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The Quantum Gauge Principle [*]

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Abstract

We consider the evolution of quantum fields on a classical background space-time, formulated in the language of differential geometry. Time evolution along the worldlines of observers is described by parallel transport operators in an infinite-dimensional vector bundle over the space-time manifold. The time evolution equation and the dynamical equations for the matter fields are invariant under an arbitrary local change of frames along the restriction of the bundle to the worldline of an observer, thus implementing a "quantum gauge principle". We derive dynamical equations for the connection and a complex scalar quantum field based on a gauge field action. In the limit of vanishing curvature of the vector bundle, we recover the standard equation of motion of a scalar field in a curved background space-time.

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1 Introduction

The concept of time in quantum field theory is derived from the structure of the underlying space-time manifold. For both flat and curved space-times, in the Heisenberg picture, the state vector of a quantum system is constant[1] and identical for all observers, and the quantum fields fulfil dynamical equations derived by means of a quantization procedure of classical equations of motion based on a classical action functional [1]. This framework is unsatisfactory for two principal reasons:

(i) The state vector is an object that describes the knowledge of an observer about a physical system. The observer, in an idealized case, moves along a worldline $\mathcal{C}$. The state vector should thus be tied to $\mathcal{C}$, and its time evolution should not be related to some globally defined time, but to the *eigenzeit* of the observer.

(ii) The observer is free to choose the basis vectors in Hilbert space. In quantum mechanics, a change of basis vectors amounts to a change of the "picture", e.g. from the Heisenberg to the Schrödinger picture. In quantum field theory, in particular in perturbation theory, the preferred picture is the interaction picture. The change of pictures is performed by means of unitary operators. If, as proposed in (i), the state vector is tied to a worldline, a change of the basis in the Hilbert space should be allowed to be observer-dependent and arbitrary at any point of the worldline.

These two requirements amount to what could be called a "local quantum relativity principle": physics is independent of the Hilbert space basis, and the basis may be chosen locally in an arbitrary way. Since a change of the basis is mediated by a unitary transformation, the local quantum relativity principle is equivalent to a local $U(\mathcal{H})$ symmetry, $U(\mathcal{H})$ being the group of unitary operators on the Hilbert space $\mathcal{H}$.

In Ref. [2], a formulation of the time evolution of quantum systems in the language of differential geometry has been given. State vectors are elements of an infinite-dimensional vector bundle, and time evolution is given by the parallel transport operator related to a connection in the bundle. In order to keep the paper self-contained, this framework is briefly reviewed in Section 2. The goal here is to propose a dynamical principle that yields the connection in the bundle and the field operators at any space-time point. The basic idea is that the geometrical formulation permits the introduction of a local gauge theory, the connection being the "gauge field", and the quantum field operators being linear-operator-valued "matter fields". Such a local gauge theory will be defined in Section 3 by means of an action functional. We wish to note that the theory does not have to be quantized, because the dynamical variables appearing in the action are already the components of the quantum operators in an arbitrary frame. It is, however, not yet clear whether in this way canonical commutation relations hold in general.

The theory distinguishes conceptually between the evolution of the state vector, given by the connection in the bundle (for short, called the "quantum connection" in the following), and the dynamical equations of the quantum fields and the quantum connection, derived from the action principle. The time evolution of the state vector, bound to the worldline of a specific observer, and the space-time dependence of the quantum fields are in principle independent, although the generator of the state-vector time evolution is a field that will appear in the equation of motion of the quantum fields. Since the action and the time evolution equation are formulated in a gauge-covariant way, the local quantum relativity principle is fulfilled.

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[1]We do not consider the problem of measurements.

The paper closes with a discussion. Some technical remarks are relegated to an appendix.

2 The Geometrical Framework

In this section, we briefly review the geometric formulation of quantum theory. For more details, we refer the reader to Ref. [2]. The appendix contains a short collection of useful relations and other technical details.

The basic ingredients of the theory are[2]:

(a) A space-time manifold M with metric $g_{\mu\nu}$.

(b) A Hermitian vector bundle $\pi : H \to M$ over M, with the fibres $H_x = \pi^{-1}(x)$ isomorphic to a Hilbert space $\mathcal{H}$. The structure group of the bundle is assumed to be the unitary group $U(\mathcal{H})$ of the Hilbert space, and Hermitian conjugation in a local trivialization is denoted by '$*$'. The metric of the bundle is denoted by G, and Hermitian conjugation with respect to G is denoted by '$\dagger$'.

(c) The bundle $H \to M$ is equipped with a connection D. In a local trivialization, the connection coefficients are denoted by K; they are anti-Hermitian operators

$$K = -K^*. \tag{1}$$

The covariant derivative of a section ψ of the bundle is defined by

$$D\psi = \mathrm{d}\psi - K\psi. \tag{2}$$

The curvature F corresponding to D is $F = D^2$. The covariant derivative of a field A of linear maps of the fibres of $H \to M$ is given by

$$DA = \mathrm{d}A - KA + AK, \tag{3}$$

and similarly the covariant derivative of the metric reads

$$DG = \mathrm{d}G - KG + GK. \tag{4}$$

It is assumed that the connection and the metric are compatible, i.e.

$$DG = 0. \tag{5}$$

The physical interpretation of these mathematical objects and some assumptions being made are:

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[2] In the following, we do not distinguish in the notation between the global, coordinate-independent quantities and their basis-dependent components in, for example, a local trivialization of the vector bundle. What is meant will be clear from the context. Moreover, we do not attempt to fulfil any standards of mathematical rigour; the focus is on the conceptual development. For example, we do not discuss the problems coming from the regularization of operator products. We also sometimes drop technical details. It is, for instance, assumed that local quantities are patched together by a partition of unity, whenever this is required.

(a′) The underlying space-time manifold M together with its metric is assumed to be fixed and given by some other theory. It would certainly be desirable that the dynamical laws governing the evolution of the metric field, i.e. the Einstein equations, could be incorporated in the formalism developed here. This could be achieved, for example, by adding the Einstein–Hilbert action S_{EH} to the quantum action S_Q to be defined later[3].

(b′) For an observer B at $x \in M$ we assume that the state vector ψ that B uses as a description of the world is an element of H_x. Observables, such as fields $\Phi(y)$, are assumed to be sections of the bundle $\pi_{\mathcal{L}H} : \mathcal{L}H \to M$, whose fibres consist of linear operators acting on the fibres of $H \to M$. We use the bracket notation for the inner product given by G; for two vectors ξ, $\eta \in H_x$ and a linear operator $A \in \mathcal{L}H_x$, we write $G(\xi, A\eta) = \langle \xi | A | \eta \rangle$. The quantity[4] $\langle \psi | \Phi(x) | \psi \rangle$, for $\psi \in H_x$, is assumed to be the expectation value of the field Φ at x, i.e. the prediction for the average of a measurement by means of a local measurement device carried by B. Predictions by an observer B at x for a measurement of Φ at y can be made by a parallel transport of the state vector along a path joining x and y (see (c)), and taking the expectation value at y. Requiring path independence of the expectation value, i.e. consistency of predictions, leads to a condition

$$[U_\alpha, \Phi(x)] \, \psi = 0 \tag{6}$$

for state vectors ψ of the physical subspace of $\mathcal{H}_x$, closed loops α attached to x and observables $\Phi(x)$. There is thus a symmetry group (the group of U_α fulfilling Eq. (6)) related to the holonomy group of the bundle; for a discussion see Ref. [2][5].

(c′) The quantum connection D can be integrated along curves $\mathcal{C}$ joining x and y to give parallel transport operators $U_{\mathcal{C}}$ mapping the fibre H_x onto the fibre H_y. The quantum connection D is assumed to govern the evolution of the state vector ψ in the direction of a tangent vector v by means of the equation $D\psi(v) = 0$. For an observer B moving along a worldline $\mathcal{C}(\tau)$, parametrized by B's eigentime τ, the evolution of the state vector $\psi(\tau)$ is thus

$$\partial_\tau \psi(\tau) = K_\mu(\mathcal{C}(\tau))\dot{\mathcal{C}}^\mu(\tau) \, \psi(\tau). \tag{7}$$

This equation is nothing but a Schrödinger equation[6] for a path-dependent Hamilton operator[7] or "quantum gauge field" K_μ. The assumption of D and G being compatible

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[3] Should a genuine quantum theory of gravity be possible, then the theory developed here could certainly no longer be applied, because in the differential geometric formulation we make use of the fact that the manifolds and bundles under consideration are smooth. What would be required in this case would be a geometry of space-time compatible with quantum gravity.

It is not clear *a priori* whether a quantum theory of gravity can be formulated by quantizing some classical action. Conceptually, space-time is the set of all possible events, and the metric, up to a conformal factor, merely encodes the causal structure. Epistemologically, these notions are much more fundamental and deeper than gauge and matter fields. It is possible that gravity should not be quantized at all.

[4] We do not include the space-time point x in the notation for the metric G.

[5] Invariant operators $\breve{A} \in \mathcal{L}H_x$ fulfilling $[U_\alpha, \breve{A}] = 0$ for all closed curves α can be constructed from an arbitrary operator $A \in \mathcal{L}H_x$ by means of a "path integral" $\breve{A} = \int \mathfrak{D}\beta \, U_\beta \, A \, U_\beta^{-1}$ over all closed curves β originating in x, if a left-invariant ($\int \mathfrak{D}\beta \, f(\alpha \circ \beta) = \int \mathfrak{D}\beta \, f(\beta)$) and normalized ($\int \mathfrak{D}\beta = 1$) measure $\mathfrak{D}\beta$ exists.

[6] It is possible to include, for example, a one-form $P(\Phi, D\Phi, F)$, polynomial in the fields Φ, the derivatives $D\Phi$ and the curvature F, in the evolution equation, such that $(D - P)\Psi(v) = 0$. This would correspond to an additional interaction term in the Schrödinger equation.

[7] To simplify the notation and suppress factors of the imaginary unit, we require the operator K to be anti-Hermitian, see Eq. (1).

means that the transition amplitude $\langle \psi(\tau)|\chi(\tau) \rangle$ of two states ψ and χ is invariant under time evolution.

We now have to discuss the question of where the quantum connection D and the dynamical equation for the quantum matter fields Φ come from. It is desirable to have a common principle for these two objects. We note that the matter fields are related to the vector bundle itself, whereas the connection is naturally related to the principal bundle. We therefore need a means to connect objects related to two different bundles. The following possibilities suggest themselves:

(A) If there is a preferred trivialization of the bundle, i.e. a canonical coordinate system, then in this particular system the quantum connection coefficients K can be defined as a function of the matter fields Φ. An example of this is the translation of the standard formulation of quantum field theory in the Heisenberg picture in Minkowski space to the geometric formulation. The bundle $H \to M$ is nothing but the direct product $M \times \mathcal{H}$ of Minkowski space M and the Hilbert space $\mathcal{H}$. There is a canonical trivialization of the bundle owing to the direct product structure. The quantum connection coefficients K are set to zero in this trivialization; D is thus simply the total differential. Consequently, the state vector is constant and the same for all observers. The metric G is inherited from the metric of $\mathcal{H}$. The dynamical law for the fields $\Phi(x)$ is the Heisenberg equation of motion

$$\partial_\mu \Phi(x) = i\,[P_\mu, \Phi(x)], \tag{8}$$

where the P_μ are the energy and momentum operators of the theory, being functions of the fields Φ. The crucial step in this construction is the assumption of a trivial bundle $M \times \mathcal{H}$, because with this a preferred trivialization of the bundle comes for free.

(B) A variant of (A) is to single out a specific coordinate system by some physical principle, the prototype being the definition of inertial frames and the application of the equivalence principle in general relativity. Unfortunately, the application of an equivalence principle based on inertial frames is not possible in our case, because this would only fix a frame in the tangent bundle of M, but not in the bundle $H \to M$.

(C) Finally, there is the possibility to postulate a dynamical law. This is well suited to the problem at hand, because the connection is essentially a differential operator on the vector bundle. This allows us to define covariant differential equations possibly derived from an action principle.

In this paper, we follow (C) by defining a gauge field action for the special case of a complex scalar field Φ. This and the derivation of the dynamical equations is done in the next section.

3 The Quantum Action and the Dynamical Equations

The action employed to derive dynamical equations for the quantum connection D and a complex scalar quantum field Φ is

$$S_Q = S_K + S_G + S_k + S_m, \tag{9}$$

where

$$S_K = \int \mathrm{d}x \, \sqrt{\sigma g}\, \frac{\alpha}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \tag{10}$$

is the action for the quantum connection coefficients,

$$S_G = \int \mathrm{d}x \, \sqrt{\sigma g} \, \mathrm{tr} \left(\lambda^\mu D_\mu G \right) \tag{11}$$

is the action implementing the constraint $DG = 0$ by means of a field of linear-operator-valued Lagrange multipliers λ^μ,

$$S_k = \int \mathrm{d}x \, \sqrt{\sigma g} \, \mathrm{tr} \left((D_\mu \Phi)^\dagger D^\mu \Phi \right) \tag{12}$$

is the action for the kinetic part of the field Φ, and

$$S_m = \int \mathrm{d}x \, \sqrt{\sigma g} \, \gamma \, \mathrm{tr} \left(\Phi^\dagger \Phi \right) \tag{13}$$

is an action reminiscent of a mass term for Φ. Here σ is the sign of the determinant

$$g = \det \left(g_{\mu\nu} \right) \tag{14}$$

of the space-time metric, α and γ are "coupling constants" to be discussed later, and $F_{\mu\nu}$ is the curvature tensor associated with the quantum connection D, defined by

$$F_{\mu\nu} = -\partial_\mu K_\nu + \partial_\nu K_\mu + [K_\mu, K_\nu]. \tag{15}$$

The unusual signs in the first two terms stem from the fact that $F = D^2$, where $D = \mathrm{d} - K$ instead of $D = \mathrm{d} + K$, as is usually assumed. The trace 'tr' is the trace operation of linear operators in a local trivialization of the vector bundle.

It can easily be checked that the action S_Q is invariant under a change of basis in the vector bundle. Owing to the Lagrange multipliers λ^ρ, the variables K_ρ, G, Φ, Φ^* and λ^ρ are independent. The action principle $\delta S_Q = 0$ then leads to the equations of motion by varying the fields[8]. In order to achieve a compact notation, we introduce the covariant derivative $\hat{D}_\mu$ for a vector t^μ and for an antisymmetric tensor $t^{\mu\nu}$ by

$$\hat{D}_\mu t^\mu = \frac{1}{\sqrt{\sigma g}} D_\mu \left(\sqrt{\sigma g} \, t^\mu \right) \tag{16}$$

and

$$\hat{D}_\mu t^{\mu\nu} = \frac{1}{\sqrt{\sigma g}} D_\mu \left(\sqrt{\sigma g} \, t^{\mu\nu} \right), \tag{17}$$

respectively. It can be shown that for vanishing torsion these expressions are covariant divergences, and transform as a scalar and as a vector, respectively.

(α) The variation of the quantum connection coefficients K_ρ leads to[9]

$$\frac{\alpha}{2} \hat{D}_\mu F^{\mu\rho} + [\lambda^\rho, G] - G \left[\Phi^\dagger, D^\rho \Phi \right] G^{-1} - \left[\Phi, (D^\rho \Phi)^\dagger \right] = 0. \tag{18}$$

In a classical gauge theory, if the vector bundle is finite-dimensional, this is the equation of motion for a gauge field coupled to a matrix-valued complex scalar field in the fundamental

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[8]The variations $i \, \delta K_\rho$, δG, $\delta \Phi$, $\delta \Phi^*$ and $\delta \lambda^\rho$ run through all infinitesimal Hermitian operators δR, so that the condition $\mathrm{tr} \left(A \, \delta R \right) = 0$ for all δR leads to $A = 0$.

[9] This is an equation for the covariant derivative $D * F$ of the dual curvature tensor $*F$. The explicit form of $D * F$ is given by Eq. (3). There is, of course, also the Bianchi identity $DF = 0$.

representation. In our case, the gauge field is related to the quantum connection D. It should be noted that this equation is different from the one obtained when quantizing, for example, a classical $SU(n)$ gauge field in conjunction with a matrix-valued matter field in the fundamental representation. In this case, for the gauge field, we would have operators $A_{\mu a}$, where a is a colour index in the adjoint representation. The matter field $\Phi^b{}_c$ would come with colour indices b and c in the fundamental representation. In Eq. (18), the operators do not carry an explicit colour index, rather the "colour indices" are the indices of the infinite-dimensional matrices, if the equations were written out in a specific Hilbert space basis.

(β) The variation of the metric G results in

$$\left(\hat{D}_\mu \lambda^\mu\right) G + \left[(D_\mu \Phi)^\dagger, D^\mu \Phi\right] + \gamma \left[\Phi^\dagger, \Phi\right] = 0. \tag{19}$$

The solution of this equation yields the Lagrange multiplier λ, eventually to be inserted into the other equations.

(γ) The variation of the complex scalar field Φ leads to

$$\hat{D}_\mu D^\mu \Phi - \gamma \, \Phi = 0 \tag{20}$$

and

$$\hat{D}_\mu \left(D^\mu \Phi\right)^\dagger - \gamma \, \Phi^\dagger = 0. \tag{21}$$

To discuss these equations, let us set the quantum gauge field in the covariant derivative to zero[10]. This can be achieved by the limit $\alpha \to \infty$, for the following reason. Defining $\alpha = 1/a^2$ and $\tilde{K}_\mu = K_\mu/a$ allows the coefficient α to be absorbed into the curvature tensor $\tilde{F}_{\mu\nu}$ of $\tilde{K}_\mu$, where the commutator term in $\tilde{F}_{\mu\nu}$ receives a factor of a. The covariant derivative is $D = \mathrm{d} - a\tilde{K}$. Setting $a = 0$ leads to the desired result. Equation (20) then reduces to

$$(\Box - \gamma)\Phi = 0, \tag{22}$$

with

$$\Box \, \Phi = \frac{1}{\sqrt{\sigma g}} \, \partial_\mu \left(\sqrt{\sigma g} \, g^{\mu\nu} \partial_\nu \Phi\right) \tag{23}$$

the wave operator on the space-time manifold M. Defining $\gamma = -m^2$, Eq. (22) is the Klein–Gordon equation for a scalar quantum field of mass m in a curved background space-time [1]. Moreover, for $a = 0$ the state vector is constant along the worldline of the observer. We are thus able to recover standard quantum field theory in curved background space-times in a certain limit.

(δ) Finally, the variation of the Lagrange multipliers λ^ρ yields the constraint that the metric in the vector bundle be consistent with the quantum connection:

$$D_\rho G = 0. \tag{24}$$

As can easily be seen, the equations of motion are all explicitly gauge covariant.

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[10]This corresponds to $F_{\mu\nu} = 0$.

4 Discussion

In this paper, we have proposed dynamical equations for the geometrical formulation of quantum field theory as defined in Ref. [2]. A classical gauge field action for a complex scalar field in an infinite-dimensional vector bundle[11] gives rise to gauge covariant equations of motion for the quantum connection and for the scalar field. A gauge transformation can be interpreted as a change of frame in the vector bundle, and thus as a space-time-dependent change of the "picture".

We wish to point out a similarity of the present theory to the quantum mechanics of a single particle coupled to an electromagnetic field. There, the requirement that the phase of the wave function have no physical meaning motivates the introduction of an Abelian gauge field, which can then be interpreted as the electromagnetic gauge potential. The Schrödinger wave function is in general a section of a complex line bundle. Unobservability of the phase can be rephrased as the independence of physics of the particular choice of basis in the line bundle, admitting arbitrary passive space-time-dependent $U(1)$ transformations. In our case, the situation is slightly different. We are not concerned with the quantum mechanics of a single particle, but with quantum field theory, where, in the geometrical formulation, the space-time dependence relates to the full state vector and not only to the amplitude at a specific space-time point. The quantum relativity principle states that physics be independent of the choice of basis in the Hilbert space, for all possible observers. Instead of the independence of physics of the phase of the Schrödinger wave function, we require that physics be invariant under arbitrary local $U(\mathcal{H})$ transformations. Since the Abelian gauge potential in the quantum mechanics case actually relates to an empirically observable field, it is tempting to speculate whether the quantum gauge connection has some counterpart in physical reality as well. The fact that a dynamical formulation involving the quantum connection coefficients, as done in this paper, is possible, and consequently a resulting set of coupled equations of motion of the quantum connection coefficients and the matter fields can be derived, suggests that quanta of the matter fields can, by quantum fluctuations, be transformed into (hypothetical) quanta of the quantum connection.

The gauge theory structure of the geometrical formulation naturally leads to some additional questions:

▷ Are there conserved Noether currents, and if so, how should these be interpreted?

▷ In the present context, "gauge fixing" means the choice of a particular local trivialization of the vector bundle. Locally, the quantum gauge field K_μ can be transformed to zero if and only if the curvature $F_{\mu\nu}$ vanishes. In general, this is not the case. However, for an observer B, it is always possible to choose a specific frame such that $K_\mu = 0$ on the restriction of the bundle to the worldline. This corresponds to a Heisenberg picture for B, since the state vector will be constant. In general, an additional condition $\partial_\mu K_\nu = 0$ along the worldline cannot be achieved. Locally, therefore, the quantum gauge field cannot be transformed away, and this raises the question of its physical significance.

Another set of questions relates to quantum field theory aspects:

▷ What are suitable initial conditions for the equations of motion?

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[11]To be more precise, for a complex scalar field in the bundle of linear operators acting on the fibres of a vector bundle.

▷ How should locality of fields be defined in the geometrical formulation?

▷ Is it possible to define a vacuum state? Is the vacuum state dependent on the state of motion of the observer (for example, does the Unruh effect lead to a different vacuum state for an accelerated observer)?

▷ Is it possible mathematically to make sense out of the theory; for example, can a perturbative expansion in the coupling constant a of the quantum connection be derived?

▷ Is there a way to introduce self-interacting scalar fields, fermions and "ordinary" gauge fields in the quantum action?

We close the discussion with a remark concerning a "global Hamiltonian". In order to recover Heisenberg type equations of motion for the quantum gauge field and for the matter fields, we need a "global Hilbert space". The bundle itself can be considered to be such an object. Elements of this space are bundle sections ξ, and a global metric can be defined by

$$\mathcal{G}\left(\xi, \eta\right) = \int \mathrm{d}x\, \sqrt{\sigma g}\, G\left(\xi(x), \eta(x)\right). \tag{25}$$

An interesting problem would be to find a global operator R, mapping sections of the vector bundle into vector-valued one-forms, such that an equation of the type

$$D\varphi = i\left[R, \varphi\right] \tag{26}$$

hold for all fields φ of the theory, including the quantum gauge field, under the assumption of suitable commutation relations.

Acknowledgements

Appendix

In this appendix, we collect some technical details related to differential geometry. General references are [3] and [4].

In a local trivialization of the vector bundle, the Hermitian conjugate $A^\dagger$ of a linear operator A with respect to the metric G is given by

$$A^\dagger = G^{-1} A^* G. \tag{27}$$

A change of frame

$$\xi' = S\, \xi \tag{28}$$

in the vector bundle, S being a unitary operator, leads to a transformation of linear operators of the form

$$A' = S\, A\, S^{-1} \tag{29}$$

and to a transformed connection $D' = \mathrm{d} - K'$ with

$$K' = S\,K\,S^{-1} - \mathrm{d}S\,S^{-1}. \tag{30}$$

The curvature $F_{\mu\nu}$ transforms as

$$F'_{\mu\nu} = S\,F_{\mu\nu}\,S^{-1}. \tag{31}$$

A variation of $\mathrm{tr}\left(A^\dagger B\right)$ with respect to the metric G, keeping A^* and B fixed, leads to

$$\delta_G\,\mathrm{tr}\left(A^\dagger B\right) = \mathrm{tr}\left(-\left[A^\dagger, B\right]G^{-1}\delta G\right). \tag{32}$$

This expression is useful for the derivation of the equations of motion.

References

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