CERN-TH/96-86
hep-ph/9604351

THE RESUMMATION OF SOFT GLUONS IN HADRONIC COLLISIONS

Stefano CATANI[1]
INFN, Sezione di Firenze and Univ. di Firenze, Florence, Italy

Michelangelo L. MANGANO[2] and **Paolo NASON**[3]
CERN, TH Division, Geneva, Switzerland

Luca TRENTADUE[1]
Univ. di Parma and INFN, Gruppo Collegato di Parma, Parma, Italy

Abstract

We compute the effects of soft gluon resummation for the production of high mass systems in hadronic collisions. We carefully analyse the growth of the perturbative expansion coefficients of the resummation formula. We propose an expression consistent with the known leading and next-to-leading resummation results, in which the coefficients grow much less than factorially. We apply our formula to Drell–Yan pair production, heavy flavour production, and the production of high invariant mass jet pairs in hadronic collisions. We find that, with our formula, resummation effects become important only fairly close to the threshold region. In the case of heavy flavour production we find that resummation effects are small in the experimental configurations of practical interest.

CERN-TH/96-86
March 1996

[1]Research supported in part by the EEC programme "Human Capital and Mobility", Network "Physics at High Energy Colliders", contract CHRX-CT93-0357(DG 12 COMA).

[2]On leave of absence from INFN, Pisa, Italy.

[3]On leave of absence from INFN, Milan, Italy.

1 Introduction

In this work we deal with the problem of the resummation of logarithmically enhanced effects in the vicinity of the threshold region in hard hadroproduction processes. Drell–Yan lepton pair production has been in the past the best studied example of this sort [1, 2, 3]. The threshold region is reached when the invariant mass of the lepton pair approaches the total available energy. A large amount of theoretical and phenomenological work has been done on this subject. References [4] and [5] summarize all the theoretical progress performed in this field. Resummation formulae have also been used in estimating heavy flavour production [6], [7]. In this case only a leading logarithmic resummation formula is known. Calculations of the next-to-leading logarithms are in progress [8].

In the present work, we will mostly be concerned with difficulties that arise when one tries to apply resummed formulae to physical processes. This is a highly nontrivial problem. What one typically finds is that resummation involves the integration of the running coupling over the Landau pole, which has to be regulated. In early works on resummation in the Drell–Yan cross section, the problem was avoided by either assuming a fixed coupling constant (Curci and Greco in ref. [1]), or by shifting the argument of α_S so as to move the position of the Landau pole to $Q^2 = 0$ (Parisi and Petronzio in ref. [1]). In ref. [9] a cut-off procedure was introduced in order to regulate these singularities. A similar approach was used in ref. [6], in the context of heavy flavour hadroproduction. In refs. [10] and [11] a principal value prescription was adopted instead, and an application to top production was proposed in ref. [7]. It was generally found that threshold corrections become quite large, long before the hadronic threshold region is reached.

Lately, the problem of the presence of an integration over the Landau pole in resummation formulae has been reexamined from the point of view of the occurrence of infrared renormalons (IR) in the QCD perturbative expansion for the Drell–Yan process [10, 12, 13]. Roughly speaking, the IR point of view relates the factorial growth of the coefficients of the perturbative expansion to the presence of power-suppressed corrections to the process in question. It is found that the ambiguities associated to the resummation of the non-convergent (asymptotic) perturbative expansion have precisely the form of a power-suppressed correction. In this context the separation of perturbative and non-perturbative effects is at best ambiguous, since it relies on a specific prescription needed to resum an asymptotic expansion. In refs. [10, 12, 13] it was argued that the Landau pole integration in the resummation formulae for threshold corrections gives the leading IR behaviour of the perturbative expansion of the Drell–Yan cross section, and that the associated factorial growth is the one corresponding to a $1/Q$ ambiguity. A recent work of Beneke and Braun [14] has however demonstrated that the approximations made in the resummation formulae for the logarithmic corrections are insufficient to correctly describe the IR structure, and that when higher order contributions are properly included no factorially growing terms associated to a $1/Q$ ambiguity do arise in the Drell–Yan cross section. This result was subsequently confirmed in refs. [15] and [16]. In the latter reference, the absence of $1/Q$ effects was shown to be a consequence of cancellations related to the Kinoshita-Lee-Nauenberg theorem.

The IR point of view teaches us a very important lesson to keep in mind: it is not enough to make sure that all the leading corrections are properly included in the perturbative expansion. We should also make sure that formally subleading terms, which are not controlled in our approximation, will not affect the asymptotic property of the expansion. In fact, if formally subleading terms happen to have a strong factorial growth, they may induce large corrections even in kinematic regions where our resummation is not justified.

In the present work we find that besides the IR problem, other, more important sources of factorial growth may be introduced, which are spurious and are by no means implied by the threshold approximation, since they are not enhanced at threshold. These large terms arise when one attempts to formulate the resummation problem in x-space, as opposed to its natural formulation in moment (or N) space. To be specific, let us consider the case of the Drell–Yan pair production. The Drell–Yan cross section can be written in x space (schematically) as

$$\sigma^{(\mathrm{DY})}(\tau) = \int_0^1 dx_1\, dx_2\, dx\; F(x_1)\, F(x_2)\; \hat{\sigma}^{(\mathrm{DY})}(x)\delta(x_1\, x_2\, x - \tau)\ . \tag{1.1}$$

In moment space a simple factorized expression follows

$$\sigma_N^{(\mathrm{DY})} = F_N\; F_N\; \hat{\sigma}_N^{(\mathrm{DY})}\ , \tag{1.2}$$

where

$$\begin{aligned} F_N &= \int_0^1 \frac{dx}{x}\, x^N\; F(x) \\ \sigma_N^{(\mathrm{DY})} &= \int_0^1 \frac{d\tau}{\tau}\, \tau^N\; \sigma^{(\mathrm{DY})}(\tau) \end{aligned}$$

$$\hat{\sigma}_N^{(\mathrm{DY})} \;=\; \int_0^1 \frac{dx}{x}\, x^N\, \hat{\sigma}^{(\mathrm{DY})}(x)\;. \tag{1.3}$$

The threshold region $\tau \to 1$ corresponds to the limit $N \to \infty$ in N-space. In this limit, soft-gluon radiation produces large logarithmic corrections of the type $\alpha_S^n \ln^m N$ that are resummed in the partonic cross section $\hat{\sigma}$. Resummation of soft-gluon effects is best expressed in N-moment space, because it leads to the exponentiation of the logarithmic corrections. Exponentiation is a consequence of dynamics and kinematics factorization. By dynamics we mean factorization of multigluon QCD amplitudes to logarithmic accuracy. By kinematics we mean factorization of the phase space: the constraint of longitudinal-momentum conservation factorizes in N-moment space.

However, the moment space formula can be turned to an x-space formula. We will see that with this transformation, by neglecting certain subleading terms, one may generate large, factorially growing corrections, which may be wrongfully attributed to the original resummation formula. In fact these subleading terms are there to compensate for the fact that exponentiation is imperfect in the x-space formulation, and should not be dropped. If they are neglected, kinematic constraints that were satisfied in the original formulae (e.g. momentum sum rules) are strongly violated, and actually diverge, in the x-space expression. A typical consequence of this procedure is that the final formula for the physical cross section receives large soft gluon corrections (actually, divergent ones) even if we are far from the threshold region. With these large factorially growing terms, a corresponding power corrections of the order of $(\Lambda_{\mathrm{QCD}}/Q)^\delta$ can be associated, where δ can be much less than 1. The usual "common sense" assumption that power corrections arise from regions of phase space where the momenta are of order Λ_{QCD} seems therefore to fail, and one is forced to use a cut-off of several GeV in order to make any sense out of the resulting formulae. We argue that all these paradoxes are simply avoided if the transformation to x-space is performed exactly.

In the present work we provide a specific prescription for the resummation of soft gluon effects that does not have any factorially growing terms in its perturbative expansion. The ambiguity associated with the perturbative expansion of our formula is therefore free of any $1/Q^M$ effects for any $M \geq 1$, and in fact turns out to be of the form $e^{-C\,Q(1-\tau)/\Lambda}$, where τ is the ratio of the squared invariant mass Q^2 of the produced heavy system and of the total CM energy squared, C is a slowly varying positive function, and Λ is the QCD scale parameter.

The fact that the ambiguities introduced by the Landau pole only arise for values of

$1-\tau$ of the order of Λ/Q is consistent with our expectation that a correct resummation of soft logarithms should allow control over the perturbative expansion down to the scale at which the coupling constant blows up, namely Λ. A successful resummation program, in fact, should work regardless of the size of the logarithmic terms being resummed, provided one can prove that the neglected terms are sufficiently small. The non-perturbative regime is not defined by the region of momenta in which higher order terms are larger than lower order ones; it is defined by the domain in which the terms which are neglected by the resummation procedure are comparable in size with those taken into account. The universality of soft gluon emission should allow full control to be maintained over the dominant contributions to the perturbative expansion even when they become large, and to correctly resum them regardless of their size. Our approach to resummation shows that this is indeed possible, confining the effects of the really non-perturbative regime of QCD to their natural scale, namely Λ.

The paper is organized as follows. In section 2 we will give a few reference formulae and establish our notation. We will mainly deal with the Drell–Yan case as an illustrative example. In section 3 we will show how large spurious corrections may arise in the computation of the resummed Drell–Yan cross sections. In section 4 we will propose an alternative resummation method, and in section 5 we discuss its implications for Drell–Yan cross sections. In sections 6 and 7 we discuss heavy flavour production, and in section 8 jet production at large transverse momenta. In section 9 we discuss a few remaining issues, and in section 10 we give our conclusions. In Appendix A we prove the absence of factorially growing terms in our resummation formula, in Appendix B we discuss some details about the numerical method we used, and in Appendix C we derive some results regarding the inverse Mellin transform in the leading logarithmic approximation.

2 Basic formulae and notation

We begin with the formula for Drell–Yan pair production

$$\sigma(\tau, Q^2) = \int_0^1 dx\, dx_1\, dx_2 F(x_1) F(x_2) \delta(x x_1 x_2 - \tau) \Delta(x, Q^2)\,. \tag{2.1}$$

At the Born level, omitting obvious factors, we have $\Delta(x, Q^2) = \delta(1-x)$. We also omit, for ease of notation, the parton indices. The cross section is given in terms of the parton densities $F(x)$ as measured in the deep inelastic processes. Defining the Mellin transform as

$$f_N = \int_0^1 \frac{dz}{z}\, z^N\, f(z) \tag{2.2}$$

we can rewrite eq. (2.1) as

$$\sigma_N(Q^2) = F_N^2(Q^2)\, \Delta_N(Q^2)\,. \tag{2.3}$$

The resummed coefficient function for the Drell–Yan process in the DIS scheme is [1, 4, 5]

$$\begin{aligned} \ln \Delta_N(Q^2) &= -\int_0^1 dx\, \frac{x^N - 1}{1-x} \left[2\int_{(1-x)^2 Q^2}^{(1-x)Q^2} \frac{dq^2}{q^2} A(\alpha_S(q^2)) + B(\alpha_S((1-x)Q^2)) \right] \\ &+ \mathcal{O}(\alpha_S(\alpha_S \ln N)^k) \ , \end{aligned} \tag{2.4}$$

with

$$A(\alpha_S) = \frac{\alpha_S}{\pi} A^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 A^{(2)} \ , \qquad B(\alpha_S) = \frac{\alpha_S}{\pi} B^{(1)} \tag{2.5}$$

where ($C_A = 3\,,\ \ C_F = 4/3\,,\ \ T_R = 1/2$ in QCD)

$$A^{(1)} = C_F \ , \quad A^{(2)} = \frac{1}{2} C_F K \ , \quad B^{(1)} = -\frac{3}{2} C_F \ , \tag{2.6}$$

and the coefficient K is given by [17]

$$K = C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{10}{9} T_R N_f \ . \tag{2.7}$$

Note that, due to the integration of the running coupling, the integral in eq. (2.4) is singular for all values of N. However, if we perform the integration to next-to-leading logarithmic (NLL) accuracy (i.e. we compute the leading $\alpha_S^n \ln^{n+1} N$ and next-to-leading $\alpha_S^n \ln^n N$ terms) [5], we find

$$\ln \Delta_N(Q^2) = \ln N\ g_1(b_0 \alpha_S \ln N) + g_2(b_0 \alpha_S \ln N) + \mathcal{O}(\alpha_S^k \ln^{k-1} N) \tag{2.8}$$

where $\alpha_S = \alpha_S(Q^2)$, b_0 and b_1 are the first two coefficients of the QCD β-function

$$b_0 = \frac{11 C_A - 4 T_R N_f}{12\pi} \ , \qquad b_1 = \frac{17 C_A^2 - 10 C_A T_R N_f - 6 C_F T_R N_f}{24\pi^2} \ , \tag{2.9}$$

and the leading and next-to-leading functions g_1 and g_2 are given by ($\gamma_E = 0.5772\ldots$ is the Euler constant)

$$g_1(\lambda) = + \frac{A^{(1)}}{\pi b_0 \lambda} \Big[(1-2\lambda)\ln(1-2\lambda) - 2(1-\lambda)\ln(1-\lambda) \Big] \ , \tag{2.10}$$

$$g_2(\lambda) = + \frac{A^{(2)}}{\pi^2 b_0^2}\Big[2\ln(1-\lambda) - \ln(1-2\lambda)\Big]$$
$$- \frac{B^{(1)}}{\pi b_0}\ln(1-\lambda) + \frac{2A^{(1)}\gamma_E}{\pi b_0}\Big[\ln(1-\lambda) - \ln(1-2\lambda)\Big] \qquad (2.11)$$
$$+ \frac{A^{(1)}b_1}{\pi b_0^3}\left[\ln(1-2\lambda) - 2\ln(1-\lambda) + \frac{1}{2}\ln^2(1-2\lambda) - \ln^2(1-\lambda)\right] .$$

We observe that the N-space formula eq. (2.8) is finite and uniquely defined up to the very large value of $N = N_L = \exp\frac{1}{2\alpha_S b_0}$, in spite of the fact that it is obtained from an expression which is formally divergent for any value of N. It was shown in ref. [5] that the expression $x^N - 1$ in formula (2.4) can be replaced to NLL accuracy by the theta function $-\theta(1 - x - e^{-\gamma_E}/N)$. The region where the integral is divergent is therefore excluded from the integration if $N < N_L$. This shows that the divergences present in the integral of eq. (2.4) are subleading for large N. They may be cancelled by other divergences of the same nature, neglected by the approximations that lead to formula (2.4). This is indeed the case for the leading $1/Q$ singularity, as shown in ref. [14].

Our result in eq. (2.8) can be expressed in terms of $\alpha_S(\mu^2)$ for an arbitrary value of the renormalization scale μ^2. We thus achieve full control over the *renormalization-scale* dependence. To do that, we must take into account the scale dependence of the next-to-leading function g_2, which is given by

$$g_2(\lambda, \mu^2) = g_2(\lambda, Q^2) + \lambda^2 g_1'(\lambda)\ln(\mu^2/Q^2) , \qquad (2.12)$$

where $g_2(\lambda, Q^2)$ is $g_2(\lambda)$ as defined above and

$$g_1'(\lambda) = -\frac{A^{(1)}}{\pi b_0 \lambda^2}\Big[\ln(1-2\lambda) - 2\ln(1-\lambda)\Big] . \qquad (2.13)$$

Also the *factorization-scheme* dependence is completely under control up to NLL accuracy [18]. In the $\overline{\rm MS}$ scheme the resummed coefficient function can still be expanded as in Eq. (2.8), that is:

$$\ln\Delta_N^{\overline{\rm MS}}(Q^2) = \ln N\; g_1^{\overline{\rm MS}}(b_0\alpha_S\ln N) + g_2^{\overline{\rm MS}}(b_0\alpha_S\ln N) + \mathcal{O}(\alpha_S^k\ln^{k-1}N) \quad , \qquad (2.14)$$

and the leading and next-to-leading functions $g_1^{\overline{\rm MS}}$ and $g_2^{\overline{\rm MS}}$ are given by

$$g_1^{\overline{\rm MS}}(\lambda) = +\frac{A^{(1)}}{\pi b_0 \lambda}\Big[2\lambda + (1-2\lambda)\ln(1-2\lambda)\Big] , \qquad (2.15)$$

$$g_2^{\overline{\rm MS}}(\lambda) = - \frac{A^{(2)}}{\pi^2 b_0^2}\Big[2\lambda + \ln(1-2\lambda)\Big] - \frac{2A^{(1)}\gamma_E}{\pi b_0} \ln(1-2\lambda) + \frac{A^{(1)} b_1}{\pi b_0^3}\left[2\lambda + \ln(1-2\lambda) + \frac{1}{2}\ln^2(1-2\lambda)\right] . \quad (2.16)$$

In this scheme, the renormalization-scale dependence is again taken into account by the function g_2 as in eq. (2.12), and the corresponding function g_1' is:

$$g_1'^{\overline{\rm MS}}(\lambda) = -\frac{A^{(1)}}{\pi b_0 \lambda^2}\Big[\ln(1-2\lambda) + 2\lambda\Big] . \quad (2.17)$$

Sometimes, as an illustration, we will also use the double log approximations (DLA) to the resummation formulae

$$g_1(\lambda) = \frac{A^{(1)}}{b_0 \pi}\lambda + \mathcal{O}(\lambda^2) , \quad (2.18)$$

$$g_1^{\overline{\rm MS}}(\lambda) = \frac{2\,A^{(1)}}{\pi b_0}\,\lambda + \mathcal{O}(\lambda^2) , \quad (2.19)$$

which give rise to the expressions

$$\ln \Delta_N(Q^2) = \frac{A^{(1)}}{\pi}\alpha_S \ln^2 N + \mathcal{O}((\alpha_S \ln N)^{k+1} \ln N) , \quad (2.20)$$

$$\ln \Delta_N^{\overline{\rm MS}}(Q^2) = \frac{2\,A^{(1)}}{\pi}\alpha_S \ln^2 N + \mathcal{O}((\alpha_S \ln N)^{k+1} \ln N) . \quad (2.21)$$

3 Problems with x-space resummation formulae

In this section we discuss the problems that may arise when turning the N-space resummation formula into an x-space formula. We begin by considering the DLA case. No running coupling effect is present in this case. Therefore, there is no Landau pole, and the result of resummation should be finite and free of ambiguities. There is a "realistic" limit that corresponds to this case, which is the limit of large colour factors and small coupling. In the case of gluon initiated processes in the $\overline{\rm MS}$ scheme, this limit is not far from reality. For the sake of definiteness, we also fix the structure function

$$F_N = \frac{6}{N(N+1)(N+2)} \quad (3.1)$$

which corresponds to the x-space structure function

$$F(x) = (1-x)^2 . \quad (3.2)$$

Our cross section formula is

$$\sigma(\tau) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} F_N^2\,\Delta_N\,\tau^{-N}\,dN\;, \tag{3.3}$$

and we now take the simplified soft gluon factor to be given by

$$\Delta_N = \exp(\,a\log^2 N\,)\;, \tag{3.4}$$

where only the double logarithmic term has been kept in Δ_N. The coefficient a is both process and scheme dependent (see Sects. 6 and 8); it is given by $a = C_F\alpha_S/\pi$ and $a = 2C_F\alpha_S/\pi$ for the $q\bar{q}$ initial state in the DIS and $\overline{\mathrm{MS}}$ scheme respectively, and by $a = 2C_A\alpha_S/\pi$ for the gg initial state in the $\overline{\mathrm{MS}}$ scheme. The integral (3.3) is not absolutely convergent for large N, since Δ_N grows faster than any power for large N. Observe, however, that if we expand Δ_N in powers of a, the integral converges order by order in perturbation theory. In the corresponding perturbative expansion, we can therefore deform the integration contour into two straight half-lines from $C-(i+\epsilon)\infty$ to C, and then to $C+(i-\epsilon)\infty$. Once the perturbative expansion is written in this way we realize that it can be resummed into the expression

$$\sigma(\tau) = \frac{1}{2\pi i}\int_{C-(i+\epsilon)\infty}^{C+(i-\epsilon)\infty} F_N^2\,\Delta_N\,\tau^{-N}\,dN\;, \tag{3.5}$$

which is now convergent, and independent of ϵ. In the following, we will always interpret the integral in the above sense, omitting the explicit reference to the ϵ.

Let us now see what happens if we try to rewrite eq. (3.3) as an x-space formula. First of all we need the inverse Mellin transform of Δ_N. We have (see Appendix C and ref. [19])

$$\begin{aligned}\Delta(x) &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \exp(a\log^2 N)\,x^{-N}\,dN \\ &= -\frac{d}{dx}\Big(\theta(1-\eta-x)\exp[a\log^2(1-x)]\Big)\times(1+\mathrm{NLL\ \ terms})\end{aligned} \tag{3.6}$$

where, according to the usual definitions, NLL terms stand for contributions of the form $\alpha_S^k\log^m 1/(1-x)$ with $k\geq 1$ and $m\leq k$. The rôle of the θ function is to define the right-hand side in a distribution sense as $\eta\to 0$, so that we have the correct normalization of the first moment

$$\int_0^1 \Delta(x)\,dx \;=\; 1\;. \tag{3.7}$$

Thus, defining

$$\mathcal{L}(z) = \int_z^1 \frac{dx_2}{x_2} \, F\left(\frac{z}{x_2}\right) \, F(x_2) \tag{3.8}$$

and neglecting NLL terms, we obtain the following expression for the cross section

$$\begin{aligned} \sigma(\tau) &= \int_0^1 dx \, dx_1 \, dx_2 \, F(x_1) \, F(x_2) \delta(x \, x_1 \, x_2 - \tau) \, \Delta(x) \\ &= \int_\tau^1 dx \, \exp[a \log^2(1-x)] \frac{d}{dx} \mathcal{L}\left(\frac{\tau}{x}\right) \,, \end{aligned} \tag{3.9}$$

where an integration by part was performed. We now see that the integral in eq. (3.9) is divergent at $x = 1$ for any value of τ, since the expression $\exp[a \log^2(1-x)]$ diverges faster than any power as $x \to 1$. Let us examine more carefully the origin of this divergence. If we expand eq. (3.9) in powers of a, each term of the expansion is integrable, but the corresponding series is divergent. Since the term $d/dx \; \mathcal{L}(\tau/x)$ is a smooth function of x as $x \to 1$, the nature of the divergence is given by the following integral

$$\int_0^1 \exp[a \log^2(1-x)] \, dx = \sum_{k=0}^{\infty} \frac{a^k}{k!} \int_0^1 \log^{2k} z \, dz = \sum_{k=0}^{\infty} \frac{a^k (2k)!}{k!} \,. \tag{3.10}$$

The asymptotic behaviour of the coefficients for large k is $(2k)!/k! \approx 4^k \, k!$. The expansion is therefore an asymptotic one. Observe that the lower limit of the integral in eq. (3.10) is irrelevant for this conclusion. It is known that factorially growing terms in the perturbative expansion are associated to power-like ambiguities in the resummed expression. In order to resum the asymptotic expansion, we should in fact truncate the series when the next term is of the same size as the current one, i.e. when $4ak = 1$. The error on the resummed expression is then of the order of the left over term

$$\delta = (4a)^k \, k! \approx (4a)^k \, k^k \, e^{-k} = e^{-\frac{1}{4a}} \,. \tag{3.11}$$

If we replace the appropriate value of a we get

$$\delta = \left(\frac{\Lambda}{Q}\right)^{\frac{\pi b_0}{2 C_F}} \qquad q\bar{q}, \quad \text{DIS}, \tag{3.12}$$

$$\delta = \left(\frac{\Lambda}{Q}\right)^{\frac{\pi b_0}{4 C_F}} \qquad q\bar{q}, \quad \overline{\text{MS}}, \tag{3.13}$$

$$\delta = \left(\frac{\Lambda}{Q}\right)^{\frac{\pi b_0}{4 C_A}} \qquad gg, \quad \overline{\text{MS}}. \tag{3.14}$$

Although it is power-suppressed, the smallness of the exponent makes this effect potentially large. Thus, for example, for $n_{\rm f} = 5$ we have for Drell–Yan in the DIS scheme a $(\Lambda/Q)^{0.72}$ power correction, and for heavy flavour production via gluon fusion we have a $(\Lambda/Q)^{0.16}$ correction, which is hardly distinguishable from a correction of order 1.

Instead of truncating the perturbative expansion, we may achieve the same goal by putting a cut-off in the integral. In fact, consider the cut-off integral

$$\int_0^{x_0} dx \, \log^{2k} \frac{1}{1-x} = \int_0^{\log \frac{1}{1-x_0}} dt \, t^{2k} \, e^{-t} \, . \tag{3.15}$$

The saddle point of the integral is at $t = 2k$. If the saddle point is within the integration range, the integral is essentially the factorial of $2k$, while for larger values of k it starts to grow like a simple power. Therefore, the cut-off acts like a truncation of the expansion. In order to have, as before, a truncation at $k = 1/(4a)$, we need to set the cut-off at $\log 1/(1-x_0) = 2k = 1/(2a)$, corresponding to

$$1 - x_0 = e^{-\frac{1}{2a}} \, . \tag{3.16}$$

In the worst case of production via gluon fusion in the $\overline{\rm MS}$ scheme, we would have $1-x_0 = (\Lambda/Q)^{0.32}$. This leads to the rather paradoxical conclusion that the cut-off on the soft radiation should be imposed at values of Q much larger than Λ. For example, for the production of a 100 GeV object we would need a cut-off of the order of 14 GeV. This would have to increase with energy. We observe that cut-offs of this kind are used in ref. [7], in formulae (114) and (115). As a matter of fact, if we take the limit of large colour factor and small coupling of the formulae given there, we recover eq. (3.16).

From the above derivation we see that the large corrections obtained have nothing to do with infrared renormalons, and it is easy to convince ourselves that they are a spurious effect. They were in fact not present in the original expression, eq. (3.3), which is finite. It is also easy to show that the perturbative expansion of eq. (3.3) has no factorially growing terms. This can be done in the following way. The $k^{\rm th}$ coefficient of the expansion is given by the integral

$$c_k = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{36}{(N(N+1)(N+2))^2} \frac{1}{k!} \log^{2k} N \;\; \tau^{-N} \, dN \, . \tag{3.17}$$

We deform the integration contour as illustrated in fig. 1. By choosing N_0 sufficiently

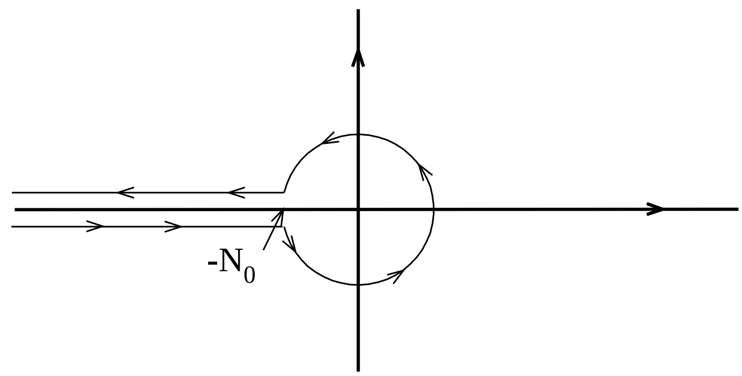

Figure 1: *Integration contour for N for the determination of asymptotic properties.*

large, we get two contributions to our integral, from the circle and from the discontinuity along the negative axis. The integral on the circle is bounded by the expression $C\log^{2k} N_0\ /k!$ for some value of C. The discontinuity integral is instead given by (replacing $N \to -N$)

$$\frac{36}{k!}\int_{N_0}^{\infty} \exp[N\log\tau - 2\log(N(N-1)(N-2))]\ \mathrm{Disc}[\log^{2k}(-N)]\ dN$$
$$\leq \frac{36}{k!}\int_{N_0}^{\infty} \exp[N\log\tau + 2k\log(\pi+\log N)]\ dN\,. \tag{3.18}$$

By saddle point integration of the right hand side we immediately see that the above expression cannot grow faster than

$$\frac{1}{k!}\left(\frac{\log 2k}{\log 1/\tau}\right)^{2k}\,. \tag{3.19}$$

Therefore the power expansion of eq. (3.3) has an infinite radius of convergence, and, *a fortiori*, does not have factorially growing terms.

Apparently, when performing the inverse Mellin transform to obtain the x-space expression of the cross section, we have simply thrown away subleading terms that would have compensated the factorial growth. In a leading-log sense we have every right to throw away subleading terms. However, if by doing so we generate unjustified factorially growing terms, we are certainly doing something wrong on physical ground, even if we are perfectly consistent with the leading-log approximation.

In the case of the full resummation formula, including the effects of the running coupling, the above illustrated problem persists. We have in this case (see Appendix C)

$$\begin{aligned}\Delta(x) &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \exp(\log N\, g_1(\alpha_S b_0 \log N))\; x^{-N}\; dN \\ &= -\frac{d}{dx}\left(\theta(1-\eta-x)\exp[l\, g_1(\alpha_S b_0 l)]\right)\times(1+\text{NLL terms}) \qquad (3.20)\end{aligned}$$

in which an unintegrable singularity is met before x reaches 1, the Landau pole singularity. In the corresponding formula for the partonic cross section, neglecting NLL terms, we get

$$\sigma(\tau) = \int_\tau^1 dx\; \exp[l\, g_1(\alpha_S b_0 l)]\frac{d}{dx}\mathcal{L}\left(\frac{\tau}{x}\right)\;. \qquad (3.21)$$

Expanding the above formula in powers of α_S, we would generate the same type of spurious factorial growth as found before. As before, the nature of the divergence is given by the integral

$$\int_0^1 dx\; \exp[l\, g_1(\alpha_S b_0 l)]\;. \qquad (3.22)$$

Using commonly available algebraic programs, it is easy to expand eq. (3.22) up to large orders, and then study numerically the factorial growth. Expanding up to α_S^{32} we have found the behaviour $k!C_{(k)}^k(b_0\alpha_S)^k$, where $C_{(k)}$ is a slowly increasing function of k. If, for large k, $C_{(k)}$ approaches a limiting value C, this corresponds to a power ambiguity of $(\Lambda/Q)^{2/C}$. For gluon fusion in the $\overline{\mathrm{MS}}$ scheme we get $C_{(32)} = 10.48$, corresponding to a power ambiguity of $(\Lambda/Q)^{0.19}$, while for $q\bar{q}$ annihilation in the $\overline{\mathrm{MS}}$ scheme we get $C_{(32)} = 4.0$, corresponding to $(\Lambda/Q)^{0.5}$. These numbers are roughly consistent with those of the exact analysis performed for the fixed coupling case.

We observe that, even if we modify the above x-space formula, by expanding it in powers of α_S and keeping only a fixed number of terms, the problem discussed earlier still persists. In fact, our discussion is relative to the case in which the exponent in formula (3.20) is expanded and truncated to order α_S.

An x-space resummation procedure, similar to the one discussed here, has indeed been adopted in the literature. In refs. [6, 20] a cut-off procedure is applied to screen the Landau pole singularity that manifests itself when x approaches 1. This cuts off both the divergence due to the Landau pole, and the spurious divergence we just described. We therefore argue that the uncertainties induced by this procedure are much larger than needed, since they introduce a divergence that is in fact not present. In ref. [7], the Landau pole singularity is dealt with by using a principal value prescription. Subsequently, a cut-off is introduced in order to screen the large subleading effects that arise when performing the Mellin transform of the partonic cross section from N to x space. Our point is precisely that if these large subleading terms had been kept, they would have cancelled the factorially growing terms arising from the leading terms after integration against the parton luminosities. In all these approaches, unphysically large cutoffs are needed in order to avoid the large corrections that arise at higher order in the perturbative expansion.

We conclude that, when proposing a resummation formula for threshold effects, it is not enough to make sure that all leading terms are included in the formula. We must also make sure that we are not introducing subleading terms that grow very fast with the order of the perturbative expansion in the final physical result. In the next section we propose a resummation formula that is correct from the point of view of the threshold approximation, but does not induce any factorial growth in the perturbative expansion.

4 The Minimal Prescription formula

Our starting point is eq. (2.4). Its perturbative expansion has the form

$$\Delta_N = \sum_{k=0}^{\infty} c_k(\log N)\, \alpha_S^k \tag{4.1}$$

where the coefficients $c_k(\log N)$ are polynomials in $\log N$. The resummed cross section can be formally written as a power expansion

$$\sigma_\tau = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \alpha_S^k \int_{C-i\infty}^{C+i\infty} F_N^2(Q^2)\, c_k(\log N)\, \tau^{-N}\, dN\ . \tag{4.2}$$

Observe that the integrals in the coefficients of the expansion are finite for $C > 2$, and no singularities occur at the right of the integration contour. The choice of $C > 2$ is motivated by the usual Regge behaviour of structure functions, which implies that we cannot have any singularity in $F_N(Q^2)$ to the right of the pomeron singularity, which is slightly above $N = 1$. The Landau pole for $N = \exp \frac{1}{2\alpha_S b_0}$ manifests itself in the fact that the series (4.2) is not convergent.

We now propose the following formula for the resummation of threshold effects in the Drell–Yan cross section

$$\sigma_{\rm res}(\tau) = \frac{1}{2\pi i} \int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty} F_N^2(Q^2)\, \Delta_N(Q^2)\, \tau^{-N}\, dN \;, \quad 2 < C_{\rm MP} < N_L \equiv \exp \frac{1}{2\alpha_S b_0} \;, \tag{4.3}$$

where $\Delta_N(Q^2)$ is given in eq. (2.8). The constant $C_{\rm MP}$ is chosen in such a way that all singularities in the integrand are to the left of the integration contour, except for the Landau singularities at $N = N_L$ and $N = N_L^2$, which lie to the far right. We will call eq. (4.3) the "Minimal Prescription" (MP) in the following. Its justification relies on the following important properties, which will be proved in Appendix A:

- The expansion (4.2) converges asymptotically to the MP formula. Observe that this would not happen if we had chosen a contour that passes to the right of the first Landau pole.
- The coefficients of the expansion (4.2) do not grow factorially.
- If we truncate the expansion (4.2) at the order at which its terms are at a minimum, the difference between the truncated expansion and the full MP formula is suppressed by a factor

  $$e^{-C\frac{Q(1-\tau)}{\Lambda}} \,, \tag{4.4}$$

  where C is a slowly varying positive function. This suppression factor is stronger than any power suppression.

We stress that with our MP formula we do not introduce any spurious factorial growth in the perturbative expansion. One may object that in this way we do not introduce any possible renormalon effect in the formula. Factorial growth due to renormalons is very likely to be present in the perturbative expansion. Our point is, however, that the leading terms in our expansion do not necessarily contain this factorial growth, and that renormalons present in a resummed expression therefore do not necessarily reflect the renormalons present in the full perturbative expansion. To be more specific, let us consider for a moment eq. (2.4). It is clear that, if we perform the x integration exactly, we are indeed integrating over the Landau pole. However, since this formula is accurate at the NLL level at most, we may integrate it in the NLL approximation, and obtain formula (2.8), which has no trace of factorial growth. In particular, it was shown in ref. [14] that the leading IR arising from naively extending Eq. (2.4) beyond the NLL level cancel in the full perturbative expansion. We take this result as a confirmation of the fact that the resummation of logarithmic effects at threshold does not teach us anything about the structure of power corrections. Resummation formulae should not, therefore, include any power correction.

5 The Drell–Yan cross section

We will not attempt to perform a detailed phenomenological analysis of resummation effects in Drell–Yan pair production in the present work. We will, however, assess the effect of resummation in the particularly simple example of the structure functions reported in Appendix B of ref. [9], for proton-antiproton collisions, and we will compare our results with the ones given there. In fig. 2 we report the cross section as a function of $\tau = Q^2/S$, normalized to the Born cross section. With this set of structure functions, which are Q^2-independent, the K-factor depends only upon the ratio Q^2/Λ^2. The dashed curve is the NL cross section, including both the $q\bar{q}$ and qg

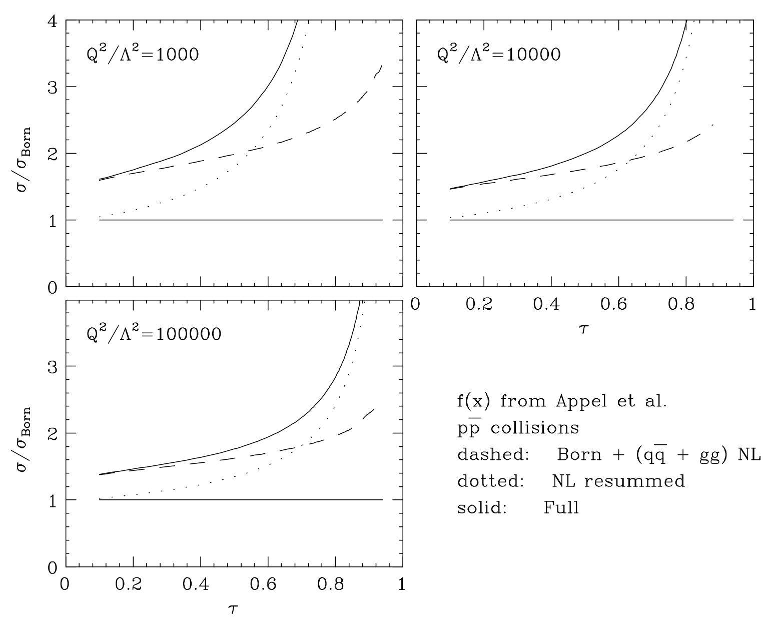

Figure 2: *Drell–Yan pair production cross section in $p\bar{p}$ collisions, normalized to the Born result.*

subprocesses (given in ref. [2]). The dotted curve is the NL-resummed cross section, without the inclusion of the exact $\mathcal{O}(\alpha_S)$ result. The full curve is obtained by adding to the NL cross section the NL-resummed contributions, after having subtracted the terms up to $\mathcal{O}(\alpha_S)$, that is to say

$$\begin{aligned}\sigma(\tau) &= \sigma^{(q\bar{q})}_{\rm Born}(\tau) + \frac{\alpha_S}{2\pi}\left[\sigma^{(q\bar{q})}_1(\tau) + \sigma^{(qg)}_1(\tau)\right] \\ &+ \left[\sigma_{\rm res}(\tau) - \sigma_{\rm res}(\tau)\Big|_{\alpha_S=0} - \alpha_S \frac{\partial}{\partial\alpha_S}\sigma_{\rm res}(\tau)\Big|_{\alpha_S=0}\right] . \qquad (5.1)\end{aligned}$$

Our result is consistent with the result of ref. [9], where analogous figures are given. There is however one important difference. The cut-off method used there to overcome the problem of the Landau singularity introduces an extra uncertainty, which is given by the spread of the various curves obtained using different cut-offs. This spread decreases roughly as $1/Q$ as Q^2 increases. Accounting for the fact that in ref. [9] also the so-called π^2 terms are exponentiated, our result is consistent with their band. It does not, however, agree with the central value, which is smaller in our case. Furthermore, even at the lowest energy, and for the very large value of $\tau = 0.5$, the fully resummed cross section is only 10% larger than the next-to-leading one.

6 Heavy flavour production

We will follow closely the notation of ref. [21]. The heavy flavour production cross section is given by the formula

$$\sigma = \int_0^1 dx_1\, dx_2 F(x_1)F(x_2)\, \hat{\sigma}\left(\frac{\rho}{x_1\, x_2}\right) , \quad \rho = \frac{4m^2}{S} , \qquad (6.1)$$

where m is the mass of the heavy quark and S is the square of the total centre-of-mass energy. As before, for notational convenience, we have dropped here the parton indices. The scale dependence is also not shown explicitly. The partonic cross section depends also upon the heavy quark mass. Here we indicate explicitly only its dependence upon ρ, which embodies its dependence upon the partonic centre-of-mass energy squared $s = x_1\, x_2\, S$. In order to include the effects of soft radiation we parallel as closely as possible the approach followed in the Drell–Yan case. The leading logarithmic (LL) soft corrections in heavy flavour production have the same structure as in the Drell–Yan case [21, 6], since there are no collinear singularities arising from final state gluon radiation. We define

$$\begin{aligned}\frac{d\sigma}{d\tau} &= \int_0^1 dx_1\, dx_2 F(x_1)F(x_2)\, \delta(\tau - x_1\, x_2\,)\hat{\sigma}\left(\frac{\rho}{\tau}\right) \\ &= \hat{\sigma}\left(\frac{\rho}{\tau}\right)\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} F_N^2\, \tau^{-N}\, dN . \qquad (6.2)\end{aligned}$$

The inclusion of soft effects can now be performed as in the Drell–Yan case

$$\frac{d\sigma^{(\rm res)}}{d\tau} = \hat{\sigma}\left(\frac{\rho}{\tau}\right) \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} F_N^2 \, \Delta_N^{HF} \, \tau^{-N} \, dN \; . \tag{6.3}$$

Using now the identity

$$\sigma = \int_\rho^\infty d\tau \, \frac{d\sigma}{d\tau} \tag{6.4}$$

and defining as usual

$$\hat{\sigma}_N = \int_0^1 \frac{dz}{z} \, z^N \, \hat{\sigma}(z) \tag{6.5}$$

we get immediately

$$\sigma^{(\rm res)} = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} F_N^2 \, \Delta_N^{HF} \, \hat{\sigma}_{N-1} \, \rho^{-(N-1)} \, dN \; . \tag{6.6}$$

The partonic cross section (after subtraction of collinear singularities) is

$$\hat{\sigma}_{ij}(s, m^2, \mu^2) \equiv \frac{\alpha_S^2(\mu^2)}{m^2} \, f_{ij}(\rho, \mu^2/m^2) \;\; , \tag{6.7}$$

where μ is the factorization scale (the renormalization scale is set equal to the factorization scale) and the dimensionless variable ρ is

$$\rho = \frac{4m^2}{s} \quad . \tag{6.8}$$

The functions f_{ij} have the following perturbative expansion

$$f_{ij}(\rho, \mu^2/m^2) = f_{ij}^{(0)}(\rho) + g_S^2(\mu^2) \left[f_{ij}^{(1)}(\rho) + \overline{f}_{ij}^{(1)}(\rho) \ln \frac{\mu^2}{m^2} \right] \quad . \tag{6.9}$$

The lowest-order terms in Eq. (6.9) are explicitly given by ($\beta \equiv \sqrt{1-\rho}$)

$$f_{q\bar{q}}^{(0)}(\rho) = \frac{\pi}{6} \frac{T_R C_F}{N_c} \, \beta\rho \, (2+\rho) \;\; , \tag{6.10}$$

$$\begin{aligned} f_{gg}^{(0)}(\rho) &= \frac{\pi}{12} \frac{T_R}{N_c^2 - 1} \, \beta\rho \left\{ 3C_F \left[(4 + 4\rho - 2\rho^2) \, \frac{1}{\beta} \, \ln \frac{1+\beta}{1-\beta} - 4 - 4\rho \right] \right. \\ &+ \left. C_A \left[3\rho^2 \, \frac{1}{\beta} \, \ln \frac{1+\beta}{1-\beta} - 4 - 5\rho \right] \right\} \;\; , \end{aligned} \tag{6.11}$$

and $f_{ij}^{(0)}(\rho) = 0$ for all the other parton channels.

The N-moments of the expressions in Eqs. (6.10) and (6.11) are as follows

$$f^{(0)}_{q\bar{q},\,N} = \frac{\pi^{\frac{3}{2}}}{4} \frac{T_R C_F}{N_c} \frac{\Gamma(N+1)}{\Gamma(N+7/2)} (N+2) \;, \tag{6.12}$$

$$\begin{aligned} f^{(0)}_{gg,\,N} &= \frac{\pi^{\frac{3}{2}}}{4} \frac{T_R}{N_c^2-1} \frac{\Gamma(N+1)}{\Gamma(N+5/2)} \frac{1}{N+3} \\ &\times \left[2C_F \frac{N^3+9N^2+20N+14}{(N+1)(N+2)} - C_A \frac{N^2+8N+11}{2N+5} \right] \;. \end{aligned} \tag{6.13}$$

The resummation effects are embodied in the factor Δ_N. While in the Drell–Yan case this resummation factor is only associated with the $q\bar{q}$ subprocess, in the heavy flavour case both $q\bar{q}$ and gg subprocesses are involved. The explicit expression of Δ_N will therefore depend upon the subprocess. To *leading* logarithmic accuracy we have ($\alpha_S \equiv \alpha_S(m^2)$)

$$\ln \Delta^{HF}_{ij,\,N}(m^2) = \ln N \; g_{ij,\,1}(b_0 \alpha_S \ln N) + \mathcal{O}(\alpha_S^k \ln^k N) \;, \tag{6.14}$$

where the functions $g_{ij,\,1}$ are related to the function g_1 in Eq. (2.10) by simple colour factors. More precisely we have

$$g_{q\bar{q},\,1}(\lambda) = g_1(\lambda) \;, \quad g_{gg,\,1}(\lambda) = \frac{C_A}{C_F} \, g_1(\lambda) \;. \tag{6.15}$$

The factorization scheme dependence is of course contained in g_1. We will therefore take eq. (6.6) with $C = C_{\rm MP}$ as our MP formula for heavy flavour resummed cross sections.

7 Heavy Flavour cross section: phenomenological results

In this section we present some phenomenological applications of the resummation formulae presented above. To start with, we present in fig. 3 the partonic cross sections for production of a pair of heavy quarks of mass $m_Q = 175$ GeV, plotted as a function of $\eta = (1-\rho)/\rho$. The figures show the Born, NLO and resummed results for both the gg and $q\bar{q}$ initial states. The resummed partonic cross section is defined as the sum of the contributions of order α_S^4 and higher from eq. (6.6) and the fixed order NLO result. These figures can be compared with similar ones in ref. [7]. Notice that while in that work the growth of the resummed cross section at small η is damped by a cut-off, in our approach the growth at small η is automatically controlled. Notice also that the resummed cross section is a smooth function of ρ down to values of $1-\rho$ of the order of 10^{-3}, which is of the order of the ratio Λ/m_Q. Since the energy of the soft radiation is of the order of $(1-\rho)m_Q$, at these values of ρ it is numerically of the order of Λ. This is the point at which the Landau pole is expected to influence the results, and non-perturbative physics to set on. We interpret this behaviour as a confirmation of the correctness of our procedure. It would make no sense to cut off the partonic cross section at values of ρ corresponding to soft gluon radiation of several GeV. For $1-\rho < 10^{-3}$ the resummed cross section starts to oscillate, but it remains integrable.

In all our phenomenological studies of resummation effects in heavy flavour production we have used the structure function set CTEQ1M [22]. The importance of the resummation effects is illustrated in figs. 4, 5 and 6, where we plot the quantities

$$\frac{\delta_{\rm gg}}{\sigma^{(gg)}_{\rm NLO}}\,, \quad \frac{\delta_{\rm q\bar{q}}}{\sigma^{(q\bar{q})}_{\rm NLO}}\,, \quad \frac{\delta_{\rm gg}+\delta_{\rm q\bar{q}}}{\sigma^{(gg)}_{\rm NLO}+\sigma^{(q\bar{q})}_{\rm NLO}}\;. \tag{7.1}$$

Here δ is equal to our MP resummed hadronic cross section in which the terms of order α_S^2 and α_S^3 have been subtracted, and $\sigma_{\rm (NLO)}$ is the full hadronic NLO cross section. The results for b at the Tevatron can be easily inferred from fig. 4, since the $q\bar{q}$ component is negligible at Tevatron energies.

For top production, we see that in most configurations of practical interest, the contribution of resummation is very small, being of the order of 1% at the Tevatron. A complete review of top quark production at the Tevatron, based upon these findings, has already been given in ref. [23]. We also observe that, for top production at the LHC, soft gluon resummation effects are negligible. Of course, in this last case, there are other corrections, not included here, that may need to be considered. Typically, since the values of x involved are small in this configuration, one may have to worry about the resummation of small-x logarithmic effects [24].

We see from the figures that in most experimental configurations of interest these effects are fully negligible. One noticeable exception is b production at HERAb, at $\sqrt{S}=39.2$, where we find a 12% increase in the cross section. This correction is however well below the uncertainty due to higher order radiative effects. For example, from the NLO calculation with the MRSA′ [25] parton densities and $m_b = 4.75\,$GeV, we get $\sigma_{b\bar{b}} = 37.3^{+11.0}_{-11.8}\,$nb, a range obtained by varying the renormalization and factorization scales from $m_b/2$ to $2m_b$. Thus we have a relative uncertainty of 30%, and an uncertainty band of 60% of the total from unknown higher order effects, while we

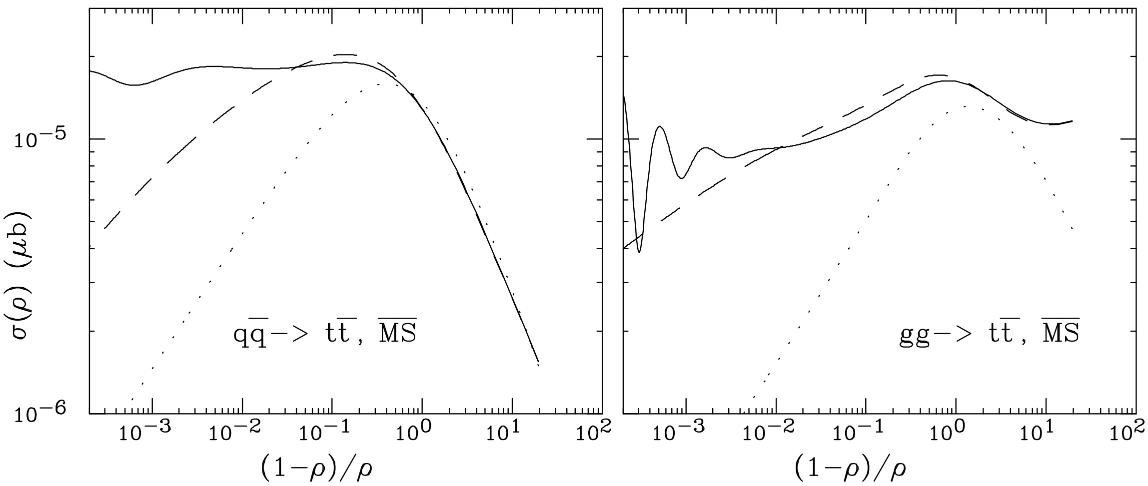

Figure 3: *Partonic cross section for the production of a 175 GeV heavy quark pair. $q\bar{q}$ initial state (left) and gg initial state (right). The dotted lines are the Born result, the dashed lines the NLO result and the solid line the resummed result.* $\Lambda_5^{\overline{MS}}$ *=152 MeV.*

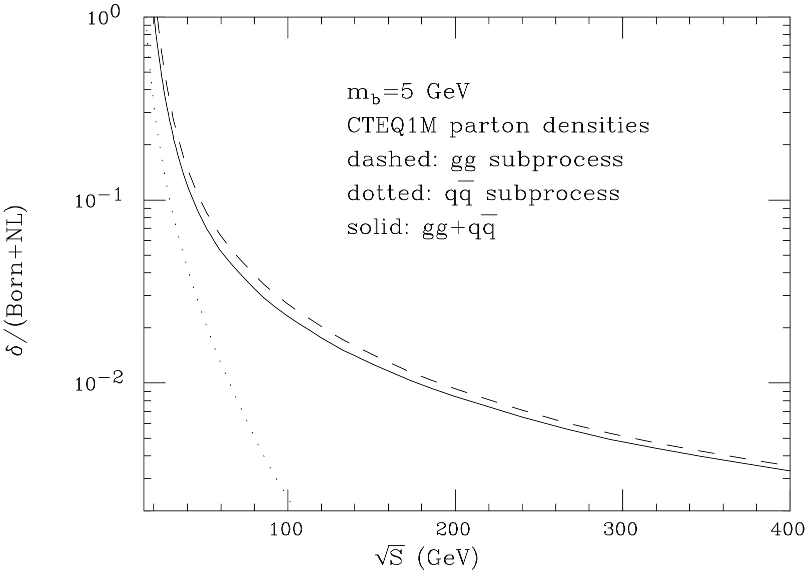

Figure 4: *Contribution of gluon resummation at order* α_S^4 *and higher, relative to the NLO result, for the individual channels and for the total, for bottom production as a function of the CM energy in pp collisions.*

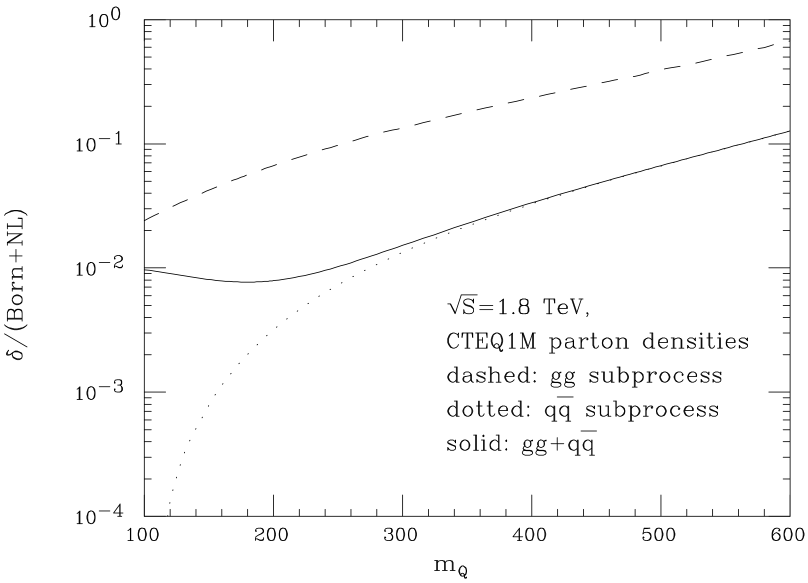

Figure 5: *Contribution of gluon resummation at order α_S^4 and higher, relative to the NLO result, for the individual subprocesses and for the total, as a function of the top mass in $p\bar{p}$ collisions at 1.8 TeV.*

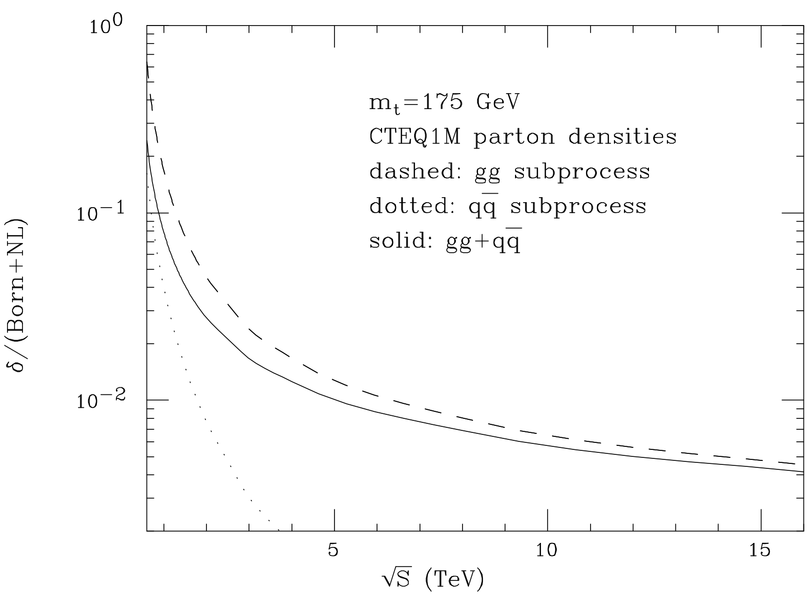

Figure 6: *Contribution of gluon resummation at order α_S^4 and higher, relative to the NLO result, for the individual channels and for the total, as a function of the CM energy in pp collisions.*

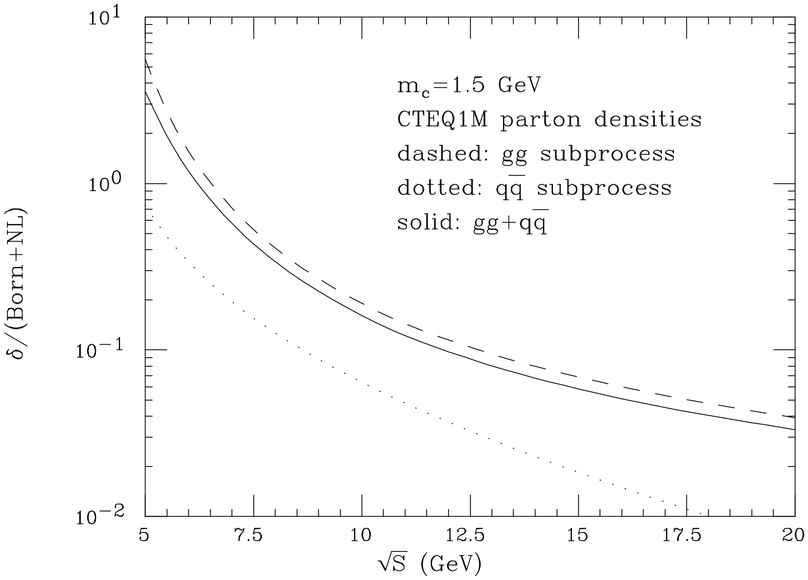

Figure 7: *Contribution of gluon resummation at order α_S^4 and higher, relative to the NLO result, for the individual channels and for the total, for charm production as a function of the CM energy in pp collisions.*

expect a 10% increase from the resummation effect. This result is much less dramatic than the results of ref. [20].

In fig. 7 we also show the effect of resummation in charm production. Due to the large uncertainties that plague charm production [26], this plot should only be considered for orientation.

We conclude with a few remarks and checks about our result. We will focus on top production by $q\bar{q}$ annihilation, with $m_t = 175$ GeV at the Tevatron. First of all, we checked that the full resummation formula that we use is well approximated by its expansion in powers of α_S. We define

$$\sigma_M^{(\mathrm{res})} = \sum_{k=2}^{M} \sigma_k \; , \tag{7.2}$$

where σ_k is the contribution of order α_S^k. We computed each term in the expansion up to order α_S^6. The results are displayed in table 1. We see that the convergence properties of the expansion are extremely good, and, up to the order we have probed, there is no sign that we are near the breakdown of the expansion. As a second observation, we notice that the term σ_3 in our resummed formula is about 7% of the

Born result. The full $\mathcal{O}(\alpha_S^3)$ correction is instead 20% of the Born term. It is easy to illustrate the contribution of the σ_3 term to the partonic cross section. We start with the full NLO cross section written in the form of eqs. (6.7,6.9)

$$\hat{\sigma}_3^{(NLO)}(\rho) = \frac{\alpha_S}{m^2}\left(f_{q\bar{q}}^{(0)}(\rho) + 4\pi\alpha_S\, h_{q\bar{q}}(\rho)\right) \tag{7.3}$$

where

$$h_{q\bar{q}}(\rho) = f_{q\bar{q}}^{(1)}(\rho) + \bar{f}_{q\bar{q}}^{(1)}(\rho)\log\frac{\mu^2}{m_t^2}\ . \tag{7.4}$$

The functions $f_{q\bar{q}}^{(0)}$, $f_{q\bar{q}}^{(1)}$ and $\bar{f}_{q\bar{q}}^{(1)}$ are defined in ref. [21], and μ is the factorization and renormalization scale. It is easy to show that the truncated resummed result at order $\mathcal{O}(\alpha_S^3)$ can be obtained by using the following partonic cross section

$$\hat{\sigma}_3^{(res)}(\rho) = \frac{\alpha_S}{m^2}\left(f_{q\bar{q}}^{(0)}(\rho) + 4\pi\alpha_S\, h'_{q\bar{q}}(\rho)\right) \tag{7.5}$$

where the function $h'_{q\bar{q}}$ can be obtained by expanding the resummation function $\Delta_{q\bar{q},N}^{HF}(m^2)$, as given in eq. (6.14), up to order α_S. After some simple algebra one obtains

$$h'_{q\bar{q}}(\rho) = \frac{C_F}{\pi^2}\int_\rho^1 dy\ f_{q\bar{q}}^{(0)}(\rho/y)\left[\frac{1}{\log 1/y}\left(\log\log 1/y + \gamma_E\right)\right]_+\ . \tag{7.6}$$

The meaning of the plus distribution is as usual

$$\int_0^1 dy\,\left[G(y)\right]_+\ F(y) = \int_0^1 dy\ G(y)\ \left(F(y) - F(1)\right)\ . \tag{7.7}$$

Notice the presence of the subleading term proportional to γ_E in the equation. This term is cancelled if we use, when performing the x integration in eq. (2.4), the relation $1-x^N = \theta(1-x-\frac{e^{-\gamma_E}}{N})$ (which is accurate up to NLL [5]) instead of the approximate one, $1-x^N = \theta(1-x-\frac{1}{N})$. This amounts to the substitution:

$$\ln\ \Delta_{q\bar{q},N}^{HF}(m^2) \to \left(1 + \gamma_E\frac{\partial}{\partial \ln N}\right)\ln\ \Delta_{q\bar{q},N}^{HF}(m^2)\ . \tag{7.8}$$

Equation (7.6) then becomes

$$h''_{q\bar{q}}(\rho) = \frac{C_F}{\pi^2}\int_\rho^1 dy\ f_{q\bar{q}}^{(0)}(\rho/y)\left[\frac{1}{\log 1/y}\log\log 1/y\right]_+\ . \tag{7.9}$$

In fig. 8 we give the contribution of the $h_{q\bar{q}}^{(1)}(\rho)$ compared with the quantity that corresponds to the full NLO correction.

$k, M =$	2	3	4	5	6
σ_k (pb)	3.71	0.256	8.27×10^{-3}	-3.24×10^{-4}	-5.88×10^{-5}
$1 - \sigma_M^{(\mathrm{res})}/\sigma^{(\mathrm{res})}$	0.066	2.0×10^{-3}	-9.63×10^{-5}	-1.48×10^{-5}	4.8×10^{-8}

Table 1: *Top pair production at the Tevatron via the $q\bar{q}$ channel, for $m_t =$ 175 GeV and CTEQ1M parton densities. Order-by-order contributions to the fully resummed formula (first line), and accuracy of the truncated perturbative expansion relative to the fully resummed result (second line).*

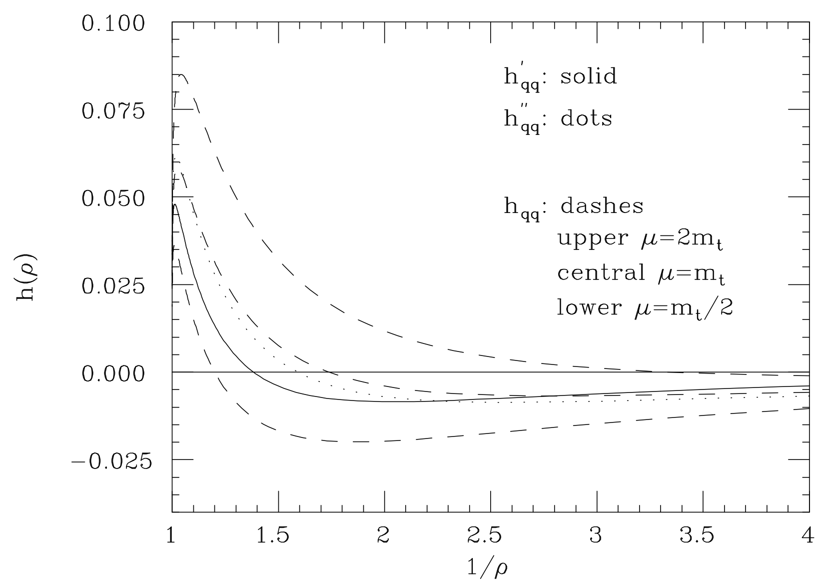

Figure 8: *Comparison of the partonic cross section at $\mathcal{O}(\alpha_S^3)$ for the MP resummation formula (solid line), for the resummation formula with the term proportional to γ_E included (dotted line), and for the exact $\mathcal{O}(\alpha_S^3)$ formula of ref. [21] with $\mu = m_t/2$, m_t and $2m_t$ (dashed lines).*

We plot the full NLO correction for $\mu = m_t/2$, m_t and $2m_t$. We see that the $h'_{q\bar{q}}$ term is consistent with the exact next-to-leading result, given the spread of the latter induced by the renormalization scale dependence. According to the choice of subleading terms, the σ_3 term can substantially change, and can be brought to almost coincide with the exact result for $\mu = m_t$. In our case, for example, inclusion of the term proportional to γ_E in formula (2.16) would cancel exactly the corresponding term appearing in formula (7.6), which then reduces to Eq. (7.9). This last formula gives a result that is very close to the exact one for $\mu = m_t$ in the most important kinematic region of $\rho \approx 1$, as can be seen from fig. 8. This is in agreement with the findings of ref. [6], where it was pointed out that the LL truncation of the $\mathcal{O}(\alpha_S^3)$ terms provide a very good approximation to the full NLO cross section, as long as $\mu = m_t$. However it should be stressed once more that this result is accidental, as it would not hold for a different choice of renormalization scale.

Notice that altering the structure of the subleading terms in the exponent of the Mellin-space coefficient function will not affect the asymptotic properties of its perturbative expansion and the integrability of its x-space MP transform. We explored the numerical impact on the contributions of order α_S^4 and higher of including the subleading terms proportional to γ_E in the exponent of the coefficient function. In the notation of Eq. (7.1) we get $\delta_{q\bar{q}}/\sigma_{\rm NL} = 0.013$, an effect which is larger than the one found previously, but still negligible. The difference from the result of fig. 5 should be taken as an estimate of the uncertainty coming from the unknown next-to-leading logarithmic terms in the exponentiated coefficient function. As such, it is a purely perturbative uncertainty, which cannot be separated from the uncertainty due to the change in renormalization scale or factorization scheme. The size of these latter uncertainties, estimated to be of the order of 10% [23], can consistently accommodate for the 1% effect we found.

8 Jet Cross Sections

In this section we present, as an additional application of our formula for the soft gluon resummation, a study of corrections to the invariant mass distribution of jet pairs produced in $p\bar{p}$ collisions at 1.8 TeV. The interest in the effects of resummation on the behaviour of jet cross sections at large energy is prompted by the discrepancy between the single-inclusive jet-$p_{\rm T}$ distribution at large $p_{\rm T}$, as measured by CDF [27], and the result of the NLO QCD predictions [28]. For simplicity we will study the effects of soft gluon resummation on the invariant mass distribution of the jet pair, which is, from a theoretical point of view, very close to the Drell–Yan pair production. Observe that other distributions, such as the $p_{\rm T}$ of the jet, have a rather different structure from the point of view of soft gluon resummation. In fact, while the jet pair mass is only affected by the energy degradation due to initial state radiation, the $p_{\rm T}$ of the jet may also be affected by the transverse momentum generated by initial state radiation, and by the broadening of the jet due to final state radiation.

A study of the jet pair mass distribution is not of purely academic interest, since also for this variable an analogous discrepancy between data and theory has been observed [29]. Studies of resummation effects in the inclusive $p_{\rm T}$ spectrum of jets are in progress (M. Greco and P. Chiappetta, private communication).

Before presenting the results, we briefly discuss the key elements of the calculation. Contrary to the case of Drell–Yan and heavy flavour production, light jets are produced at Born level from all possible initial states, gg, qg, qq and $q\bar{q}$. The resummed invariant mass distribution of dijets, at LL order, is therefore given by

$$\frac{d\sigma^{(\rm res)}}{d\tau} = \sum_{i,j\in\{q,\bar{q},g\}} \hat{\sigma}_{ij}\left(\hat{s},\theta^*_{min}\right) \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} F^{i,N} F^{j,N}\, \Delta^{J}_{ij,N}\, \tau^{-N}\, dN \,, \tag{8.1}$$

where $\hat{s} = S\tau = m^2_{JJ}$. The functions $\Delta^J_{ij,N}$ are given by eqs. (6.14) and (6.15), supplemented by

$$\ln \Delta^{J}_{qg,\,N}(m^2_{JJ}) = \ln N \; g_{qg,\,1}(b_0\alpha_S \ln N) + \mathcal{O}(\alpha_S^k \ln^k N) \tag{8.2}$$

$$g_{qg,\,1}(\lambda) = \frac{C_A + C_F}{2C_F} g_1(\lambda) \,. \tag{8.3}$$

The partonic Born cross section $\hat{\sigma}_{ij}\left(\hat{s},\theta^*_{min}\right)$ depends on the range of integration for the partonic centre-of-mass scattering angle θ^* (θ^* is related to the rapidity difference $\eta^* = (\eta_1 - \eta_2)/2$ of the two jets by $\sin\theta^* = 1/\cosh\eta^*$). To avoid the Rutherford singularity, we will keep this angle strictly larger than zero. While the absolute production rate depends on the choice of θ^*_{min}, we will now show that the K-factor, i.e. the ratio of the resummed cross section to the LO one is to good approximation independent of it. Therefore the results for the K-factor will be rather independent of the details of the experimental cuts. It is a well-known fact [30] that the LO amplitudes for parton-parton scattering processes are related to one another, in the small scattering angle limit, as follows

$$d\hat{\sigma}_{gg} \; : \; d\hat{\sigma}_{qg} \; : \; d\hat{\sigma}_{qq} \; = \; 1 \; : \; \frac{4}{9} \; : \; \left(\frac{4}{9}\right)^2 \; , \tag{8.4}$$

where the indices refer to the pair of partons in the initial state, q being an arbitrary quark (or antiquark) flavour. Even at 90° this approximation is good to about 10%, becoming better and better when the cross sections are integrated over a larger and larger range of the scattering angle. Using eq. (8.4), we can rewrite the expression for the resummed jet invariant mass cross section as follows

$$\frac{d\sigma^{(\mathrm{res})}}{d\tau} = \hat{\sigma}_{gg}\left(\hat{s}, \theta^*_{min}\right) \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} H_N^2 \tau^{-N} \, dN \; , \tag{8.5}$$

with

$$H_N = F_{g,N} N^{\frac{C_A}{2C_F} g_1(\lambda)} + \frac{4}{9} \sum_{i \in \{q_f, \bar{q}_f\}} F_{i,N} N^{\frac{1}{2} g_1(\lambda)} \; . \tag{8.6}$$

The K-factor is therefore given by the following expression, in which all dependence upon the acceptance cut θ^* has disappeared

$$\frac{\dfrac{d\sigma^{(\mathrm{res})}}{d\tau}}{\dfrac{d\sigma^{(\mathrm{LO})}}{d\tau}} = \frac{\dfrac{1}{2\pi i} \displaystyle\int_{C-i\infty}^{C+i\infty} H_N^2 \tau^{-N} \, dN}{\dfrac{1}{2\pi i} \displaystyle\int_{C-i\infty}^{C+i\infty} E_N^2 \tau^{-N} \, dN} \; . \tag{8.7}$$

E_N is the Mellin transform of the standard *effective structure function*, defined by

$$E_N = F_{g,N} + \frac{4}{9} \sum_{i \in \{q_f, \bar{q}_f\}} F_{i,N} \; . \tag{8.8}$$

While a more accurate implementation of the experimental acceptance will eventually be necessary[4], our approximation is more than adequate to provide a first estimate of the resummation effects.

In fig. 9 we show the following quantities

$$\frac{\delta^{(3)}_{\mathrm{gg}}}{\sigma^{(2)}} , \quad \frac{\delta^{(3)}_{\mathrm{qg}}}{\sigma^{(2)}} , \quad \frac{\delta^{(3)}_{\mathrm{q\bar{q}}}}{\sigma^{(2)}} , \quad \frac{\delta^{(3)}_{\mathrm{gg}} + \delta^{(3)}_{\mathrm{qg}} + \delta^{(3)}_{\mathrm{q\bar{q}}}}{\sigma^{(2)}} \tag{8.9}$$

where $\delta^{(3)}$ is equal to our MP resummed hadronic cross section in which the terms of order α_S^2 have been subtracted, and $\sigma^{(2)} = \sigma^{(2)}_{\mathrm{gg}} + \sigma^{(2)}_{\mathrm{qg}} + \sigma^{(2)}_{\mathrm{q\bar{q}}}$ is the full hadronic LO cross section (of order α_S^2). We use as a reference renormalization and factorization scale for our results $\mu = M_{jj}/2$. Notice that for large invariant masses the effects of

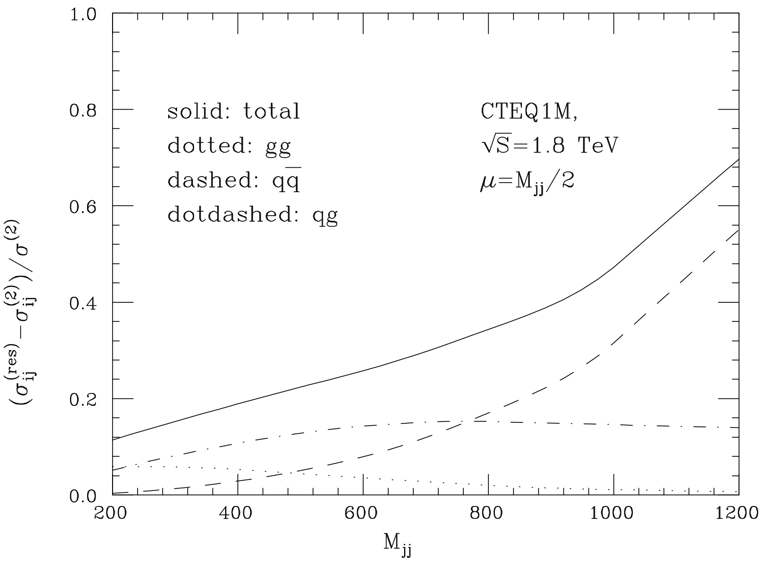

Figure 9: *Contribution of gluon resummation at order α_S^3 and higher, relative to the LO result, for the invariant mass distribution of jet pairs at the Tevatron.*

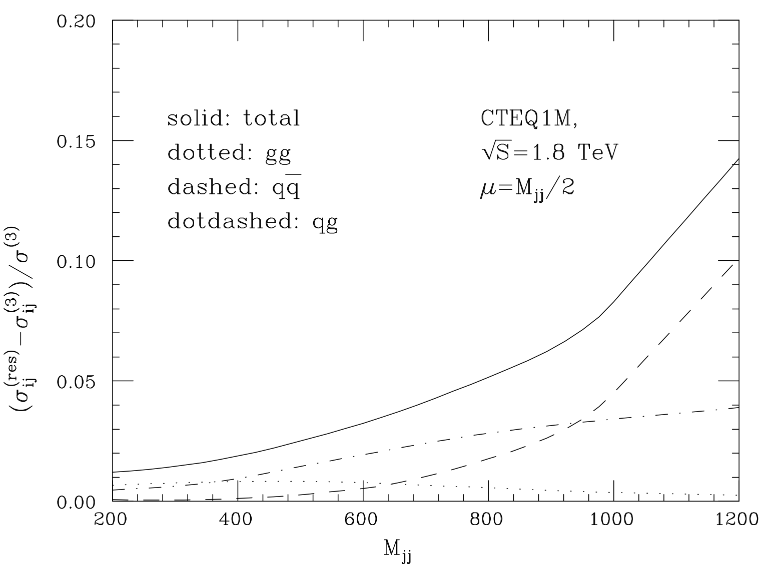

Figure 10: *Contribution of gluon resummation at order α_S^4 and higher, relative to the truncated $\mathcal{O}(\alpha_S^3)$ result, for the invariant mass distribution of jet pairs at the Tevatron.*

higher orders are large. To understand how much is due to the first order corrections (which are exactly calculable [31]) and how much is due to corrections of order α_S^4 and higher, we show the following quantities in fig. 10

$$\frac{\delta^{(4)}_{\rm gg}}{\sigma^{(3)}}\,,\quad \frac{\delta^{(4)}_{\rm qg}}{\sigma^{(3)}}\,,\quad \frac{\delta^{(4)}_{\rm q\bar{q}}}{\sigma^{(3)}}\,,\quad \frac{\delta^{(4)}_{\rm gg}+\delta^{(4)}_{\rm qg}+\delta^{(4)}_{\rm q\bar{q}}}{\sigma^{(3)}}\,, \tag{8.10}$$

where $\delta^{(4)}$ is now equal to the MP resummed hadronic cross section with terms of order α_S^3 subtracted, and $\sigma^{(3)}$ is an approximation to the full NL cross section, summed over all subprocesses, obtained by truncating the resummation formula at order α_S^3. This figure shows that indeed most of the large K factor is due to the pure NLO corrections, with the resummation of higher-order soft gluon effects contributing only an additional 10% at dijet masses of the order of 1 TeV. Removing the subleading term proportional to γ_E from the exponent of the coefficient function, in the spirit of the discussion at the end of the previous section, we found only a slight increase of the contribution from the terms of order α_S^4 and higher (an additional absolute 5% for dijet masses of 1 TeV).

As we will discuss in more detail in the next section, these results should only be taken as an indication of the order of magnitude of the correction, since we have not included here a study of the resummation effects on the determination of the parton densities. From this preliminary study it seems however unlikely that the full 30–50% excess reported by CDF for jet $p_{\rm T}$'s in the range 300–450 GeV could be explained by resummation effects in the hard process. It is possible that the remaining excess is due to the poor knowledge of the gluon parton densities at large x, an idea pursued by the CTEQ group [32], but challenged in ref. [33].

9 A few additional remarks

A fully consistent treatment of the resummation effects requires the use of parton densities that

(i) are extracted from low-energy data by taking into account the resummation effects for the corresponding scattering process;

(ii) are evolved in Q^2 using resummed anomalous dimensions.

[4]For instance, at small θ^* a more refined resummation procedure would lead to the replacement $M_{jj}^2 \to M_{jj}^2 \sin^2(\theta^*/2)$ in the hard scale of the resummed coefficient functions $\Delta^J_{ij,N}$ in eqs. (8.1,8.2).

In both steps (*i*) and (*ii*), the Minimal Prescription should be implemented.

In the case, for example, of jet production, it is quite possible that resummation effects significantly influence the determination of the large-x structure functions from low energy data. The Tevatron jets of the highest energy probe the partonic densities at values of x of the order of 0.5, which one could argue are far enough from the $x \to 1$ region to make the Sudakov effects small. However, the large values of Q^2 involved (of the order of 10^6 GeV^2) are such that the Q^2 evolution of the parton densities from low energy is significant, and we are therefore sensitive to the input structure functions measured at low Q^2 in regions of x significantly larger than 0.5. It would therefore be important to reexamine the extraction of the large-x non-singlet structure functions, in the light of the resummation results for the DIS process, before firmer conclusions can be drawn on the significance of the present jet cross section discrepancy.

In the Q^2 evolution of the parton densities, one has to use the equation

$$\frac{dF_{i,N}(Q^2)}{d\ln Q^2} = \sum_j \gamma_{ij,N}(\alpha_S(Q^2)) \; F_{j,N}(Q^2) \;\; , \tag{9.1}$$

with anomalous dimensions $\gamma_{ij,N}$ that include the resummation of logarithmic contributions to the same accuracy as in the coefficient function. For instance, the analogue of Eq. (2.8) is the following expansion

$$\gamma_{ij,N}(\alpha_S) = \gamma_{ij}^{(1)}(b_0\alpha_S \ln N) + \alpha_S \; \gamma_{ij}^{(2)}(b_0\alpha_S \ln N) + \dots \;\; . \tag{9.2}$$

The leading and next-to-leading resummed anomalous dimensions $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$ have, in general, singularities related to the Landau pole and these singularities (like those in Eqs. (2.10), (2.11), and (2.15), (2.16)) depend on the factorization scheme in which the parton densities are defined. Thus, for the purpose of implementing the Minimal Prescription, it is particularly convenient to use the $\overline{\rm MS}$ definition of the parton densities. Indeed, in the $\overline{\rm MS}$ factorization scheme the resummed anomalous dimensions have no singularity associated with the Landau pole and, more precisely, they have the following simple form [5, 34]

$$\gamma_{qq,N}^{\overline{\rm MS}}(\alpha_S) \;\; = \;\; - \left[A(\alpha_S) + \mathcal{O}(\alpha_S^3) \right] \ln N + \mathcal{O}(1) \;\; , \tag{9.3}$$

$$\gamma_{gg,N}^{\overline{\rm MS}}(\alpha_S) \;\; = \;\; - \frac{C_A}{C_F} \; \left[A(\alpha_S) + \mathcal{O}(\alpha_S^3) \right] \ln N + \mathcal{O}(1) \;\; , \tag{9.4}$$

where $A(\alpha_S)$ is given in Eq. (2.5) and $\gamma_{ij,N}^{\overline{\rm MS}}(\alpha_S) = \mathcal{O}(1/N)$ for $N \to \infty$ if $i \neq j$.

Note, however, that the main feature of the Minimal Prescription, i.e. the absence of factorially-growing coefficients, remains valid in any factorization scheme. The only difference in using different schemes is in the fact that one can introduce non-perturbative corrections which, although suppressed by more than any power of Λ/Q at fixed $1-\tau$, can actually have a different overall magnitude. As long as Q^2 is sufficiently perturbative and τ sufficiently far from the hadronic threshold, this difference should not sizeably affect the actual value of the cross section. Obviously, approaching the essentially non-perturbative regime ($Q^2 \to 1$ GeV$^2, \tau \to 1$), a physically motivated treatment of non-perturbative effects has to be introduced. This is beyond the scope of the present paper.

10 Conclusions

In our study of resummation procedures for partonic-threshold corrections, we have found that the factorial growth due to the infrared renormalon is only a minor problem, when compared with the large spurious factorial growth that is generated when the x space resummation formulae are limited to the LL or NLL level. Although we consider the recent indication of the cancellation of $1/Q$ renormalon effects in Drell–Yan pair production [14] as an important progress, we find this spurious factorial growth to be a much more serious problem in practice. In fact, while a $1/Q$ power ambiguity in the Tevatron top production cross section should be below the per cent level, the ambiguity induced by the spurious factorial growth is at the level of 10%.

In the present paper we have proposed the Minimal Prescription for the resummation of partonic-threshold effects in hadronic processes. This formula has a perturbative expansion free of factorially growing terms. The ambiguity arising from its asymptotic nature is in fact exponentially suppressed, behaving as $\exp(-C(1-x)Q/\Lambda)$. We would also like to remark that certain kinematic constraints are respected by the MP formula. For example, for $N=1$ the MP formula gives no resummation corrections. In general, sum rules associated to low moments receive small corrections in our procedure.

As far as our phenomenological results are concerned, we have found that in heavy flavour production, in current experimental configurations, resummation effects are negligible. The process of high mass dijet production at the Tevatron, being much closer to threshold, has instead non-negligible corrections. We wish to remind the reader that, in this last case, in order to perform a reliable phenomenological prediction, resummation formulae should be applied not only to the production process in question, but also to the processes that have been used to extract the parton densities and to their evolution equations.

Acknowledgements

We wish to thank Harris Contopanagos for useful discussions.

APPENDIX A: Asymptotic nature of the resummation formula

In this appendix we will study the asymptotic nature of our resummation formula. We are dealing with the LL function

$$G(\alpha_S) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} dN \, \tau^{-N} \, h_N(\lambda) \, , \quad h_N(\lambda) = \exp[\log N \, f(\lambda)] \, , \quad \lambda = \alpha_S b_0 \, \log N \, . \tag{A.1}$$

Observe that we will consider $h_N(\lambda)$ as an independent function of N and λ. We will focus upon the $\overline{\mathrm{MS}}$ case, in which

$$f(\lambda) = \frac{c}{\lambda} [(1 - 2\lambda) \log(1 - 2\lambda) + 2\lambda] \tag{A.2}$$

with $c = A^{(1)}/(\pi b_0)$, but the proof can be easily generalized, since it only relies upon the analyticity properties of $f(\lambda)$, and the fact that it does not grow too strongly at infinity. Because of the same reason, the proof can also be generalised to subleading logarithmic accuracy.

We introduce the notation

$$\begin{aligned} h_N(\lambda)|_n &= \sum_{i=0}^{n} h_N^{(i)}(0) \frac{\lambda^i}{i!} . \\ G(\alpha_S)|_n &= \frac{1}{2\pi i} \int dN \, \tau^{-N} \, h_N(\lambda)|_n \, . \end{aligned} \tag{A.3}$$

It is clear that $G(\alpha_S)|_n$ is the truncated expansion of $G(\alpha_S)$ to order n in α_S. We will show that $G(\alpha_S)|_n$ is asymptotic to $G(\alpha_S)$, that is to say that

$$\delta G(\alpha_S) = G(\alpha_S) - G(\alpha_S)|_{n-1} = \mathcal{O}(\alpha_S^n). \tag{A.4}$$

We begin by stating a few useful results for analytic functions. For a generic analytic function $F(z)$ we have the identity

$$F(z)-\sum_{k=0}^{n-1}F^{(k)}(0)\,\frac{z^k}{k!} \;=\int_0^z dx_1\int_0^{x_1}dx_2\ldots\int_0^{x_{n-1}}dx_n F^{(n)}(x_n)$$
$$=\frac{1}{(n-1)!}\int_0^z (z-x)^{n-1}F^{(n)}(x)\,dx\,. \tag{A.5}$$

where we assume that the integrations are performed along a straight path from 0 to z. We then use Cauchy's inequality, which states that for a function $F(z)$, analytic in a circle of radius r around the point z_0 we have

$$|F^{(n)}(z_0)|\le\frac{n!}{r^n}\max_{|z-z_0|<r}|F(z)|\,. \tag{A.6}$$

Assuming that $F(z)$ is analytic in a circle of radius R around $z=0$, from eq. (A.6) it follows that

$$|F^{(n)}(z)|\le\frac{n!}{(R-|z|)^n}\max_{|w|<R}|F(w)|\,, \tag{A.7}$$

which together with eq. (A.5) yields the following bound

$$\left|F(z)-\sum_{k=0}^{n-1}F^{(k)}(0)\,\frac{z^k}{k!}\right|\;\le\; n\,\max_{|w|<R}|F(w)|\,\int_0^{|z|}\frac{(|z|-x)^{n-1}}{(R-x)^n}\,dx\;, \tag{A.8}$$

which holds for $|z|<R$. We now use the relation

$$\int_0^t\frac{(t-y)^{n-1}}{(R-y)^n}\,dy=\left(\frac{t}{R}\right)^n\int_0^1\frac{(1-y)^{n-1}}{(1-yt/R)^n}\,dy<\frac{1}{n}\left(\frac{t}{R}\right)^n\frac{R}{R-t}\,, \tag{A.9}$$

(where the last inequality becomes an identity in the limit $n\to\infty$) and get the bound

$$\left|F(z)-\sum_{k=0}^{n-1}F^{(k)}(0)\,\frac{z^k}{k!}\right|\;\le\;\max_{|w|<R}|F(w)|\,\frac{R}{R-z}\,\left(\frac{z}{R}\right)^n\,, \tag{A.10}$$

which we will apply in the following.

Consider now our expression for δG

$$\delta G(\alpha_S)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}dN\,\tau^{-N}\,\left(h_N(\lambda)-h_N(\lambda)|_{n-1}\right)\,. \tag{A.11}$$

We will apply the bound (A.10) to $h_N(\lambda)$, considered as a function of λ at fixed N. This function is analytic up to the Landau pole at $\lambda=1/2$, so that in our case $R=1/2$. We bent the N contour towards the negative real axis, as depicted in fig. 1, with N_0 of order 1. Choosing any real $0 < r < 1$ (close to 1) we split our integral into a near ($|\log N| < r/(2\alpha_S b_0)$) and far ($|\log N| > r/(2\alpha_S b_0)$) part,

$$\begin{aligned} G(\alpha_S) &= G_<(\alpha_S) + G_>(\alpha_S) \\ G_{<>}(\alpha_S) &= \frac{1}{2\pi i} \int_{|\log N| <> r/(2\alpha_S b_0)} dN\, \tau^{-N}\, h_N(\lambda)\,. \end{aligned} \tag{A.12}$$

For the near part, using eq. (A.10), we obtain immediately

$$|\delta G_<| = \left| \int_{|\log N| < r/(2\alpha_S b_0)} dN\, \tau^{-N}\, (h_N(\lambda) - h_N(\lambda)|_{n-1}) \right| \le$$

$$\int_{|N| \le e^{r/(2\alpha_S b_0)}} |dN\, \tau^{-N}|\, \frac{\exp[C|\log N|]}{1-r}\, (2\alpha_S b_0\, |\log N|)^n \le g_n \alpha_S^n\,, \tag{A.13}$$

where

$$g_n = \int |dN\, \tau^{-N}|\, \frac{\exp[C|\log N|]}{1-r}\, (2b_0\, |\log N|)^n \tag{A.14}$$

and

$$C = \max_{|\lambda| < r/2} |f(\lambda)|. \tag{A.15}$$

By inspection it is easy to see that the far part is exponentially suppressed. In fact, we can write

$$\begin{aligned} |\delta G_>| &= \left| \int_{|\log N| > r/(2\alpha_S b_0)} dN\, \tau^{-N}\, (h_N(\lambda) - h_N(\lambda)|_{n-1}) \right| \\ &\le \int_{|\log N| > r/(2\alpha_S b_0)} |dN| e^{-\log\frac{1}{\tau}|N|}\, |h_N(\lambda) - h_N(\lambda)|_{n-1}|\ . \end{aligned} \tag{A.16}$$

Since $f(\lambda)$ is logarithmically bounded in the far region, $h_N(\lambda)$ grows at most as $(N)^{\ln \ln N}$. Instead, $h_N(\lambda)|_n$ is polynomial in $\log N$. Therefore, the dominant factor in the integrand is the exponential, which is of the order of $\exp(-\exp(r/(2\alpha_S b_0)) \log 1/\tau)$. We have therefore proved that the MP formula is asymptotic to its formal α_S expansion.

It is easy to find, using saddle point methods, the rate of growth of g_n. We find, for $n \ll \log 1/\tau$, the following power behaviour

$$g_n \approx \left(2b_0 \log \frac{C}{\log 1/\tau} \right)^n\,, \tag{A.17}$$

and for $n \gg \log 1/\tau$ we obtain the coefficients

$$g_n \approx \left(2b_0 \log \frac{n}{\log 1/\tau} \right)^n\,, \tag{A.18}$$

whose growth is faster than any power, but much slower than factorial. We can estimate the value of n at which $\alpha_S^n g_n$ has a minimum from the equation

$$\frac{\alpha_S^{n+1} g_{n+1}}{\alpha_S^n g_n} \approx \left(\frac{\log \frac{n+1}{\log 1/\tau}}{\log \frac{n}{\log 1/\tau}} \right)^n 2\alpha_S b_0 \, \log \frac{n+1}{\log 1/\tau} \approx e^{\frac{1}{\log \frac{n}{\log 1/\tau}}} \, 2\alpha_S b_0 \, \log \frac{n+1}{\log 1/\tau} = 1 \, . \tag{A.19}$$

The value at the minimum is obtained from

$$\alpha_S^n g_n \approx \exp\left(n \log \left[2\alpha_S b_0 \, \log \frac{n}{\log 1/\tau} \right] \right) \tag{A.20}$$

by using eq. (A.19). We get

$$\alpha_S^n g_n \approx \exp\left(- \frac{n}{\log \frac{n}{\log 1/\tau}} \right) \, . \tag{A.21}$$

The value of n at the minimum is

$$n \approx e^{\frac{1}{2\alpha_S b_0}} \, \log \frac{1}{\tau} \, , \tag{A.22}$$

which inserted in eq. (A.21) yields

$$\alpha_S^n g_n \approx \exp\left(-2\alpha_S b_0 \, e^{\frac{1}{2\alpha_S b_0}} \, \log \frac{1}{\tau} \right) \, . \tag{A.23}$$

Remembering that $\alpha_S \, b_0 = 1/\log(Q^2/\Lambda^2)$ we see that, taking $r \to 1$, these suppression factors correspond to exponentials of the form (neglecting logarithmic factors in the exponent)

$$e^{-C \frac{Q(1-\tau)}{\Lambda}} \, , \tag{A.24}$$

which is suppressed by more than any power of Q. Notice that as $Q(1-\tau)$ approaches Λ the above expression becomes of order 1. In fact, in this limit, the total mass of the radiation that accompanies the production of the object with mass Q becomes of order Λ. This is clearly a regime over which we do not have any perturbative control.

As a last point, we discuss the inclusion of other N-dependent factors in the integrand like the partonic cross section in the case of heavy flavour production, or the parton luminosities. Partonic cross sections have a negative-power behaviour in N for large N, so they will appear in the integrand as factors of $\exp[-M \log N]$, which simply modifies the value of the constant C. Similar effects would be given by common parametrizations of structure functions. Of course, we cannot guarantee that the structure functions themselves do not have a Sudakov like behaviour at large x. The only statement we can make is that in the $\overline{\mathrm{MS}}$ scheme, if such a behaviour is not present in the initial conditions, it will not arise because of evolution (see sec. 9).

APPENDIX B: Numerical implementation of the MP formula

In principle, there are no difficulties in the numerical implementation of the MP formula. However, the modern sets of parton densities are usually provided in terms of numerical codes or of parametrized expressions which, in practice, cannot be used to evaluate their N-moments in analytic form for arbitrary complex values of N. To overcome this practical difficulty we have to rewrite the MP formula in terms of an x-space convolution of the parton densities and the inverse Mellin transformation of the coefficient factor Δ_N. Since $\Delta_N(Q^2)$ in formula (4.3) has singularities to the right of the integration contour, much care has to be taken when turning formula (4.3) to an x-space integral. We begin by observing that, for large N, $\Delta_N(Q^2)$ (in the $\overline{\rm MS}$ scheme) is suppressed by more than any power of N as $N \to \infty$. We will also assume that F_N is suppressed by some powers of N as $N \to \infty$. We will limit the present discussion to the Drell–Yan case, although the extension to other cases is straightforward.

We rewrite formula (4.3) by setting

$$F_N^2(Q^2) = \int_0^1 \frac{dx}{x}\, x^N\, \mathcal{L}(x,Q^2) \tag{B.1}$$

where

$$\mathcal{L}(x,Q^2) = \int_x^1 F(x/z)\, F(z)\, \frac{dz}{z}\,. \tag{B.2}$$

We obtain

$$\begin{aligned} \sigma(\tau) &= \frac{1}{2\pi i}\int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty} \tau^{-N}\, \Delta_N(Q^2)\, \left[\int_0^1 \frac{dx}{x}\, x^N \mathcal{L}(x,Q^2)\right] dN \\ &= \int_0^1 \frac{dx}{x}\, \mathcal{L}(x,Q^2)\, \frac{1}{2\pi i}\int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty} \left(\frac{\tau}{x}\right)^{-N} \Delta_N(Q^2)\, dN\,. \end{aligned} \tag{B.3}$$

Observe that the x integration extends from 0 to 1, not from τ to 1. This is because of the fact that, due to the Landau singularity to the right of the contour, the Mellin transform of $\Delta_N(Q^2)$ does not vanish when its argument is greater than 1. It does however vanish very fast when its argument is above 1 by an amount greater than Λ/Q, so that the basic parton model assumptions are not violated. Let us in fact call $z = \tau/x$. For $z > 1$ we have

$$\frac{1}{2\pi i}\int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty} \Delta_N(Q^2)\, z^{-N}\, dN = \frac{1}{\pi}\int_{N_L}^{\infty} e^{-N\log z}\, {\rm Im}\,\Delta_{N+i\epsilon}(Q^2)\, dN\,. \tag{B.4}$$

Using the formula

$$e^{-Nt} = \sum_{k=0}^{\infty} \frac{(-1)^k}{N^{k+1}}\, \delta^{(k)}(t)\ , \tag{B.5}$$

where $\delta^{(k)}(t)$ is the $k^{\rm th}$ derivative of the δ function, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty} \Delta_N(Q^2)\, z^{-N}\, dN \;&=\; \delta(\log z)\frac{1}{\pi}\int_{N_L}^{\infty} {\rm Im}\,\Delta_{N+i\epsilon}(Q^2)\,\frac{dN}{N} \\ -\delta^{(1)}(\log z)\frac{1}{\pi}\int_{N_L}^{\infty} {\rm Im}\,\Delta_{N+i\epsilon}(Q^2)\,\frac{dN}{N^2} + \ldots\ .\end{aligned} \tag{B.6}$$

Using saddle point arguments it is possible to show that the first term is of order 1, while the subsequent terms are suppressed by powers of Λ/Q. Let us in fact consider the integral

$$I_m = \frac{1}{\pi}\int_{N_L}^{\infty} {\rm Im}\,\Delta_{N+i\epsilon}(Q^2)\,\frac{dN}{N^{m+1}} = \frac{1}{2\pi i}\int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty} \Delta_N(Q^2)\,\frac{dN}{N^{m+1}}\ . \tag{B.7}$$

The N integration can be turned into a $\lambda = \alpha_S b_0 \log N$ integration, where the contour $\mathcal{C}$ comes from $\infty - \epsilon i$, encircles clockwise the Landau singularity at $\lambda = 1/2$, and then goes to $\infty + \epsilon i$. In the $\overline{\rm MS}$ scheme we have

$$I_m = \frac{1}{b_0\alpha_S}\frac{1}{2\pi i}\int_{\mathcal{C}} \exp\left(\frac{r}{\alpha_S b_0}\left[2\lambda - \frac{m}{r}\lambda + (1-2\lambda)\log(1-2\lambda)\right]\right)\, d\lambda\ , \tag{B.8}$$

where $r = A^{(1)}/(b_0\pi)$. The integration contour can be deformed into a steepest descent contour, where the saddle point is located at $\lambda = \frac{1}{2}(1-e^{-\frac{m}{2r}})$, and the saddle point estimate of the integral is

$$\begin{aligned} I_m \;&\approx\; \exp\left(\frac{r}{\alpha_S b_0}\left[1 - \frac{m}{2r} - e^{-\frac{m}{2r}}\right]\right)\frac{1}{b_0\alpha_S}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left(\frac{2r}{\alpha_S b_0} e^{\frac{m}{2r}}\,\bar{\lambda}^2\right)\, d\bar{\lambda} \\ &=\; e^{-\frac{m}{4r}}\sqrt{\frac{1}{8\pi r\alpha_S b_0}}\,\exp\left(\frac{r}{\alpha_S b_0}\left[1 - \frac{m}{2r} - e^{-\frac{m}{2r}}\right]\right)\ .\end{aligned} \tag{B.9}$$

We have verified numerically that the above estimate is in fact quite good, although terms down by a power of α_S have been neglected in the saddle point approximation. It is apparent from the formula that, except for $m = 0$, I_m is power-suppressed, with an exponent increasing with m. In our philosophy, however, all the terms of the expansion should be kept, since we have chosen not to include any power-suppressed effects in our definition.

In principle there is nothing wrong in the integral (B.3), it is convergent, and it should be possible to perform it numerically. In practice, however, this is not convenient. Even if a careful importance sampling is implemented in order to correctly treat the region around $z = 1$, the integrand performs a few oscillations, in which large cancellations occur, so that a direct numerical integration proves in this case to be too time-consuming. In order to overcome this problem we use the following trick. We introduce a fake luminosity $\mathcal{L}^{(0)}(x,\tau)$, defined as

$$\mathcal{L}^{(0)}(x,\tau) = A\,(1-x)^{\alpha}\,x^{-\beta}\ \left(1 - P(x)\right)\ . \tag{B.10}$$

where $P(x)$ is a polynomial in x. We then define

$$x_i \equiv \tau + (i-1)\eta\ , \quad l_i \equiv \mathcal{L}^{(0)}(x_i, Q^2)\ , \tag{B.11}$$

where $i = 1, \ldots, 4$ and η is a small quantity, usually taken to be of the order of Λ/Q. We then choose

$$P(x) = B\,(x-x_1)\,(x-x_2)\,(x-x_3)\ , \tag{B.12}$$

and A, B, α, β in such a way that

$$\mathcal{L}^{(0)}(x_i,\tau) = l_i\ , \quad i = 1, \ldots, 4\ . \tag{B.13}$$

These conditions can be easily solved to yield

$$\begin{aligned}
\alpha &= \frac{\log\dfrac{l_1}{l_3}\log\dfrac{x_1}{x_2} - \log\dfrac{l_1}{l_2}\log\dfrac{x_1}{x_3}}{\log\dfrac{x_1}{x_2}\log\dfrac{1-x_1}{1-x_3} - \log\dfrac{1-x_1}{1-x_2}\log\dfrac{x_1}{x_3}} \\
\beta &= \frac{\log\dfrac{l_1}{l_3}\log\dfrac{1-x_1}{1-x_2} - \log\dfrac{l_1}{l_2}\log\dfrac{1-x_1}{1-x_3}}{\log\dfrac{x_1}{x_2}\log\dfrac{1-x_1}{1-x_3} - \log\dfrac{1-x_1}{1-x_2}\log\dfrac{x_1}{x_3}} \\
A &= \frac{l_1}{(1-x_1)^{\alpha}\,x_1^{-\beta}} \\
B &= \frac{A\,(1-x_4)^{\alpha}\,x_4^{-\beta}\ - l_4}{A\,(1-x_4)^{\alpha}\,x_4^{-\beta}\ (x_4-x_1)\,(x_4-x_2)\,(x_4-x_3)}\ .
\end{aligned} \tag{B.14}$$

We now rewrite formula (B.3) in the following way

$$\begin{aligned}
\sigma(\tau) &= \int_0^1 \frac{dx}{x}\left(\mathcal{L}(x,Q^2) - \mathcal{L}^{(0)}(x,\tau)\right)\frac{1}{2\pi i}\int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty}\left(\frac{\tau}{x}\right)^{-N}\ \Delta_N(Q^2)\,dN \\
&\quad + \frac{1}{2\pi i}\int_{C_{\rm MP}-i\infty}^{C_{\rm MP}+i\infty}\ \tau^{-N}\ \mathcal{L}_N^{(0)}(\tau)\ \Delta_N(Q^2)\,dN\ ,
\end{aligned} \tag{B.15}$$

where as usual

$$\mathcal{L}_N^{(0)}(\tau) = \int_0^1 \frac{dx}{x}\, x^N\, \mathcal{L}^{(0)}(x,\tau)\ , \tag{B.16}$$

which is easily performed analytically for possibly complex values of N. The difficult region of $x \to \tau$ is now strongly suppressed in the first integral of eq. (B.15), and the whole formula can be implemented numerically with no further difficulties. The only problem one encounters is instabilities with structure function packages that perform piecewise interpolation of tables, since they do not give a smooth structure function. Thus, we found it much easier to use the CTEQ sets, which are analytic in x, rather than the MRS sets, which use an interpolation procedure.

APPENDIX C: Some formulae and theorems on the Mellin transforms

Let us begin by discussing the basic result

$$\int_{C-i\infty}^{C+i\infty} \frac{dN}{N}\, e^{-N\log x + \log N\, F(\alpha_S \log N)} = \theta(1-x)\, e^{l\,F(\alpha_S l)} \times [1 + E(\alpha_S, l)] \tag{C.1}$$

where

$$\begin{aligned} l &= -\log(-\log x) \approx \log\frac{1}{1-x}\ , \\ F(z) &= \sum_{k=1}^{\infty} f_k\, z^k \\ E(\alpha_S, l) &= \sum_{k=1}^{\infty} \alpha_S^k \sum_{j=0}^{k} e_{kj}\, l^j\ . \end{aligned} \tag{C.2}$$

Therefore, the E term is a NLL correction, since its power expansion has no terms with more powers of l than of α_S. Equation (C.1) should be interpreted in the sense of its formal power expansion in α_S. A simple proof has been given in ref. [19]. For completeness, we shortly report here the basic argument. We rewrite the integral in terms of $z = -N\log x$

$$\begin{aligned} \int \frac{dN}{N}\, e^{-N\log x + \log N\, F(\alpha_S \log N)} &= \int \frac{dz}{z}\, e^{z + [\log z + l]\, F(\alpha_S[\log z + l])} \\ &= e^{l\,F(\alpha_S l)} \times \int \frac{dz}{z}\, e^{z + G(\log z, \alpha_S, l)}\ , \end{aligned} \tag{C.3}$$

where

$$G(\log z, \alpha_S, l) = [\log z + l]\, F(\alpha_S[\log z + l]) - l\, F(\alpha_S l) \ = \sum_{k=1}^{\infty} \alpha_S^k \sum_{j=0}^{k} g_{kj}(\log z)\, l^j \ , \quad \text{(C.4)}$$

where the coefficients $g_{kj}(\log z)$ are polynomials in $\log z$. Therefore G represents a NLL correction. Expanding its exponential, and integrating term by term in z, we get precisely an expression of the form $1 + E(\alpha_S, l)$, which is still a NLL correction. By taking the derivative of both sides of eq. (C.1) with respect to $\log x$ we get

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} dN\, e^{-N \log x + \log N\, F(\alpha_S \log N)} = x \frac{d}{dx} \left\{ \theta(1-x)\, e^{l\, F(\alpha_S l)} \times [1 + E(\alpha_S, l)] \right\} \ , \quad \text{(C.5)}$$

which has the form used in section 3.

In section 3, we encounter the Mellin transforms

$$\frac{1}{2\pi i} \int dN\, x^{-N} \ \log^m N \ . \quad \text{(C.6)}$$

We have

$$R_{m,k}(x) \equiv \int \frac{dN}{2\pi i}\, x^{-N} \frac{\log^m N}{N^k} = (-)^m \frac{d^m}{dk^m}\, T(k, x), \quad \text{(C.7)}$$

where

$$T(k, x) = \int \frac{dN}{2\pi i} x^{-N} e^{-k \log N} = \frac{1}{\Gamma(k)} \left(\log \frac{1}{x} \right)^{k-1} \ , \quad \mathrm{Re}\, k > 0\, . \quad \text{(C.8)}$$

To prove the above formula, observe that it can always be interpreted according to a contour distortion, as the one given in fig. 1. If $k < 1$, the contour can be made to coincide with the negative real axis, without convergence problems:

$$\begin{aligned} I(k) &= -\int dt\ x^t\ \frac{e^{-k(i\pi + \log t)} - \mathrm{cc}}{2\pi i} \\ &= \frac{\sin(k\pi)}{\pi} \int dt e^{-t \log \frac{1}{x}} t^{-k} \\ &= \left(\log \frac{1}{x} \right)^{k-1} \frac{\sin(k\pi)\, \Gamma(1-k)}{\pi} \\ &= \frac{1}{\Gamma(k)} \left(\log \frac{1}{x} \right)^{k-1} \end{aligned} \quad \text{(C.9)}$$

(using $\Gamma(1-x)\Gamma(x)\sin(\pi x) = \pi$). This proof is valid for $k < 1$. On the other hand, $I(k)$ can also be given in a form that is manifestly analytical for all $\mathrm{Re}\, k > 0$, by distorting the contour away from $N = 0$. By analytic continuation we can therefore extend the above result to all the $\mathrm{Re}\, k > 0$ region. We finally get the result

$$R_{m,k}(x) = \left(-\frac{\partial}{\partial k}\right)^m e^{(k-1)\log\log\frac{1}{x} - \log\Gamma(k)} \; . \tag{C.10}$$

From the above formula we also obtain the property

$$\int_0^1 dx\, R_{m,k}(x) = 0 \; , \tag{C.11}$$

valid for $k, m > 0$, which follows from

$$\int_0^1 \log^{k-1}\left(\frac{1}{x}\right) \, dx \; = \; \Gamma(k) \, . \tag{C.12}$$

Notice now that

$$R_{m,1}(x) = \left(-\log\log\frac{1}{x}\right)^m - m\left(-\log\log\frac{1}{x}\right)^{m-1} \gamma_E + \ldots \; . \tag{C.13}$$

If we keep only the leading logarithmic term in the expansion, we get

$$\int_0^1 R_{m,1}(x)\, dx \approx \int_0^1 \left(-\log\log\frac{1}{x}\right)^m \, dx \approx m! \; . \tag{C.14}$$

Observe however that also the integral of the first subleading term is proportional to $m!$, with opposite sign. In fact, if we keep all subleading terms we must get zero, as shown earlier. These are precisely the dangerous, factorially growing terms that arise when neglecting next-to-leading terms in the x-space resummation formulae.

References

[1] Yu.L. Dokshitzer, D.I. Dyakonov and S.I. Troyan, *Phys. Rep.* **58**(1980)271;
G. Parisi and R. Petronzio, *Nucl. Phys.* **B154**(1979)427;
G. Parisi, *Phys. Lett.* **B90**(1980)295;
G. Curci and M. Greco, *Phys. Lett.* **B92**(1980)175; *Phys. Lett.* **B102**(1981)280;
D. Amati et al., *Nucl. Phys.* **B173**(1980)429;
M. Ciafaloni and G. Curci, *Phys. Lett.* **B102**(1981)352;
P. Chiappetta, T. Grandou, M. Le Bellac and J.L. Meunier, *Nucl. Phys.* **B207**(1982)251.

[2] G. Altarelli, R.K. Ellis and G. Martinelli, *Nucl. Phys.* **B157**(1979)461;
J. Kubar-André and F.E. Page, *Phys. Rev.* **D19**(1979)221.

[3] R. Hamberg, W.L. van Neerven, and T. Matsuura, *Nucl. Phys.* **B359**(1991)343;
W.L. van Neerven and E.B. Zijlstra, *Nucl. Phys.* **B382**(1992)11.

[4] G. Sterman, *Nucl. Phys.* **B281**(1987)310.

[5] S. Catani and L. Trentadue, *Nucl. Phys.* **B327**(1989)323, *Nucl. Phys.* **B353**(1991)183.

[6] E. Laenen, J. Smith and W.L. van Neerven, *Nucl. Phys.* **B369**(1992)543; *Phys. Lett.* **B321**(1994)254.

[7] E. Berger and H. Contopanagos, *Phys. Lett.* **B361**(1995)115; ANL-HEP-PR-95-82, hep-ph/9603326.

[8] G. Sterman, talk presented at the 10^{th} Rencontres de Physique de la Vallée d'Aoste, La Thuile, Val d'Aosta, March 3-9, 1996;
N. Kidonakis and G. Sterman, ITP-SB-97-7, hep-ph/9604234;
H. Contopanagos, E. Laenen and G. Sterman, ANL-HEP-25, hep-ph/9604313.

[9] D. Appel, G. Sterman and P. Mackenzie, *Nucl. Phys.* **B309**(1988)259.

[10] H. Contopanagos and G. Sterman, *Nucl. Phys.* **B419**(1994)77;
L. Alvero and H. Contopanagos, *Nucl. Phys.* **B436**(1995)184.

[11] H. Contopanagos, L. Alvero and G. Sterman, ANL-HEP-CP-94-53, hep-ph/9408393;
L. Alvero and H. Contopanagos, ANL-HEP-PR-94-59, hep-ph/9411294;
L. Alvero, ITP-SB-94-67, hep-ph/9412335.

[12] G.P. Korchemsky and G. Sterman, *Nucl. Phys.* **B437**(1995)415.

[13] R. Akhoury and V.I. Zakharov, *Phys. Lett.* **B357**(1995)646.

[14] M. Beneke and V.M. Braun, *Nucl. Phys.* **B454**(1995)253, hep-ph/9506452.

[15] Yu.L. Dokshitzer, G. Marchesini and B.R. Webber, preprint CERN-TH-95-281, hep-ph/9512336.

[16] R. Akhoury and V.I. Zakharov, UM-TH-95-33, Dec 1995, hep-ph/9512433

[17] J. Kodaira and L. Trentadue, *Phys. Lett.* **B112**(1982)66, *Phys. Lett.* **B123**(1983)335 and SLAC-PUB-2934(1982), unpublished;
S. Catani, E. d'Emilio and L. Trentadue, *Phys. Lett.* **B211**(1988)335.

[18] S. Catani, G. Marchesini and B.R. Webber, *Nucl. Phys.* **B349**(1991)635.

[19] S. Catani, L. Trentadue, G. Turnock and B.R. Webber, *Nucl. Phys.* **B407**(1993)3.

[20] N. Kidonakis and J. Smith, ITP-SB-95-16, hep-ph/9506253.

[21] P. Nason, S. Dawson and R. K. Ellis, *Nucl. Phys.* **B303**(1988)607-633.

[22] J. Botts et al., *Phys. Lett.* **B304**(1993)159.

[23] S. Catani, M.L. Mangano, P. Nason and L. Trentadue, CERN-TH/96-21, hep-ph/9602208, to appear in Phys. Lett. B.

[24] R.K. Ellis and D.A. Ross, *Nucl. Phys.* **B345**(1990)79;
S. Catani, M. Ciafaloni and F. Hautmann, *Phys. Lett.* **B242**(1990)97, *Nucl. Phys.* **B366**(1991)135, *Nucl. Phys. (Proc. Suppl.)* **23A**(1991)328;
J.C. Collins and R.K. Ellis, *Nucl. Phys.* **B360**(1991)3.

[25] A.D. Martin, R.G. Roberts and W.J. Stirling, *Phys. Rev.* **D50**(1994)6734.

[26] S. Frixione, M.L. Mangano, P. Nason and G. Ridolfi, *Nucl. Phys.* **B431**(1994)453.

[27] F. Abe et al., CDF Coll., FNAL-Pub-96/020-E.

[28] F. Aversa, P. Chiappetta, M. Greco and J.P.Guillet, *Nucl. Phys.* **B327**(1989)105, *Phys. Rev. Lett.* **65**(1990)401;
S. Ellis, Z. Kunszt and D. Soper, *Phys. Rev.* **D40**(1989)2188, *Phys. Rev. Lett.* **64**(1990)2121.

[29] E. Buckley-Geer, for the CDF Collaboration, FERMILAB-CONF-95-316-E, Sep 1995, presented at International Europhysics Conference on High Energy Physics (HEP 95), Brussels, Belgium, 1995.

[30] B.L. Combridge and C.L. Maxwell, *Nucl. Phys.* **B239**(1984)429;
F. Halzen and P. Hoyer, *Phys. Lett.* **B130**(1983)326.

[31] S. Ellis, Z. Kunszt and D. Soper, *Phys. Rev. Lett.* **69**(1992)1496;
W.T. Giele, E.W.N. Glover and D.A. Kosower, *Phys. Rev. Lett.* **73**(1994)2019.

[32] J. Huston et al., MSU-HEP-50812, hep-ph/9511386.

[33] E.N. Glover, A.D. Martin, R.G. Roberts and W.J. Stirling, DTP/96/22, hep-ph/9603327.

[34] G.P. Korchemsky , *Mod. Phys. Lett.* **A4**(1989)1257.