Low-Energy Analysis of M and F Theories

on Calabi-Yau Threefolds

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Abstract

We elucidate the interplay between gauge and supersymmetry anomalies in six-dimensional $N = 1$ supergravity with generalized couplings between tensor and vector multiplets. We derive the structure of the five-dimensional supergravity resulting from the S_1 reduction of these models and give the constraints on Chern-Simons couplings that follow from duality to M theory compactified on a Calabi-Yau threefold. The duality is supported only on a restricted class of Calabi-Yau threefolds and requires a special type of intersection form. We derive five-dimensional central-charge formulas and discuss briefly the associated phase transitions. Finally, we exhibit connections with F-theory compactifications on Calabi-Yau manifolds that admit elliptic fibrations. This analysis suggests that F theory unifies Type-IIb three branes and M-theory five branes.

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1 Introduction

Six-dimensional $N = 1$ supergravity theories with an arbitrary number of tensor multiplets [1, 2] arise very naturally in open descendants [3] of Type-IIb K_3 reductions [4]. They also correspond to non-perturbative heterotic vacua [5, 6], and play a role in orbifold compactifications of M theory [7, 8] and in Calabi-Yau (CY) compactifications of F theory [9, 10]. In this paper we discuss the low-energy field theory of these models both from a five and from a six-dimensional viewpoint, relating them to eleven-dimensional supergravity [11] (the low-energy limit of M theory) compactified on CY manifolds, as well as to F theory.

Upon circle compactification, six-dimensional $N = 1$ supergravity with n_T tensor multiplets, n_V vector multiplets and n_H hypermultiplets yields five-dimensional simple supergravity with $n_T + n_V + 1$ vector multiplets and n_H hypermultiplets. The resulting five-dimensional interactions of the vector multiplets are entirely specified by the Chern-Simons couplings of the $n_T + n_V + 2$ vector fields [12],

$$C_{\Lambda\Sigma\Delta} \int A^{\Lambda} \wedge F^{\Sigma} \wedge F^{\Delta} \quad . \tag{1.1}$$

In compactifications of eleven-dimensional supergravity on CY threefolds, the symmetric constants $C_{\Lambda\Sigma\Delta}$ are the intersection numbers. The vector fields arise from the expansion of the three-form tensor in the $H^{1,1}$ cohomology, whose dimension is related to the number of massless multiplets by $h^{1,1} = n_V + n_T + 2$. The CY volume deformation belongs to a five-dimensional universal hypermultiplet, so that $n_H = h^{2,1} + 1$. This is also the counting for six-dimensional vacua obtained from F theory on the same CY manifold. Moreover, the vanishing condition for the irreducible part of the six-dimensional gravitational anomaly, $n_H - n_V + 29\, n_T = 273$, requires that

$$n_T = 9 + \frac{1}{60}\, \chi \quad , \tag{1.2}$$

where $\chi = 2(h^{1,1} - h^{2,1})$ is the Euler characteristic of the CY threefold. This relation holds when all six-dimensional vectors are abelian.

When the five-dimensional theory is obtained by S_1 compactification from six dimen-

sions, the intersection numbers $C_{\Lambda\Sigma\Delta}$ are subject to certain restrictions, that should be satisfied in F-theory constructions and are necessary constraints on CY threefolds in order that a six-dimensional interpretation be possible. In particular, CY threefolds that are elliptic fibrations [10] should fall in this class. The restrictions generalize those that associate M-theory compactifications with heterotic duals to CY threefolds that are K_3 fibrations. In these cases, the effective field theory contains a preferred vector that may be turned into an antisymmetric tensor of the dual heterotic theory. In the more general case of $n_T > 1$, we shall see that the restrictions on the CY threefold allow a total of n_T such preferred vectors. The restrictions overlap with those resulting from K_3 fibrations, but in general they differ. In particular, when $n_V = 0$ the five-dimensional vector multiplet moduli space is $O(1,1) \times \frac{O(1,n_T)}{O(n_T)}$, a result inherited from the six-dimensional couplings of [1]. This property will play a crucial role in the F-theory interpretation of the $(11,11)$ CY threefold, the simplest example exhibiting these features. When $n_V > 0$ and $n_T \geq 1$, the moduli space is no longer homogeneous (aside from the special case $n_T = 1$, and in the absence of certain Chern-Simons couplings for vector multiplets, where one can obtain $O(1,1) \times \frac{O(1,n_V+1)}{O(n_V+1)}$), and the theory will generally undergo phase transitions for finite values of the moduli [2, 5], as already occurs in six dimensions.

The plan of this paper is as follows. In Section 2 we extend the results of [2] on the supersymmetric coupling of tensor multiplets and YM multiplets. In particular, we elucidate the interplay between gauge and supersymmetry anomalies in these theories and show that, already at the lowest order in the fermi fields, the anomalous supersymmetry Ward identities lead to Yang-Mills currents different from those of [2], that embody the consistent form of the gauge anomaly. For the case of a single tensor multiplet coupled to supergravity, or more generally for (non-supersymmetric) models with several tensors not restricted by (anti)self-duality, the resulting equations follow from a lagrangian, while the field equations for several tensor multiplets coupled to supergravity are nicely determined by the $O(1, n_T)$ symmetry. Despite the presence of anomalies, supersymmetry retains its predictive power, since the Wess-Zumino consistency conditions [13] link gauge and supersymmetry anomalies. A corresponding phenomenon occurs in globally

supersymmetric four-dimensional models [14]. In Section 3 we consider the reduction to five dimensions. This results in the standard form of five-dimensional supergravity [12], since the left-over anomaly, cohomologically trivial, may be disposed of by a local counterterm involving solely the n_V vector multiplets. In Section 4 we derive central-charge formulas for five-dimensional electric (point-like) and magnetic (string-like) states. The six-dimensional phase transitions are accompanied by additional ones that are briefly discussed. In Section 5 we compare our results to F-theory constructions and analyze the effective lagrangians of some CY threefolds. Certain models (*e.g.* in the (11,11) CY threefold) are suggestive of dual descriptions related by the interchange of $(n_T - 1)$ and n_V. Finally, in Section 6 we display some geometrical twelve-dimensional couplings[1] that suggest that F theory unifies the Type-IIb three brane with the M-theory five brane.

2   Tensor and Vector Multiplets in Six-Dimensional Supergravity

In this Section we extend the construction of [2], elucidating some features of the resulting field equations that may play a role in future constructions. In particular, we display a new phenomenon in supergravity, whereby gauge anomalies reflect themselves in suitable supersymmetry anomalies, fully determined by the Wess-Zumino consistency conditions [13]. The analogue of this phenomenon in globally supersymmetric models was discussed in [14].

A generalized Green-Schwarz mechanism [15] in six-dimensional supergravity, involving several tensor and vector multiplets, was motivated by the systematic appearance, first noted in [4], of several tensor multiplets in the open descendants [3] of Type-IIb K_3 compactifications. The coupling of several tensor multiplets to simple six-dimensional supergravity had been studied previously by Romans [1] as an interesting generalization of the methods of [16] to models with scalar fields, while the coupling of vector multiplets

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[1]Geometrical couplings do not involve the space-time metric. Topological couplings are a subset of these that give vanishing results upon integration over topologically trivial manifolds.

to six-dimensional supergravity with a single tensor multiplet was originally considered in [17]. Couplings related to [2] have been recently displayed in [6] for models with a single tensor multiplet, where a lagrangian can be explicitly constructed.

After all tadpole constraints were enforced in the open descendants of [4], the residual anomaly polynomial revealed, upon diagonalization, an $O(1, n_T)$-like structure of the type

$$A \sim \sum_{r,s} \eta_{rs} \ c^{rz} \ c^{sz'} \ Tr_z(F^2) \ Tr_{z'}(F^2) \quad . \tag{2.1}$$

As in [2], η_{rs} denotes a Minkowski metric of signature $(1 - n_T)$ and z labels the various simple factors of the gauge group, while the c_z^r are rescaled in all component expressions. In all the models of [4] that have been analyzed, the gravitational contribution is confined to the $r = 0$ term. It is of higher order in the derivatives, and thus we shall ignore it as in [2], though we shall return to it at the end of this Section.

Whereas the conventional Green-Schwarz mechanism [15] does not apply directly to these models, the residual anomaly may be removed by the combined action of several antisymmetric tensors, as dictated by the constants c_z^r. In diagonal rational open-string models these have a microscopic interpretation in terms of the S matrix of the conformal theory [2], while in more general rational models they are related to the tensors A introduced in [18]. One is thus led to consider generalized Chern-Simons couplings of antisymmetric tensors valued in the vector representation of $O(1, n_T)$, with field strengths

$$H^r \ = \ dB^r \ - \ c^{rz} \ \omega_z \quad , \tag{2.2}$$

where ω_z are Chern-Simons three forms for the gauge fields of the vector multiplets. As usual, the gauge invariance of H^r [19] demands that B^r change under vector gauge transformations according to

$$\delta B^r \ = \ c^{rz} \ Tr_z(\Lambda dA) \quad . \tag{2.3}$$

It should be appreciated that in these models the Chern-Simons couplings are induced by the residual gauge anomaly. This feature will be reflected in the resulting field equations, that embody the residual gauge anomaly. As we shall see, the latter is quadratic in the c^{rz}, that draw their origin naturally from genus-$\frac{1}{2}$ open-string amplitudes [2].

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Together with the antisymmetric tensor fields B^r and the gauge fields A, the low-energy field theory includes the vielbein and the scalar field coordinates v_r of $\frac{O(1,n_T)}{O(n_T)}$, that satisfy the quadratic constraint

$$\mathcal{V} \;=\; \eta_{rs}\, v^r\, v^s \;=\; v^r\, v_r \;=\; 1 \quad . \tag{2.4}$$

The additional elements of the scalar $O(1, n_T)$ matrix, denoted by x_r^m in [1, 2], satisfy the constraints

$$v_r\, v_s \;-\; x_r^m\, x_s^m \;=\; \eta_{rs} \qquad\qquad \eta^{rs}\, x_r^m\, x_s^n \;=\; -\,\delta^{mn} \tag{2.5}$$

and enter the composite $O(n_T)$ connection. The fermionic fields are a pair of left-handed gravitini ψ_μ, n_T pairs of right-handed spinors χ^m and pairs of left-handed gaugini λ, all satisfying symplectic Majorana-Weyl conditions. As in [2], we work with a space-time metric of signature $(1-5)$, restricting our attention to terms of lowest order in the fermi fields.

In order to simplify the equations for the bosonic fields, naturally neutral with respect to the local composite $O(n_T)$ symmetry, it is convenient to introduce the matrix

$$G_{rs} \;=\; -\,\frac{1}{2}\,\partial_r\partial_s\,\log(v^r\,v_r)\,|_\mathcal{V} \;=\; 2\,v_r\,v_s \;-\; \eta_{rs} \quad , \tag{2.6}$$

since it reduces the (anti)self-duality constraints for the antisymmetric tensors to

$$G_{rs}\,H^{s\mu\nu\rho} \;=\; \eta_{rs}\,*H^{s\mu\nu\rho} \;=\; \frac{1}{6e}\,\epsilon^{\mu\nu\rho\alpha\beta\gamma}\,H_{r\alpha\beta\gamma} \quad . \tag{2.7}$$

The second-order tensor equation of [2],

$$\nabla_\mu\,(G_{rs}H^{s\mu\nu\rho}) \;=\; -\,\frac{1}{4e}\,\epsilon^{\nu\rho\alpha\beta\gamma\delta}\,c_r^z\,Tr_z\,(F_{\alpha\beta}\,F_{\gamma\delta}) \quad , \tag{2.8}$$

also takes a simpler form, since now its source does not involve the scalar fields. This equation follows from the Bianchi identity for H^r resulting from eq. (2.2) and from the (anti)self-duality condition of eq. (2.7). Tensor current conservation is implied by the Bianchi identity for the vector field strengths.

To lowest order, the fermionic field equations are

$$\gamma^{\mu\nu\rho}\,D_\nu\,\psi_\rho \;+\; v_r\,H^{r\mu\nu\rho}\,\gamma_\nu\,\psi_\rho \;-\; \frac{i}{2}\,x_r^m\,H^{r\mu\nu\rho}\,\gamma_{\nu\rho}\,\chi^m \tag{2.9}$$

$$+\; \frac{i}{2}\,x_r^m\,\partial_\nu v^r\,\gamma^\nu\,\gamma^\mu\,\chi^m \;-\; \frac{1}{\sqrt{2}}\,\gamma^{\sigma\tau}\,\gamma^\mu\,v_r\,c^{rz}\,\mathrm{tr}_z(F_{\sigma\tau}\,\lambda) \;=\; 0$$

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$$\gamma^\mu \, D_\mu \, \chi^m \quad - \quad \frac{1}{12} \, v_r \, H^r{}_{\mu\nu\rho} \, \gamma^{\mu\nu\rho} \, \chi^m \, - \, \frac{i}{2} \, x^m_r \, H^{r\mu\nu\rho} \, \gamma_{\mu\nu} \, \psi_\rho \tag{2.10}$$

$$- \, \frac{i}{2} \, x^m_r \, \partial_\nu v^r \, \gamma^\mu \, \gamma^\nu \, \psi_\mu \, - \, \frac{i}{\sqrt{2}} \, x^m{}_r \, c^{rz} \, \mathrm{tr}_z(\gamma^{\mu\nu}\lambda \, F_{\mu\nu}) \; = \; 0$$

$$\gamma^\mu \, D_\mu(v_r \, c^{rz} \, \lambda) \quad + \quad \frac{1}{2\sqrt{2}} \, (v_r \, c^{rz}) \, F_{\lambda\tau} \, \gamma^\mu \, \gamma^{\lambda\tau} \, \psi_\mu \tag{2.11}$$

$$+ \quad \frac{i}{2\sqrt{2}} \, (x^m{}_r \, c^{rz}) \, \gamma^{\mu\nu} \, \chi^m \, F_{\mu\nu} \, - \, \frac{1}{6} \, c^{rz} \, H^r{}_{\mu\nu\rho} \, \gamma^{\mu\nu\rho} \, \lambda \; = \; 0 \quad ,$$

where the last term in eq. (2.11) is (inexplicably) absent in [2].

All bosonic couplings in the bosonic field equations may obtained from the supersymmetry variations of eqs. (2.9), (2.10) and (2.11), making use of the first and second-order tensor equations. Thus, the scalar field equation is

$$x^m_r \, \nabla_\mu \, \partial^\mu \, v^r \, + \, \frac{2}{3} \, x^m_r \, v_s \, H^{r\mu\nu\rho} H^s{}_{\mu\nu\rho} \, - \, x^m_r \, c^{rz} \, Tr_z(F_{\alpha\beta} \, F^{\alpha\beta}) \; = \; 0 \quad , \tag{2.12}$$

where the overall x^m_r reflects the constraint of eq. (2.4), while Einstein's equations are

$$R_{\mu\nu} \quad - \quad \frac{1}{2} \, g_{\mu\nu} \, R \, + \, \partial_\mu v^r \, \partial_\nu v_r \, - \, \frac{1}{2} \, g_{\mu\nu} \, \partial^\rho v^r \, \partial_\rho v_r$$

$$- \quad G_{rs} \, H^r{}_{\mu\rho\sigma} \, H^s{}_\nu{}^{\rho\sigma} \, + \, 4 \, v_r \, c^{rz} \, \mathrm{tr}_z(F_{\lambda\mu} \, F^\lambda{}_\nu \, - \, \frac{1}{4} g_{\mu\nu} \, F^2) \; = \; 0 \quad . \tag{2.13}$$

Finally, the vector field equation is

$$D_\mu(v_r \, c^{rz} \, F^{\mu\nu}) \, - \, c^{rz} \, G_{rs} \, H^{s\nu\rho\sigma} \, F_{\rho\sigma} \; = \; 0 \quad . \tag{2.14}$$

Realizing the supersymmetry algebra on the fields requires a modification of the tensor transformation [2], and the resulting supersymmetry transformations are

$$\delta_\epsilon \, e_\mu{}^m \; = \; - \, i \, \bar\epsilon \, \gamma^m \, \psi_\mu$$

$$\delta_\epsilon \, \psi_\mu \; = \; D_\mu \, \epsilon \, + \, \frac{1}{4} \, v_r \, H^r{}_{\mu\nu\rho} \, \gamma^{\nu\rho} \epsilon$$

$$\delta_\epsilon \, B^r{}_{\mu\nu} \; = \; i \, v^r \, \bar\psi_{[\mu}\gamma_{\nu]}\epsilon \, + \, \frac{1}{2} \, x^{rm} \, \bar\chi^m \, \gamma_{\mu\nu}\epsilon \, - \, 2 \, c^{rz} \, \mathrm{tr}_z(A_{[\mu} \, \delta_\epsilon A_{\nu]})$$

$$\delta_\epsilon \, \chi^m \; = \; + \, \frac{i}{2} \, \partial_\mu v^r \, x^m_r \, \gamma^\mu \, \epsilon \, + \, \frac{i}{12} \, x^m_r \, H_{r\mu\nu\rho}{}^m \, \gamma^{\mu\nu\rho}\epsilon$$

$$\delta_\epsilon \, v_r \; = \; x^m{}_r \, \bar\epsilon \, \chi^m$$

$$\delta_\epsilon \, \lambda \; = \; - \, \frac{1}{2\sqrt{2}} \, F_{\mu\nu} \, \gamma^{\mu\nu}\epsilon$$

$$\delta_\epsilon \, A_\mu \; = \; - \, \frac{i}{\sqrt{2}}(\bar\epsilon \, \gamma_\mu \, \lambda) \tag{2.15}$$

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Both the field equations and the supersymmetry transformations should be completed by the addition of higher-order spinor terms. At any rate, one may verify that, on all bosonic fields, the commutator of two supersymmetry transformations closes on *all* local symmetries

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \;=\;& \delta_{\mathrm{gct}}(\xi^\mu = -i\bar{\epsilon}_1\gamma^\mu\epsilon_2) + \delta_{\mathrm{tens}}(\Lambda^r_\mu = -\frac{1}{2}v^r\xi^\mu - \xi^\nu B^r_{\mu\nu}) + \delta_{\mathrm{vect}}(\Lambda = -\xi^\mu A_\mu) \\
+\;& \delta_{\mathrm{susy}}(\zeta = -i\xi^\mu\psi_\mu) + \delta_{\mathrm{Lorentz}}(\Omega^{mn} = \xi^\mu(\omega^{mn}_\mu - v_r H^{rmn}_\mu)) \quad .
\end{aligned} \tag{2.16}
$$

Again, in deriving this result one must use the first-order equation for the tensor fields. In addition, the torsion equation contains a contribution from the gaugini:

$$
\begin{aligned}
D_\mu \, e^m_\nu \;-\; D_\nu \, e^m_\mu \;+&\; i\,(\bar{\psi}_\mu \, \gamma^m \, \psi_\nu) - \frac{i}{4}\,(\bar{\chi}^m_\mu \, \gamma_{\mu\nu}{}^m \, \chi^m) \\
+&\; \frac{i}{2}\, v_r \, c^{rz} \, Tr_z(\bar{\lambda} \, \gamma_{\mu\nu\rho} \, \lambda) \;=\; 0 \quad .
\end{aligned} \tag{2.17}
$$

Eq. (2.14) is quite peculiar. First of all, as noted in [2], it could be derived from the action

$$
\mathcal{L}_v \;=\; -\frac{e}{2}\, v_r \, c^{rz} \, Tr_z(F^{\mu\nu}F_{\mu\nu}) \quad , \tag{2.18}
$$

and thus the kinetic terms are positive *only* if the scalar fields are restricted to particular subregions of the moduli space, delimited by boundaries where the effective gauge couplings diverge. This, however, is a blessing in disguise, since as proposed in [5], the singularity signals a phase transition, that as in [6, 20] may be ascribed to tensionless strings, and helps one gain insight into the structure of six-dimensional vacua. The second unusual feature of eq. (2.14) is that the vector gauge currents

$$
J^\mu \;=\; 2\, c^{rz} \, G_{rs} \, H^{s\mu\rho\sigma} \, F_{\rho\sigma} \tag{2.19}
$$

that result from the coupling to the tensor multiplets are in general *not* conserved, as pertains to a theory with an anomaly that is to be disposed of by fermion loops. This is the reducible part of the six-dimensional gauge anomaly, the portion left-over after tadpole conditions are enforced in open-string loop amplitudes as in [4]. Indeed, taking the divergence of eq. (2.19) and making use of eq. (2.8) one finds

$$
D_\mu \, J^\mu \;=\; -\frac{1}{2e}\, \epsilon^{\mu\nu\alpha\beta\gamma\delta} \, c^{rz} \, c^{z'}_r \, F_{\mu\nu} \, Tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \quad , \tag{2.20}
$$

that may be recognized as the covariant form of the residual anomaly. It involves the $O(1, n_T)$ lorentzian product of pairs of c_z^r coefficients, a natural measure of the chirality of the tensor couplings.

It is interesting to ask whether one can arrive at a vector equation embodying the consistent form of the residual anomaly. That this should be possible is suggested by the long-held expectation that covariant and consistent anomalies are related by local counterterms [21]. Still, the latter form is more satisfactory, and pursuing this issue is quite instructive, since the solution of the problem rests on peculiar properties of the supersymmetry algebra. These have already emerged in globally supersymmetric models [14]. The basic observation is that these field equations embody a vector gauge anomaly and, as is usually the case in component formulations, the commutator of two supersymmetry transformations in eq. (2.16) involves the anomalous vector gauge transformations.

Even without a lagrangian formulation, combining the field equations with the corresponding supersymmetry variations one can retrieve the total variation of the effective action under local supersymmetry. Denoting by $\mathcal{A}_\Lambda$ the vector gauge anomaly and by $\mathcal{A}_\epsilon$ the supersymmetry anomaly, one has the Wess-Zumino consistency conditions

$$\delta_{\epsilon_1} \mathcal{A}_{\epsilon_2} \ - \ \delta_{\epsilon_2} \mathcal{A}_{\epsilon_1} \ = \ \mathcal{A}_{\Lambda(\epsilon_1, \epsilon_2)} \quad ,$$

$$\delta_\Lambda \mathcal{A}_\epsilon \ = \ \delta_\epsilon \mathcal{A}_\Lambda \quad , \tag{2.21}$$

the first of which clearly requires a non-vanishing supersymmetry anomaly. Here we shall confine our attention to the bosonic contributions to the vector current, though a systematic use of eqs. (2.21) would also determine the fermionic terms in $\mathcal{A}_\Lambda$ and $\mathcal{A}_\epsilon$, some of which may be anticipated from [14].

With the expected residual bosonic contribution to the vector gauge anomaly

$$\mathcal{A}_\Lambda \ = \ \gamma \ \epsilon^{\mu\nu\alpha\beta\gamma\delta} \ c^{rz} \ c_r^{z'} \ Tr_z(\Lambda \ \partial_\mu A_\nu) \ Tr_{z'}(F_{\alpha\beta} F_{\gamma\delta}) \quad , \tag{2.22}$$

one may verify that the second of eqs. (2.21) determines the relevant part of the supersymmetry anomaly,

$$\mathcal{A}_\epsilon \ = \ - \ \gamma \ \epsilon^{\mu\nu\alpha\beta\gamma\delta} \ c^{rz} \ c_r^{z'} \ Tr_z(\ A_\mu \delta_\epsilon A_\nu) \ Tr_{z'}(F_{\alpha\beta} F_{\gamma\delta})$$

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$$- \quad 4\,\gamma\,\epsilon^{\mu\nu\alpha\beta\gamma\delta}\,c^{rz}\,c_r^{z'}\,Tr_z(\delta_\epsilon A_\nu F_{\mu\alpha})\,\omega_{z'\beta\gamma\delta} \quad . \tag{2.23}$$

Whereas the other Wess-Zumino consistency condition of eq. (2.21) would fix this overall factor as well, it is simpler to fix it from the divergence of the gauge current. Indeed, while combining eq. (2.14) with the other field variations would result in an effective action invariant under supersymmetry, demanding that the total variation be $\mathcal{A}_\epsilon$ alters the gauge current, and in particular the choice $\gamma = -1/4$ reproduces precisely the consistent anomaly of eq. (2.22) ,

$$D_\mu\,\hat{J}^\mu \;=\; -\,\frac{1}{4e}\,\epsilon^{\mu\nu\alpha\beta\gamma\delta}\,c^{rz}\,c_r^{z'}\,\partial_\mu A_\nu\,Tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \quad . \tag{2.24}$$

This hints to a simpler way of obtaining these results, as well as to the rather general nature of the phenomenon, with potential applications in other contexts. As in the ten-dimensional Green-Schwarz mechanism [15], one is embodying in the dynamics a portion of the gauge anomaly that the antisymmetric tensors make cohomologically trivial. Differently from that case, however, the resulting modifications are already present at the lowest-order in the derivative expansion. The construction of field theories with these anomalies thus acquires some interest of its own, and one may dispense momentarily with the restriction to (anti)self-dual tensors introduced by six-dimensional supersymmetry to consider lagrangians of the type

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{e}{12}\,G_{rs}\,H^{r\mu\nu\rho}\,H^s{}_{\mu\nu\rho} - \frac{e}{2}\,v_r\,c^{rz}\,Tr_z(F_{\mu\nu}F^{\mu\nu}) \\
& - \frac{1}{8}\,\epsilon^{\mu\nu\alpha\beta\gamma\delta}\,\tilde{c}_r^z\,B_{\mu\nu}^r\,Tr_z(F_{\alpha\beta}F_{\gamma\delta}) - \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta\gamma\delta}\,c_r^z\,\tilde{c}^{rz'}\,\omega_{z\mu\nu\alpha}\,\omega_{z'\beta\gamma\delta}
\end{aligned} \tag{2.25}$$

for unconstrained tensor fields, with Chern-Simons couplings as in eq. (2.2) and geometrical couplings with coefficients $\tilde{c}^r$ in general different from the c^r that enter the tensor Bianchi identities. The relative normalization of the last two contributions is fixed by symmetry, and the resulting vector gauge current has the consistent anomaly

$$\mathcal{A}'_\Lambda \;=\; -\,\frac{1}{8}\,\epsilon^{\mu\nu\alpha\beta\gamma\delta}\,(c^{rz}\,\tilde{c}_s^{z'} + \tilde{c}^{rz}\,c_s^{z'})\,Tr_z(\Lambda\,\partial_\mu A_\nu)\,Tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \quad . \tag{2.26}$$

The last term in eq. (2.25) was considered in [22]. Though clearly vanishing for a single gauge factor or for c_z^r proportional to $\tilde{c}_z^r$, it restores the symmetry of the anomaly. For

a single tensor multiplet coupled to supergravity, the two different terms in eq. (2.25) reflect the existence of two second-order invariant tensors in $O(1,1)$, η_{rs} and ϵ_{rs}, a property not shared by larger $O(1, n_T)$ groups, where the restriction to (anti)self-dual tensors would imply the equality of the two sets of couplings and the vanishing of the additional contribution.

Let us conclude this Section by noticing that the inclusion of gravitational Chern-Simons couplings would result in the presence, in eq. (2.25), of additional terms involving the gravitational curvature. These higher-derivative terms could be accommodated extending the range of z and treating the gravitational sector as an additional factor of the gauge group, with corresponding c^r coefficients clearly displayed in the residual anomaly polynomial. In all the models of [4] that have been analyzed explicitly, the gravitational contribution is confined to the $r = 0$ term of the anomaly polynomial and couples to the sum of the vector contributions.

3 Reduction to Five Dimensions and M Theory on a CY Threefold

The five-dimensional action resulting from a CY compactification of M theory includes geometrical interactions between the $h^{1,1}$ vectors arising from the eleven-dimensional three-form [23, 24, 25, 26]

$$I_5^{geom} = -\frac{1}{12}\, C_{\Lambda\Sigma\Delta} \int_{\mathcal{M}_5} A_1^\Lambda \wedge F_2^\Sigma \wedge F_2^\Delta \quad . \tag{3.1}$$

The intersection numbers $C_{\Lambda\Sigma\Delta}$ determine the metric on the vector moduli space [12].

The requirement that this theory be dual to the reduction of a six-dimensional theory with an arbitrary number of tensor and vector multiplets restricts the form of this interaction, and thus the intersection form of the CY manifold. Special properties of the intersection form simplify the analysis significantly. First, it is a polynomial, and the different contributions are additive. Different special regimes can be analyzed separately and then joined together in the general expression. Second, it is a set of numbers, moduli

independent, and thus all scalar fields can be set to constant values when analyzing it.

Our strategy will be to analyze in some detail the reduction of the $n_T = 1$ lagrangian to then extend the results in an $O(1, n_T)$ covariant fashion to $n_T > 1$ case. In order to simplify the comparison with M theory, one can perform the reduction at a generic point of the moduli space where all vector fields are abelian. The coefficients c_z^r now become c_{xy}^r, where x, y $= 1, .., n_V$. For $n_T = 1$, v_{xy} and $\tilde{v}_{xy}$ are $c_{0xy} \pm c_{1xy}$ respectively and, following [6], we define the $O(1, 1)$ "light-cone" decomposition

$$
\begin{aligned}
\omega &= \sum v^i \, \omega_i \quad , \\
\tilde{\omega} &= \sum \tilde{v}^i \, \omega_i \quad .
\end{aligned}
\tag{3.2}
$$

The standard compactification ansatz for the vielbein,

$$
e_{\hat{\mu}}^{\hat{a}} = \begin{pmatrix} e_\mu^a & rZ_\mu \\ 0 & r \end{pmatrix} \quad ,
\tag{3.3}
$$

determines the part of the five-dimensional lagrangian originating from the six-dimensional tensor multiplets and their interactions,

$$
\begin{aligned}
\mathcal{L}_5 &= \frac{r}{2} e^{-2\phi} (H - H^6 \wedge Z) \wedge *(H - H^6 \wedge Z) + \frac{1}{2r} e^{-2\phi} H^6 \wedge *H^6 \\
&+ H \wedge \tilde{\omega}^6 - H^6 \wedge \tilde{\omega} + \frac{1}{2} \omega \wedge \tilde{\omega}^6 - \frac{1}{2} \omega^6 \wedge \tilde{\omega} \quad .
\end{aligned}
\tag{3.4}
$$

Here we use slightly different conventions with respect to the previous Section, namely a form language and a space-time metric of signature $(5-1)$. H^6 and ω^6 denote the internal parts of the corresponding forms. Adding a Lagrange multiplier $H^0 \wedge F = H \wedge F + \omega \wedge F$ (locally, the solution for F is $F = dC$), one gets

$$
*H = *(H^6 \wedge Z) + \frac{e^{2\phi}}{r} \hat{F} \quad ,
\tag{3.5}
$$

where $\hat{F} = F - \tilde{\omega}^6$. The dualized lagrangian is then

$$
\begin{aligned}
\tilde{\mathcal{L}}_5 &= \frac{1}{2r} e^{2\phi} \hat{F} \wedge *\hat{F} + \frac{1}{2r} e^{-2\phi} H^6 \wedge *H^6 - H^6 \wedge Z \wedge \hat{F} \\
&- \omega \wedge \hat{F} - H^6 \wedge \tilde{\omega} - \frac{1}{2} \omega \wedge \tilde{\omega}^6 - \frac{1}{2} \omega^6 \wedge \tilde{\omega} \quad .
\end{aligned}
\tag{3.6}
$$

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In order to compare $\tilde{\mathcal{L}}_5$ to the standard form of five-dimensional simple supergravity, one needs also to perform the following redefinitions of the abelian gauge fields:

$$\hat{A}^{\mathrm{x}}_\mu = A^{\mathrm{x}}_\mu - a^{\mathrm{x}} Z_\mu \ ,$$
$$\hat{B}_\mu = B_{\mu 6} - v_{\mathrm{xy}} a^{\mathrm{x}} \hat{A}^{\mathrm{y}}_\mu \ ,$$
$$\hat{C}_\mu = C_\mu - \tilde{v}_{\mathrm{xy}} a^{\mathrm{x}} \hat{A}^{\mathrm{y}}_\mu \ . \tag{3.7}$$

The resulting lagrangian contains three types of Chern-Simons couplings,

$$Z \, d\hat{B} \qquad \hat{C} \, v_{\mathrm{xy}} \, d\hat{A}^x \, d\hat{A}^y \qquad \text{and} \qquad \hat{B} \, \tilde{v}_{\mathrm{xy}} \, d\hat{A}^x \, d\hat{A}^y \ . \tag{3.8}$$

They may be compared to the geometrical interaction (3.1), and thus to the intersection numbers $C_{\Lambda\Sigma\Delta}$ of the Calabi-Yau manifold. The $(n_T + n_V + 1)$ scalar fields parametrize the hypersurface $\mathcal{V} = 1$, where $\mathcal{V} = C_{\Lambda\Sigma\Delta} X^\Lambda X^\Sigma X^\Delta$, with $X^\Lambda(\phi)$ a set of $(n_T + n_V + 2)$ special coordinates. In the case of a single tensor multiplet, the intersection form is

$$\mathcal{V} = z \, b \, c - \frac{1}{2} \, b \, \tilde{v}_{\mathrm{xy}} \, a^{\mathrm{x}} \, a^{\mathrm{y}} - \frac{1}{2} \, c \, v_{\mathrm{xy}} \, a^{\mathrm{x}} \, a^{\mathrm{y}} \ . \tag{3.9}$$

The generalization to $n_T > 1$ is fully determined by the $O(1, n_T)$ symmetry. Reverting to the vector $O(1, n_T)$ notation of Section 2 and defining $b^r = (\frac{b+c}{2}, \frac{c-b}{2}, b^{r'})$, with $(r' = 1, ..., n_T - 1)$, one finally obtains

$$\mathcal{V} = z \, b^r \, \eta_{rs} \, b^s - b^r \, c_{r\mathrm{xy}} \, a^{\mathrm{x}} \, a^{\mathrm{y}} \ , \tag{3.10}$$

in terms of the special coordinates $X^\Lambda = (z, b^r, a^{\mathrm{x}})$. This structure of the intersection form is consistent with the condition that the manifold admits elliptic fibrations [9, 10].

Cubic terms in any of the moduli should generally be allowed in eq. (3.10), since they are compatible with five-dimensional supersymmetry. In six dimensions, these terms may correspond to a topological term $\int F \wedge F \wedge F$ for three six-dimensional vectors (see [26] for a recent discussion), and would not contribute to the field equations. The only prerequisite for having a^3 terms in $\mathcal{V}$ is thus the existence of a symmetric tensor d_{xyz} for the gauge group. In this case, the intersection form (3.10) can be augmented by a term $d_{\mathrm{xyz}} \, a^{\mathrm{x}} a^{\mathrm{y}} a^{\mathrm{z}}$. On the other hand, the absence in six dimensions of cubic interactions for

antisymmetric tensors alone implies that in general eq. (3.10) can not be modified by cubic terms in the b^r moduli, consistently with its general $O(1, n_T)$ structure.

Instanton contributions arising in the S_1 reduction from six to five dimensions may lead to additional terms in eq. (3.10). If these contributions are compatible with supersymmetry and modify the scalar kinetic terms, they must also modify the Chern-Simons couplings of fields other than b^r moduli. In the twelve-dimensional setting, these contributions can be seen to arise from two-brane instantons [27] when the two brane is wrapped around $S_1 \times \gamma_2^\alpha$, where γ_2^α are two-cycles on the CY manifold not on the base.

For $n_V = 0$ the moduli space reduces to

$$O(1,1) \times \frac{O(1, n_T)}{O(n_T)} \quad, \tag{3.11}$$

the original six-dimensional moduli space augmented with the radial mode. In a similar fashion, for $n_T = 1$ and $\tilde{c}_{rxy} = 0$ the space reduces to

$$O(1,1) \times \frac{O(1, n_V + 1)}{O(n_V + 1)} \quad, \tag{3.12}$$

where the second factor is the Narain lattice [38] associated with the S_1 reduction of the dual heterotic theory on K_3, while the first factor is the six-dimensional moduli space for $n_T = 1$. When both $n_T \geq 1$ and $n_V > 0$ the space is no longer homogeneous and kinetic terms may have singularities corresponding to phase transitions, thus extending the phenomenon of [2, 5]. This will be discussed in more detail in the following Section.

The final point we would like to stress is that, as discussed in the previous Section, the lagrangian in eq. (3.6) is not invariant under the vector gauge transformation. Consequently, the bosonic action obtained by reduction from six dimensions inherits an anomalous gauge variation. In five dimensions, however, this anomaly can be canceled by a local counterterm $\int A^{\mu x} J_{\mu x}$, since the current

$$*J_x \sim c_{rxy} \, c_{rzw} \left(F^z \wedge F^w \wedge a^y \, + \, 2 \, F^z \wedge a^w \wedge F^y \right) \tag{3.13}$$

is gauge invariant.

4 Five-Dimensional Central Charges and Phase Transitions

Phase transitions in six dimensions have received considerable attention lately, and have been studied extensively in [6, 10, 20, 26, 28, 29, 30, 31]. The analysis rests on the six-dimensional formula for the central charge, uniquely determined by the $O(1, n_T; Z)$ duality symmetry,

$$Z(\phi) = v^r(\phi) \, n_r \quad . \tag{4.1}$$

For $n_T = 1$, solving the constraint of eq. (2.4) gives

$$Z(\phi) = e^\phi \, n_e + e^{-\phi} \, n_m \quad , \tag{4.2}$$

with a phase transition at $e^{-2\phi} = -\frac{n_e}{n_m}$. Four-dimensional instanton configurations of the gauge group G_{x} with instanton numbers n_{x} would give $n_e = n_{\mathrm{x}}\tilde{v}^{\mathrm{x}}$, $n_m = n_{\mathrm{x}}v^{\mathrm{x}}$, thus leading to a central charge

$$Z(\phi) = n_{\mathrm{x}} \left(\tilde{v}^{\mathrm{x}} \, e^\phi + v^{\mathrm{x}} \, e^{-\phi} \right) \quad . \tag{4.3}$$

As discussed in detail in [6], $Z \to 0$ precisely when the gauge kinetic term of eq. (2.14) becomes singular. This is related to a new phenomenon, whereby a string becomes tensionless.

In turning to five dimensions, six-dimensional strings may wrap around S_1, giving rise to point-like objects, or else they may simply reduce to five-dimensional strings. The contribution of four-dimensional instantons to the five-dimensional charges may be dispelled from the equations

$$d * \left(\frac{1}{\lambda^2} e^{-2\phi} \, dB \right) = \tilde{v}^{\mathrm{x}} \, Tr_{\mathrm{x}}(F^2)$$
$$d * \left(\frac{1}{\lambda^2} e^{2\phi} \, dC \right) = v^{\mathrm{x}} \, Tr_{\mathrm{x}}(F^2) \quad . \tag{4.4}$$

These follow from the dimensionally-reduced kinetic terms and from the Chern-Simons couplings, once one sets $a^{\mathrm{x}} = 0$. The relevant part of the action is then

$$\Delta\mathcal{L} = R + \frac{1}{2} \lambda^4 \, dZ \wedge *dZ + \frac{1}{2} \lambda \left(e^{-\phi} v^{\mathrm{x}} + e^\phi \tilde{v}^{\mathrm{x}} \right) Tr_{\mathrm{x}}(F \wedge *F)$$
$$+ \frac{1}{\lambda^2} \left(e^{-2\phi} \, dB \wedge *dB + e^{2\phi} \, dC \wedge *dC \right) \quad , \tag{4.5}$$

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where $\lambda = r^{2/3}$ and ϕ denotes the six-dimensional dilaton. The central charges for electric (point-like) and magnetic (string-like) states [24] are completely fixed by five-dimensional simple supergravity, and are

$$Z_e = X^\Lambda \, e_\Lambda \qquad\qquad Z_m = X_\Lambda \, m^\Lambda \quad , \qquad\qquad (4.6)$$

where X^Λ are five-dimensional special coordinates and $X_\Lambda = C_{\Lambda\Sigma\Delta} \, X^\Sigma X^\Delta$ are "dual" coordinates.

Bearing in mind the structure of the intersection form (3.10), at $a^{\mathrm{x}} = 0$ the components of the gauge-field kinetic metric for $U(1)^3 \times \prod_{\mathrm{x}} G_{\mathrm{x}}$ are

$$G_{zz} = b^2 \, c^2 \qquad G_{bb} = \frac{1}{b^2} \qquad G_{cc} = \frac{1}{c^2} \qquad G_{\mathrm{xx}} = c \, v^{\mathrm{x}} + b \, \tilde{v}^{\mathrm{x}} \quad . \qquad (4.7)$$

Letting

$$b = \lambda \, e^\phi \qquad c = \lambda \, e^{-\phi} \qquad z = \frac{1}{bc} = \frac{1}{\lambda^2} \quad , \qquad\qquad (4.8)$$

one finds

$$\begin{aligned} Z_e &= \frac{1}{\lambda^2} \, e_2 + \lambda \left(e^\phi \, e_1 + e^{-\phi} \, e_0 \right) = \frac{1}{bc} \, e_2 + b \, e_1 + c \, e_0 \\ Z_m &= \lambda^2 \, m_2 + \frac{1}{\lambda} \left(e^{-\phi} \, m_1 + e^\phi \, m_0 \right) = bc \, m_2 + \frac{1}{b} \, m_1 + \frac{1}{c} \, m_0 \; . \end{aligned} \qquad (4.9)$$

The non-perturbative phase-transition points are

$$\begin{aligned} Z_e &= 0 \qquad \left(e_2 = 0, \quad \forall \lambda, \quad e^{2\phi} = -\frac{e_0}{e_1} = -\frac{v}{\tilde{v}} \right) \\ Z_m &= 0 \qquad \left(m_2 = 0, \quad \forall \lambda, \quad e^{2\phi} = -\frac{m_1}{m_0} = -\frac{v}{\tilde{v}} \right) \quad . \end{aligned} \qquad (4.10)$$

Thus tensionless strings give rise to five-dimensional massless particles as well as to tensionless strings [26]. These are non-perturbative singularities occurring at non-vanishing instanton numbers. In the zero instanton sector, $n_{\mathrm{x}} = 0$, one also expects perturbative BPS states that become massless.

$$\begin{aligned} Z_e &= 0 \qquad \left(e_0 = 0, \ \forall \phi, \ b^2 c = -\frac{e_2}{e_1} \right) \\ Z_m &= 0 \qquad \left(m_0 = 0, \ \forall \phi, \ b^2 c = -\frac{m_1}{m_2} \right) \quad . \end{aligned} \qquad (4.11)$$

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The non-perturbative singularities have an F-theory [9] interpretation in terms of the moduli space of the base manifold, since they involve ϕ, while the perturbative singularities are associated with the fiber, since they involve the radial modulus λ. In the general case of $a^{\mathrm{x}} \neq 0$ ($e^{\mathrm{x}}, m^{\mathrm{x}} \neq 0$), one expects more complicated singularities, that may also be understood from the study of the CY threefolds. Still, it is worth stressing that the general formula for the central charges, fully determined by the intersection form (3.10), allows one to study these phase transitions in arbitrary models in a rather general fashion.

5 Examples of Manifolds with $\chi = 0$

As seen from eq. (1.2), a vanishing Euler characteristic implies that the number of tensor multiplets in six dimensions is $n_T = 9$, while anomaly cancellation requires $n_H - n_V = 12$. An interesting feature of these models is that the reducible part of the gravitational anomaly vanishes as well. Indeed, $I(R) = I_{\frac{3}{2}} - (n_H + n_T - n_V)I_{\frac{1}{2}} - (n_T - 1)I_A$ vanishes identically when the coefficients of the second and third terms are 21 and 8 respectively [32]. We will consider two such models that provide an interesting laboratory for testing duality conjectures.

Another motivation for considering these examples is to gain a better understanding of the connection between the present construction and F theory that has already surfaced in our discussion. Since our field content is the same as that obtained from a CY compactification of F theory, one expects that the conditions for the existence of F theory on the same CY manifold be encoded in the intersection form (3.10) that can be lifted to six dimensions. It was shown in [9, 10] that, in order to obtain this six-dimensional spectrum, one must consider compactifications of F theory on manifolds that admit an elliptic fibration. The structure of the intersection form obtained from M theory agrees with that result. Moreover, it displays the relation between the number n_T of tensor multiplets and the number k of Kähler deformations that does not change the Kähler class of the elliptic fiber [9] : $k = n_T + 1$. In eq. (3.10), the b coordinates are moduli of the base manifold, $k = h^{1,1}(B)$, including the volume modulus that is in the hypermultiplet sector

[10]. Note that the Hodge numbers of the CY manifold related to the five-dimensional model have a six-dimensional interpretation, thus making the lifting to a hypothetical F theory [9] in twelve dimensions possible. Indeed,

$$h^{1,1} + h^{2,1} = n_T + 1 + n_V + n_H \quad , \tag{5.1}$$

where $(n_T + 1)$ is the number of (anti)self-dual tensors in six dimensions and n_V and n_H are the numbers of vector multiplets and hypermultiplets respectively. Moreover,

$$h^{2,1} = n_H - 1 \tag{5.2}$$

since, as discussed in [23], one of the $h^{1,1}$ moduli is the CY volume deformation.

The $(11, 11)$ *model*

This model was discussed in [33] in the context of heterotic-Type II duality, and more recently in [10] in a context related to this. A six-dimensional model with $n_T = 9$ and $n_H = 12$, a field content apparently in proper correspondence with the Hodge numbers, can be obtained from the Type-IIb string compactified on K_3 with a freely acting Z_2. Presumably, it can also be obtained from the completely higgsed $(3, 243)$ model with one tensor multiplet un-higgsing eight tensors. Our analysis is similar to the one in [33], where a CY manifold with Hodge numbers $(11, 11)$ was constructed as an orbifold of $K_3 \times T^2$. Upon compactification on K_3, the Type IIb string gives an $N = (2, 0)$ chiral theory in six dimensions with a supergravity multiplet that includes self-dual tensor fields in the 5 of $USp(4)$ coupled to 21 $N = 2$ tensor multiplets [35]. In this model the scalar fields parametrize $\frac{O(5,21)}{(O(5) \times O(21))}$ [1]. The theory can be truncated to $N = (1, 0)$ resorting to the involution σ of K_3. Under Z_2, $USp(4) \to USp(2) \times USp(2)$, with $4 \to (2, 1)^+ + (1, 2)^-$, and one of the gravitini changes sign. Moreover, $5 \to (1, 1)^+ + (2, 2)^-$, and since σ has eigenvalues $[(+1)^{10}; (-1)^{10}]$ when acting on $H^{1,1}$ and -1 when acting on $H^{(2,0)}$ and $H^{(0,2)}$ [33], $21 \to 9^+ + 12^-$. The moduli space of the $N = 1$ theory is the subspace even under the involution:

$$\mathcal{M}_T \times \mathcal{M}_H = \frac{O(1, 9)}{O(9)} \times \frac{O(4, 12)}{O(4) \times O(12)} \quad . \tag{5.3}$$

The result of [33] that the five-dimensional moduli space for the vector multiplets (the Kähler moduli space of the Enriques surface at fixed volume, together with the radial modulus) is $\mathcal{M}_V = O(1,1) \times \frac{O(1,9)}{O(9)}$ can now be understood in terms of the $n_T = 9$ tensor moduli space. Note that the tensor multiplets in this model come from the untwisted sector. In contrast to the construction of [3, 4, 36, 37], here the Z_2 involution does not require the open-string sector, since in this case $n_V = 0$. The theory can be thought of as a compactification of F theory on a CY manifold whose base is the Enriques surface ($h^{1,1} = 10$) [10], and in some sense is the simplest compactification of F theory.

Upon further reduction to five dimensions, one obtains a theory dual to M theory on a CY manifold with an intersection form

$$\mathcal{V} = z\,(b\,c \,-\, b_{r'}\,b^{r'}) \qquad (r' = 1, ..., n_T - 1) \quad . \tag{5.4}$$

The dual heterotic theory with $n_T = 1$ and $n_V = 8$ [33] is described by the same (exact) moduli space with an intersection form

$$\mathcal{V} = c(bz - a^x a^y d_{xy}) \quad . \tag{5.5}$$

Note that in latter case the a^3 term is absent, and we find a complete symmetry between these intersection forms. This symmetry interchanging vectors from six-dimensional vector multiplets and vectors from six-dimensional tensor multiplets is not present in a generic five-dimensional theory.

The $(19, 19)$ model

Let us now turn to another model discussed in [10] - a manifold with $h^{1,1} = h^{2,1} = 19$. The field content $n_T = 9$, $n_V = 8$ and $n_H = 20$ has been obtained via an orbifold construction of M theory[2] [7] and as an orientifold of the Type-IIb theory [37]. The model has also a dual perturbative heterotic description [34] if the heterotic string is first compactified on S_1, so that the gauge group is broken to $U(1)^{16}$, and is then compactified on K_3 to yield 17 vector multiplets and 20 hypermultiplets.

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[2]Other $n_T = 9$ models may be obtained from M theory on orientifolds of $K_3 \times S_1$ [8] and yield $n_V = 1, 2, 4$ and, correspondingly, $n_H = 13, 14, 18$.

The choice of a dilaton is naturally accompanied by the replacement of a pair of constrained tensors, its antiself-dual partner and the self-dual one in the gravity multiplet, with a single unconstrained tensor. After making a choice for the dilaton, one is thus left with equal numbers of (anti)self-dual tensors and vectors. Still, the nature of the six-dimensional geometrical (and topological) interactions does not allow a symmetry in the intersection form between the a and b moduli. Thus, although in five dimensions tensor multiplets are identical to vector multiplets, the theory "remembers" the origin of the vector fields. It would be of interest to find models that are dual under the interchange $(n_T - 1, n_V) \leftrightarrow (n'_T - 1 = n_V, n'_V = n_T - 1)$.

6 Twelve-Dimensional Interpretation of Low-Energy Couplings

In Section 2 we have discussed the six-dimensional origin of certain geometrical couplings, that upon reduction to five dimensions as in Section 3 may be related to the intersection form of suitable (elliptically fibered) CY reductions of M theory. The intersection forms are then in direct correspondence with the $F_4 \wedge F_4 \wedge A_3$ geometrical interaction of eleven-dimensional supergravity [11], the low-energy effective field theory of M theory.

In this Section we would like to return to our six-dimensional viewpoint in order to shed some light on the low-energy field equations of F theory. The resulting speculations are in the spirit of ref. [39], where the Seiberg-Witten construction [40] was reinterpreted in terms of a six-dimensional theory of (anti)self-dual tensor multiplets on $\mathcal{M}_4 \times C_r$, with C_r a genus r hyperelliptic Riemann surface. A recent realization of that proposal was presented in [31].

Let us begin by considering the twelve-dimensional geometrical coupling

$$T_{12} \;=\; \int_{\mathcal{M}_{12}} A_4 \wedge F_4 \wedge F_4 \quad , \tag{6.1}$$

where A_4 is the four-form potential of Type-IIb ten-dimensional supergravity lifted to twelve dimensions and F_4 is the four-form field strength of eleven-dimensional supergravity

lifted to twelve dimensions. If one assumes that the harmonic expansions of A_4 and A_3 on $M_6 \times CY$ are

$$A_4 = \sum_{i=1}^{h^{11}(B)=n_T+1} b_2^i \wedge \tilde{V}_2^i \quad , \tag{6.2}$$

where the $\tilde{V}_2^i$ are elements of $H^{1,1}(B)$, the cohomology of the base, and

$$A_3 = \sum_{\alpha=1}^{h^{11}-h^{11}(B)-1=n_V} b_2^\alpha \wedge \hat{V}_2^\alpha \quad , \tag{6.3}$$

where the $\hat{V}_2^\alpha$ are another subset of elements of the H^2 cohomology of the CY manifold, eq. (6.1) induces six-dimensional couplings of the form

$$T_6 = \int_{\Sigma_6} C_{rxy} \, b_2^r \wedge F^x \wedge F^y \quad . \tag{6.4}$$

We thus learn that the coupling of eq. (6.1) must be present in F theory. Interestingly, this term joins the three form of eleven-dimensional supergravity and the four-form of ten-dimensional Type-IIb supergravity, both lifted to twelve dimensions. This also suggests that the five brane of M theory arises as a magnetic three brane in F theory, a property implicit in the coupling of eq. (6.1). In the presence of five-brane sources, T_{12} is not gauge invariant, but its gauge variation is confined to the seven-dimensional world-volume of the six-brane sources, and

$$\delta \, T_{12} = \int_{\Sigma_7} \Lambda_3 \wedge F_4 \quad . \tag{6.5}$$

In a similar fashion, one is led to consider a coupling

$$T'_{12} = \int_{\mathcal{M}_{12}} A_3 \wedge F_5 \wedge F_4 \quad , \tag{6.6}$$

naively identical to the previous one, but different in the presence of mixed couplings between the three form and the four form, whose gauge variation

$$\delta \, T'_{12} = \int_{\Sigma_7} \Lambda_2 \wedge F_5 + \int_{\Sigma_6} \Lambda_2 \wedge F_4 \tag{6.7}$$

may be canceled by the six-brane world-volume term

$$B_7 = \int_{\Sigma_7} A_3 \wedge A_4 \tag{6.8}$$

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and by the five-brane world-volume coupling familiar from M theory [41, 42],

$$\int_{\Sigma_6} A_3 \wedge T_3 \quad , \tag{6.9}$$

where T_3 is the antiself-dual field strength on the five-brane world volume. Note that the six-brane couples to a composite seven-form field. The other terms needed to reproduce the six-dimensional couplings are

$$L_1 = \int_{\mathcal{M}_{12}} F_5 \wedge \omega_7(R) \quad , \tag{6.10}$$

$$L_2 = \int_{\mathcal{M}_{12}} F_4 \wedge d\omega_7(R) \quad , \tag{6.11}$$

and

$$L_3 = \int_{\mathcal{M}_{12}} F_4 \wedge F_4 \wedge F_4 \quad . \tag{6.12}$$

L_1 may be responsible for the six-dimensional tensor-gravity coupling, while the topological terms L_2 and L_3 become gravitational and Chern-Simons [43] couplings of M theory, once one takes $\mathcal{M}_{11} = \partial \mathcal{M}_{12}$. In five dimensions, L_2 describes the coupling of a linear combination of $n_V + 1$ vectors to TrR^2 [25], while the reduction of L_3 to six dimensions gives rise to a term $d_{xyz} \, a^x \, a^y \, a^z$ in the intersection form $\mathcal{V}$, as discussed in Section 3.

While all these terms may be regarded as a convenient bookkeeping for six-dimensional couplings, their natural twelve-dimensional form is strongly suggestive of an F-theory rationale for the CY compactification. In particular, the coupling of eq. (6.1) implies that F theory should accommodate both the Type-IIb three brane and the M-theory five brane. According to eqs. (6.2) and (6.3), however, A_3 and A_4 are not independent, since they are both needed to get the entire $H^{1,1}$ cohomology. Indeed[3], only $H^{1,1}(B)$ contributes to A_4, while the rest contributes to A_3. Moreover, A_4 should become self-dual when restricted to the Type-IIb theory in ten dimensions.

Let us conclude by observing that the twelve-dimensional couplings we have thus far identified do not require a twelve-dimensional metric, a pleasing feature since the twelve-dimensional theory can not be an ordinary gravitational theory. Still, it is remarkable

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[3]The restrictions on the cohomological expansion of A_3 and A_4 may be regarded as induced from the yet unknown F-theory dynamics.

that some insight can be gained on the explicit couplings of F theory from these simple considerations.

Acknowledgments

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