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UNIVERSAL ASPECTS OF STRING PROPAGATION ON CURVED BACKGROUNDS

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ABSTRACT

String propagation on D–dimensional curved backgrounds with Lorentzian signature is formulated as a geometrical problem of embedding surfaces. When the spatial part of the background corresponds to a general WZW model for a compact group, the classical dynamics of the physical degrees of freedom is governed by the coset conformal field theory $SO(D-1)/SO(D-2)$, which is universal irrespective of the particular WZW model. The same holds for string propagation on D–dimensional flat space. The integration of the corresponding Gauss–Codazzi equations requires the introduction of (non–Abelian) parafermions in differential geometry.

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1 Introduction

String propagation on a given background defines an embedding problem in differential geometry. Choosing, whenever possible, the temporal gauge one may solve the Virasoro constraints and consider the non–linear dynamics governing the physical degrees of freedom of the string. Simple counting shows that for D–dim backgrounds the physical degrees of freedom satisfy a coupled system of $D-2$ differential equations, which are defined on the 2–dim string world–sheet and they are non–linear due to the quadratic form of the Virasoro constraints. Our primary aim is to investigate the integrability of these equations and explore some of their universal aspects for a wide class of backgrounds.

Lund and Regge considered this problem several years ago for string propagation on flat 4–dim Minkowski space, in the presence of a Kalb–Ramond field as well [1]. This geometrical approach was subsequently generalized to $D \geq 5$ [2]. It became clear more recently [3] that the dynamics of the physical degrees of freedom in the $D = 4$ case admits a Lagrangian formulation as an $SO(3)/SO(2)$ gauged WZW model. However, for $D \geq 5$ an analogous Lagrangian description using cosets to model the dynamics of the $D - 2$ physical degrees of freedom has been lacking. The technical problem that arises here is finding the appropriate non–local field variables to integrate the underlying Gauss–Codazzi equations of the embedding. We solve this problem by introducing, just from purely geometrical considerations, the non–Abelian parafermions of the coset model $SO(D-1)/SO(D-2)$ and show that the chiral equations they obey [4] are equivalent to the Gauss–Codazzi embedding equations. Hence, string dynamics on D–dim flat Minkowski space, after we solve the Virasoro constraints, is governed by the semi–classical geometry of the conformal field theory coset $SO(D-1)/SO(D-2)$ [4, 5].

An interesting generalization of this program includes Lorentzian backgrounds of the product form $R \otimes K_{D-1}$, where K_{D-1} is a WZW model for a semi–simple compact group. The integration of the Gauss–Codazzi equations for these backgrounds is similar to flat space in that $SO(D-1)/SO(D-2)$ parafermions are also used, thus exhibiting a universal behavior irrespectively of the particular WZW model K_{D-1}. The coset space structure of the physical degrees of freedom of the free string is rather remarkable, leading to the world–sheet integrability of the underlying non–linear equations. Using the parafermion variables of the Gauss–Codazzi equations one may easily find chiral W_∞ symmetries as hidden on–shell symmetries of the classical theory. Our results shed new light into the differential geometry of embedding surfaces using concepts and field variables, which so far have been natural only in conformal field theory.

The organization of this paper is as follows: In section 2 we set up the Gauss–Codazzi equations for string propagation on D–dim curved space and determine a wide class of backgrounds that allow for their integration. We expose the universal aspects of string dynamics for Lorentzian backgrounds whose spatial part is either flat space or a WZW model based on a general compact group. In section 3 we use the $SO(D-1)/SO(D-2)$ WZW model to describe systematically the dynamics of the physical degrees of freedom and present explicit results for $D = 4$ and $D = 5$. Finally, in section 4 we comment on various other generalizations and the quantization of strings before or after solving the classical Virasoro constraints. Connections with reduced σ–models [6]–[9] and the associated systems of symmetric space sine–Gordon models are also discussed.

2 String dynamics and embedding surfaces

We first review relevant parts from the theory of embedding surfaces in the context of Riemannian geometry (see for instance [10]). Then we consider classical string propagation on backgrounds with Lorentzian signature and we formulate the problem of determining the dynamics of the physical modes as a geometrical problem of surface embedding, after solving the Virasoro constraints in the temporal gauge. At the end we specialize to backgrounds with spatial part corresponding to flat space or WZW models based on general semi–simple compact groups.

Gauss–Codazzi equations: Generalities

Consider a D–dim space M_D with line element ($\equiv$ fundamental quadratic form) given by

$$ds_D^2 = G_{\mu\nu}(y)dy^\mu dy^\nu \ , \qquad \mu,\nu = 1,\dots,D \ . \tag{2.1}$$

A d–dim subspace M_d of M_D with local coordinates x^i, $i = 1,\dots,d$ may be considered as an embedded surface with defining equations $y^\mu = y^\mu(x^1,\dots,x^d)$. The line element in M_d will be denoted by

$$ds_d^2 = g_{ij}(x)dx^i dx^j \ , \qquad i,j = 1,\dots,d \ . \tag{2.2}$$

The restriction of (2.1) in M_d should be equivalent to (2.2). Thus we have the relation[1]

$$g_{ij}(x) = G_{\mu\nu}(y)y^\mu_{,i}y^\nu_{,j} \ . \tag{2.3}$$

The embedded surface is completely specified by the set of vectors $\{\xi^\mu_\sigma,\ \sigma = d+1,\dots,D\}$ normal to it. These are chosen to satisfy the orthonormalization conditions

$$G_{\mu\nu}\xi^\mu_\sigma\xi^\nu_\tau = \delta_{\sigma\tau} \ , \tag{2.4}$$

and by definition are also orthogonal to the tangent vectors to the surface $y^\mu_{,i}$:

$$G_{\mu\nu}y^\mu_{,i}\xi^\nu_\sigma = 0 \ . \tag{2.5}$$

The set of vectors $\{y^\mu_{,i},\xi^\mu_\sigma\}$ satisfy the completeness relation in M_D:

$$g^{ij}y^\mu_{,i}y^\nu_{,j} + \xi^\mu_\sigma\xi^\nu_\tau\delta^{\sigma\tau} = G^{\mu\nu} \ . \tag{2.6}$$

[1] We will use the notation $y^\mu_{,i} \equiv \frac{\partial y^\mu}{\partial x^i}$. Covariant derivatives on M_D and on M_d will be denoted by D_μ and D_i respectively. The $y^\mu(x)$'s are scalars with respect to covariant differentiation on M_d, i.e., $D_i y^\mu = y^\mu_{,i}$.

The dynamics of the embedded surface is determined from the evolution of the vectors $y^\mu_{,i}$ and ξ^μ_σ as functions of the variables x^i in M_d. The corresponding equations are determined by repeated covariant differentiations of (2.3)–(2.5) and subsequent algebraic manipulations. Here we will only present the result leaving out the detailed proofs which can be found in [10]. We recall the concept of the second fundamental quadratic form with components defined as

$$\Omega^\sigma_{ij} = G_{\mu\nu}\xi^\mu_\sigma\Big(D_i D_j y^\nu + \Gamma^\nu_{\lambda\alpha} y^\lambda_{,i} y^\alpha_{,j}\Big) \ . \tag{2.7}$$

It is obvious that it is a symmetric tensor in M_d, i.e., $\Omega^\sigma_{ij} = \Omega^\sigma_{ji}$. We also define the torsion ($\equiv$ third fundamental form) in M_d

$$\mu^{\sigma\tau}_i = G_{\mu\nu}\xi^\mu_\sigma\Big(\xi^\nu_{\tau,i} + \Gamma^\nu_{\lambda\alpha}\xi^\lambda_\tau y^\alpha_{,i}\Big) \ . \tag{2.8}$$

Though not immediately obvious it can be shown that it is antisymmetric, i.e., $\mu^{\sigma\tau}_i + \mu^{\tau\sigma}_i = 0$. With the above definitions the evolution equations can be written as

$$D_i D_j y^\mu = \Omega^\sigma_{ij}\xi^\mu_\sigma - \Gamma^\mu_{\nu\lambda} y^\nu_{,i} y^\lambda_{,j} \ , \tag{2.9}$$

and

$$\xi^\mu_{\sigma,i} = -\Omega^\sigma_{ij} g^{jk} y^\mu_{,k} + \mu^{\tau\sigma}_i \xi^\mu_\tau - \Gamma^\mu_{\lambda\alpha} y^\lambda_{,i}\xi^\alpha_\sigma \ . \tag{2.10}$$

The careful reader will notice that for curves ($d = 1$) in 3–dim Euclidean space, the equations (2.9) and (2.10) reduce to the well known Serret–Frenet formulae.

It is a quite straightforward but tedious procedure to derive the necessary conditions for the existence of solutions to (2.9) and (2.10). The resulting compatibility equations are given by

$$R_{ijkl} = R_{\mu\nu\alpha\beta} y^\mu_{,i} y^\nu_{,j} y^\alpha_{,k} y^\beta_{,l} + \Omega^\tau_{k[i}\Omega^\tau_{j]l} \ , \tag{2.11}$$

$$D_{[k}\Omega^\sigma_{j]i} = \mu^{\tau\sigma}_{[k}\Omega^\tau_{j]i} + R_{\mu\nu\alpha\beta} y^\mu_{,i} y^\alpha_{,j} y^\beta_{,k}\xi^\nu_\sigma \ , \tag{2.12}$$

and

$$D_{[k}\mu^{\sigma\tau}_{j]} + \mu^{\rho\sigma}_{[j}\,\mu^{\rho\tau}_{k]} + \Omega^\sigma_{l[j}\Omega^\tau_{k]i} g^{li} + R_{\mu\nu\alpha\beta} y^\mu_{,j} y^\nu_{,k}\xi^\alpha_\sigma\xi^\beta_\tau = 0 \ . \tag{2.13}$$

Equations (2.11) and (2.12) for the case of a 2–dim surface in 3–dim Euclidean space are known as the Gauss–Codazzi equations, whereas (2.13) for the case of a surface immersed in Euclidean space is known as the Ricci equation. In general, the number of unknown functions in the embedding equations of a space M_d in M_D exceeds the number of equations. However, the extra functions may be eliminated using the freedom to perform local transformations in the normal space to the surface that rotate Ω^σ_i and $\mu^{\sigma\tau}_i$, also using any additional information that might be in our disposal. The precise mechanism, for the cases of interest in this paper, will be considered in detail in the next subsection.

String evolution in $M_D = R \otimes K_{D-1}$

We consider classical propagation of closed strings on a D–dim background that is the direct product of the real line R (contributing a minus in the signature matrix) and a general manifold (with Euclidean signature) K_{D-1}, i.e., $M_D = R \otimes K_{D-1}$. The corresponding target space variables are $y^0(\sigma^+, \sigma^-)$ and $y^\mu(\sigma^+, \sigma^-)$ with $\mu = 1, \ldots, D-1$. Here $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$, where τ and σ are the natural time and spatial variables on the world–sheet Σ. Then, the 2–dim σ–model action is given by

$$S = \frac{1}{2} \int_\Sigma Q^+_{\mu\nu} \partial_+ y^\mu \partial_- y^\nu - \partial_+ y^0 \partial_- y^0 \ , \qquad Q^+_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} \ , \tag{2.14}$$

where G, B are the metric and antisymmetric tensor fields corresponding to the non–trivial part of the string background. The classical equations of motion are given by

$$\delta y^0 : \qquad \partial_+ \partial_- y^0 = 0 \ , \tag{2.15}$$

$$\delta y^\mu : \qquad \partial_+ \partial_- y^\mu + (\Gamma^-)^\mu_{\nu\lambda} \partial_+ y^\nu \partial_- y^\lambda = 0 \ , \tag{2.16}$$

where $(\Gamma^\pm)^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} \pm \frac{1}{2} H^\mu_{\nu\lambda}$ are the generalized connections that include the string torsion $H_{\mu\nu\lambda} \equiv \partial_{[\mu} B_{\nu\lambda]}$. We have implicitly imposed the conformal gauge in writing (2.14). Hence, the classical equations of motion are supplied with the constraints

$$T_{\pm\pm} \equiv \frac{1}{4} G_{\mu\nu} \partial_\pm y^\mu \partial_\pm y^\nu - \frac{1}{4} \partial_\pm y^0 \partial_\pm y^0 = 0 \ . \tag{2.17}$$

The conformal gauge allows for transformations $\sigma^\pm \to f^\pm(\sigma^\pm)$, which can be used in a way consistent with the equations of motion (2.15), (2.16). We choose the so called temporal gauge, where $y^0 = \tau$. Then (2.15) is trivially satisfied whereas (2.16) remains unaffected since G and B are independent of y^0. The constraints (2.17) take the form

$$G_{\mu\nu} \partial_\pm y^\mu \partial_\pm y^\nu = 1 \ . \tag{2.18}$$

For later use we define an angular variable θ via the relation

$$G_{\mu\nu} \partial_+ y^\mu \partial_- y^\nu = \cos\theta \ . \tag{2.19}$$

The Euclidean signature of K_{D-1} warrants the reality of θ.

Clearly in the temporal gauge we may restrict our analysis entirely on K_{D-1} and on the projection of the string world–sheet Σ on the $y^0 = \tau$ hyperplane, following the spirit of the Lund–Regge analysis [1]. The resulting 2–dim surface S has Euclidean signature with metric given by

$$\begin{aligned} ds^2 &= G_{\mu\nu} dy^\mu dy^\nu \\ &= G_{\mu\nu} \Big(\partial_+ y^\mu \partial_+ y^\nu d\sigma^{+2} + \partial_- y^\mu \partial_- y^\nu d\sigma^{-2} + 2 \partial_+ y^\mu \partial_- y^\nu d\sigma^+ d\sigma^- \Big) \ . \end{aligned} \tag{2.20}$$

Using the constraints (2.18) and the definition (2.19) we obtain from (2.20) the expression

$$ds^2 = d\sigma^{+2} + d\sigma^{-2} + 2 \cos\theta d\sigma^+ d\sigma^- \ . \tag{2.21}$$

Thus, for $y^0 = \tau$, determining the classical evolution of the string is equivalent to the problem of embedding the 2–dim surface S with metric (2.21) on the $(D-1)$–dim space K_{D-1}. Hence, the general analysis we have presented in the previous subsection becomes relevant to string theory at this point.

For further convenience we present the expressions for the non–vanishing Christoffel symbols and the Riemann curvature of the metric (2.21):

$$\Gamma^{\pm}_{\pm\pm} = \cot\theta \partial_\pm \theta \ , \qquad \Gamma^{\pm}_{\mp\mp} = -\frac{1}{\sin\theta}\partial_\mp \theta \ , \qquad R_{+-+-} = -\sin\theta \partial_+\partial_-\theta \ . \tag{2.22}$$

Contracting (2.16) with $G_{\mu\alpha}\xi^\alpha_\sigma$, where $\sigma = 3,\dots,D-1$, and using (2.4) we obtain

$$\Omega^\sigma_{+-} = \Omega^\sigma_{-+} = \frac{1}{2} H_{\mu\nu\lambda}\xi^\mu_\sigma \partial_+ y^\nu \partial_- y^\lambda \ , \qquad \sigma = 3,\dots,D-1 \ . \tag{2.23}$$

Contracting with $G_{\mu\alpha}\partial_\pm y^\alpha$ and using (2.18) we obtain instead an identity and thus have no additional restrictions. Hence, the information contained in the $D-1$ classical equations (2.16) is entirely encoded in the components of the second fundamental form (2.23) and in the two constraints (2.18). It will be convenient to modify the torsion $\mu^{\sigma\tau}_\pm$ defined by (2.8), using a term that includes the string torsion for $i = \pm$:

$$\begin{aligned} M^{\sigma\tau}_\pm &\equiv \mu^{\sigma\tau}_\pm \pm \frac{1}{2} H_{\mu\nu\lambda}\xi^\mu_\sigma \xi^\nu_\tau \partial_\pm y^\lambda \\ &= G_{\mu\nu}\xi^\mu_\sigma \Big(\partial_\pm \xi^\nu_\tau + (\Gamma^\pm)^\nu_{\lambda\alpha}\xi^\lambda_\tau \partial_\pm y^\alpha\Big) \ . \end{aligned} \tag{2.24}$$

It is evident that, similarly to $\mu^{\sigma\tau}_\pm$, $M^{\sigma\tau}_\pm$ is also antisymmetric, and thus non–trivial only for target spaces with dimension $D \geq 5$. After some tedious algebraic manipulations, equations (2.11)–(2.13) for the remaining components of the second fundamental form $\Omega^\sigma_{\pm\pm}$ and for the modified torsion $M^{\sigma\tau}_\pm$ can be cast into the following form:

$$\Omega^\tau_{++}\Omega^\tau_{--} + \sin\theta\partial_+\partial_-\theta = -R^+_{\mu\nu\alpha\beta}\partial_+ y^\mu \partial_+ y^\alpha \partial_- y^\nu \partial_- y^\beta \ , \tag{2.25}$$

$$\partial_\mp \Omega^\sigma_{\pm\pm} - M^{\tau\sigma}_\mp \Omega^\tau_{\pm\pm} - \frac{1}{\sin\theta}\partial_\pm\theta \Omega^\sigma_{\mp\mp} = R^\mp_{\mu\nu\alpha\beta}\partial_\pm y^\mu \partial_\pm y^\alpha \partial_\mp y^\beta \xi^\nu_\sigma \ , \tag{2.26}$$

and

$$\partial_+ M^{\sigma\tau}_- - \partial_- M^{\sigma\tau}_+ - M^{\rho[\sigma}_- M^{\tau]\rho}_+ + \frac{\cos\theta}{\sin^2\theta}\Omega^{[\sigma}_{++}\Omega^{\tau]}_{--} = (R^-_{\mu\nu\alpha\beta} - D^-_\mu H_{\nu\alpha\beta})\partial_+ y^\mu \partial_- y^\nu \xi^\alpha_\sigma \xi^\beta_\tau \ , \tag{2.27}$$

where the curvatures are defined using the generalized connections $(\Gamma^\pm)^\mu_{\nu\lambda}$,

$$R^\pm_{\mu\nu\alpha}{}^\beta = -\partial_{[\mu}(\Gamma^\pm)^\beta_{\nu]\alpha} + (\Gamma^\mp)^\gamma_{\alpha[\mu}(\Gamma^\pm)^\beta_{\nu]\gamma} \ , \tag{2.28}$$

and similarly for the covariant derivatives D^-_μ and D^+_μ.

Next, counting the number of the embedding equations in (2.25)–(2.27) we find that there are $1 + 2(D-3) + \frac{1}{2}(D-3)(D-4)$ of them, whereas the number of the unknown functions θ, $\Omega^\sigma_{\pm\pm}$ and $M^{\sigma\tau}_\pm$ is $1 + 2(D-3) + (D-3)(D-4)$. Hence, for $D \geq 5$ there are $\frac{1}{2}(D-3)(D-4)$ more unknown functions than equations. Notice, however, that the system (2.25)–(2.27) is invariant under local transformations on the world–sheet generated by

$$\xi^\mu \to \Lambda^{-1}\xi^\mu \ , \qquad \Omega_{\pm\pm} \to \Lambda^{-1}\Omega_{\pm\pm} \ , \qquad M_\pm \to \Lambda^{-1}(M_\pm + \partial_\pm)\Lambda \ , \tag{2.29}$$

where $\Lambda = \Lambda(\sigma^+, \sigma^-)$ is an orthogonal matrix of $SO(D-3)$. This gauge invariance accounts for the extra (gauge) degrees of freedom in (2.25)–(2.27) and can be used to eliminate them (gauge fix).

WZW backgrounds K_{D-1}

It seems an enormous task to make further progress with the embedding system of equations (2.25)–(2.27) as it stands in all generality. There are two major difficulties. First, the presence of source–like terms depending explicitly on $\partial_\pm y^\mu$ and ξ^μ_σ seems to prohibit us from integrating them, even partially. Second, a Lagrangian description from which (2.25)–(2.27) can be derived as equations of motion is also lacking.

It is rather remarkable that both problems can be solved by considering for K_{D-1} either flat space or any WZW model based on a semi–simple compact group G, with $\dim(G) = D-1$. This is due to the identities [11]

$$R^\pm_{\mu\nu\alpha\beta} = D^\pm_\mu H_{\nu\alpha\beta} = 0 \ , \tag{2.30}$$

which are generally valid for any WZW model. Then we completely get rid of the bothersome terms on the right hand side of (2.25)–(2.27).[2] In order to show that a Lagrangian description exists, we first extend the range of definition of Ω^σ_{++} and $M^{\sigma\tau}_\pm$ by appending new components defined as:

$$\Omega^2_{++} = \partial_+\theta \ , \qquad M^{\sigma 2}_+ = \cot\theta\,\Omega^\sigma_{++} \ , \qquad M^{\sigma 2}_- = -\frac{1}{\sin\theta}\Omega^\sigma_{--} \ . \tag{2.31}$$

Then equations (2.25)–(2.27) can be recast into the suggestive form

$$\partial_-\Omega^a_{++} + M^{ab}_-\Omega^b_{++} = 0 \ , \tag{2.32}$$

$$\partial_+ M^{ab}_- - \partial_- M^{ab}_+ + [M_+, M_-]^{ab} = 0 \ , \tag{2.33}$$

where the new index $a = (2, \sigma)$. Notice that if we treat Ω^a_{++} not as a row of the bigger matrix M^{ab}_+, as suggested by (2.31), but as an independent vector, then the number of unknown functions in (2.32) and (2.33) is augmented by $D-3$ compared to the same number of functions in equations (2.25)–(2.27). However, there is a simultaneous enlargement of the local gauge symmetry from $SO(D-3)$ to $SO(D-2)$ that takes care of it. Such a gauge symmetry enlargement can only be achieved if the underlying change

[2] Actually, the same result is obtained by demanding the weaker conditions $R^-_{\mu\nu\alpha\beta} - D^-_\mu H_{\nu\alpha\beta} = 0$ and using the general identity $R^-_{\mu\nu\alpha\beta} - D^-_\mu H_{\nu\alpha\beta} = R^+_{\mu\nu\alpha\beta} - D^+_\nu H_{\mu\alpha\beta}$ and the property $R^+_{\mu\nu\alpha\beta} = R^-_{\alpha\beta\mu\nu}$. It would be interesting to find explicit examples where these weaker conditions hold.

of variables is non–local. This will become more clear soon after the introduction of parafermions in the next section.

Equation (2.33) is a zero curvature condition for the matrices $M_\pm$ and it is solved (without worrying here about global issues related to the world–sheet topology) by $M_\pm = \Lambda^{-1}\partial_\pm\Lambda$, where $\Lambda \in SO(D-2)$. Then (2.32) can be written as

$$\partial_-(\Lambda^{ab}\Omega^b_{++}) = \partial_-(\Lambda^{a2}\partial_+\theta + \partial_+\Lambda^{a2}\tan\theta) = 0 \ . \tag{2.34}$$

The vector Λ^{a2} has unit length, i.e., $\Lambda^{a2}\Lambda^{a2} = 1$. We can incorporate this constraint by defining $Y^a = \Lambda^{a2}\sin\theta$. Then (2.34) assumes the form

$$\partial_-\left(\frac{\partial_+ Y^a}{\sqrt{1-Y^2}}\right) = 0 \ , \qquad Y^2 \equiv Y^b Y^b \ , \qquad a,b = 2,3,\ldots,D-1 \ . \tag{2.35}$$

These equations were derived before in [2], while describing the dynamics of a free string propagating in D–dimensional *flat* space–time. It is remarkable that these equations remain unchanged even if the flat $(D-1)$–dim space–like part is replaced by a curved background corresponding to a general WZW model. In retrospect, we may attribute this unexpected result to the fact that a group space is parallelizable, and it can be made flat in the sense of (2.30) with the addition of the appropriate amount of torsion. It should be emphasized that although the compatibility equations are universal, the actual evolution equations of the normal and tangent vectors to the surface are given by specializing (2.9) and (2.10) to K_{D-1}; they are certainly different from those of the flat space free string.

As we have already mentioned, it would be advantageous if (2.35) (or an equivalent system) could be derived as classical equations of motion. The key that will enable us next to construct the corresponding Lagrangian is the observation that (2.35) imply chiral conservation laws, which are reminiscent of the equations obeyed by classical parafermions in coset models [12].In fact (2.35) were derived as classical string equations for gauged WZW models corresponding to $SO(D-1)/SO(D-2)$ cosets in [4]; they are analytic continuations of the models $SO(D-3,2)/SO(D-3,1)$ that give rise to string propagation in backgrounds with Lorentzian signature [13]. We mention for completeness that they also arise in the massless limit of the $SO(D)/SO(D-1)$ symmetric space sine–Gordon models, which were recently formulated as integrable perturbations of the $SO(D-1)/SO(D-2)$ gauged WZW models [9]. Since (2.35) themselves do not correspond to a Lagrangian system of equations, our strategy in the following will be to perform a non–local change of variables that maps them into Lagrangian form. This non–local change of variables is highly non–intuitive in differential geometry, and only the correspondence with parafermions makes it natural.

3 Dynamics of physical degrees of freedom

In this section we first briefly discuss some general aspects of gauged WZW models in connection with the associated coset conformal field theories. Then we restrict our attention to $SO(D-1)/SO(D-2)$ coset models and establish a relation between the chiral conservation laws obeyed by the corresponding parafermions and the embedding equations (2.35). At the end we present explicit results for $D=4$ and $D=5$.

Lagrangian description and parafermions

Recall that the gauged WZW action is [14, 15]

$$S = I_{wzw}(g) + \frac{k}{\pi}\int \mathrm{Tr}\Big(A_+\partial_- g g^{-1} - g^{-1}\partial_+ g A_- + A_+ g A_- g^{-1} - A_+ A_-\Big) \ , \qquad (3.1)$$

where $g \in G$ and $A_\pm$ are gauge fields valued in the Lie algebra of a subgroup $H \subset G$. The corresponding field strength is $F_{+-} = \partial_+ A_- - \partial_- A_+ - [A_+, A_-]$. We also split indices as $A = (a, \alpha)$, where $a \in H$ and $\alpha \in G/H$. Variation of (3.1) with respect to all fields gives the classical equations of motion

$$\delta A_+ \ : \qquad D_- g g^{-1}|_H = 0 \ , \qquad (3.2)$$

$$\delta A_- \ : \qquad g^{-1} D_+ g|_H = 0 \ , \qquad (3.3)$$

$$\delta g \ : \qquad D_-(g^{-1} D_+ g) + F_{+-} = 0 \ . \qquad (3.4)$$

Imposing (3.3) on (3.4) yields the zero curvature condition $F_{+-} = 0$ on–shell, and

$$D_-(g^{-1} D_+ g)|_{G/H} = 0 \ . \qquad (3.5)$$

There are two commuting copies of an affine algebra corresponding to a WZW action for a group G, one for each chiral sector [16]. A remnant of this algebra is also present in the gauged WZW model. We parametrize the gauge fields as $A_\pm = (\partial_\pm h_\pm) h_\pm^{-1}$, where $h_\pm \in H$. Thus, $h_\pm$ are given in terms of $A_\pm$ as

$$h_+^{-1} = \mathrm{P} e^{-\int^{\sigma^+} A_+} \ , \qquad h_-^{-1} = \mathrm{P} e^{-\int^{\sigma^-} A_-} \ , \qquad (3.6)$$

where P stands for path ordering. Using the gauge invariant group element

$$f = h_+^{-1} g h_+ \in G \ , \qquad (3.7)$$

and the on–shell zero curvature condition $F_{+-} = 0$, we write equation (3.5) as

$$\partial_- \Psi_+ = 0 \ , \qquad \Psi_+ = \frac{ik}{\pi} f^{-1}\partial_+ f \in G/H \ . \qquad (3.8)$$

Thus, the coset valued matrix Ψ_+ is chirally conserved.

In fact, Ψ_+ are nothing but the classical parafermions [12]. Since they have Wilson lines attached to them (cf. (3.7), (3.6)) they are non–local objects. This is also reflected in the algebra they obey [12] (we drop $+$ as a subscript and denote σ^+ by x or y),

$$\begin{aligned} \{\Psi_\alpha(x), \Psi_\beta(y)\} \ &= \ -\frac{k}{\pi}\delta_{\alpha\beta}\delta'(x-y) - f_{\alpha\beta\gamma}\Psi_\gamma(y)\delta(x-y) \\ &\quad -\frac{\pi}{2k} f_{c\alpha\gamma} f_{c\beta\delta}\ \epsilon(x-y)\Psi_\gamma(x)\Psi_\delta(y) \ , \end{aligned} \qquad (3.9)$$

where the antisymmetric step function $\epsilon(x-y)$ equals $+1$ (-1) if $x>y$ $(x<y)$. The last term in (3.9) is responsible for their non–trivial monodromy properties and unusual statistics. In addition, conformal transformations are generated by $T_{++}=-\frac{\pi}{2k}\Psi_\alpha\Psi_\alpha$.

The 2–dim σ–model having the above infinite dimensional symmetry is obtained by first choosing a unitary gauge by fixing $\dim(H)$ variables among the total number of $\dim(G)$ parameters of the group element g. Hence, there are $\dim(G/H)$ remaining variables, which will be denoted by X^μ. Then, we eliminate the gauge fields in (3.1) using their equation of motion (3.2), (3.3)

$$\begin{aligned} A_+^a &= +i(C^T-I)^{-1}_{ab}L^b_\mu\partial_+X^\mu\ , \\ A_-^a &= -i(C-I)^{-1}_{ab}R^b_\mu\partial_-X^\mu\ , \end{aligned} \tag{3.10}$$

where the appropriate short–hand definitions are

$$L^a_\mu=-i\mathrm{Tr}(t^a g^{-1}\partial_\mu g)\ ,\qquad R^a_\mu=-i\mathrm{Tr}(t^a\partial_\mu g g^{-1})\ ,\qquad C^{ab}=\mathrm{Tr}(t^a g t^b g^{-1})\ . \tag{3.11}$$

Finally, the σ-model action is given by

$$S=I_{wzw}(g)-\frac{k}{\pi}\int_\Sigma R^a_\mu(C^T-I)^{-1}_{ab}L^b_\nu\partial_+X^\mu\partial_-X^\nu\ . \tag{3.12}$$

$SO(D-1)/SO(D-2)$ coset structure

We specialize now to the $SO(D-1)/SO(D-2)$ gauged WZW model and show that (2.35) is equivalent to the parafermion equation (3.8). We will essentially follow the analysis of [4] adopted to our present purposes.

The group element $g\in SO(D-1)$ in the right coset decomposition can be written as $g=\tilde{h}t$, where

$$\tilde{h}=\begin{pmatrix}1 & 0\\ & \\ 0 & h\in SO(D-2)\end{pmatrix} \tag{3.13}$$

and $t\in SO(D-1)/SO(D-2)$ is parametrized by a $(D-2)$–dim vector $\vec{X}$ as

$$t=\begin{pmatrix} b & X^j\\ & \\ -X^i & \delta_{ij}-\frac{1}{b+1}X^iX^j\end{pmatrix}\ ,\qquad b\equiv\sqrt{1-\vec{X}^2}\ . \tag{3.14}$$

The range of the parameters X^i is restricted by $\vec{X}^2\le 1$ and the value of b is such that the matrix t is an element of $SO(D-1)$ obeying $t^{-1}=t^T$. Then we compute

$$dtt^{-1}\ =\ \begin{pmatrix} 0 & dX^j+\frac{\vec{X}\cdot d\vec{X}}{b(b+1)}X^j\\ & \\ -dX^i-\frac{\vec{X}\cdot d\vec{X}}{b(b+1)}X^i & \frac{1}{b+1}dX^{[i}X^{j]}\end{pmatrix}\ ,$$

$$t^{-1}dt \;=\; \begin{pmatrix} 0 & dX^j + \frac{\vec{X}\cdot d\vec{X}}{b(b+1)}X^j \\ & \\ -dX^i - \frac{\vec{X}\cdot d\vec{X}}{b(b+1)}X^i & -\frac{1}{b+1}dX^{[i}X^{j]} \end{pmatrix} . \tag{3.15}$$

To find an explicit expression for the parafermions in (3.8) we first rewrite the constraint (3.3) as

$$(f^{-1}\partial_+ f)_{ij} = (T^{-1}\partial_+ T)_{ij} + (T^{-1}H^{-1}\partial_+ HT)_{ij} = 0 \ , \tag{3.16}$$

where $H = h_+^{-1}hh_+$ and $T = h_+^{-1}th_+$. The explicit form of T is as in (3.14) with $X^i \to Y^i \equiv X^j(h_+)^{ji}$. Notice that since $\vec{Y}^2 = \vec{X}^2$, the Y^i are gauge invariant. Then we solve for

$$(H^{-1}\partial_+ H)_{ij} = \frac{1}{b(b+1)}\partial_+ Y^{[i}Y^{j]} \ . \tag{3.17}$$

The parafermion in (3.8) is computed by explicitly writing out $\Psi^i \equiv \frac{ik}{\pi}(f^{-1}\partial_+ f)_{0i}$ and utilizing (3.17). The final result is [4]

$$\begin{aligned} \Psi^i &= \frac{ik}{\pi}\frac{\partial_+ Y^i}{\sqrt{1-\vec{Y}^2}} = \frac{ik}{\pi}\frac{1}{\sqrt{1-\vec{X}^2}}(D_+X)^j h_+^{ji} \ , \\ (D_+X)^j &= \partial_+ X^j - A_+^{jk}X^k \ . \end{aligned} \tag{3.18}$$

Thus, the corresponding equation $\partial_-\Psi^i = 0$ is precisely (2.35). The Y^i are related to the σ-model variables non–locally as

$$Y^i = X^j(h_+)^{ji} \ , \qquad h_+^{-1} = \mathrm{P}e^{-\int^{\sigma^+} A_+} \ , \tag{3.19}$$

where the gauge field A_+ is given by (3.10). This provides the necessary non–local change of variables that transform (2.35) into a Lagrangian system of equations.

The representation matrices for $SO(D-1)$ are $(t_{AB})_{CD} = \delta_{C[A}\delta_{B]D}$, where the indices split as $A = (0,i)$ with $i = 1,2,\dots,D-2$. Then the algebra of the parafermions (3.9) becomes

$$\{\Psi^i(x),\Psi^j(y)\} = \frac{k}{2\pi}\delta_{ij}\delta'(x-y) - \frac{\pi}{2k}\epsilon(x-y)\Big(\delta_{ij}\Psi(x)\cdot\Psi(y) - \Psi_j(x)\Psi_i(y)\Big) \ . \tag{3.20}$$

The absence of linear terms in Ψ^i on the right hand side is due to the simple fact that $SO(D-1)/SO(D-2)$ is a symmetric space. Thus, structure constants involving only coset space indices are zero.

It remains to choose a gauge and explicitly compute A_+ and the $(D-2)$–component σ–model action (3.12). This has been done in another context for $SO(3)/SO(2)$ (the only Abelian case) in [12, 17], for $SO(4)/SO(3)$ in [4] and for $SO(5)/SO(4)$ in [5]. Here, for the time being we proceed with a unified treatment of all $SO(D-1)/SO(D-2)$ models. It is convenient to distinguish between the cases of D even or odd integers.

$\underline{D = 2N+2 = \text{even}}$: We have enough gauge freedom to cast the orthogonal matrix $h \in SO(2N)$ and the vector $\vec{X}$ into the form

$$h = \begin{pmatrix} \cos 2\phi_1 & \sin 2\phi_1 & 0 & \cdots & 0 & 0 \\ -\sin 2\phi_1 & \cos 2\phi_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cos 2\phi_N & \sin 2\phi_N \\ 0 & 0 & 0 & \cdots & -\sin 2\phi_N & \cos 2\phi_N \end{pmatrix} , \quad \vec{X} = \begin{pmatrix} 0 \\ X_2 \\ 0 \\ X_4 \\ \vdots \\ 0 \\ X_{2N} \end{pmatrix} . \tag{3.21}$$

The total number of independent variables in h and $\vec{X}$ is $2N = D-2$, as it should be.

$\underline{D = 2N+3 = \text{odd}}$: In such cases the orthogonal matrix $h \in SO(2N+1)$ and the vector $\vec{X}$ can be gauge fixed into the form

$$h = \begin{pmatrix} \cos 2\phi_1 & \sin 2\phi_1 & 0 & \cdots & 0 & 0 & 0 \\ -\sin 2\phi_1 & \cos 2\phi_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cos 2\phi_N & \sin 2\phi_N & 0 \\ 0 & 0 & 0 & \cdots & -\sin 2\phi_N & \cos 2\phi_N & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} , \quad \vec{X} = \begin{pmatrix} 0 \\ X_2 \\ 0 \\ X_4 \\ \vdots \\ 0 \\ X_{2N} \\ X_{2N+1} \end{pmatrix} . \tag{3.22}$$

Again the total number of the remaining independent variables is $2N+1 = D-2$, as it should be.

Using the above gauge fixing together with (3.15) and the Polyakov–Wiegman formula, we find that the WZW action (3.12) contributes to the total line element

$$ds^2_{wzw} = d\vec{\phi}^2 + \frac{1}{2(1+b)} d\vec{X}^2 + \frac{1+2b}{4b^2(1+b)^2} (\vec{X} \cdot d\vec{X})^2 , \tag{3.23}$$

and has zero contribution to the total antisymmetric tensor. The contribution of the second term of (3.12) is more complicated and will not be presented here in all generality; of course, its effect will be taken into account in the specific examples below.

Examples

We will work out all the technical details in two examples. The first one is the Abelian coset $SO(3)/SO(2)$ [12]. In terms of our original problem it arises after solving the Virasoro constraints for strings propagating on 4–dim Minkowski space or on the direct product of the real line R and the WZW model for $SU(2)$, which is the only 3–dim non–Abelian group for which a WZW action exists. The second example is the simplest non–Abelian coset based on $SO(4)/SO(3)$ and was considered in [4]. In our context it arises in string propagation on 5–dim Minkowski space or on the direct product of the real line R and the WZW model based on $SU(2) \otimes U(1)$.

$\underline{SO(3)/SO(2)}$: Using (3.21) with $X_2 = \sin 2\theta$, we find that the solution for the gauge fields is

$$A_\pm = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (1 \mp \cot^2\theta)\partial_\pm\phi \ , \tag{3.24}$$

and that the corresponding background has metric [12]

$$ds^2 = d\theta^2 + \cot^2\theta d\phi^2 \ . \tag{3.25}$$

Using (3.18), the corresponding Abelian parafermions $\Psi_\pm = \Psi_2 \pm i\Psi_1$ assume the familiar form

$$\Psi_\pm = (\partial_+\theta \pm i\cot\theta\partial_+\phi)e^{\mp i\phi \pm i\int \cot^2\theta\partial_+\phi} \ , \tag{3.26}$$

up to an overall normalization.

The emergence of the $SO(3)/SO(2)$ parafermions can also be seen directly from the original system of embedding equations (2.25)–(2.27). Since the indices σ, τ take only one value, the torsion matrix is $\mu_\pm = 0$. Then equation (2.27) is trivially satisfied, whereas (2.25) and (2.26) give (after setting $\Omega_{\pm\pm} = \cot\frac{\theta}{2}\partial_\pm\phi$) the following two equations:

$$\begin{gathered}
\partial_+\left(\cot^2\frac{\theta}{2}\partial_-\phi\right) + \partial_-\left(\cot^2\frac{\theta}{2}\partial_+\phi\right) = 0 \ , \\
\partial_+\partial_-\theta + \frac{\cos\frac{\theta}{2}}{2\sin^3\frac{\theta}{2}}\partial_+\phi\partial_-\phi = 0 \ .
\end{gathered} \tag{3.27}$$

These are the classical equations of motion of the $SO(3)/SO(2)$ coset with metric (3.25) (up to rescaling of θ, ϕ by a factor of 2) having the parafermions (3.26) as natural chiral objects. In the present geometrical context equations (3.27) were first derived in [1], whereas in [3] it was subsequently realized that they admit the $SO(3)/SO(2)$ coset interpretation we have just mentioned. It should be pointed out that for $D \geq 5$ a Lagrangian description for the embedding equations (2.25)–(2.27) cannot be possibly found in general without first making contact with parafermions, due to the fact that the torsion matrix $\mu_\pm$ (or $M_\pm$) is non–trivial.

$\underline{SO(4)/SO(3)}$: We parametrize $X_2 = \sin 2\theta\cos\omega$ and $X_3 = \sin 2\theta\sin\omega$ and use the basis of $SO(3)$ representation matrices

$$t_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad t_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ , \quad t_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \ . \tag{3.28}$$

Using (3.22) and the expansion for the gauge fields $A_\pm = \sum_{i<j} A^{ij}_\pm t_{ij}$ we find the solution

$$A^{12}_+ = -\left(\frac{\cos 2\theta}{\sin^2\theta\cos^2\omega} + \tan^2\omega\frac{\cos^2\theta - \cos^2\phi\cos 2\theta}{\cos^2\theta\sin^2\phi}\right)\partial_+\phi - \cot\phi\tan\omega\tan^2\theta\partial_+\omega \ ,$$

$$
\begin{aligned}
A_+^{13} &= \tan\omega \frac{\cos^2\theta - \cos^2\phi\cos 2\theta}{\cos^2\theta\sin^2\phi}\partial_+\phi + \cot\phi\tan^2\theta\partial_+\omega \ , \\
A_+^{23} &= \cot\phi\tan\omega\frac{\cos 2\theta}{\cos^2\theta}\partial_+\phi - \tan^2\theta\partial_+\omega \ .
\end{aligned} \tag{3.29}
$$

It turns out that an analogous expression for A_-^{ij} can be found from (3.29) by writing all θ–dependence in terms of $\cos 2\theta$ and replacing $\cos 2\theta$ by $1/\cos 2\theta$. Then, the background metric is [4]

$$
ds^2 = d\theta^2 + \tan^2\theta(d\omega + \tan\omega\cot\phi d\phi)^2 + \frac{\cot^2\theta}{\cos^2\omega}d\phi^2 \ , \tag{3.30}
$$

and the antisymmetric tensor is zero. The parafermions of the $SO(4)/SO(3)$ coset are non–Abelian given by (3.18) with covariant derivatives (omitting an overall factor of 2)

$$
\begin{aligned}
\frac{(D_+X)^1}{\sqrt{1-\vec{X}^2}} &= \frac{\cot\theta}{\cos\omega}\partial_+\phi \ , \\
\frac{(D_+X)^2 \pm i(D_+X)^3}{\sqrt{1-\vec{X}^2}} &= e^{\pm i\omega}\Big(\pm i\tan\theta(\tan\omega\cot\phi\partial_+\phi + \partial_+\omega) + \partial_+\theta\Big) \ .
\end{aligned} \tag{3.31}
$$

As a check, one may verify that $\Psi^i\Psi^i = \frac{1}{1-\vec{X}^2}(D_+X)^i(D_+X)^i$ is indeed proportional to the T_{++}–component of the energy momentum tensor corresponding to a σ–model with metric (3.30).

In addition to the two examples above, there also exist explicit results for the coset $SO(5)/SO(4)$ [5]. This would correspond in our context to string propagation on a 6–dim Minkowski space or on the background R times the $SU(2)\otimes U(1)^2$ WZW model. It should be pointed out that there is no reason to demand conformal invariance for the backgrounds with metrics (3.25) and (3.30) because they arise in a different context describing the geometry of the physical degrees of freedom.

4 Conclusions

We have investigated some universal aspects of classical string dynamics by integrating the Gauss–Codazzi equations of the corresponding embedding problem. We found for the class of D–dim backgrounds $R\otimes K_{D-1}$, where K_{D-1} is R^{D-1} or the WZW model for a general $(D-1)$–dim semi–simple compact group, that there are $D-2$ physical degrees of freedom whose dynamics is governed by the $SO(D-1)/SO(D-2)$ coset conformal field theory. The parafermion variables of this coset arise naturally in the present geometrical context, and so our results could be viewed as a link between conformal field theory techniques and the classical differential geometry of embedding surfaces.

There are two obvious extensions one can further make. First, suppose we start with a D–dim string background with signature $(2, D-2)$. The "spatial" part of this background is now Lorentzian, and therefore one has to consider suitable analytic continuation of the previous results. In particular, instead of the coset $SO(D-1)/SO(D-2)$ we find that the dynamics of the physical degrees of freedom is now given by the non–compact coset $SO(D-3,2)/SO(D-3,1)$. The simplest version of this for $D=4$ has already been considered in [18]. Second, it is also interesting to consider various supersymmetric generalizations of the present framework.

There are many similarities between classical string dynamics and the theory of ordinary 2–dim σ–models. The latter can also be viewed as describing the embedding of 2–dim surfaces into a group or coset space manifold, which in turn is embedded in flat space. Exploiting classical conformal invariance, which is similar to choosing the orthonormal gauge in string theory, amounts to reducing ordinary σ–models to the so called symmetric space sine–Gordon models (SSG) [6]–[8]. The SSG models have been described as perturbations of conformal field theory cosets [9]; for example, the reduced $S^n = SO(n+1)/SO(n)$ σ–model yields an integrable sine–Gordon perturbation of the $SO(n)/SO(n-1)$ coset conformal field theory. Hence, apart from the potential terms, and in the absence of string self–interactions, the structure of the kinetic terms is the same for the two classes of embedding problems. It is interesting to note that other reduced σ–models for general symmetric spaces have been described using appropriately chosen gauged WZW cosets (plus perturbations). Therefore, the parafermion variables of the corresponding coset conformal field theories (at and away from the conformal point) also play a key role in the integration of the embedding equations.

Finally, an interesting issue is the quantization of string theory. There are two different methods of quantizing constrained systems, either by solving the classical constraints and then quantize directly the physical degrees of freedom, or by quantizing the unconstrained degrees of freedom and then impose the constraints as quantum conditions on the physical states. It is well known that in general these two methods of quantization are not equivalent, in particular when the constraints have quadratic form as in string theory. Quantization of string theory usually proceeds using the second method, but in the present framework the physical degrees of freedom should be quantized directly using the quantization of the associated parafermions. Exploring this issue further is an interesting problem.

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