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Self-Dual Strings and N=2 Supersymmetric Field Theory

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We show how the Riemann surface Σ of $N = 2$ Yang-Mills field theory arises in type II string compactifications on Calabi-Yau threefolds. The relevant local geometry is given by fibrations of ALE spaces. The 3-branes that give rise to BPS multiplets in the string descend to self-dual strings on the Riemann surface, with tension determined by a canonically fixed Seiberg-Witten differential λ. This gives, effectively, a dual formulation of Yang-Mills theory in which gauge bosons and monopoles are treated on equal footing, and represents the rigid analog of type II-heterotic string duality. The existence of BPS states is essentially reduced to a geodesic problem on the Riemann surface with metric $|\lambda|^2$. This allows us, in particular, to easily determine the spectrum of *stable* BPS states in field theory. Moreover, we identify the six-dimensional space $\mathbb{R}^4 \times \Sigma$ as the world-volume of a five-brane and show that BPS states correspond to two-branes ending on this five-brane.

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1. Introduction

It is becoming increasingly clear that dualities in field theory and string theory are very strongly interrelated. A particular case of this, and perhaps the one with both interesting physics and exactly computable vacuum structure, is that of $N = 2$ supersymmetric theories in 4-dimensions. On the field theory side one has the results of Seiberg and Witten [1] and its generalizations. On the string theory side we have the $N = 2$ type II/heterotic duality proposed in [2,3] and further explored in [4].

Since one can consider the point particle limit of strings (by considering $\alpha' \to 0$ limit), one would expect to rederive the non-perturbative field theory results from string theory. This was partially done for some classes of examples in [5]. There is one basic puzzle: The field theory results are naturally phrased in terms of a Riemann surface, and in some of the examples considered in [5] (for instance, one with an $SU(3)$ gauge symmetry) this did not appear. Here we remedy this by adopting a slightly different viewpoint and show how one can obtain the Riemann surface more canonically from the Calabi-Yau space. In particular, we find that the Riemann surface times $\mathbb{R}^4$ can be viewed in the string language as a symmetric five-brane and the $N = 2$ effective field theory corresponds to the low energy lagrangian of this five-brane theory.

Furthermore, we use the string theory technology of D-branes to shed light on the BPS states of field theory. This is a refinement of the field theory results in that, in this context alone, it is extremely difficult to find the spectrum of the *stable* BPS states,[1] even though one can find the quantum numbers and masses of the allowed ones. We show that the BPS states of field theory can be best understood as the two-branes whose boundaries are self-dual strings on the Riemann surface. Moreover the differential one-form on the Riemann surface can be viewed, roughly speaking, as the tension of this string. Considering geodesics on the Riemann surface with the metric determined by this one-form allows one to explicitly study the spectrum of stable BPS states.

In short, the moral is that the natural arena of the Seiberg-Witten theory is string theory, where the Riemann surface has a concrete physical meaning (this is in the spirit of refs. [7,8,9]). The BPS states correspond to self-dual[2] strings [10,9,11,12] that wind

[1] However within SW theory the stable BPS states can be determined using symmetry arguments [6].

[2] Note that these self-dual strings are not the usual critical strings involving gravity that needs to be decoupled at some point, but rather are *non-critical* strings without gravity that give a "dual" formulation of gauge theory.

geodesically around the homology cycles. The relationship between such self-dual strings and ordinary Yang-Mills field theory is the rigid analog of the duality [2] between type II and heterotic strings, and is, as we will show, actually a consequence of it.

The organization of this paper is as follows: In section 2 we review an example of type II/heterotic duality that was studied in [5] and show how one can deduce in this and many similar cases the existence of a Riemann surface anticipated from field theory. In section 3 we show how the Riemann surface can be used to give us insight into the structure of three-cycles on the Calabi-Yau, which allows us to formulate the condition for having stable BPS states directly in terms of the Riemann surface and the differential form on it. Moreover, we show the relation of the effective $N = 2$ SYM field theory with the field theory living on the five-brane, and the relation between two-branes ending on five-branes and the BPS states. In section 4 we apply the corresponding results to study the spectrum of stable BPS states for pure $SU(2)$ gauge theory.

2. Local Seiberg-Witten Geometry and Fibrations of ALE Spaces

2.1. K3-fibrations revisited

We begin by explicitly illustrating our point by considering a simple example, namely the $K3$-fibration threefold $X_{24}(1,1,2,8,12)$ with Hodge numbers $h_{1,1} = 3, h_{2,1} = 243$. This is one of the basic examples of heterotic-type IIA string duality [2,4]. Equivalently, we consider the type IIB theory on the mirror manifold with $h_{2,1} = 3, h_{1,1} = 243$, whose defining polynomial can be written as

$$\begin{aligned} W^* \;&\equiv\; \frac{1}{24}(x_1^{24}+x_2^{24}) + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2 \\ &\qquad - \psi_0(x_1x_2x_3x_4x_5) - \frac{1}{6}\psi_1(x_1x_2x_3)^6 - \frac{1}{12}\psi_2(x_1x_2)^{12} \;=\; 0\ . \end{aligned} \tag{2.1}$$

Introducing a suitable parametrization, $a = -{\psi_0}^6/\psi_1,\ b = {\psi_2}^{-2}$ and $c = -\psi_2/{\psi_1}^2$, we exhibit the $K3$-fibration by setting $x_1/x_2 = \zeta^{1/12}\, b^{-1/24}$ and ${x_1}^2 = x_0\zeta^{1/12}$:

$$\begin{aligned} W^*(\zeta,a,b,c) \;&\equiv\; \frac{1}{24}(\zeta + \frac{b}{\zeta} + 2){x_0}^{12} + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2 \\ &\qquad + \frac{1}{6\sqrt{c}}(x_0x_3)^6 + \big(\frac{a}{\sqrt{c}}\big)^{1/6} x_0x_3x_4x_5 \;=\; 0\ . \end{aligned} \tag{2.2}$$

Here, ζ is the coordinate on the base space $\mathbb{P}^1$ and $-\log b$ corresponds to the volume of the $\mathbb{P}^1$ in the type IIA formulation. Regarding ζ as a parameter, (2.2) represents a $K3$ with discriminant

$$\begin{aligned}\Delta_{K3} &= \left(2\,\zeta+\zeta^2+b\right)\left(2\,\zeta\,c+\zeta^2\,c+b\,c-2\,\zeta\right)\times\\ &\quad\left(4\,\zeta\,a-2\,\zeta\,a^2+2\,\zeta\,c+\zeta^2\,c+b\,c-2\,\zeta\right)\\ &\equiv\prod_{i=1}^{6}\left(\zeta-e_i(a,b,c)\right)\end{aligned}\tag{2.3}$$

Over points e_i in the base $\mathbb{P}^1$ where $\Delta_{K3}=0$ the $K3$ fiber is singular; note that there is a symmetry under exchanging e_i with $1/e_i$. The total space, ie. the threefold, is non-singular, unless zeros e_i coincide:

$$\Delta_{CY} = \prod_{i<j}\left(e_i-e_j\right)^2 \;\propto\; (b-1)\left((1-c)^2-b\,c^2\right)\left(\left((1-a)^2-c\right)^2-b\,c^2\right)\,.$$

We now investigate the fibration in the local neighborhood of the Seiberg-Witten regime in the moduli space. Specifically, we consider the theory near its $SU(3)$ point by setting [5]

$$\begin{aligned}&a=-2\epsilon u^{3/2}\\ &b=\epsilon^2\Lambda^6\\ &c=1-\epsilon(-2u^{3/2}+3\sqrt{3}v)\end{aligned}$$

for $\epsilon\equiv(\alpha')^{3/2}\to 0$ (the $SU(2)$ line at $c=1$ and the $SU(2)\otimes SU(2)$ point at $c=1, a=2$ can be treated in exactly the same way). Here, u and v are the gauge invariant Casimir variables of $SU(3)$. Expanding in ϵ, we get for the singular points on $\mathbb{P}^1$:

$$\begin{aligned}e_0 &= 0\,,\quad e_\infty = \infty\\ e_1^{\pm} &= 2u^{\frac{3}{2}}+3\sqrt{3}v\pm\sqrt{\left(2u^{\frac{3}{2}}+3\sqrt{3}v\right)^2-\Lambda^6}\\ e_2^{\pm} &= -2u^{\frac{3}{2}}+3\sqrt{3}v\pm\sqrt{\left(2u^{\frac{3}{2}}-3\sqrt{3}v\right)^2-\Lambda^6}\end{aligned}$$

up to some irrelevant rescalings. These are precisely the branch points (in the z-plane) of the $SU(3)$ Seiberg-Witten curve Σ, when written in the form [13,14]

$$z+\frac{\Lambda^6}{z}+2P_{A_2}(x,u,v) \;=\; 0\,.\tag{2.4}$$

Here, $P_{A_2} = x^3 - ux - v$ is the simple singularity [15] associated with $SU(3)$; replacing $z \to y - P$ gives back the original form of the curve given in [16,17]. The structure of the curve given by (2.4) can easily be related to the Calabi-Yau manifold described by (2.2), by considering a local neighborhood of the singularity in the fibration. That is, we expand around the singular point of $W^*(\zeta = 0, a, b = 0, c)$, and going to the patch $x_0 = 1$ this gives (modulo trivial redefinitions):

$$W^* = \epsilon \Big(z + \frac{\Lambda^6}{z} + 2P_{A_2}(x, u, v) + y^2 + w^2 \Big) \; + \mathcal{O}(\epsilon^2) \tag{2.5}$$

where $\zeta = \epsilon\, z$. This is of the same singularity type as (2.4), which means that the local geometry of the threefold in the SW regime of the moduli space is equivalent to the one of the Seiberg-Witten curve. This point will be elaborated further in section 3.

The appearance of local SW geometry can be seen to hold for other $K3$-fibrations as well. This is obvious for type IIA compactifications on K3-fibered threefolds in [18] that are of Fermat form, whose type IIB mirrors can be written as

$$W^* = \frac{1}{2k} \Big({x_1}^{2k} + {x_2}^{2k} + \frac{2}{\sqrt{b}} (x_1 x_2)^k \Big) + \tilde{W}\big(\frac{x_1 x_2}{b^{1/2k}}, x_3, x_4, x_5, u_k\big) \ . \tag{2.6}$$

Writing $x_1/x_2 = \zeta^{1/k}\, b^{-1/2k}$ and ${x_1}^2 = x_0 \zeta^{1/k}$ one immediately obtains

$$W^*_{K3} = \frac{1}{2k} \Big(\zeta + \frac{b}{\zeta} + 2 \Big) {x_0}^k + \tilde{W}^*(x_0, x_3, x_4, x_5, u_k) \ . \tag{2.7}$$

The piece of W^*_{K3} that is independent of ζ and b describes the underlying $K3$ in some parametrization. Going to the patch $x_0 = 1$ and assuming that the $K3$ is singular of type A_{n-1} in some neighborhood in the vector moduli space, we can expand the $K3$ around the critical point and thereby replace it by the ALE normal form of the singularity:

$$\begin{aligned} \frac{1}{k} + \tilde{W}^* \ &= \ \epsilon \left[2\, P_{A_{n-1}}(x, u_k) + y^2 + w^2 \right] + \mathcal{O}(\epsilon^2) \ , \\ P_{A_{n-1}}(x, u_k) \ &\equiv \ x^n - \sum_{k=2}^{n} u_k\, x^{n-k} \ . \end{aligned} \tag{2.8}$$

Rescaling $y = \epsilon^2 \Lambda^{2n}$ and $\zeta = \epsilon\, z$, we obtain the $SU(n)$ generalization of (2.5), ie., the naive fibration of the corresponding ALE space.

For non-Fermat threefolds the story is quite similar. The mirrors can always be represented in terms of a quasi-homogenous "Landau-Ginzburg" polynomial, only that the weights w_i^* of the mirror will in general be different as compared to the original weights

w_i. Moreover it is shown in ref. [19] for a much larger class of $K3$-fibrations constructed in toric varieties that the mirror generically takes the form

$$W^* = \Big(\zeta + \frac{b}{\zeta} + 2\Big) + \tilde{W}^* \tag{2.9}$$

in some appropriate coordinate patch. Thus, the same arguments as above can be applied.

Here we have concentrated mainly on pure $N = 2$ Yang-Mills theory, but the situation is not much different for theories with extra matter; the local ADE singularity will still be the same, but the fibering data over the ζ plane will be different [19]. Nevertheless the arguments developed in the present paper also apply to those cases.

2.2. Geometrical Interpretation

We can understand what happens in geometrical terms if we view the SW curves as fibrations as well, namely fibrations over $\mathbb{P}^1$ with fibers given by the"spectral set" that characterizes *classical* Yang-Mills theory. More specifically, the spectral set is given by the set of points

$$V \;=\; \big\{x : P_G^{\mathcal{R}}(x, u_k) = 0\big\}\ ,$$

where

$$P_G^{\mathcal{R}}(x, u_k) = \det(x - \Phi_0)$$

is the characteristic polynomial of the Higgs field, Φ_0, evaluated in some representation $\mathcal{R}$ of the gauge group G. For $G = SU(n)$, the picture is particularly simple: if we write $\Phi_0 = a_i(\lambda_i \cdot H)$, where λ_i are the weights of the defining fundamental representation[3] and H are the generators of the CSA, then gauge symmetry enhancement occurs whenever $a_i - a_j = 0$ for some i and j. Furthermore,

$$P_{SU(n)}^{\mathbf{\underline{n}}}(x, u_k) = \prod_{i=1}^{n} \big(x - a_i(u_k)\big) \;\equiv\; P_{A_{n-1}}(x, u_k) \tag{2.10}$$

It is useful to think of the (base-pointed) homology, $H_0(V, \mathbb{Z})$, of V, which is generated by the formal differences $a_{i+1} - a_i$ and which may be identified with the root lattice of $SU(n)$: $H_0(V, \mathbb{Z}) \cong \Lambda_R$ [15]. Symmetry enhancement of the classical theory is thus equivalent to having a vanishing 0-cycle in V [20]. Note that describing gauge symmetry enhancement for the A_n series in terms of coinciding points also has a natural interpretation in terms of D-branes [21], which we will make use of in the next section.

[3] As explained in [14], the choice of the representation is actually irrelevant.

The spectral surface of $N=2$ *quantum* Yang-Mills theory in the form [13,14]

$$W_{SW} \ = \ z + \frac{\Lambda^{2n}}{z} + 2P_{A_{n-1}}(x, u_k) \ = \ 0 \ , \tag{2.11}$$

can then simply be viewed as fibration of the classical spectral set V over the $\mathbb{P}^1$ base defined by z (this is depicted for pure $G = SU(3)$ gauge theory in Fig.1.). More precisely, the curve can be seen as a multi-sheeted cover (foliation) over the z-plane constructed so that $x(z)$ becomes a meromorphic function. The sheets of this foliation are in one-to-one correspondence with the weights λ_i of the representation $\mathcal{R}$. Note that the surface has a symmetry $z \to \frac{\Lambda^{2n}}{z}$, which means that branch points on the z-sphere will naturally come in pairs $e_i^{\pm}(u_k)$, $i = 1, .., rank(G)$. To make contact to the curves given in [16,17], e_i^- should be linked by cuts to the branch point $e_0 \equiv 0$, while e_i^+ get connected to $e_\infty \equiv \infty$.

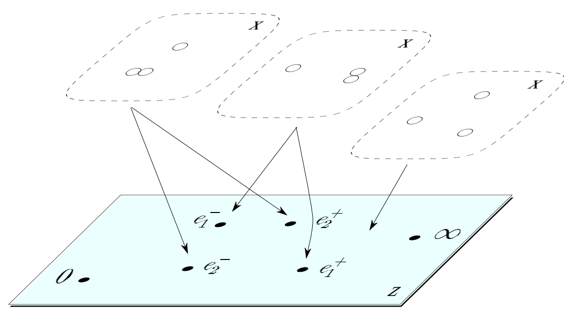

Fig.1. *The Seiberg-Witten curve can be understood as fibration of a weight diagram over* $\mathbb{P}^1$. *Pairs of singular points in the base are associated with vanishing* 0*-cycles in the fiber, i.e., to root vectors* $a_i - a_j$. *In string theory, the local fibers are replaced by appropriate ALE spaces with corresponding vanishing two-cycles. As will be explained in the next section, this picture has a natural interpretation in terms of D-branes.*

In the present context of threefold $K3$-fibrations, we have found that locally $K3$ is fibered over $\mathbb{P}^1$ in a very similar manner. The structure of the $K3$ fibers is locally constant, and so we may view this as giving us a fibration of $H_2(K3, \mathbb{Z})$ over $\mathbb{P}^1$. In other words, the transition from Yang-Mills to string theory essentially amounts to simply replacing $H_0(V)$ by $H_2(K3)$ in the fiber, where locally $H_2(ALE) \cong H_0(V)$.

To explicitly show that the local geometry of the threefold indeed reproduces the Seiberg-Witten periods, remember that in string theory the relevant periods are those of the holomorphic 3-form, Ω. On the other hand, in the supersymmetric gauge theory the corresponding quantities can be expressed as integrals of the meromorphic 1-form,

$$\lambda \;=\; x\,\frac{dz}{z}\;, \tag{2.12}$$

over the cycles of the Riemann surface [1]. Given that $W^*_{K3}=0$ differs from the equation of the Seiberg-Witten curve (2.11) by "trivial" quadratic pieces, one should expect to be able to relate Ω and λ directly by integrating Ω over homology cycles in the $K3$ fiber. It is indeed easy to demonstrate this explicitly.

In the coordinate patch defined above, the (un-normalized) holomorphic 3-form can be written as

$$\Omega \;=\; \frac{dz}{z}\wedge\left[\frac{dy\wedge dx}{\frac{\partial W^*}{\partial w}}\right]\,. \tag{2.13}$$

To isolate the non-trivial 2-cycles on the $K3$, it is useful to recall that for the singularity $y^2+w^2+x^2\,=\,1$, the non-trivial 2-cycle is simply the 2-sphere obtained by taking the y,w,x to be real. Equivalently, taking $w=\sqrt{(1-x^2)-y^2}$, one maps out the two-sphere by first fixing x and running around the cut in the y-plane, and then varying x between limits where the x-cut collapses to a point. For fibered ALE spaces of the local form

$$W^* \;=\; z+\frac{\Lambda^{2n}}{z}\;+\;2P_{A_{n-1}}(x,u_k)+y^2+w^2\;, \tag{2.14}$$

the surface $W^*=0$ has $n-1$ independent two-spheres in each $K3$ fiber: If one fixes the z and x and solves $W^*=0$ for y, then the latitude circles of the spheres circulate around the w-cuts. The poles (as in North and South, as opposed to as in singularity) of the spheres occur when these circles (or cuts) collapse, that is, when $W_{SW}\,=\,0$ (2.11). For fixed z there are n such values of x, any pair of which defines a homology 2-sphere. Thus the Seiberg-Witten Riemann surface may be thought of as defining the poles (in the x direction) of the homology 2-spheres in the $K3$-fibration.

To integrate Ω over these spheres, one solves for w using $W^*=0$, and substitutes into (2.13). The integral over y around each latitude, or cut, is trivial, and is equal to 2π. This leaves us with the two form (up to constant factors)

$$\int_y \Omega=\frac{dx\,dz}{z}=d(\frac{x\,dz}{z}) \tag{2.15}$$

We can integrate this further between the limits of x that are pairs of roots of (2.11): that is, integrating Ω over the fiber one is left with the difference of the values of $(x\,dz)/z$ for any pair of roots of (2.11).

We mentioned above that the Riemann surface can be thought of as a n-sheeted foliation over the z-sphere, with each leaf corresponding to a root of (2.11). Consider now a closed path in the base space (z-space), and imagine lifting this to the various sheets in the foliation. Integrating the difference between the values of $(x\,dz)/z$ for pair of sheets will produce a non-zero result if and only if the path circulates around a piece of Riemann surface plumbing that connects the two sheets. Putting this all together one sees that the integral of Ω on a 3-cycle of the Calabi-Yau collapses directly to an integral of λ over the cycles of the Riemann surface (2.11).

It is perhaps less obvious that similar arguments apply when the fiber develops a D_n or E_n-type singularity – this will be addressed in ref. [19].

The description of the three-cycles in the fibration in terms of the Riemann surface and its projection onto the z-plane will be discussed in more detail in the next section.

3. Riemann Surfaces, p-Branes and the Calabi-Yau three-Fold

We have seen that in type IIB string theory, the Seiberg-Witten regime in the Calabi-Yau three-fold is locally[4] equivalent to an ALE space[5] (characterized by ADE type) that is fibered over the complex z-plane. Furthermore, the moduli of the ALE space vary holomorphically with z. Clearly, in the rigid $N = 2$ field theory in four dimensions, all the geometry should be understood just from these local fibration data. In particular, the relation between the coupling constants of the gauge fields is simply special geometry applied in this particular limit [5]. Moreover, the BPS states of the $N = 2$ effective field theory should arise as particular limits of three-branes wrapped around the three-cycles of the Calabi-Yau [23].

In order to have a better understanding of the effective $N = 2$ system, we want to find a simpler system that replaces the Calabi-Yau in this limit but captures the geometry of the relevant three-cycles. This system is ought to reproduce the field theory properties such as the spectrum of BPS states or the gauge coupling constants. The discussion will lead us to the usefulness of the Riemann surface discussed in the previous section, and will make the connection between string theory concepts and field theory states more concrete.

The three-cycles in the Calabi-Yau can be viewed, roughly speaking, as a combination of two-cycles coming from the ALE space and a one-cycle from the z-plane. As we vary

[4] Note that the limit $\epsilon \to 0$ corresponds to $\alpha' \to 0$ and thus to switching off gravity.

[5] D-branes on ALE spaces have recently been considered in [22].

the z-parameter, the ALE space varies, and the two-cycles of the ALE space will vanish at some points $e_i^{\pm}$ in the z-plane. Let us denote the totality of vanishing two-cycles by $\mathcal{C}$, and denote the ADE group of the ALE space by G and its Weyl group by $W(G)$. If we consider a vanishing cycle $C \in \mathcal{C}$, then as we go around a curve γ on the z-plane, C in general transforms to another vanishing cycle, given by $g(\gamma)C$ where $g \in W(G)$.

It is convenient to define a Riemann surface Σ using these data: namely by definition Σ is the Riemann surface associated with the given monodromies in the z-plane, with the property that curves γ on Σ get mapped to curves on the z-plane such that $g(\gamma) = 1$.

To be concrete, let us consider the case where the ALE space is of type A_{n-1}. The Riemann surface is then of course precisely the one given in eq. (2.11). As discussed in the previous section, the local description of the Calabi-Yau manifold is given by (2.14), which in view of (2.10) can be represented by

$$\prod_{i=1}^{n}(x - a_i(z)) + y^2 + w^2 \;=\; 0 \;,$$

where $a_i(z) \equiv a_i(u_2, \ldots, u_n - \frac{1}{2}(z + \frac{1}{z}\Lambda^{2n}))$. Note that in this form of the surface, the equation for the Calabi-Yau is well defined but the functions $a_i(z)$ are not single-valued as functions of z; only $\prod_{i=1}^{n}(x - a_i(z))$ is well defined over z. As any two a_i approach each other, we get a vanishing two-cycle (for a discussion of this, see for example [21]). As we go around in the z-plane, the set of a_i comes back to itself, but the individual $a_i(z)$ do not necessarily come back to themselves. In general they are permuted by an element of S_n, which is the Weyl group $W(A_{n-1})$. Moreover, the action on the vanishing cycles is also clear, since each vanishing cycle is associated with a pair of a_i.

The Riemann surface Σ defined above is simply the surface defined by

$$\Sigma : \qquad \prod_{i=1}^{n}(x - a_i(z)) = 0 \;, \tag{3.1}$$

which has genus $g = n - 1$. Clearly this Riemann surface projects onto the z-plane, and moreover it has the property that any curve on it corresponds on the z-plane to a curve with trivial monodromy action on the a_i.

We will now see why this Riemann surface, which has been constructed using the data of how the Calabi-Yau is locally described as an ALE fiber space over the z-plane, leads to a tremendous insight into the three-cycles of the Calabi-Yau in the rigid limit.

Let us recall some aspects of our discussion from the previous section. For a fixed value of z there are n points on Σ, i.e., the map is n to 1; these points are given by $x = a_i(z)$ (see Fig.1.). Moreover, as noted above, a two-cycle in the ALE space corresponds to a pair of points in the x-plane. In particular, for a fixed z, the image of a two-cycle on the Riemann surface is a 0-cycle consisting of the class $[a_i] - [a_j]$. Consider a three-cycle C_3 in the Calabi-Yau. The image of this three-cycle on the z-plane will be a curve, which can in principle be of two types: either it is an open curve or it is a closed curve. This will depend on what C_3 precisely is.

For example, if the three-cycle is $S^2 \times S^1$, where S^2 can be identified with a vanishing two-cycle of the ALE space, then the image of this on the z-plane is a circle; moreover this circle also lifts to a closed curve on the Riemann surface, because the S^2 comes back to itself as we go around this circle. Let us consider the vanishing two-cycle associated with a_i, a_j and parameterize the S^1 by θ. From what we have said it follows that that the image of the three-cycle on the Riemann surface can be viewed as the class $[a_i(\theta)] - [a_j(\theta)] = [C_i] - [C_j]$, where C_i, C_j are two closed curves on the Riemann surface. If the class $[C_i] \neq [C_j]$ we get a non-trivial three-cycle of the Calabi-Yau.

On the other hand, the three-cycle on the Calabi-Yau might be an S^3, which can be viewed from the ALE perspective by slicing S^3 into S^2's given by going from the 'north pole' of S^3, corresponding to a vanishing S^2, to the 'south pole' which again corresponds to a vanishing S^2. The image of this three-cycle on the z-plane will then be an open curve, with boundaries at the points $e_i^{\pm}$ in the z-plane where the Riemann surface is branched over and where pairs of the a_i come together. On the Riemann surface this corresponds to a cycle which starts from the pre-image of a branch point where two sheets come together and ends on another branch point where the same two sheets meet again. Independently of which of these two types of three-cycles we consider, we thus see that we have a map

$$f: \qquad H_3(M) \to H_1(\Sigma)$$

Now recall from the previous section that on the Riemann surface Σ there is a one-form λ with the property that the integral over the holomorphic three-form Ω of the Calabi-Yau over a three-cycle C_3 is equivalent to

$$\Omega(C_3) = \lambda(f(C_3)) \ ,$$

where $f(C_3)$ is the one-cycle on the Riemann surface discussed above.

We now argue that the kernel of the map f is trivial for the relevant classes of three-cycles in the rigid limit and that this implies that the Riemann surface Σ faithfully represents all the data about three-cycles of interest. This is essentially clear when we recall that over a trivial cycle C_1 on the Riemann surface, the one-form λ, being meromorphic with only second order poles, will integrate to zero over it and thus Ω integrated over the pre-image $f^{-1}(C_1)$ also vanishes; this implies (generically) the triviality of the three-cycle. It is also easy to see that by the map f the canonical bilinear form on $H_3(M)$ gets mapped to the canonical bilinear form on $H_1(\Sigma)$.

3.1. Type IIB Perspective

The importance of three-cycles for type IIB theories is that three-branes can wrap around them and thereby give rise to BPS states [23]. The three-branes wrapped around cycles of type $S^2 \times S^1$ can in principle give vector- or a hypermultiplets [21,24,25]; in our case they give rise to vector multiplets. Remember that the images of these cycles on the z-plane are closed curves. On the other hand, the three-branes wrapped around S^3 are of the type discussed in [23] and correspond to hypermultiplets. We have seen that the images of these cycles on the z-plane are open curves that end on the branch points $e_i^{\pm}$. Moreover, given the above map between the three-cycles on the Calabi-Yau and the one-cycles on Σ, we can view the three-branes wrapped around the A-cycles of Σ as electrically charged and those wrapped around the B-cycles as magnetically charged states. Note that the mass of any BPS state corresponding to a one-cycle C_1 on Σ is simply given by

$$M = \Big| \int_{C_1} \lambda \Big| \ ,$$

which is the familiar BPS formula.

So far we have discussed how the three-cycles of the Calabi-Yau manifold are represented through curves on the Riemann surface, together with a projection on the z-plane. From the physics point of view it is crucial to know whether we really do get a BPS state, or not, from wrapping a three-brane around a given three-cycle in the threefold. In other words, we would like to find the spectrum of BPS states in the theory. Here is where for the first time the advantage of the string perspective on the SW theory becomes clear: A three-brane partially wrapped around an S^2 of the ALE space becomes a self-dual string [10,9,11,12] on the z-plane. In the present case the tension of this string will depend on where on the z-plane we are, since the volume of S^2 varies over the z-plane.[6]

[6] This is in contrast to $N = 4$ Yang Mills theory considered in [9], where the compactification manifold, $K3 \times T_2$, is a direct product.

More specifically, consider a point on the z-plane and consider a three dimensional space given by a vanishing two-cycle S_{ij}, corresponding to the pair $(a_i(z), a_j(z))$, plus an interval dz on the z-plane. Let us ask what the mass of this string is. To find the tension, we first have to integrate the holomorphic three-form over the two-sphere S_{ij} corresponding to this pair, and, as discussed in the previous section, this is nicely summarized in terms of a one-form λ. For a given point on the z-plane, the one-form λ has n-different values (as λ is well-defined only over the Riemann surface Σ and Σ is an n-fold covering of the z-plane). Namely $\lambda = x dz/z$, and so for a fixed z the pre-images of x are given by $a_i(z)$. The integral of the three-form over the two-cycle S_{ij} is thus given by

$$\Omega_{S_{ij}} = \Delta_{ij}\lambda = \Delta_{ij}(x)\frac{dz}{z} = (a_i(z) - a_j(z))\frac{dz}{z} \ . \tag{3.2}$$

Therefore the tension of an $i-j$ type of self-dual string, which by definition is the leftover piece of the three-brane wrapped around the two-sphere S_{ij}, is given by

$$T_{ij} = \left|a_i - a_j\right| \ , \tag{3.3}$$

where the metric on the z-plane is given by $|\frac{dz}{z}|^2$. In other words, an $i-j$ type of string stretched between z and $z+dz$ has mass $T_{ij}|dz/z|$. We will give an explanation of the simple formula (3.3) for the tension of the $i-j$ string when we will talk below about the type IIA interpretation of all this.

In order to make our points a bit more concrete, let us concentrate on the A_1 case. There is then only one two-cycle, S_{12}, and only one type of self-dual string. With the coordinates we have chosen, we have $a_1 = -a_2$, so the energy of a piece of an infinitesimal piece of string is simply $2\left|\lambda\right|$.

Now we come to the point of what concrete advantage the string viewpoint has over mere field theory. What we are effectively interested in is constructing minimal-volume three-cycles. For each point over the z-plane, the fiber has a minimal two-sphere, which is thus part of the minimal volume three-cycle. To minimize the whole three-volume, we can thus equivalently minimize the mass of the string on the z-plane, whose tension is given by $2\left|\lambda\right|$. This is equivalent to looking for the geodesics on the z-plane, for the metric given by

$$g_{z\bar{z}} = 4\lambda_z\bar{\lambda}_{\bar{z}} \ . \tag{3.4}$$

Moreover, as discussed above, depending on whether we are interested in hypermultiplets or vector multiplets, we should look for open geodesics that end at the branch points $e_i^{\pm}$ in the z-plane (HM), or for closed geodesics on the z-plane which lift to closed curves on the

Riemann surface (VM). If we cannot find a (primitive) geodesic in each class this simply means that the corresponding three-cycle does not give rise to a BPS state. This gives us a method to find which BPS states are occupied in the field theory and which are not; we will exemplify this method in section 4 below.

Note that the metric (3.4) is flat because $\partial\bar{\partial}\log g = 0$. This implies that the geodesic lines can be found by simply integrating λ (i.e. by going to the special coordinates where the flat metric is in the canonical form):

$$\int^{z} \lambda = \alpha t + \beta \ , \tag{3.5}$$

for arbitrary constants α and β, where t parameterizes the geodesic. For the open geodesics that correspond to hypermultiplets we thus expect to find a discrete number of primitive curves, corresponding to stable BPS states.

For the closed geodesics corresponding to vector multiplets, given the fact that the metric is flat, we will get a family of such curves and we will then need to quantize the moduli space of this family, as is familiar from similar examples for D-branes [26]. For simplicity, we will mainly concentrate on the hypermultiplet spectrum in this paper, postponing the study of vector multiplets for future work.

Note that the relation between the BPS mass and charge simply follows from the fact that the absolute value of the integral of λ around the corresponding cycle on Σ corresponds to the mass of the string, and that is in turn fixed by the meromorphicity of λ in terms of cohomology classes.

In section 4 we study solutions to this equation for the A_1 case and confirm the spectrum of hypermultiplet BPS states anticipated for this theory. Clearly the above picture easily generalizes to A_n, where the role of 2λ for A_1 case is played by λ_{ij}.

3.2. Type IIA Perspective

So far we have been discussing type IIB string theory near an fibered ALE space with A_{n-1} singularity. We would also like to discuss the type IIA perspective. There are two such perspectives. One is simply by going back to study type IIA on the original manifold. This turns out not to be particularly helpful. Instead we will consider a further T-duality transform, now acting on the ALE fiber instead of on the base, which will give us another type IIA description of the same limit of the compactification: It was shown in [27] that type IIB (IIA) on an A_{n-1} ALE space is equivalent to type IIA (IIB) near n symmetric fivebranes. More specifically, the n fivebranes are described by

$$w = y = 0 \ , \qquad x = a_i \ .$$

This was used in [21] to map the A_{n-1} gauge symmetry enhancement in type IIA theory near an A_{n-1} singularity to type IIB with n-symmetric fivebrane which, by strong/weak duality, becomes the statement that n coincident Dirichlet five-branes have an enhanced $SU(n)$ gauge symmetry [28]. This transforms the two-cycles wrapped around the vanishing S^2's of A_{n-1} to elementary strings going between the five-branes in the type IIB dual description. Also the similarity of the description of the open strings stretched between pairs of a_i and the vanishing two-cycles was explained there.

In our case we are in the opposite situation because we are considering type IIB near an A_{n-1} singularity, which is equivalent to type IIA with n symmetric fivebranes [27]. It was observed in [11] that IIB three-branes partially wrapped around the vanishing two-spheres (giving the non-critical strings [9]) correspond in IIA to Dirichlet two-branes that end on the symmetric five-brane. Specifically, the left-over one-brane piece of the three-brane corresponds to the boundary of the two-brane living on the five-brane. Note that if we consider a Dirichlet two-brane, of which a one-brane piece is stretched between a_i and a_j, we are left with a self-dual string in six dimensions with tension $|a_i - a_j|$; this is a simple explanation of the tension formula (3.3).

Let us recall that a_i vary holomorphically over z. This implies, if we take the non-trivial monodromy properties of the z-plane into account, that the world-volume of the n five-branes located at the a_i effectively forms a *single* five-brane given by

$$\Sigma \times \mathbb{R}^4 \ ,$$

where the $\mathbb{R}^4$ is the uncompactified spacetime and Σ is the Riemann surface discussed above.

Note that in this way we can make immediate contact with the low energy description of the rigid $N = 2$ field theory. We have to recall [29] that as far as the low energy (bosonic) fields of the symmetric fivebrane is concerned, we have an antisymmetric two-form $B_{\mu\nu}$ with self-dual field strength, plus in addition 5 scalars. Similar to the considerations of [26], out of these scalars 2 are twisted and correspond to one-forms on Σ, while the other three remain ordinary scalars[7]. The gauge fields of the $N = 2$ low-energy lagrangian on $\mathbb{R}^4$ originate from the zero modes corresponding to harmonic one-forms ω on Σ with

$$B_{\mu\nu} = \omega_\mu A_\nu^{(4)} \ .$$

Taking into account the self-duality of B, this implies that on $\mathbb{R}^4$ we have as generic gauge group $U(1)^g$, with A- and B-cycles corresponding to electric and magnetic states, respectively. Moreover, in this language the BPS states now correspond to Dirichlet two-branes that end on the Riemann surface. In particular, the three-cycles of the Calabi-Yau threefold now get mapped to discs whose boundaries lie as one-cycles on the Riemann surface. This is shown in Fig. 2.

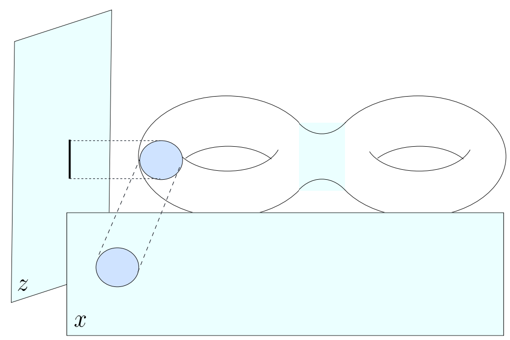

Fig.2. *Projecting a self-dual string that winds around the SW curve produces an open string in the z-plane. On the x-plane we see a projection of the Dirichlet two-brane.*

In other words, we can view the two-brane, which consists of one-branes stretching between the points in the x-plane and ending on the Riemann surface, as 'filling' the cycle of the Riemann surface into a disc. The boundary of this two-brane disc is indeed a string on the Riemann surface. Moreover, the electric/magnetic charge of this two-brane, given the coupling [11,30,31] of the boundary of the two-brane living on Σ to $B_{\mu\nu}$ and its relation to A_ν defined above, is obvious.

Note that the two scalars that correspond to one-forms on Σ have $2g$ zero modes in one-to-one correspondence with the g independent A-cycles above. These are to be identified with the scalars in the $N = 2$ vector multiplet. Changing the expectation value of these scalars corresponds to changing the complex structure of the ALE space on the type IIB

[7] These scalars do not have zero modes because the Riemann surface with its natural metric has infinite volume.

side. From this viewpoint it is natural to identify λ defined above as the expectation value of the scalars:

$$< \phi_z >= \lambda_z$$

This is in line with the fact that variation of λ with respect to the zero mode of ϕ_z (corresponding to varying in the Coulomb phase of $N = 2$ YM) gives rise to harmonic forms on the Riemann surface [1].

Summarizing, the main message of this discussion is that instead of considering the $N = 2$ SYM field theory, we can consider a five-brane given by $\Sigma \times \mathbb{R}^4$ living in the 8-dimensional space $(x, z, \mathbb{R}^4)$. Moreover, the metric on the x-plane is the flat metric and on the z-plane the metric is cylindrical, given by $|dz/z|^2$. The BPS states correspond to two-branes that live in the (x, z) space, whose boundaries lie on the Riemann surface as non-trivial cycles. Moreover, the minimal two-branes correspond to ruled surfaces (straight lines on the x-plane) which bound non-trivial cycles on the Riemann surface and whose surface tension is given by $|dxdz/z|$. As we will see in the next section, these facts allow us to perform explicit computations to obtain results for the spectrum of BPS states in rigid $N = 2$ Yang-Mills theory.

4. BPS states in $SU(2)$ Yang-Mills Theory

To demonstrate the power of the techniques hinted at in the previous section, we will consider the example of pure $SU(2)$ $N = 2$ Yang-Mills theory [1]. It is crucial to use the precise form of the one-form differential λ as given in (2.12), and not just some modification of it that gives the same periods, because the geodesics that we will study will depend on the choice of the differential. It is quite satisfying to see that string theory has picked a canonical form of λ, which enters via the metric of the five-brane world-volume theory. For pure $SU(2)$ SYM it is given by

$$\lambda \ = \ \sqrt{2u - z - \frac{1}{z}}\,\frac{dz}{z}\ .$$

According to the discussion in the previous section, the geodesics of the self-dual strings on the $SU(2)$ curve (2.11) are governed by the following differential equation:

$$\sqrt{2u - z - \frac{1}{z}}\left(z^{-1}\frac{\partial}{\partial t}z\right) \ = \alpha\ . \tag{4.1}$$

If we want to study trajectories emanating from, say, the first branch point, we impose the boundary condition $z(0) \ = \ e_1^-(u)$, where $e_1^{\pm}(u) \equiv u \pm \sqrt{u^2-1}$. Different choices of

α correspond to different angles of the straight trajectories (3.5) in the Jacobian, so up to an overall factor we can take $\alpha = g - \frac{2a(u)}{a_D(u)} q$ for a dyon with charges (g, q).

The first order equation (4.1) can easily be solved numerically, and some of the resulting trajectories on the Seiberg-Witten curve are depicted in Fig.3. For real $u > 1$, the generic form of the trajectories is easy to understand: in the regime $z + \frac{1}{z} \ll u$, the leading behavior is $z(t) = e^{\alpha t}$ and this yields a monotonically increasing oscillatory behavior for dyons with non-vanishing electric charge, whereas the gauge bosons correspond to closed loops (cf., Fig.3b). The branch points $e_i^{\pm}$ are outside the regime of validity of this argument, but our numerical analysis confirms that nothing drastic happens at the branch points.

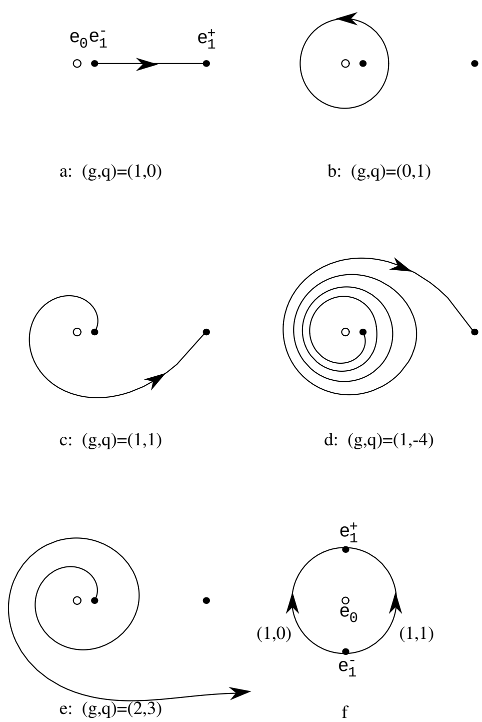

Fig.3. *Geodesics that represent actual minimal-tension self-dual strings on the SW curve. The trajectories (a,c,d,f) of stable BPS dyons with charges (g, q) run between the branch points e_1^- and e_1^+ in the z-plane, whereas the gauge bosons (b) correspond to closed loops. The counter-clockwise winding number around e_0 measures positive electric charge units. In contrast, the trajectories of unstable states (e) never close on the branch points $e_1^{\pm}$ but rush off to infinity. All trajectories shown correspond to real $u > 1$, except (f) where $u = 0$.*

We can in this way easily reproduce the expected stable dyon spectrum in the Higgs regime, given by $(g,q)=(1,n)$, $n\in\mathbf{Z}$, by finding that the corresponding trajectories close on $e_1^\pm$ (cf., Fig.3a, c and d). In contrast, for non-stable dyons the trajectories do not close but wander off to infinity (cf., Fig.3e). Viewing the world-brane theory in Hamiltonian formulation, such trajectories correspond to infinite time and do not represent physical BPS states.

On the other hand, we expect the situation to be quite different when u is on or inside of the curve of marginal stability [1,32]. Obviously, on this curve where $a_D(u)/a(u)$ is real, the Jacobian lattice degenerates, so that for all (g,q) the trajectories are on top of each other (looping through $e_1^\pm$). This means in particular that the closed trajectory of the gauge boson $(0,1)$ cannot be distinguished from the trajectory of the dyon $(1,1)$ from e_1^- to e_1^+ plus the trajectory of the monopole $(-1,0)$ from e_1^+ to e_1^-. That is, the string representation of the Yang-Mills BPS states degenerates for real $a_D(u)/a(u)$, and we see the "decay" of the gauge boson (and other BPS dyons) into the monopole/dyon pair in a very simple and direct way. We thus have, in fact, mapped the jumping phenomenon in four dimensions [1] back to two dimensions [33].

Inside of the curve of marginal stability the spectrum of BPS states will be quite different. This can be easily seen from the possible trajectories for $u=0$. We parametrize $z(t)=e^{i\theta(t)}$ to rewrite (4.1) as $\sqrt{2}\int\sqrt{\cos\theta}\,d\theta=-\alpha t$; only for α real or purely imaginary one can have a real solution[8] for $\theta(t)$, which means that $z(t)$ runs with some parametrization along the unit circle. In fact, one obtains a semi-circular trajectory running from $e_1^-=-i$ to $e_1^+=i$ that is associated with the monopole with charges $(1,0)$, and, by symmetry, another semi-circle associated with the dyon of type $(1,1)$, cf., Fig.3f. This confirms the statements about the BPS spectrum from consistency [1,32] and symmetry [6] considerations.

[8] For $u=0$, eq. (4.1) can easily be integrated in terms of standard elliptic functions, see e.g. [34].

5. Outlook

Note how easy it is to make non-perturbative statements about $N = 2$ gauge theory by using a "dual" string formulation, in which gauge bosons and monopoles are treated on equal footing! With ordinary field theoretic methods, statements about the stability of quantum BPS states are much harder to derive; see for example Sen's work [35] on $N = 4$ Yang-Mills theory, or the highly non-trivial computation of BPS states with magnetic charge 2 in some $N = 2$ systems [36,37]. Obviously, many interesting questions can now be very directly addressed, like for example the appearance and decay of BPS states in theories with extra matter multiplets.

On the more abstract level, there is a known connection with integrable field theories [13,14,38]. As remarked above, the analysis of the BPS states *crucially* depends on using precisely $\lambda = xdz/z$, without modifications by exact pieces. This particular form of the differential is very natural in Toda theory: it is the Hamilton-Jacobi function of the system. It thus seems possible that the $\Sigma \times \mathbb{R}^4$ world-brane dynamics of the five-brane can be described in terms of an integrable Toda theory. At any rate, we have seen that for the Yang-Mills theory the existence of BPS geodesics corresponds to the existence of semi-classical "states" in the complexified Toda theory. Thus we are finding a much more direct connection between integrable theories and $N = 2$ supersymmetric QCD.

More generally, we are once again discovering that string theory is not only an intrinsically interesting subject, but that it can give us new insights into fundamental issues in field theory.

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Note: As we were finishing this paper, we obtained a pre-release draft [39] that addresses related issues.

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