mathbf{x}(\tau)$. The induced metric g_{ab} on this surface and its determinant g are given by

$$\begin{aligned} g_{ab} &= \frac{\partial x_\mu}{\partial \sigma^a} \frac{\partial x_\mu}{\partial \sigma^b} , \\ g &= \frac{1}{2} X_{\mu\nu} X_{\mu\nu} . \end{aligned} \tag{4.12}$$

Inserting (4.11) into (4.7) and carefully taking the local limit as explained above yields

$$S_{\text{ind}}^E = \frac{\Lambda^2 K_0(\epsilon m_\theta)}{4\pi} \int d^2\sigma \sqrt{g} + \frac{e^2 m_\theta f(\epsilon m_\theta)}{8\pi^2} \int d\tau \sqrt{\frac{dx_\mu}{d\tau} \frac{dx_\mu}{d\tau}} - i \frac{\pi}{n} \nu , \tag{4.13}$$

where $f(x) \equiv \int_x^\infty \frac{dz}{z} K_1(z)$ and K_0 and K_1 are modified Bessel functions. Here ϵ is a short-distance (ultraviolet) cutoff satisfying $\epsilon m_\theta \geq O(1)$ and

$$\nu = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \epsilon^{\mu\nu\alpha\beta} g^{ab} \partial_a t_{\mu\nu} \partial_b t_{\alpha\beta} , \tag{4.14}$$

is the self-intersection number of the surface $x(\sigma)$ in four Euclidean dimensions, defined in terms of $t_{\mu\nu} \equiv X_{\mu\nu}/\sqrt{g}$.

The first term in the induced action (4.13) is the Nambu -Goto term measuring the *area* of the surface enclosed by $\mathbf{x}(\tau)$. It shows that at large distances the potential between a particle-antiparticle pair is linear and identifies the monopole condensate phase as a *confinement phase*. The second term is a correction term measuring the length of the boundary of the surface. The third is the ‘spin’ term [20]. Note that the string becomes ‘fermionic’ for $\theta/2\pi = n = 1$ which corresponds to the condensation of elementary dyons. Let us also mention that the above computation can be generalized. Had we started with an $(h-1)$ -form coupled to a closed $(h-1)$ -dimensional hypersurface in $(d+1)$ -dimensional Euclidean space, the dominant piece of S_{ind}^E at long distances in the phase with a condensate of topological defects would have been the h -dimensional hypervolume enclosed by the $(h-1)$ -dimensional closed hypersurface. This shows that the Julia–Toulouse mechanism for the phase with a condensate of topological defects describes the natural generalization of a *confinement phase*.

5. The Julia–Toulouse mechanism as the dual Higgs mechanism

5.1. Standard Duality

It is by now well known that in $(d + 1)$ -dimensional space-time an $(h - 1)$ -form ϕ_{h-1} is *dual* to a $(d - h)$ -form $\tilde{\phi}_{d-h}$. Indeed, starting from the master action ²

$$S_{\text{master}} = \int \frac{(-1)^{h-1}}{e^2} \Phi_h \wedge \Phi_h^* + \Phi_h \wedge d\tilde{\phi}_{d-h} , \quad (5.1)$$

our action (3.1) can be recovered by integrating out the Lagrange multiplier $\tilde{\phi}_{d-h}$ and solving the resulting constraint as $\Phi_h = d\phi_{h-1}$. On the other hand we can also integrate out Φ_h to obtain an action for the dual form $\tilde{\phi}_{d-h}$:

$$\tilde{S} = \int \frac{(-1)^{d-h} e^2}{4} d\tilde{\phi}_{d-h} \wedge (d\tilde{\phi}_{d-h}) . \quad (5.2)$$

This action describes

$$N_{\tilde{\phi}_{d-h}} = \binom{d}{d-h} - \binom{d}{d-h-1} + \binom{d}{d-h-2} + \dots + (-1)^{d-h} \binom{d}{0} = \binom{d-1}{d-h} = N_{\phi_{h-1}} \quad (5.3)$$

massless degrees of freedom. The two actions S and \tilde{S} are related by a functional Fourier transform and constitute two different representations of the same physics.

5.2. Duality including topological defects

The duality transformation can be extended to *compact* antisymmetric tensor theories. In particular, we shall consider both the gauge field ϕ_{h-1} and its dual $\tilde{\phi}_{d-h}$ to be compact, so that the model admits two types of topological defects: in modern parlance the original $(d - h - 1)$ -branes and their dual $(h - 2)$ -branes [11]. Some particular examples of duality in presence of topological defects have already been considered in [32].

The best way to formulate the extended duality is to treat these topological defects explicitly. We thus start from the master action

$$S_{\text{master}} = \int \frac{(-1)^{h-1}}{e^2} (\Phi_h - q V_h) \wedge (\Phi_h - q V_h)^* + \Phi_h \wedge (d\tilde{\phi}_{d-h} - \tilde{q} \tilde{V}_{d+1-h}) , \quad (5.4)$$

² Throughout this section we will consider only the gauge sector, leaving out couplings to conserved currents.

where V_h and \tilde{V}_{d+1-h} are singular h - and $(d+1-h)$ -forms such that

$$\begin{aligned} j_{d-h} &= (-1)^{d+h^2} (dV_h)^* , \\ \tilde{j}_{h-1} &= (-1)^{(d+1)(d-h)} \left(d\tilde{V}_{d+1-h} \right)^* , \end{aligned} \quad (5.5)$$

represent the conserved ‘currents’ of the $(d-h-1)$ -branes and their dual $(h-2)$ -branes. Here q and \tilde{q} are constants of canonical dimension $\pm(d-2h+1)/2$ respectively; they play the role of the units of charge for the topological defects and their duals. A useful representation is given in terms of V^* and \tilde{V}^* as follows:

$$\begin{aligned} V_h^* &= T_{d-h+1} , \\ \tilde{V}_{d-h+1}^* &= -(-1)^{h(d+1-h)} \tilde{T}_h , \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} T_{d-h+1}^{\mu_1 \cdots \mu_{d-h+1}} &= \int \delta^{d+1} (x - y(\sigma)) dy^{\mu_1} \wedge \cdots \wedge y^{\mu_{d-h+1}} , \\ \tilde{T}_h^{\mu_1 \cdots \mu_h} &= \int \delta^{d+1} (x - \tilde{y}(\tilde{\sigma})) d\tilde{y}^{\mu_1} \wedge \cdots \wedge \tilde{y}^{\mu_h} , \end{aligned} \quad (5.7)$$

and $y(\sigma)$ and $\tilde{y}(\tilde{\sigma})$ represent open hypersurfaces bounded by the world-hyperlines of the topological defects and their duals.

By integrating out $\tilde{\phi}_{d-h}$ in (5.4) we obtain the action

$$S = \int \frac{(-1)^{h-1}}{e^2} (d\phi_{h-1} - qV_h) \wedge (d\phi_{h-1} - qV_h)^* + \tilde{q}\phi_{h-1} \wedge \tilde{j}_{h-1}^* . \quad (5.8)$$

The singular V_h can be absorbed into $d\phi_{h-1}$ by considering it as a singularity due to the compactness of the gauge field and represents a solitonic $(d-h-1)$ -brane. The dual $(h-2)$ -branes, instead, appear as elementary non-dynamical matter and couple ‘minimally’ to ϕ_{h-1} .

By integrating out Φ_h , instead, we obtain the dual action

$$\begin{aligned} \tilde{S} &= \int \frac{(-1)^{d-h} e^2}{4} \left(d\tilde{\phi}_{d-h} - \tilde{q}\tilde{V}_{d+1-h} \right) \wedge \left(d\tilde{\phi}_{d-h} - \tilde{q}\tilde{V}_{d+1-h} \right)^* \\ &\quad + q\tilde{\phi}_{d-h} \wedge j_{d-h}^* - q\tilde{q}V_h \wedge \tilde{V}_{d+1-h} . \end{aligned} \quad (5.9)$$

In this formulation of the theory it is the $(h-2)$ -branes, represented by \tilde{B}_{d+1-h} , which appear as topological defects, while the original $(d-h-1)$ -branes enter as elementary non-dynamical matter ‘minimally’ coupled to $\tilde{\phi}_{d-h}$. The last term in (5.9) represents a

generalized Aharonov–Bohm interaction between the topological defects. Using the representations (5.6), (5.7), it is easy to see that it contributes to the partition function a term:

$$\exp \left\{ i(-1)^{d+1} q \tilde{q} I(y, \tilde{y}) \right\} , \quad (5.10)$$

where I is the signed intersection number of the two hypersurfaces y and \tilde{y} in $(d+1)$ -dimensional space-time. Requiring that this term does not contribute to the partition function leads to a generalized Dirac quantization condition [3]:

$$q \tilde{q} = 2\pi p, \quad p \in \mathbb{Z}. \quad (5.11)$$

The above exact dualities are generically broken in the presence of dynamical matter.

5.3. Duality of Higgs and confinement phases

In the following we shall perform an analogous duality transformation on the finite density action S_{d-h-1} in (3.3).

To this end we start again from a master action,

$$\begin{aligned} S_{\text{master}} = & \int \frac{(-1)^h}{\Lambda^2} \Omega_{h+1} \wedge \Omega_{h+1}^* + \Omega_{h+1} \wedge \left(d\tilde{\omega}_{d-h-1} - \tilde{\phi}_{d-h} \right) \\ & + \frac{(-1)^{h-1}}{e^2} (\omega_h - d\phi_{h-1}) \wedge (\omega_h - d\phi_{h-1})^* - \omega_h \wedge d\tilde{\phi}_{d-h} , \end{aligned} \quad (5.12)$$

formulated in terms of the dual couples ϕ_{h-1} , $\tilde{\phi}_{d-h}$ and ω_h , $\tilde{\omega}_{d-h-1}$ and the additional master field Ω_{h+1} . This action possesses two gauge symmetries under transformations

$$\begin{aligned} \omega_h & \rightarrow \omega_h + d\psi_{h-1} , \\ \phi_{h-1} & \rightarrow \phi_{h-1} + \psi_{h-1} , \end{aligned} \quad (5.13)$$

and transformation

$$\begin{aligned} \tilde{\omega}_{d-h-1} & \rightarrow \tilde{\omega}_{d-h-1} + \tilde{\psi}_{d-h-1} , \\ \tilde{\phi}_{d-h} & \rightarrow \tilde{\phi}_{d-h} + d\tilde{\psi}_{d-h-1} . \end{aligned} \quad (5.14)$$

Clearly, these two gauge symmetries are also dual to each other. Both have to be gauge-fixed.

Let us first integrate out the fields $\tilde{\phi}_{d-h}$, $\tilde{\omega}_{d-h-1}$ and Ω_{h+1} . To this end we first absorb $d\tilde{\omega}_{d-h-1}$ into a redefinition of $\tilde{\phi}_{d-h}$: the remaining divergent integration over $\tilde{\omega}_{d-h-1}$ is

cancelled out by the gauge fixing. The integration over the Lagrange multiplier $\tilde{\phi}_{d-h}$ enforces the constraint

$$\Omega_{h+1} = (-1)^h d\omega_h . \quad (5.15)$$

The last integration over Ω_{h+1} then yields

$$S_{d-h-1} = \int \frac{(-1)^h}{\Lambda^2} \Omega_{h+1} \wedge \Omega_{h+1}^* + \frac{(-1)^{h-1}}{e^2} (\omega_h - d\phi_{h-1}) \wedge (\omega_h - d\phi_{h-1})^* , \quad (5.16)$$

with Ω_{h+1} as in (5.15), which is exactly the action (3.3) for the confinement phase at finite density of topological defects in the case $T_h = 0$.

On the other hand, we can integrate out the fields ϕ_{h-1} , ω_h and Ω_{h+1} to obtain an action for the dual variables. The integration over ϕ_{h-1} is eliminated by gauge-fixing after absorbing $d\phi_{h-1}$ into a redefinition of ω_h . The two remaining integrations over ω_h and Ω_{h+1} then give the action

$$\begin{aligned} \tilde{S}_{d-h-1} = & \int \frac{(-1)^{d-h} e^2}{4} d\tilde{\phi}_{d-h} \wedge \left(d\tilde{\phi}_{d-h} \right)^* \\ & + \frac{(-1)^{d-h-1} \Lambda^2}{4} \left(\tilde{\phi}_{d-h} - d\tilde{\omega}_{d-h-1} \right) \wedge \left(\tilde{\phi}_{d-h} - d\tilde{\omega}_{d-h-1} \right)^* , \end{aligned} \quad (5.17)$$

where gauge fixing of the dual gauge symmetry is always understood.

For the particular case $h = d - 1$, $\tilde{\phi}_{d-h}$ is a 1-form and $\tilde{\omega}_{d-h-1}$ is a 0-form: in this case \tilde{S}_0 embodies the familiar Higgs mechanism, in which a vector gauge field ‘eats’ a scalar thereby acquiring a mass, in its pristine Stückelberg formulation [17]. As a natural generalization we shall call the theory described by \tilde{S}_{d-h-1} the Higgs phase for a $(d - h)$ -form $\tilde{\phi}_{d-h}$. As shown by the above computation this (generalized) Higgs mechanism is the dual of the Julia–Toulouse mechanism, embodied by S_{d-h-1} and describing the confinement phase for the dual $(h - 1)$ -form ϕ_{h-1} . In this duality transformation all coupling constants are reversed: in particular, for $\Lambda \rightarrow 0$ we recover the familiar duality between $(h - 1)$ -forms and $(d - h)$ -forms described at the beginning of the section. Thus we reach the conclusion that the same physics can be described as a confinement phase for ϕ_{h-1} or a Higgs phase for $\tilde{\phi}_{d-h}$: this generalizes the well known ’t Hooft–Mandelstam duality [18] to any compact p -form theory in $(d + 1)$ space-time dimensions.

So far we have two pairs of dual actions, namely the pair (3.1) –(5.2) describing the ‘Coulomb phase’ and the pair (5.16) –(5.17) describing the confinement phase for ϕ and the Higgs phase for $\tilde{\phi}_{d-h}$. We may wonder if there is a dual phase which would be a Higgs phase for ϕ and a confinement phase for $\tilde{\phi}_{d-h}$. Clearly we expect such a phase to

be generated by the condensation of the dual $(h - 2)$ -branes. The best way to see this is to apply the Julia–Toulouse mechanism to the dual action (5.2). To this end we can just repeat the steps leading to equations (5.16) –(5.17) with the exchanges:

$$\begin{aligned} \phi_{h-1} &\leftrightarrow \tilde{\phi}_{d-h} , & h &\leftrightarrow d - h + 1 , \\ e^2 &\leftrightarrow 4/e^2 , & \Lambda^2 &\leftrightarrow \tilde{\Lambda}^2 . \end{aligned} \tag{5.18}$$

We arrive at the two dual actions:

$$\begin{aligned} S_{h-2} &= \int \frac{(-1)^{d-h+1}}{\tilde{\Lambda}^2} d\rho_{d-h+1} \wedge (d\rho_{d-h+1})^* \\ &\quad + \frac{(-1)^{d-h} e^2}{4} \left(\rho_{d-h+1} - d\tilde{\phi}_{d-h} \right) \wedge \left(\rho_{d-h+1} - d\tilde{\phi}_{d-h} \right)^* , \\ \tilde{S}_{h-2} &= \int \frac{(-1)^{h-1}}{e^2} d\phi_{h-1} \wedge (d\phi_{h-1})^* + \frac{(-1)^{h-2} \tilde{\Lambda}^2}{4} (\phi_{h-1} - d\tilde{\rho}_{h-2}) \wedge (\phi_{h-1} - d\tilde{\rho}_{h-2})^* . \end{aligned} \tag{5.19}$$

Both describe $N_{\rho_{d-h+1}} = \binom{d}{d-h+1} = \binom{d}{h-1}$ massive degrees of freedom. From the second expression we can explicitly see that in this phase it is the field ϕ_{h-1} , rather than $\tilde{\phi}_{d-h}$, that gets a mass $m = e\tilde{\Lambda}/2$. Therefore we have *three* pairs of dual actions describing three phases of each of the two dual theories. We will identify the phase described by (3.1) –(5.2) as ‘the Coulomb phase’, the one described by (5.16) –(5.17) as ‘the confinement phase’ and that described by (5.19) as ‘the Higgs phase’.

For the particular case, $h = (d + 1)/2$ the Higgs and confinement phases are described by the same type of tensors, since ω_h and ρ_{d-h+1} have the same rank. In this case also the two dual types of topological defects have the same dimensionality, since $d - h - 1 = h - 2$. Although we cannot prove it in our formalism, one would therefore expect that the condensation dynamics, embodied by the functions $\Lambda^2(e)$ and $\tilde{\Lambda}^2(e)$, respects the duality $e^2 \leftrightarrow 4/e^2$ of the Coulomb phase, which means

$$\tilde{\Lambda}^2(e) = \Lambda^2\left(\frac{2}{e}\right) . \tag{5.20}$$

If this is the case, we can immediately conclude that the whole phase diagram must be symmetric around the self-dual point $e^2 = 2$.

There are two ways of studying the condensation dynamics: either to take into account also the higher modes, neglected in the low-energy effective theory, or else to rely on lattice calculations. The structure of the phase diagram obtained in these lattice analyses for $d = 1, 3$ is as follows [26,27]: there is a confinement phase at strong coupling ($e^2 \gg 2$),

dual to a Higgs phase at weak coupling ($e^2 \ll 2$); in between there is a self-dual, massless Coulomb phase. Either the Higgs phase or the confinement phase disappear if only one type of topological defect is taken into account.

Note also that in the case $h = (d + 1)/2$ one can add to the original low-energy theory a topological term as in (3.11). This would highly increase the complexity of the phase diagram, leading also to *oblique confinement* phases and $SL(2, Z)$ duality [27].

Our results can be easily generalized to field theories of several, coupled antisymmetric tensors. Besides the fact that these couplings appear naturally in string theories, this is also of interest in view of the fact that in several cases low-energy fermions can be represented in terms of antisymmetric tensors, a procedure that goes by the name of higher-dimensional bosonization [33] and which is of relevance both to particle and condensed matter physics [34].

For example, we could consider a theory containing an $(h - 1)$ -form ϕ_{h-1} and a $(d - h + 1)$ -form ψ_{d-h+1} coupled by a generalized BF-term $\phi_{h-1} \wedge d\psi_{d-h+1}$. In this case ψ_{d-h+1} plays the same role as the field V in the compact QED example of section 4, with the only difference that its dynamics is specified directly in terms of the kinetic term $d\psi_{d-h+1} \wedge (d\psi_{d-h+1})^*$. In the confinement phase for ϕ_{h-1} , driven by the condensation of $(d - h - 1)$ -branes, the low-energy induced action for ψ_{d-h+1} describes a Higgs phase and viceversa.

As a concrete example we would like to mention the $(2+1)$ -dimensional theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2\pi} A_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha B_\nu - \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu}, \quad (5.21)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. This theory has recently been the focus of many investigations in connection with planar strongly correlated electron systems and Josephson junction arrays [35]. Our results for this theory are in accordance with the lattice analysis of ref. [36].

Let us finish this section with the following remark: we know that in the limit $\Lambda \rightarrow 0$, the pair of equations (5.16) –(5.17) reduces to the dual pair (3.1) –(5.2). The same can be said for the pair (5.19) in the limit $\tilde{\Lambda} \rightarrow 0$. In both limits the corresponding masses vanish and we recover the Coulomb phase. It is curious to note that in the confinement phase, the mass also vanishes if we take the (strong coupling) limit $e \rightarrow \infty$; however in this case we *do not* recover the original Coulomb phase for the field ϕ_{h-1} . We instead obtain a

Coulomb phase for the higher rank field ω_h , with the coupling constant determined by Λ , as can easily be seen from equations (3.6) and (5.16). On the other hand, the dual version (5.17), will describe in this limit a massless field of rank $d-h-1$ dual to the massless field ω_h . Similarly, in the Higgs phase, the mass vanishes in the limit $e \rightarrow 0$, which gives the Coulomb phase for an antisymmetric tensor of rank $h-2$ and its dual, of rank $d-h+1$ with coupling given by $\tilde{\Lambda}$. Therefore, for a given space-time dimension $D = d+1$, we can see that the different phases of antisymmetric tensor theories of any rank $r = 0, 1, \dots, d$ ³ may *all* be connected by changing the different parameters of each theory, as long as there are topological defects that can condense.

6. Examples

We will discuss now some particular examples that illustrate our generic results. We have already seen in section 4 that the case $d = 3, h = 2$ corresponds to compact QED. Since this is a case for which $h = (d+1)/2$, the Higgs and confinement phases can both be described in terms of a massive vector, dual to a massive two-index tensor. The massive vector can be either a ‘magnetic’ or ‘electric’ massive photon. We will now describe some other examples, probably less familiar.

6.1. The puzzle of the axion mass

Let us consider the case $d = h = 3$. In this case ϕ_{h-1} is the standard two-index tensor of string theory $B_{\mu\nu}$ [1] and $\tilde{\phi}_{d-h}$ is the pseudoscalar axion field a [37]. The two dual formulations of the Coulomb phase are well understood in terms of a single *massless* degree of freedom, implied by the existence of a Peccei–Quinn symmetry [37]: $a \rightarrow a + \text{constant}$ in the dual theory:

$$\begin{aligned} S &= \int d^4x \frac{3}{f^2} (\partial_{[\mu} B_{\nu\alpha]} - K_{\mu\nu\alpha}) \left(\partial^{[\mu} B^{\nu\alpha]} - K^{\mu\nu\alpha} \right) , \\ \tilde{S} &= \int d^4x \frac{1}{2} \partial_\mu a \partial^\mu a - a \frac{1}{16\pi^2 f} \text{Tr} F_{\mu\nu} F^{\mu\nu*} , \end{aligned} \tag{6.1}$$

³ The Coulomb phase for a rank d antisymmetric tensor is only given by a cosmological constant term, since such a massless tensor does not have propagating degrees of freedom.

where we have included the coupling to the QCD sector, f is a mass parameter ⁴ and $K_{\mu\nu\alpha}$ is the dual of the Chern–Simons current, defined by

$$\epsilon^{\mu\nu\alpha\beta}\partial_\mu K_{\nu\alpha\beta} = -\frac{1}{16\pi^2}\text{Tr}F_{\mu\nu}F^{\mu\nu*} . \quad (6.2)$$

The dual formulation \tilde{S} was originally introduced to solve the strong CP problem; it was immediately recognized that the QCD instantons generate a potential $V(a)$, thereby giving the axion a mass. In the dilute instanton gas approximation, in which one considers only a dilute gas of pointlike instantons, this potential is easily computed to be $V(a) = \Lambda^4(1 - \cos(a/f))$, where Λ^4 is the average density of instantons in 4D Euclidean space. The axion mass in this formulation is thus $m_a = \Lambda^2/f$.

Up until recently the origin and description of the axion mass in the formulation S were a puzzle. This was disturbing since it is exactly this formulation of the axion which is obtained in 4D string models. Last year two independent investigations solved this problem. In ref.[23] it was pointed out that the QCD instantons do indeed generate a mass for the $B_{\mu\nu}$ field via the Polyakov mechanism, but not making it massive itself (a massive two-index tensor in 4D has three degrees of freedom and cannot be equivalent to a massive (pseudo)scalar). The authors concluded that also in this formulation there is a physical massive particle without spin but they could not write an effective action describing this degree of freedom. Nevertheless they were still able to find the short-range correlation function for the field $\epsilon^{\mu\nu\alpha\beta}\partial_{[\nu}B_{\alpha\beta]}$. In ref.[22] the same question was approached by investigating gaugino condensation in a supersymmetric version of the string model S . It was found that in such a model the massive axion must be described by a massive *three-index antisymmetric tensor* $H_{\mu\nu\alpha}$ which in 4D has one degree of freedom. The correlation function found in ref.[23] corresponds to the propagator for the dual field $\epsilon^{\mu\nu\alpha\beta}H_{\nu\alpha\beta}$.

Let us now explain how these two aspects are unified by the results obtained in the present paper. In the dilute instanton gas approximation the actions in (6.1) take exactly the form of the dual actions (5.8) and (5.9), with $\tilde{j}_2 = \tilde{V}_1 = 0$ and with the instanton density $-(1/16\pi^2)\text{Tr}F_{\mu\nu}F^{\mu\nu*}$ playing the role of j_0 and the Chern–Simons term $K_{\mu\nu\alpha}$ playing the role of V_3 . Note that, in this approximation, the QCD instantons get identified with the topological defects of a 4D two-index antisymmetric tensor, the so-called *axionic instantons* [38]. Using our results we can immediately conclude that a condensation of

⁴ Note that here we chose $B_{\mu\nu}$ to have canonical dimension (mass)².

these instantons drives a transition to a confinement phase, in which the low-energy action is written in terms of a three-index antisymmetric tensor:

$$S_{-1} = \int d^4x \left\{ -\frac{3}{4\Lambda^4} \partial_{[\mu} H_{\nu\alpha\beta]} \partial^{[\mu} H^{\nu\alpha\beta]} + \frac{3}{f^2} H_{\mu\alpha\beta} H^{\mu\alpha\beta} \right\}. \quad (6.3)$$

This action describes one massive degree of freedom of mass $m_H = \Lambda^2/f$ dual to the massive axion:

$$\tilde{S}_{-1} = \int d^4x \left\{ \frac{1}{2} \partial_\mu a \partial^\mu a - \frac{\Lambda^4}{2f^2} a^2 \right\}. \quad (6.4)$$

Thus it is indeed the condensation of instantons that generates a mass in the string formulation, as pointed out in ref.[23]. And the massive phase must indeed be formulated in terms of a three-index antisymmetric tensor, as pointed out in ref.[22]. The circle is closed when one realizes that gaugino condensation is also expected to be driven by a condensation of instantons. These results seem to indicate also the possibility to formulate a supersymmetric version of the Julia–Toulouse mechanism.

Notice that this example goes beyond our prescription in two ways. First, the two dual effective actions (6.3) and (6.4), can be explicitly *derived* in the condensing phase (see the appendix for a sketch of the derivation in the gaugino condensation case). Secondly, they do agree with our prescription but only in the approximation of small scalar field, for which the $\cos a/f$ potential reduces to the quadratic term. From the combination of Julia-Toulouse mechanism and duality, we could only arrive at the mass term for the axion missing the fact that since a is a periodic variable, the potential should also be periodic. We may trace this deficiency in providing the full scalar potential either to the fact that the Julia–Toulouse prescription gives only the Lagrangian up to two derivatives in the fields or to the corresponding duality between a massive antisymmetric tensor and a massive scalar which can be performed when the path integrals are Gaussian (see however the appendix).

Let us finally mention the Higgs phase ($d = 3, h = 1$). This would correspond to the condensation of one-dimensional objects, strings or vortices. In this case the $B_{\mu\nu}$ field itself acquires a mass, dual to a massive vector, each carrying three degrees of freedom, just as in the compact QED case.⁵ This phase was also explored in the past by studying in detail the condensation of strings in 4D string theory [39]. The end result was identical to ours.

⁵ See the comments at the end of the previous section for a relation among this phase and the Higgs-confinement phase of compact QED.

6.2. Compact QED and $O(2)$ models in 3D

Three-dimensional QED corresponds to $d = h = 2$, dual to a massless scalar (compact $O(2)$ model). The relevant topological defects are instantons ($d - h - 1 = -1$). Their condensation would generate the confinement phase described by a massive two-index tensor carrying one degree of freedom in 3D. The Higgs phase is obtained by the condensation of monopoles ($h - 2 = 0$, vortices from the 4D point of view) and it is described by a massive vector, the massive photon, dual to a massive scalar, coinciding with the results of Polyakov in 3D [20]. It is interesting to mention that in this case, it has been shown [20], that the system is never in a Coulomb phase. Furthermore, this is also the case for the $B_{\mu\nu}$ field in 4D [20] and for any case in which $h = d$ [6]. This suggests that the 4D axion (and then the string dilaton too) might always be in a massive phase, although additional couplings might play an important role.

As we mentioned at the end of the previous section, a massless limit in the confinement phase would give rise to a massless two-index tensor which has no dynamics. This completes all the possibilities for 3D.

6.3. Strings on a Circle

This is the $d = h = 1$ case. The Coulomb phase is a free 2D scalar, therefore we can see that this is relevant for the worldsheet action of string theory:

$$S = \frac{1}{4\pi\alpha'} \int d^2z \{ (G_{MN} + B_{MN}) \partial_\mu X^M \partial^\mu X^N + \dots \}. \quad (6.5)$$

Where X^M , $M = 1, \dots, D$ are the coordinates of the D dimensional target space, G_{MN} is the metric in target space, B_{MN} the antisymmetric tensor and α' is the inverse string tension. If both are constant we have the standard T duality $(G_{MN} + B_{MN}) \rightarrow (G_{MN} + B_{MN})^{-1}$ [21]. The case of our interest is the compactification on a circle of radius R . For which the action is:

$$S = \frac{1}{4\pi\alpha'} \int d^2z \{ \partial_\mu X \partial^\mu X + \dots \} \quad (6.6)$$

Where X is the coordinate of the circle which is identified with $X + 2\pi R$ ⁶ and R is the radius of the circle, the ellipsis refer to the extra coordinates of the string, as in (6.5),

⁶ Notice that we are absorbing the coupling constant $1/R$ into the definition of the field, unlike the previous sections; this is why R now appears in the periodicity conditions. We change our conventions here because the variable X , rather than $\Theta \equiv X/R$, is the standard in string theory.

which play no role in our discussion. Duality in the ‘Coulomb’ phase amounts to the famous $R \leftrightarrow \alpha'/R$ duality of the 2D action. We will write the dual action as:

$$\tilde{S} = \frac{1}{4\pi\alpha'} \int d^2z \left\{ \partial_\mu \tilde{X} \partial^\mu \tilde{X} + \dots \right\} \quad (6.7)$$

Where now \tilde{X} is identified with $\tilde{X} + 2\pi p\alpha'/R$, $p \in \mathbb{Z}$. The integer parameter p arises from the Dirac quantization condition (5.11) between the charge unit of the topological defects and their duals, as can be easily seen by comparing with (5.10) and with lattice formulations of this model [27]. If only one type of topological defect is taken into account, p is irrelevant and can just be absorbed into R . Instead, it plays a crucial role if both types of instantons are taken into account [40]. Notice that p appears in the periodicity condition for the dual variable and so it corresponds to only allowing quantized momenta that are multiples of p/R . One-loop modular invariance in string theory would then require that also winding states should be restricted indicating that the radius R should be redefined by $R \leftarrow R/p$ and the effective result would reduce to the $p = 1$ case. We will present the general case for arbitrary p below, keeping in mind that the only modular invariant string case would be $p = 1$.

This is also a case in which $h = (d+1)/2$. The confinement and Higgs phases are both driven by the condensation of instantons and both can be described either in terms of a massive scalar or a massive vector. For simplicity we shall adopt the former description.

Following the terminology introduced in section 5, we call the phase driven by the condensation of X -instantons the confinement phase. In this phase, the low-energy action in the scalar formulation must be written in terms of the dual coordinate \tilde{X} :

$$\tilde{S}_{-1} = \frac{1}{4\pi\alpha'} \int d^2z \left\{ \partial_\mu \tilde{X} \partial^\mu \tilde{X} - \frac{\Lambda^2 R^2}{\alpha'^2} \tilde{X}^2 + \dots \right\}, \quad (6.8)$$

although we expect that, similar to the axion case in 4D, the quadratic potential in (6.8) will complete to the periodic form:

$$V(\tilde{X}) = \Lambda^2 \left(1 - \cos \left(\frac{R}{\alpha'} \tilde{X} \right) \right). \quad (6.9)$$

This is the well known sine-Gordon model. The mass term moves this phase away from a conformal field theory: the new phase is confining in the sense that the map from the worldsheet to target space collapses to a singular map in which antipodal target space points are mapped into a single worldsheet point (in the compact dimension), although

this is best seen in the massive vector formulation with a computation similar to the one for compact QED in section 4. Thus we could say that it is the target space coordinates themselves that get confined: any space-time interpretation is no longer possible since the compact coordinate simply disappears.

Contrary to previous examples, in this case the phase transition point is known. Indeed the condensation of instantons corresponds to the famous Berezinsky–Kosterlitz–Thouless (BKT) phase transition for which the renormalization group flow has been computed analytically [13,14]. Adapting the BKT results to our notation we find that the transition occurs at $R = R_c \equiv 2\sqrt{\alpha'}$. For $R > R_c$ we have the Coulomb (conformal) phase; for $R < R_c$ we have the confinement phase. This result, and the corresponding interpretation of R_c as a minimum radius for compactified strings were already obtained in ref. [41] and confirmed by a matrix model computation in ref. [42]. In addition to their results, we can provide an action for the confining phase: note that the corresponding interpretation of the confining mechanism as the disappearance of a dimension is in agreement with ref. [42], who interpreted it as a reduction in one unit of the central charge. Notice also that, if X is originally a time coordinate, then R would be an inverse temperature: the phase transition would be a high temperature transition with critical temperature coinciding with the Hagedorn temperature [41], [43].

The existence of a minimum radius was interpreted as a breakdown of T -duality. As possible ways out, it was suggested a mechanism to discard the vortices (instantons for us) [42], or that another conformal phase might exist for $R < \sqrt{\alpha'}/2$ [41]. The solution is actually another. Indeed, up to now, we have neglected the dual \tilde{X} -instantons. A condensation of these topological defects drives in fact a transition to a Higgs phase with action:

$$\tilde{S}_{-1} = \frac{1}{4\pi\alpha'} \int d^2z \left\{ \partial_\mu X \partial^\mu X - \frac{\tilde{\Lambda}^2 \alpha'}{R^2} X^2 + \dots \right\}, \quad (6.10)$$

formulated in terms of the original coordinate X . Following Polyakov [44], we interpret the short-range correlations $\langle \partial X \partial X \rangle$ in this phase as an indication that the compact dimension is crumpled. Therefore, also in the Higgs phase we lose the space-time interpretation of the compact coordinate, but for a different reason.

In addition, the presence of the dual instantons also changes dramatically the phase diagram. The new renormalization group flow has been computed in [26,27]. Adapting

their results to our notation we find the following phase structure:

$$\begin{aligned}
p < 4 &\rightarrow \begin{cases} \frac{R^2}{p\alpha'} < 1, & \text{confinement phase,} \\ \frac{R^2}{p\alpha'} > 1, & \text{Higgs phase,} \end{cases} \\
p > 4 &\rightarrow \begin{cases} \frac{R^2}{p\alpha'} < \frac{4}{p}, & \text{confinement phase,} \\ \frac{4}{p} < \frac{R^2}{p\alpha'} < \frac{p}{4}, & \text{Coulomb (conformal) phase,} \\ \frac{R^2}{p\alpha'} > \frac{p}{4}, & \text{Higgs phase.} \end{cases}
\end{aligned} \tag{6.11}$$

Note that the familiar $R \rightarrow \alpha'/R$ duality in absence of topological defects is changed to a $R \rightarrow p\alpha'/R$ duality which is realized as follows. Only the Coulomb (conformal) phase is self-dual; the Higgs and confinement phases are instead interchanged under the duality transformation. The Coulomb (conformal) phase exists only for $p > 4$. In this case we have both a minimal and a maximal radius

$$\begin{aligned}
R_{\min} &= 2\sqrt{\alpha'}, \\
R_{\max} &= \frac{p}{2}\sqrt{\alpha'}.
\end{aligned} \tag{6.12}$$

A space-time interpretation is possible only for $R_{\min} < R < R_{\max}$. As we mentioned before, only the $p = 1$ case is modular invariant therefore modular invariant bosonic strings live only in the Higgs and confinement phases which are dual to each other. Note that these results can be generalized to the case of several compact dimensions. For a torus of dimension n , the modular group is $O(n, n; \mathbb{Z})$ [21] and the phase diagram becomes much more complex with the possibility of several conformal windows and oblique confinement, as found by Cardy for the $SL(2, \mathbb{Z})$ case [27]. In case the compact coordinate is the time coordinate the maximal radius would correspond to a minimal temperature dual to the Hagedorn temperature. An extension of these results to the heterotic string is under current investigation [40].

6.4. Higher-dimensional generalizations

For 10D string theory, an interesting case would be $d = 9, h = 3$. Topological defects have dimension $d - h - 1 = 5$ which are usually called five-branes [11]. We claim that their condensation will give rise to a new phase, described by a massive three-index tensor dual to a massive six-index tensor in 10D. The Higgs phase would again be generated by the condensation of strings which give a mass to the two-index field dual to a massive seven-index field in 10D. These are the extensions to 10D of the axionic instanton results in 4D. The Higgs and confinement phases have not been previously studied.

The existence of five-brane solitons in string theory has been explicitly shown by using the low-energy effective action. This can easily be generalized to any dimension D starting with an antisymmetric tensor $A_{M_1 M_2 \dots M_{h-1}}$, the metric G_{MN} and the dilaton field Φ , with the effective action [11]:

$$S = \int d^D x \sqrt{-G} \left(R - \frac{1}{2} (\partial\Phi)^2 - \frac{1}{2h!} e^{-a(h)\Phi} F_h^2 \right), \quad (6.13)$$

where R is the curvature scalar, $a(h)$ a constant and $F_h = dA_{h-1}$. Notice that the A_{h-1} dependence of this action is only through F_h and therefore it can be dualized. This action has regular solitonic solutions of dimension $d - h - 1$ and its dual has regular solitonic solutions of dimension $h - 2$. Therefore it provides us with explicit examples in any dimension, where the topological defects assumed previously are present and, if they condense, we have all the different phases mentioned before.⁷ The Higgs and confinement phases of these theories have not been considered previously.

Actually a word of care is due at this point. In our previous examples all the massless degrees of freedom of the Coulomb phase are described by just one single antisymmetric tensor. The situation is different in (6.13), which contains in addition also a massless dilaton and a massless graviton. Extending our results to this case we can predict the antisymmetric tensor content of the Higgs and confinement phases. However it is also crucial to know the fate of the dilaton and the graviton in these phases. Supersymmetry will probably help answering these questions, which are under current investigation. Various possibilities are discussed in the next section.

7. Final Remarks

We would like to mention the possible relevance of the present discussion to string theory. We know that string theory has two main problems, namely how to break supersymmetry and how to lift the large vacuum degeneracy, especially due to the existence of

⁷ An interesting case is 11D supergravity (effective theory of a yet unknown M -theory): the bosonic spectrum consists of the metric and a three-index tensor ($h=4$). This tensor may lead to the condensation of five-branes. The confinement phase would correspond to massive four-index tensors. It is believed that due to the existence of a Chern–Simons like coupling, there is no dual version of this theory in terms of massless six-index tensors, so the Higgs phase may not be well defined.

fields, such as the dilaton, that have flat potentials to all orders in perturbation theory. The increasing evidence for the existence of a strong–weak coupling duality in string theory has raised the hope that the answer to these questions may well be within reach [9,10]. There is mounting evidence that all string theories are related by such a strong–weak coupling duality so, strong coupling effects can in principle be understood by knowing weak coupling string theory. Even though this is a great step forward, it cannot be the full story. The question is that if the strong coupling domain of a string theory is determined by the weak coupling of a different string, and this is given by string perturbation theory, the problems mentioned above will not be solved because they were unsolved in perturbation theory.

We hope that the ‘new’ phases we are describing here could provide a new insight into these questions. The reason being that if in the confinement and Higgs phases, the antisymmetric tensors get a mass, then by supersymmetry also the dilaton will get a mass, since the dilaton is always in a supersymmetric multiplet with the antisymmetric tensor $B_{\mu\nu}$; thus the dilaton vev seems to be fixed in these phases as we wanted, or supersymmetry is broken, which is also well taken, or both, which may be even better! For many cases the graviton is also in a multiplet with antisymmetric tensors and if supersymmetry is unbroken then the graviton itself will get a mass, breaking invariance under general coordinate transformations! Nevertheless, in the phenomenologically interesting case of 4D $N = 1$ supersymmetry, the graviton belongs to a different multiplet and we may have massive antisymmetric tensors with massless gravitons. Still, it will be crucial to understand under which circumstances supersymmetry is broken in the condensing phases, or if it is possible at all.

From our results and by analogy with the different examples we studied in the last section, we may interpret the recently found strong–weak coupling dualities among the different 10D string theories, as relations among the different Coulomb phases of a fundamental theory, with many confining and Higgs phases in between, described by massive antisymmetric tensors. Furthermore, due to the observation at the end of section 5: for a given dimension, we can start with any massless antisymmetric tensor theory and reproduce all the phases of all the antisymmetric tensors of arbitrary rank. This may be consistent with the recent claims about p -brane democracy [45].

As we already mentioned, we cannot be very specific about these issues yet. We can say nothing about the dynamics that causes the condensation of the topological defects, nor we can deduce yet the complete effective theory in the new phases. Nevertheless

we believe that the unified view we are providing in our discussion and the qualitative description of the phases, may be a good starting point to understand the dynamics of these theories and the complete phase structure. This may turn out to be crucial in solving the outstanding questions of string theory. Probably, the recent developments in terms of Dirichlet-branes [46] may provide a useful tool towards a more concrete investigation of the process of condensation of the $(d - h - 1)$ -branes. It would also be very interesting to find a supersymmetric generalization to the Julia–Toulouse mechanism ⁸ so that the supersymmetry-breaking question could be approached in a more quantitative fashion.

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Appendix A.

Gaugino condensation and duality

Since the process of gaugino condensation in $N = 1$ supersymmetric theories is relatively simple to describe, it is possible in this case to *derive* the effective action in the confinement phase. This effect is known to be triggered by the existence of gauge field instantons breaking the Peccei-Quinn symmetry and therefore the effective theory below condensation scale coincides with the ‘confinement’ phase with finite density of instantons. For the sake of completeness we will now sketch the main steps of this derivation.

In 4D, $N = 1$ supersymmetric strings, the antisymmetric tensor belongs, together with the dilaton and the dilatino, to a linear superfield L defined by the constraint $\overline{\mathcal{D}}\mathcal{D}L = 0$. This constraint, when expressed in components, implies the symmetry $B_{\mu\nu} \rightarrow B_{\mu\nu} +$ a closed two-form. The simplest scenario to study gaugino condensation is to consider the

⁸ The results of refs. [22,24] may be already a step forward for the 4D case.

couplings of L to a gauge multiplet of a non-Abelian gauge group G in global supersymmetry. The most general action is then the D -term of an arbitrary function $\Phi(L)$:

$$\mathcal{L}_L = \int d^4x \left(\Phi(\hat{L}) \right)_D \quad (\text{A.1})$$

with $\hat{L} \equiv L - \Omega$ and Ω the Chern–Simons superfield, satisfying $\overline{\mathcal{D}\mathcal{D}}\Omega = W_\alpha W^\alpha$, W_α is the gauge field strength superfield, containing in its components, the gauginos λ_α and the gauge field strength $F_{\mu\nu}$. Expressing this action in components implies, for instance, that the gauge coupling is given by $\partial\Phi/\partial L$ and the field L then provides the field dependent gauge coupling, usual in string theory. The action (A.1) can be obtained from a first-order (master) Lagrangian, which can be seen as the supersymmetric generalization of (5.1):

$$\mathcal{L}(V, S) = (\Phi(V))_D + (S\overline{\mathcal{D}\mathcal{D}}(V + \Omega))_F, \quad (\text{A.2})$$

where V is an arbitrary vector superfield and S a Lagrange multiplier chiral superfield ($\overline{\mathcal{D}}S = 0$). The components of S are the dilaton, the axion a and their fermionic superpartner. Integrating out S , implies $\overline{\mathcal{D}\mathcal{D}}(V + \Omega) = 0$ or $V = L - \Omega \equiv \hat{L}$, giving back the original theory. On the other hand, integrating first V gives the dual theory in terms of S and A (the gauge superfield). This is the situation above the condensation scale.

If the gauge group is asymptotically free, it is expected that at lower energies the gauge coupling becomes stronger and at a given scale (the renormalization group invariant scale of G), the gauginos may condense $\langle \lambda_\alpha \lambda_\alpha \rangle \neq 0$. In the supersymmetric language this is equivalent to requiring $\langle \text{Tr}W_\alpha W^\alpha \rangle \neq 0$. To investigate if condensation takes place we have to construct the effective action for $\langle \text{Tr}W_\alpha W^\alpha \rangle$. We choose to do it in the first order Lagrangian (A.2). We couple an external current J to the operator we want the expectation value, namely $\text{Tr}W_\alpha W^\alpha$:

$$\exp \{i\mathcal{W}(J)\} = \int DA DS DV \exp \left\{ i \int d^4x \left(\mathcal{L}(V, S) + (JW_\alpha W^\alpha)_F \right) \right\}. \quad (\text{A.3})$$

Following the standard effective action procedure [47], we first define the *classical* field $U \equiv \delta\mathcal{W}/\delta J = \langle \text{Tr}W_\alpha W^\alpha \rangle$. Integrating first the gauge field A , the effective action is a function of the other variables (S, V) and the classical superfield U , $\Gamma(U, V, S) \equiv \mathcal{W} - \int d^4x (UJ)_F$. The important result is that since \mathcal{W} depends on S and J only through the combination $S + J$, we can see that $\delta\Gamma/\delta S = \delta\mathcal{W}/\delta S = \delta\mathcal{W}/\delta J = U$ so $\Gamma(U, S, V) = US + \Xi(U, V)$,

where $\Xi(U, V)$ can be fixed by the symmetry $U \rightarrow e^{i\alpha}U, S \rightarrow S + i\alpha$. Therefore S appears only linearly in the path integral and its integration gives again a δ -function, imposing now $\overline{\mathcal{D}\mathcal{D}}V = -U$ instead of the constraint $\overline{\mathcal{D}\mathcal{D}}(V + \Omega) = 0$ above the condensation scale. We can then see that since the constraint on V is different, there is no linear multiplet implied by this new constraint. This is an indication that the $B_{\mu\nu}$ field is no longer in the spectrum.

The effective action in components can be easily written [22]. Considering two condensing groups, freezing the dilaton degree of freedom and eliminating the auxiliary fields by their field equations, we end up with a Lagrangian of the form:

$$\mathcal{L} = H_{\mu\nu\rho}^2 - a\epsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} + (a - \theta)^2 + (1 - \cos\theta) + f(\theta)\partial_\mu\theta\partial^\mu\theta, \quad (\text{A.4})$$

where θ is the difference in phases of the two classical fields (U_1 and U_2) representing the condensate. The function $f(\theta)$ defines the kinetic term for θ . The important point here is that θ , unlike $H_{\mu\nu\rho}$ and a is not a variable to be integrated in the path integral, it is only a field to be eliminated by its field equations. We can easily see that integrating out a and setting θ at the minimum of its potential $\theta = 0$ we obtain:

$$\mathcal{L} = H_{\mu\nu\rho}^2 - \frac{1}{4}(\partial_\mu H_{\nu\rho\sigma})^2 \quad (\text{A.5})$$

whereas integrating out $H_{\mu\nu\rho}$ and setting θ at the minimum of its potential $\theta = a$ we arrive at the Lagrangian for the axion

$$\mathcal{L} = -\frac{1}{4}(\partial a)^2 + (1 - \cos a). \quad (\text{A.6})$$

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